STATISTICS OF THE VISITS TO LAMINAR SETS IN CHAOTIC SYSTEMS PRELIMINARY VERSION 20 DEC19

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Abstract.

1. Introduction

In this paper we will consider a diffeomorphism T with singularities, defined on a multidimensional manifold M^* with singularities. It preserves a probability measure μ defined on the Borelian sets of M^*. The system (M^*, T, μ) is mixing. The set M^* is divided in a "fine" set M and a "bad" set B. Let F be the map induced by T on M; it preserves the measure $\nu = \frac{\mu}{\mu(M)}$. We will suppose the system (M, T, ν) satisfies exponential decay of correlations (see C2 in section 2) and B can be thought as a region where the dynamical system is regular, laminar, not so chaotic. Let $I_K = \{x \in M : T^i(x) \in B, i = 1, ..., k \geq K\}$ and we suppose that $\nu(I_K) \cong K^{-\alpha}$, $\alpha > 1$.

We study the visit to sets directly related with dynamics of higher dimension hyperbolic systems that have regions with strong mixing properties and regions with laminar evolutions. The present paper follows closely a paper by P. Collet and A. Galves [10] on one dimensional expanding systems with indifferent points. Our definitions will follow their notation.

For each $x \in M$, let be $f^1_K(x) = \inf\{n \geq 0 : T^n(x) \in I_K\} + 1$; this is the number of steps the trajectory starting from x takes to enter in B and staying there for at least K iterations.

We will estimate the asymptotic statistics of $f^1_K(x)$. The distribution we get is exponential for some scale-normalizer $\beta_K > 0$; this means that the process $\beta_K^{-1}(f^1_K)$ loose memory when K increases. In a precise formulation (see Theorems 3.1, 3.2) one of the results is

$$\lim_{K\to\infty} \nu\{f^1_K > \beta_K\, t\} = e^{-t}.$$

The main motivation for our work were billiards. See [12] for a related work in billiards. The general framework of the present work are inspired in previous works on this prototypical kind of systems with singularities. As applications of our abstract results we consider these cases: a) Bunimovich stadium; b) Bunimovich-type billiards with non parallel straight lines [7]; c) *tanga billiards* (dispersing billiards with cusps) studied by Chernov and Markarian [4]; d) billiards studied by Chernov and Zhang [5, 6, 9].

Stadium. In order to understand the main motivation and methods it is very important to have in mind the studies of ergodicity and decay of correlations of the Bunimovich stadium in Chapter 8 of the book [3].

A stadium is a table bounded by two line segments tangential to two circular arcs. Bunimovich's straight stadium is made by two equal semicircles and two parallel lines. It has C^1 but not C^2, boundary. It remains chaotic no matter how short the parallel segments are; but if they vanish, the stadium turns into a disk where the billiards is completely integrable. Thus one gets continuous families of billiard tables where a transition from a completely regular region to total chaos occurs instantly.

Let $\mathcal{D} \subset \mathbb{R}^2$ be a stadium. Its boundary can be decomposed as $\partial\mathcal{D} = \partial^0\mathcal{D} \cup \partial^-\mathcal{D}$ where $\partial^-\mathcal{D} = \Gamma_1 \cup \Gamma_2$ denotes the union of two arcs, and $\partial^0\mathcal{D} = \Gamma_3 \cup \Gamma_4$ consists of straight sides of $\mathcal{D}$. Then $M^* = \partial\mathcal{D} \times [-\pi/2, \pi/2]$, the collision space is the standard cross-section of the billiard dynamics. Canonical coordinates on M^* are r and ϕ, where r is the arc length parameter on ∂D and $\phi \in [-\pi/2, \pi/2]$ is the angle of reflection of the outcoming trajectory with the normal to the boundary. The first return map $T : M^* \to M^*$ is called the collision map or the billiard map, it preserves smooth measure $d\mu = \cos\phi dr d\phi$ on M^*. The collision space can be naturally divided into focussing and neutral parts: $M^*_- = \{(r, \phi) : r \in \partial^-\mathcal{D}\}$, $M^*_0 = \{(r, \phi) : r \in \partial^0\mathcal{D}\}$.

$M \subset M^*_-$ consists of only first collisions at the semicircles. It is formed by two parallelograms corresponding to both half circular arcs. One of them has vertices in $(A, \pi/2), (B, 0), B * -\pi/2), (A, 0)$ where the first coordinate indicates the

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endpoints of a circular arc. See Section 8.4 in [3]. The bad set B is the union of M_0^* and the part of M_-^* corresponding to successive trajectories on the circular arcs.

The sets I_K in the stadium are bounded by the singularity lines of F, close to the vertices E and C (and its symetric points B and D in Figures 8.11, 8.13 and 8.15 in [3]. There ν measures are respectively $\cong K^{-3}$ and $\cong K^{-2}$. The reason of this difference is that close to E the density $\cos\phi \cong 1/K$. For more details, see [7].

