

COUNTING PROBLEMS FOR SPECIAL-ORTHOGONAL ANOSOV REPRESENTATIONS

LEÓN CARVAJALES

ABSTRACT. Let ρ be a projective Anosov representation of a word hyperbolic group Γ into $G := \text{PSO}(p, q)$ and X_G be the Riemannian symmetric space of G . Let $o \subset \mathbb{R}^{p+q}$ be a line on which the quadratic form defining G is negative. We define the totally geodesic sub-manifold $S^o \subset X_G$ consisting of negative definite q -dimensional subspaces of \mathbb{R}^{p+q} containing o . For certain choices of o we prove that

$$\#\{\gamma \in \Gamma : d_{X_G}(S^o, \rho(\gamma)S^o) \leq t\}$$

is finite for every $t \geq 0$ and show a purely exponential asymptotic for this function as $t \rightarrow \infty$. An interpretation in the pseudo-Riemannian hyperbolic space $\mathbb{H}^{p, q-1}$ is also provided.

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1. INTRODUCTION

Let X be a proper non compact metric space and o a point in X . Given a discrete group Δ of isometries of X , consider the following *orbital counting function*:

$$(1.1) \quad t \mapsto \#\{g \in \Delta : d_X(o, g.o) \leq t\}$$

where $t \geq 0$. The *orbital counting problem* consists on the study of the asymptotic behaviour of (1.1) as $t \rightarrow \infty$.

When $X = \mathbb{R}^2$ and $\Delta = \mathbb{Z}^2$ this is known as the *Gauss circle problem*. For X a negatively curved complete simply connected Riemannian manifold and Δ co-compact this problem was studied by Margulis in his PhD Thesis (see [36]): the author shows a purely exponential asymptotic for (1.1), the exponent being the topological entropy of the geodesic flow of the quotient space $\Delta \backslash X$. Many authors have generalized the work of Margulis to different contexts, see Roblin [50] and references therein for a complete picture in the negatively curved setting.

Research partially funded by CSIC MIA (2017), CSIC Iniciación C068 (2018) and ANR-16-CE40-0025.

When X is a (not necessarily Riemannian) symmetric space associated to a semisimple Lie group G and $\Delta < G$ is a lattice, these kind of problems were studied notably by Eskin & McMullen [17] and Duke & Rudnick & Sarnak [16]. In the non lattice case but restricted to Riemannian symmetric spaces, one also finds the work of Quint [49] and Sambarino [52]. Quint deals with the case on which Δ is a Schottky group (in the sense of Benoist [3]). Sambarino treats more generally the case of Anosov subgroups (in the full flag variety of G) introduced by Labourie [32].

Counting functions different from (1.1) have also been studied by many authors. We will not be exhaustive on this point and we only mention the work of Oh & Shah [41, 42, 43] which is related with some of the results of the present paper: see Subsection 1.2.1.

Let $d := p + q$ where $p, q \geq 1$ and $\langle \cdot, \cdot \rangle_{p,q}$ be the form on \mathbb{R}^d defined by

$$\langle (x_1, \dots, x_d), (y_1, \dots, y_d) \rangle_{p,q} := \sum_{i=1}^p x_i y_i - \sum_{i=p+1}^d x_i y_i.$$

We denote by $G := \text{PSO}(p, q)$ the group of projectivized matrices in $\text{SL}(d, \mathbb{R})$ preserving $\langle \cdot, \cdot \rangle_{p,q}$. The goal of this paper is to study counting problems in the following symmetric spaces associated to G :

- **The Riemannian symmetric space X_G of G :** We study the growth of the orbit of a copy of the Riemannian symmetric space of $\text{PSO}(p, q - 1)$ inside X_G .
- **The pseudo-Riemannian hyperbolic space**

$$\mathbb{H}^{p,q-1} := \{o = [\hat{o}] \in \mathbb{P}(\mathbb{R}^d) : \langle \hat{o}, \hat{o} \rangle_{p,q} < 0\}$$

of signature $(p, q - 1)$: We study the growth of the orbit of a point in $\mathbb{H}^{p,q-1}$.

In the case $q = 1$ one has $\mathbb{H}^p = X_G = \mathbb{H}^{p,q-1}$ and our results correspond to classical and well-known counting theorems, so we are mostly concerned with the case $q > 1$.