The billiard return map $F : M \to M$ is mixing and has exponential decay of correlations. It corresponds to $\mathcal{F}_{\clubsuit} : \mathcal{M}_{\clubsuit} \to \mathcal{M}_{\clubsuit}$ in Section 8.15 of [3].

Recently some results were obtained on the convergence to the Poisson law for different systems related with billiard systems. In all of them the visit to small balls is considered. We refer to three of these recent papers: [1], [13] and [14].

2. Basic definitions. Decay of correlations. Stopping times

Let M' be a d-dimensional compact C^∞-Riemannian manifold, perhaps with boundary and corners, $S_1 \subset M'$ be the union of C^1-compact submanifolds of positive co-dimension. that includes the boundaries of M'. Then $M^* = M' \setminus S_1$ is an open and dense subset of M'. We assume that $T : M^* \to M'$ is a $C^r, r \geq 1$, diffeomorphism of M^* onto $T(M^*)$. All the iterations of T are defined on $N = \cap_{n=-\infty}^{\infty} T^n(M^*)$. Assume that T preserves a probability non atomic measure μ on M' and $\mu(N) = 1$. M^* is partitioned in two measurable sets B and M. The set M is a region with good mixing properties. In billiard systems (that motivated this work) B is the set where the expansion on unstable manifolds is close to 1.

C1. Denote $I_K = \{x \in M : T^i(x) \in B, i = 1, ..., k \geq K\}$. Observe that $I_{k+1} \subset I_K$ and the interior of $I_K \setminus I_{K+1}, K \geq 1$ are disjoint sets. We assume that $\mu(I_K) \cong K^{-\alpha}$ for some $\alpha > 1$ ($a \cong b$ means that there exists a constant $D > 1$ such that $D^{-1} < a/b < D$.)

For each $x \in M^*$ let be

$$U(x) = \inf\{i \geq 1 \,:\, T^i(x) \in M\}.$$

In fact, if $x \in M \setminus I_1$ then $U(x) = 1$; if $x \in I_1$, then $U(x)$ is one plus the number of iterations $i \geq 1$, such that, $T^i(x)$ is not in M before returning to it; if $x \in B$, then $U(x)$ is exactly the number of 'bad' iterations before leaving B for the first time $(x, Tx, \cdots T^{U(x)-1}x \in B, T^{U(x)}x \in M$. In particular, $I_{K-1} = \{x \in M : U(x) > K\}$ and $\nu\{U > K\} = C(K-1)^{-\alpha}$.

C2. We assume that M is an open subset such that, $\overline{M}$ is compact, and let $\Gamma \subset \overline{M}$ be the union of C^1-compact submanifolds of positive co-dimension that includes the boundaries of the sets I_K and S_1; $M^0 = M \setminus \Gamma$.

We define the induced first return transformation by $F(x) = T^{U(x)}$. It preserves the measure $\nu = \mu/\mu(M)$. $F : M\setminus\Gamma \to M$ is a $C^r, r \geq 1$, diffeomorphism of M^0 onto its image. The set Γ will be referred to as the singularity set for F. $\cap_{n=-\infty}^{\infty} T^n(M^0)$ is the set of points where all the iterations by F are defined.

We will assume that $\int \log^+ ||D_x F^{\pm 1}|| d\nu(x) < \infty$ (Oseledets condition) and that F is fully hyperbolic in terms of fields of cones: there exist two families of cones C_x^u, C_x^s in the tangent space $T_x M, x \in \overline{M}$, such that, $DF(C_x^u) \subset C_{F(x)}^u$, $C_{F(x)}^s \subset DF(C_x^s)$, whenever DF exists, and there exists $\Lambda > 1$, such that,

$$|DF(v)|_p \geq \Lambda\, |v|_p,\ \forall v \in C_x^u \text{ and } |DF^{-1}(u)|_p \geq \Lambda\, |v|_p,\ \forall u \in C_x^s,$$

where $|\,.\,|_p$ is a pseudo metric which conveniently describes expansions and contractions of tangent vectors. Its definition depends of the specific dynamical system under consideration.

These families of cones are continuous in connected components of $\overline{M}$, the angles between C_x^u and C_x^s are bounded away from zero and the sum of the dimensions of their axes is d=dim M. As a consequence Lyapunov exponents are non zero on M^0 and local stable (LUM) and stable (LSM) manifolds are defined ν-almost everywhere in M.

We will suppose that the dynamical system (M, ν, F) is mixing and satisfies exponential decay of correlations: for every pair of dynamically Hölder continuous function $f, g \in \mathcal{H}$ and $n \geq 0$

$$\left| \int f(g \circ F^n) d\nu - \int f\, d\nu \int g d\nu \right| \leq B_{f,g}\, \theta_{f,g}^n, \tag{1}$$

where $\theta_{f,g} < 1$, $B_{f,g} = C_0(K_f||g||_\infty + K_g||f||_\infty + ||f||_\infty||g||_\infty)$, and C_0 is a constant. See Sections 7.5 - 7.7 in [3] for definitions of the constants.

Remark 2.1. We will always assume that f and g are piecewise Hölder continuous with singularities included in Γ. For example, in billiard systems the free path between successive reflections is one such function. See [4] , page 730, and Remark 8.49 in [3]. If f and g are the characteristic functions of sets bounded by singularity manifolds of F then, $B = B_{f,g}, \theta = \theta_{f,g}$ are constants not depending on the sets.

C4. We will distinguish the iteration of each $x \in M$ by T in two classes: roughly speaking they will be "bad", if $Tx \in B$, and "fine", otherwise.

Definition 2.2. For each $x \in M$, let be

$$f_K^1(x) = \inf\{n \geq 0 \,:\, T^n(x) \in I_K\} + 1.$$

This is the number of steps the trajectory starting from $x \in M$ takes to enter "profoundly" for the first time in the "bad" region B (after entering it will take more than K iterations to return to $M \,:\, T^{f_K^1(x)+h} \in B$ for $h = 0, 1, \cdots s$ for some finite $s \geq K$). The sequence of successive entrance times in I_K is defined by

$$f_K^2(x) = \inf\{n \geq f_K^1(x) + U(T^{f_K^1(x)}x) \;;\; T^n x \in I_K\} + 1,$$

and, inductively, for $j > 1$,

$$f_K^j(x) = \inf\{n \geq f_K^{j-1}(x) + U(T^{f_K^{j-1}(x)}x) \;;\; T^n x \in I_K\} + 1,$$

We remark that not all iterations $T^i x$, where $f_K^{j-1} < i < f_K^j$, are in M; there can be "short" (less than K) periods in B.

Definition 2.3. For $x \in M, K \geq 1$ we denote by $\tau_K^j(x), j \geq 1$ the time of the j-visit to the interval $U^{-1}\{[K, \infty)\}$ of the orbit by x under the induced map F.

$$\tau_K^1(x) = \inf\{n \geq 0 \,:\, U(F^n x) \geq K\},$$

and

$$\tau_K^j(x) = \inf\{n > \tau_K^{j-1} \,:\, U(F^n x) \geq K\}, j > 1.$$

All these functions are stopping times; they are finite almost everywhere (with respect to the ergodic measure ν).It results that up to the j-visit the "fine" iteration period has a total length

$$f_K^j(x) = 1 + \sum_{n=0}^{\tau_K^j(x)-1} (U \circ F^n(x)). \tag{2}$$

It is also very useful to observe that $\nu\{\tau_K^1 < z\} \leq z\,\nu\{U \geq K\}$. See Lemma 3.3 (i).

We remark that neither τ_K^j nor f_K^j discriminate points in M or B, but do it with points that are (or, will be) profoundly in B (more than K times). The required discrimination is done in the following way:

Definition 2.4. The random variable $N_t(x)$, $x \in M$, counts the number of returns (iterations by F) of the path starting at x to the set M until t iterations by T:

$$N_t(x) = \sup\{j \geq 0 \,:\, \sum_{i=0}^{j} (U \circ F^i)\, x \;\leq t\}.$$

Obviously, $N_t(x) \leq t$, for any $x \in M$, $t \geq 1$. $t - N_t(x) + 1$ is the number of iterations that $x, Tx, T^2x, \cdots, T^t x$ stays in M.

Definition 2.5. We define the "power law scale of time" by

$$\beta_K = \min\{n \in \mathbb{N} \,:\, \nu\{f_K^1 \geq n\} \leq e^{-1}\}.$$

As a consequence of the assumed ergodicity of T the value β_K is finite for each K.

3. Statistics of visits to the laminar set. Main results. Preliminay estimates

In this Section we state the main results and some estimates that are necessary for their proofs. They will be given in the next Section. We would like to point out that we will consider a bidimensional dynamical system and not one-dimensional systems. We also mention that the measure considered in [10] is σ-finite but our ν is finite (in fact we will consider it normalized as a probabilty measure). The mixing rate in our case is much more better.

We will estimate the asymptotic statistics of $f_K^j(x)$. Our result claims that in the scale of time $\beta_K\ t$ the distribution of f_K^1 is almost exponential when K is large; this means that the process $\beta_K^{-1} f_K^1$ loose memory when K increases.

Theorem 3.1. *For any positive real number t the following limit holds*

$$\lim_{K\to\infty} \nu\{\beta_K^{-1} f_K^1 \,>\, t\} = e^{-t}.$$

Moreover, β_K satisfies

$$\lim_{K\to\infty} \beta_K^{-1} \int f_K^1 \, d\nu = 1$$

The value β_K is a kind of scale-normalizer to get an exponential process with distribution e^{-t}. The Theorem says that the time needed to perform the first visit to the interval $[K,\infty)$ is much more longer that the typical mixing times which is needed to loose memory from the initial condition in M.