1.1. Notations and reminders on projective Anosov representations. In this paper the discrete group $\Delta < G$ to which we associate counting functions will always be the image of a word hyperbolic group by an *Anosov representation*. Anosov representations are (a stable class of) faithful and discrete representations from word hyperbolic groups into semisimple Lie groups that share many geometrical and dynamical features with holonomies of convex co-compact hyperbolic manifolds. They were introduced by Labourie [32] in his study of the Hitchin component and further extended to arbitrary word hyperbolic groups by Guichard & Wienhard in [21]. After that Anosov representations had been object of intensive research in the field of geometric structures on manifolds and their deformation spaces (see for instance the surveys of Kassel [29] or Wienhard [54] and references therein).

In order to state our results we recall, very informally, the definition and basic facts on (projective) Anosov representations into G . Precisions are given in Section 6.

Let $P_1^{p,q}$ be the stabilizer of an isotropic line in \mathbb{R}^d , i.e. a line on which $\langle \cdot, \cdot \rangle_{p,q}$ equals zero. Then $P_1^{p,q}$ is a parabolic subgroup of G and the quotient space $\partial \mathbb{H}^{p,q-1} := G/P_1^{p,q}$, called the *boundary* of $\mathbb{H}^{p,q-1}$, identifies with the set of isotropic lines in \mathbb{R}^d .

Fix Γ a non elementary word hyperbolic group and $\partial_\infty \Gamma$ its Gromov boundary. Every infinite order element γ in Γ has a unique attracting (resp. repelling) fixed point in $\partial_\infty \Gamma$ denoted by γ_+ (resp. γ_-). Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation. By definition this means that there exists a continuous¹ equivariant map

$$\xi : \partial_\infty \Gamma \rightarrow \partial \mathbb{H}^{p,q-1}$$

with the following properties:

- **Transversality:** Let ${}^\perp_{p,q}$ denote the orthogonal complement with respect to the form $\langle \cdot, \cdot \rangle_{p,q}$. Then the map $\eta := \xi^\perp_{p,q}$ satisfies $\xi(x) \oplus \eta(y) = \mathbb{R}^d$ for every $x \neq y$ in $\partial_\infty \Gamma$.

¹In fact Hölder continuous (see Bridgeman & Canary & Labourie & Sambarino [11, Lemma 2.5]).

- **Uniform hyperbolicity:** Some flow associated to ρ satisfies a uniform contraction/dilation property (see Labourie [32] and Guichard & Wienhard [21] or Section 6 for precisions).

When ρ is $P_1^{p,q}$ -Anosov all infinite order elements in $\rho(\Gamma)$ are *proximal*. This means they act on $\mathbb{P}(\mathbb{R}^d)$ with a unique attractive fixed line and a unique repelling hyperplane. The *limit set* of ρ is, by definition, the closure of the set of attractive fixed lines of proximal elements in $\rho(\Gamma)$. It is denoted by $\Lambda_{\rho(\Gamma)}$ and coincides with the image of ξ .

Our counting theorems will be related to the choice of a point in the set

$$\Omega_{\rho} := \{o = [\hat{o}] \in \mathbb{H}^{p,q-1} : \langle \hat{o}, \hat{\xi} \rangle_{p,q} \neq 0 \text{ for all } \xi = [\hat{\xi}] \in \Lambda_{\rho(\Gamma)}\}.$$

These kind of sets were considered by Guichard & Wienhard [21] and by Kapovich & Leeb & Porti [26] in their study of the existence of co-compact domains of discontinuity in flag manifolds associated to Anosov representations. An important class of Anosov representations for which Ω_{ρ} is non empty is given by $\mathbb{H}^{p,q-1}$ -convex co-compact subgroups in the sense of Danciger & Guéritaud & Kassel [14]. However in our results we only assume that $\Omega_{\rho} \neq \emptyset$ (see Example 7.1).

1.2. Counting in X_G . The Riemannian symmetric space X_G of G is the space q -dimensional subspaces in \mathbb{R}^d on which $\langle \cdot, \cdot \rangle_{p,q}$ is negative definite, endowed with the G -invariant Riemannian metric induced by the Killing form of $\mathfrak{so}(p, q)$ (in Subsection 2.1 we recall basic facts on this space). Given a point $o \in \mathbb{H}^{p,q-1}$ we set

$$S^o := \{\tau \in X_G : o \subset \tau\}.$$

Observe that S^o is a totally geodesic sub-manifold of X_G isometric to the Riemannian symmetric space of $\text{PSO}(p, q-1)$.

The main results of this paper are Theorem A and Theorem B. The first one concerns counting distances in X_G between S^o and $\rho(\gamma)S^o$ for elements γ in Γ . For the second one we choose a point $\tau \in S^o$ and count distances between τ and $\rho(\gamma)S^o$.