Theorem 3.2. *For any positive integer n and any sequence of positive real numbers $s_1, s_2, ...$ the following limit holds*

$$\lim_{K\to\infty} \mu\{f_K^1 > \beta_K s_1, f_K^2 - f_K^1 > \beta_K s_2, ..., f_K^n - f_K^{n-1} > \beta_K s_n\} = \prod_{j=1}^{n} e^{-s_j}.$$

Therefore, for large values of K, every unsuccessful trial to overrun level K after the process starts is approximately like a new run from the origin.

In the proof of these theorems we will need some important estimates. Their statements and proofs will occupy the last part of this Secion.

Lemma 3.3. *(i) For any positive integer k, we get*

$$\nu\{\tau_K^1 \le k\} \le k\ \nu\{U \ge K\};$$

(ii) There exists a constant $C > 0$, such that,

$$C^{-1} \le \nu\{U \ge K\} \int \tau_K^1 \, d\nu \le C.$$

Proof: (i) From the definition of τ_K^1 we have that

$$I_{\{\tau_K^1 \le k\}} = I_{\{U\ge K\}} + \sum_{n=1}^{k} I_{\{U\ge K\}} \circ F^n \,(\Pi_{l=0}^{n-1} I_{\{U<K\}} \circ F^l) \le I_{\{U\ge K\}} + \sum_{n=1}^{k} I_{\{U\ge K\}} \circ F^n, \tag{3}$$

The result follows from the integration of this formula with respect to ν and the invariance of ν by F.

(ii) If in (3) we substitute k by ∞, integrating with respect to ν we obtain

$$1 = \nu\{\tau_K^1 < \infty\} = \nu\{U \ge K\} + \sum_{n=1}^{\infty} \int I_{\{U\ge K\}} \circ F^n \,(\Pi_{l=0}^{n-1} I_{\{U<K\}} \circ F^l) \, d\nu.$$

As a consequence of (1), each term in the sum of the right hand side is bounded above) by

$$\int I_{\{U\ge K\}} \int I_{\{\tau_K^1 = n\}} + B\theta^n.$$

Summing up in n and making a similiar computation for the lower bound, we obtain

$$C^{-1} \le \nu\{U \ge K\} \sum_{n=0}^{\infty} \nu\{\tau_K^1 = n\} \le C.$$

Since $\int \tau_K^1 d\nu = \sum_{n=0}^{\infty} \nu\{\tau_K^1 = n\}$ we get (ii). □

Corollary 3.4. *(i) For any positive integer k, we get*

$$\nu\{\tau_K^1 \le k\} \le Ck(K-1)^{-\alpha};$$

(ii)

$$\nu\{\tau_K^1 \ge k\} \le \frac{C}{k\ \nu\{U \ge K\}} \cong \frac{C(K-1)^{\alpha}}{k}.$$

Proof. (i) follows from (i) in the previous Lemma and estimates in **C1** of Section 1. (ii) Use Markov inequality for any random variable X ($P(X \ge \lambda) \le EX^t/\lambda^t$) and (ii) of the previous Lemma. □

Lemma 3.5. *If $L(K) \cong K^{\frac{\alpha+1}{2}}$, then,*

$$\limsup_{K\to\infty, \bar{s}\ge 0} \nu\{\bar{s} \le f_K^1 \le \bar{s} + L(K)\} = 0.$$

Proof: Note that $\lim_{K\to\infty} L(K)K^{-\alpha} = 0$.

The proof is based in the following elementary observation:

$$\{\bar{s} \le f_K^1 \le \bar{s} + L(K)\} \subset \cup_{j=s}^{\bar{s}+L} T^{-j}(I_K) \cap M.$$

Then, as $\alpha > 1$ we get that $\nu\{\bar{s} \le f_K^1 \le \bar{s} + L(K)\} \le L(K)\,\nu(I_K) \le$ constant $K^{\frac{1-\alpha}{2}} \to 0$, if $K \to \infty$. □

Corollary 3.6. $\lim_{K\to\infty} \beta_K/K = \infty$.

Proof. By definition $\nu\{x : f_K^1(x) < \beta_K\} > 1 - e^{-1}$. As a consequence of Lemma 3.5 (with $\bar{s} = 0$), for K large enough, we have that $L(K) \le \beta_K$. Then, $\beta_K/K \ge L(K)/K >$ constant $K^{(\alpha-1)/2}$. □

4. Proofs of the main results

In this section we will present the proofs of our main theorems. We will follow closely the proofs of [10] adapting their methods to our system (M, ν, F)

Proof of theorem 3.1

We have to prove that the function $G_K(t)$ given by

$$G_K(t) = \nu\{f_K^1\, \beta_K^{-1} \geq t\},$$

satisfies the factorization property

$$\lim_{K\to\infty}(G_K(t+s) - G_k(t)\, G_K(s)) = 0.$$

Indeed, among the distribution functions this factorization property is satisfied just by the exponential function (or by the trivial functions 0 or 1). We will explain more carefully this point later.

First, we introduce some notation. For K, t fixed, we define the partition $A_k = A_k(t, K) = \{x : N_{\beta_k\, t}(x) = k\}$. he number of elemets of the partition is finite because $k \leq \beta_K\, t$. Define $B(K) = \{x : \tau_K^1(x) \leq m+1\}$. Finally observe that $m = m(K) = (L(K)/K)^{1/2}$ goes to ∞, as $K \to \infty$ and that for large K, $\beta_K - mK >\ K^{(\alpha+1)/2} - K^{(\alpha+3)/4}$ that also goes to infinity with K; then $\beta_K - mK \cong \beta_K$.

First we will show

$$\lim_{K\to\infty}(G_K(t+s) - G_k(t)\, G_K(s)) \geq 0.$$

We begin by verifying the following inclusion: for each $k, t, s, K > 0$

$$\{f_K^1(x) \geq \beta_K(s+t)\} \cap A_k \supset \{f_K^1 \geq \beta_K\, t\} \cap \{f_K^1 \circ F^{m+k} \geq \beta_K s - m\} \cap \{F^{-k}(B(K)^c)\} \cap A_k. \tag{4}$$

It results from the following observations (the value of m is not essential for the following computation):

a) if $x \in \{F^{-k}(B(K)^c)\}$ then $\tau_K^1 \circ F^k(x) > m+1$. This means that $T^k x$ 'must wait' more than $m+1$ iterations by F for being in I_K; but as it is also in A_k and being $N_{\beta_K t}$ a supreme, $F^{k+m}x$ will not yet be in I_K.

b) As f_K^1 counts iterates of T, if

$$x \in \{f_K^1 \geq \beta_K\, t\} \cap A_k \cap \{F^{-k}(B(K)^c)\} \cap F^{-m-k}\{f_K^1 \circ F^{m+k} \geq \beta_K s - m\}$$

then $f_K^1(x) \geq f_K^1 \circ F^{m+k}(x) + \beta_K t + m$.

c) Finally, if x is in the set at the right hand of (4) we have

$$f_K^1(x) \geq f_K^1(F^{m+k}(x)) + \beta_K\, t + m \geq \beta_K s - m + \beta_K\, t + m = \beta_K(t+s).$$

Summing up the measures of the sets in (4) we get

$$G_K(s+t) \geq \sum_k [\, \nu\,(\{f_K^1 \geq \beta_K t\} \cap A_k \cap \{f_K^1 \circ F^{m+k} \geq \beta_K s - m\}\,)\ -\ \nu(B(K) \cap A_k)\,].$$

Next we prove

$$\lim_{K\to\infty}\sum_k \nu(B(K) \cap A_k) = \lim_{K\to\infty}\nu(B(K)) = 0. \tag{5}$$

Indeed, from Lemma 3.3 (i), the definition of $m = m(K)$ and the value of $L(K)$ in Lemma 3.5, we have

$$\nu(B(K)) = \nu(\{\tau_K^1(x) \leq m+1\}) \leq (m+1)\ \nu\{U \geq K\} \leq$$

$$\text{const.}(L(K)/K)^{1/2}\, K^{-\alpha} \leq \text{const.}K^{-(3\alpha+1)/4}.$$

Using the exponential decay of correlations of the induced transformation F (**C2** with $n = m+k$) we have, for any fixed m, K, t, s, and each fixed $k \leq \beta_K t$,

$$\left|\nu\left(\left[\{f_K^1 \geq \beta_K t\} \cap A_k\right] \cap \{f_K^1 \circ F^{m+k} \geq \beta_K s - m\}\right) - \nu(\{f_K^1 \geq \beta_K t\} \cap A_k)\ \ \nu\{f_K^1 \geq \beta_K s - m\}\right| \leq B\theta^{m+k}$$

Summing on k (for fixed K, t, the partition A_k is finite since $k \leq \beta_K t$.) we obtain

$$\sum_k \nu\,(\,\{f_K^1) \geq \beta_K t\} \cap A_k \cap \{f_K^1 \circ F^{m+k} \geq \beta_K s - m\}\,) \geq$$

$$\sum_k \left[\nu\,(\,\{f_K^1 \geq \beta_K t\} \cap A_k)\, \nu\{f_K^1 \geq \beta_K s - m\}\ - B\theta^{m+k}\right]\ \ \geq$$

$$\nu\{f_K^1 \geq \beta_K s - m\} \sum_k \nu\,(\,\{f_K^1 \geq \beta_K t\} \cap A_k) - C\theta^m \geq \nu(\{f_K^1 \geq \beta_K s\})\, \nu\,(\,\{f_K^1 \geq \beta_K t\}) - C\theta^{m+1}.$$

When $K \to \infty$, then $m = m(K) \to \infty$; therefore,

$$\lim_{K\to\infty} \sum_k \nu\,(\,\{f_K^1) \geq \beta_K t\} \cap A_k \cap \{f_K^1 \circ F^{m+k} \geq \beta_K s - m\}\,) \geq G_K(t) G_K(s).$$

This finish the proof of

$$\lim_{K\to\infty} (G_K(t+s) - G_K(t)G_K(s)) \;\geq\; 0. \tag{6}$$

Now we will prove the inequality $\lim_{K\to\infty}(G_K(t+s) - G_K(t)G_K(s)) \;\leq\; 0$ which will require the use of Lemma 3.5.

Once again we begin with an inclusion that can be proved as the previous one (we will not give the details): for each fixed $k, t, s, K, m > 0$ we have

$$\{f_K^1 \geq \beta_K(s+t)\} \cap A_k \subset A_k \,\cap\, \left[(\,\{f_K^1 \geq \beta_K t\} \cap \{f_K^1 \circ F^{m+k} \geq \beta_K s - (m+1)\,K\} \cup F^{-k}(B(K))\right] \tag{7}$$

Similarly to the previous case, summing up the measures of this sets, and using the exponential decay of correlations, we obtain $G_K(t+s) \leq$

$$\sum_k \nu\,(\,A_k \cap \{f_K^1) \geq \beta_K t\} \cap \{f_K^1 \circ F^{(m+k)} \geq \beta_K s - (m+1)K\}\,) + \sum_k \nu(F^{-k}(B(K) \cap A_k) \leq$$

$$\sum_k \,\nu(\{A_k \cap f_K^1 \geq \beta_K t\})\;\nu\{f_K^1 \geq \beta_K s - (m+1)K\}\,) + C\theta^m + \sum_k \nu(F^{-k}(B(K) \cap A_k)$$

The second sum goes to zero as K goes to infinity, as in the previous computation. Consider now the first sum, whose term not depending on k can be decomposed in the following way

$$\nu\{f_K^1 \geq \beta_K s - (m+1)K\} = \nu\{\beta_K s - (m+1)K \leq f_K^1 < \beta_K s\} + \nu\{f_K^1 \geq \beta_K s\}.$$

Since $m = m(K) = (L(K)/K)^{1/2} \to \infty$, then $(m+1)\,K < L(K)$, and applying Lemma 3.5 with $\bar{s} = \beta_K s - (m+1)K$, we obtain that

$$\nu\{\beta_K s - (m+1)K \leq f_K^1 < \beta_K s\} = \nu\{\bar{s} \leq f_K^1 < \bar{s} + (m+1)K\} \leq \nu\{\bar{s} \leq f_K^1 < \bar{s} + L(K)\}$$

goes to zero. Therefore,

$$\lim_{K\to\infty} \sum_k (\,\nu(\{f_K^1 \geq \beta_K t\} \cap A_k)\;\;\nu\,\{f_K^1 \geq \beta_K s - (m+1)K\}\,) =$$

$$\lim_{K\to\infty} \sum_k (\,\nu(\{f_K^1 \geq \beta_K t\} \cap A_k)\;\;\nu\,\{f_K^1 \geq \beta_K s\}\,) = \lim_{K\to\infty} G_K(t) G_K(s).$$

PROOF THAT $\sum_k \nu(F^{-k} B(K))$ GOES TO ZERO AS $K \to \infty$. Since the measure ν preserves F, we have that $\sum_k \nu(F^{-k}B(K)) = \sum_k \nu(B(K))$ It was previously proved -see (5)- that $\nu(B(K)) \leq \text{const.} K^{-(3\alpha+1)/4}$ goes to zero as $K \to \infty$. $k \leq \beta_{Kt},\;\; \beta_{Kt} > \text{const.}\; K^{\frac{\alpha+1}{2}}$ (proof of Lemma 3.5).

This finish the proof of

$$\lim_{K\to\infty} (G_K(t+s) - G_K(t)G_K(s)) \;\leq\; 0 \qquad \text{and} \qquad \lim_{K\to\infty} (G_K(t+s) - G_K(t)G_K(s)) \;=\; 0.$$

Now we claim that

$$\lim_{K\to\infty} G_K(1) = \lim_{K\to\infty} \nu\{f_k^1 \geq \beta_K\} = e^{-1}.$$

This is so because on one hand, by definition of β_K, $\nu\{f_k^1 \geq \beta_K - 1\} > e^{-1} \geq \nu\{f_K^1 \geq \beta_K\}$, and on the other hand $\lim_{K\to\infty} \nu\{f_k^1 \geq \beta_K\} \geq e^{-1}$ because $\lim_{K\to\infty} \nu\{\beta_K - 1 \leq f_k^1 < \beta_K\} = 0$ (see the proof of Lemma 3.5). This finish the proof of the claim and the first statement of Theorem 3.1.

Now, we will prove the second part of Theorem 3.1: $\lim_{K\to\infty} \beta_K^{-1} \int f_K^1 d\nu = 1$. The proof is similar to the one in [10], p.473. We give some more details. Let be $H_u = \{x : f_K^1 \geq u\}$; then, making the change of variable $u = t\beta_K$ we have

$$\int_0^\infty G_K(t)dt = \int_0^\infty \nu\{H_{t\beta_K}\} = \int_0^\infty \int_M I_{H_{t\beta_K}} d\nu dt = \beta_K^{-1} \int_0^\infty \int_M I_{H_u} d\nu du.$$

Then, applying a version of Fubini's Theorem and considering the function defined on the positive real line, that is one for $0 \leq u \leq f_K^1(x)$ and zero otherwise, we obtain that the last term equals

$$\beta_K^{-1} \int_M \int_0^\infty I_{H_u} du d\nu = \beta_K^{-1} \int_M f_K^1 d\nu.$$

Since $G_K(t) \to e^{-t}$ as $K \to \infty$, the result will follow from Lebesgue dominated theorem if wew prove that $G_K(t)$ is uniformly bounded on K for all $t > 1$.

Using inequality (6) with $s = 1$ and t substituted by $t-1$, we get

$$G_K(t) \leq G_K(t-1)\nu\{f_K^1 > \beta_K) \geq G_K(t-1)\nu\{f_K^1 > \beta_K - (m+1)K\}.$$

We know that when $K \to \infty$ the term $\nu\{f_K^1 > \beta_K - (m+1)K\}$ goes to e^{-1}. Consider $\lambda < 1$ such that $\nu\{f_K^1 > \beta_K - (m+1)K\} < \lambda$ for K large. Then $G_K(t) \leq G_K(1)\lambda^{[t]}$, uniformly on t for such large K and the result follows.

This is the end of the proof of Theorem 3.1. □

Proof of theorem 3.2

The proof is by induction.

The case $n = 1$ was proven in Theorem 3.1.

Now we will use the induction hypothesis: the result holds up to the integer n. Let be

$$D_n = \{f_K^1 > \beta_K s_1, f_K^2 - f_K^1 > \beta_K s_2, \ \cdots \ , f_K^n - f_K^{n-1} > \beta_K s_n\}.$$

Using formula (2) we obtain that

$$f_K^{n+1}x - f_K^n = \sum_{j=\tau_K^n}^{\tau_K^{n+1}-1} U(F^j x) = U(F^{\tau_K^n} x) + \sum_{k=0}^{\tau_K^{n+1}-\tau_K^n-2} U(F^k(F^{\tau_K^n+1}(x)).$$

Also, from Definition 2.3 we obtain $\tau_K^1(F^{\tau_K^1}x) = \tau_K^{n+1}(x) - \tau_K^n(x) - 1$, and using once again formula (2) we obtain that

$$\sum_{k=0}^{\tau_K^{n+1}-\tau_K^n-2} U(F^k(F^{\tau_K^n+1}(x)) = f_K^1(F^{\tau_K^n+1}x) - 1$$

and

$$D_{n+1} = D_n \cap \{x : f_K^1(F^{\tau_K^n+1}x) + U(F^{\tau_K^n}x) > \beta_K s_{n+1} + 1\}.$$

a) We will prove first un upper bound for the measure of D_{n+1}.

The set D_{n+1} is contained in the union of

$$D_n \cap \{x : (U(F^{\tau_K^n}x) \leq mK\} \cap \{x : \ f_K^1(F^{\tau_K^n+1}x) > \beta_K s_{n+1} - mK\} \tag{8}$$

and $\{x : U(F^{\tau_K^n}x) > mK\}$, for the value of m fixed at the begining of the proof of Theorem 3.1 ($\beta_K s_{n+1} - 2mK > 0$).

We will obtain un upper bound for the measure of the set in (8). By definition $\tau_K^1 < m$ implies $U(F^j x) < K$ for $j = 0, \cdots \tau_K^1(x)$ and so $f_K^1(x) = 1 + \sum_{n=0}^{\tau_K^1-1} U(F^n x) < 1 + mK$. Then

$$\{x : \ \tau_K^1(F^{\tau_K^n+1}(x)) < m\} \subset \{x : f_K^1(F^{\tau_K^n+1}(x)) < mK\}.$$

For large enough K we obtain $mK < \beta_K s_{n+1} - mK$ and we conclude that the set in (8) is contained in

$$D_n \cap \{x : (U(F^{\tau_K^n}x) \leq mK\} \cap \{x : \ f_K^1(F^{\tau_K^n+m}x) > \beta_K s_{n+1} - 2mK\}.$$

Then, using formula (1) we obtain that the measure of the last set is

$$\leq \nu(D_n \cap \{x : (U(F^{\tau_K^n}x) \leq mK\})\nu(\{x : \ f_K^1 x) > \beta_K s_{n+1} - 2mK\}) + B\theta^{\tau_K^n+m} \leq$$

$$\nu(D_n)\nu(\{x : \ f_K^1 x) > \beta_K s_{n+1} - 2mK\} + B\theta^{\tau_K^n+m}.$$

Using the induction hypothesis, Theorem 3.1 and the remark after the definition of m at the begining of the proof of Theorem 3.1 we obtain that the last number is bounded above by $\prod_{j=1}^{n+1} e^{-s_j}$.

We now prove that the remaining term in the union that contains D_{n+1} converges to 0 as K goes to ∞. Note that

$$\nu(\{x : (U(F^{\tau_K^n}x) \leq mK\}) = \mu(I_{mK-1})/\mu(M) \cong (mK-1)^{-\alpha}/\mu(M). \tag{9}$$

that goes to zero because mK goes to ∞ with K.

b) We now find a lower bound for the measure of D_{n+1}. Since by definition $U \circ F^{\tau_K^n(x)}(x) \geq K$ we have that D_{n+1} contains the set

$$D_n \cap \{f_K^1 \circ F^{\tau_K^n+1} > \beta_K s_{n+1} - K\}.$$

The following set is even smaller:

$$D = D_n \cap \{f_K^1 \circ F^{\tau_K^n+m} > \beta_K s_{n+1} - K\} \cap \{\tau_K^1 \circ F^{\tau_K^n+1} > m\}.$$

Then,

$$\nu(D) \geq \nu(D_n \cap \{f_K^1 \circ F^{\tau_K^n+m} > \beta_K s_{n+1} - K\}) - \nu\{\tau_K^1 \circ F^{\tau_K^n+1} \leq m\}.$$

Usingformula (2) we obtain that

$$\nu(D_n \cap \{f_K^1 \circ F^{\tau_K^n+m} > \beta_K s_{n+1} - K\}) \geq \nu(D_n)\nu\{f_K^1 > \beta_K s_{n+1} - K\} - B\theta^{\tau_K^n+m}.$$

Then, since $K/\beta_K \to 0$, Theorem 3.1 and the induction hypothesis imply that the last term converges to $\prod_{j=1}^{n+1} e^{-s_j}$, as K and m go to ∞.

On the other hand, Lemma 3.3(i) implies

$$\nu\{\tau_K^1 \circ F^{\tau_K^n+1} \leq m\} \leq m\,\nu\{U \geq K\} \cong mK^{-\alpha} \to 0,$$

as $K \to \infty$.

This concludes the proof of Theorem 3.2. □

The Main Theorem in the paper by Collet and Galves is a nonequilibirum result: the sequence of normalized stopping times $f_K^j \beta_K^{-1}$ converges in law to a mean-one homogeneous Poisson point process on R^+, when $K \to \infty$,. Its proof is presented with full details at the end of [10].

It includes the following result: for any $g : M \to R^+$ of bounded variation and $\int d\lambda = 1$, we have

$$\lim_{K\to\infty} \left(\nu(D_n) - \int I_{D_n}(x) g(x) \right) d\lambda = 0, \tag{10}$$

where λ is the Lebesgue measure on M.

References

[1] J.-R. Chazottes and P. Collet, Poisson approximation for the number of visits to balls in non-uniformly hyperbolic dynamical systems. Ergod. Th. & Dynam. Sys. 24 (2013) 33, 49–80
[2] N. I. Chernov, Decay of correlations and dispersing billiards. J. Statist. Phys. 94 (1999) 513-556
[3] N. I. Chernov and R. Markarian, Chaotic Billiards. AMS, Mathematical Surveys and Monographs 127 (2006).
[4] N. I. Chernov and R. Markarian, Decay of correlations of billiards with cusps. Comm. Math. Phys. 270 (2007), 727—758.
[5] N. Chernov and H-K. Zhang, Billiards with polynomial mixing rates. Nonlinearity 18 (2005), no. 4, 1527—1553.
[6] N. Chernov and H.-K. Zhang, A family of chaotic billiards with variable mixing rates. Stochastics and Dynamics, 5 (2005), 535-553.
[7] N. Chernov and H.-K. Zhang, Regularity of Bunimovich's stadia. Regular and Chaotic Dynamics, 12 (2007), 335-356.
[8] N. Chernov and H.-K. Zhang, Improved estimates for correlations in billiards. Communications in Mathematical Physics, 277 (2008), 305-321.
[9] N. Chernov and H.-K. Zhang, On statistical properties of hyperbolic systems with singularities. Journal of Statistical Physics, 136 (2009), 615-642
[10] P. Collet and A. Galves, Statistics of close visits to the indifferent fixed point of an interval map. J. Statist. Phys. 72 (1993), no. 3-4, 459-–478.
[11] M. Hirata, B. Saussol and S. Vaienti, Statistics of return times: a general framework and new applications. Comm. Math. Phys. 206 (1999), no. 1, 33—55.
[12] R. Markarian, Billiards with polynomial decay of correlations. Ergod. Th. & Dynam. Sys. 24 (2004), no. 1, 177—197.
[13] J. Milhazes Freitas, N. Haydn and M. Nicol, Convergence of rare event point processses to the Poisson process for planar billiards. Nonlinearity 27 (2014) 1669–1687
[14] F. Pène and B. Saussol, Poisson law for some non-uniformly hyperbolic dynamical systems with polynomial rate of mixing. Ergod. Th. & Dynam. Sys. 36 (2016) 2602–2626

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