Theorem A. *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation and $o \in \Omega_{\rho}$. Then for every $t \geq 0$ one has*

$$\#\{\gamma \in \Gamma : d_{X_G}(S^o, \rho(\gamma)S^o) \leq t\} < \infty.$$

Moreover, there exists positive constants h and M such that

$$M e^{-ht} \#\{\gamma \in \Gamma : d_{X_G}(S^o, \rho(\gamma)S^o) \leq t\} \rightarrow 1$$

as $t \rightarrow \infty$.

Theorem B. *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation, a point $o \in \Omega_{\rho}$ and $\tau \in S^o$. Then for every $t \geq 0$ one has*

$$\#\{\gamma \in \Gamma : d_{X_G}(\tau, \rho(\gamma)S^o) \leq t\} < \infty.$$

Moreover, there exists positive constants h and M' such that

$$M' e^{-ht} \#\{\gamma \in \Gamma : d_{X_G}(\tau, \rho(\gamma)S^o) \leq t\} \rightarrow 1$$

as $t \rightarrow \infty$.

The constant h is the same in both Theorems A and B and it is independent on the choice of o in Ω_{ρ} (and τ in S^o). It is the *entropy* of ρ introduced by Bridgeman & Canary & Labourie & Sambarino [11]:

$$h = h_{\rho} := \limsup_{t \rightarrow \infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho(\gamma)) \leq t\}}{t}$$

where $[\gamma]$ denotes the conjugacy class of γ and $\lambda_1(\rho(\gamma))$ denotes the logarithm of the spectral radius of $\rho(\gamma)$.

1.2.1. Relationship with the work of Oh & Shah. Motivated by the study of *Apollonian circle packings* on the Riemann sphere, Oh & Shah [43] studied a particular case of our counting results. Indeed, let $p = 1$ and $q = 3$. Then $\mathbb{H}^{1,2}$ identifies with the space of circles of the Riemann sphere or, equivalently, the space of totally geodesic isometric copies of \mathbb{H}^2 inside \mathbb{H}^3 . In [43, Theorem 1.5] the cited authors proved that for a well-chosen $S^o \cong \mathbb{H}^2 \subset \mathbb{H}^3$ and any point $\tau \in \mathbb{H}^3$ then

$$\#\{g \in \Delta : d_{\mathbb{H}^3}(\tau, gS^o) \leq t\} \sim Me^{ht}$$

as $t \rightarrow \infty$. Hence Theorem B can be interpreted as a higher rank generalization of this result. We note however that Oh & Shah's Theorem apply to a much more general class of discrete groups than the one being Anosov (e.g. some geometrically finite Kleinian groups) and that they obtained also a very explicit description of the constant M (see [43] and references therein for precisions). A slightly different counting theorem in $\mathbb{H}^{1,2}$ was obtained by the cited authors in [41]. Effective versions of Oh & Shah's results have been obtained by Lee & Oh [34] and Mohammadi & Oh [40].

Since the work of Margulis [36], in order to obtain a counting result one usually studies the ergodic properties of a well chosen dynamical system. The approach by Oh & Shah is similar to the one of Eskin & McMullen [17]: they study the equidistribution, with respect to certain measures, of the orthogonal translates of S^o under the geodesic flow (see Oh & Shah [42] for precisions). Here we use different techniques. Following Sambarino [51] we find a cocycle over $\partial_\infty \Gamma$ from which we build a reparametrization of the Gromov geodesic flow of Γ . Our results will follow from the study of spatial distribution of periodic orbits of this reparametrization.

1.3. Counting in $\mathbb{H}^{p,q-1}$. Both Theorems A and B have very natural geometrical interpretations in the pseudo-Riemannian hyperbolic space $\mathbb{H}^{p,q-1}$, which we now state.

Geodesics in $\mathbb{H}^{p,q-1}$ are intersection of projectivized 2-dimensional subspaces of \mathbb{R}^d with $\mathbb{H}^{p,q-1}$ and they are classified in three types, depending on the sign of $\langle \cdot, \cdot \rangle_{p,q}$ on its derivative (see Subsection 2.2.3). We are mainly interested on *space-like geodesics*, i.e. geodesics associated to planes on which $\langle \cdot, \cdot \rangle_{p,q}$ has signature $(1,1)$. Let $o, o' \in \mathbb{H}^{p,q-1}$ be two points joined by a space-like geodesic and let $\ell_{o,o'}$ be the length of this geodesic segment (c.f. Subsection 2.2.5). We denote by $\mathcal{C}_o^>$ the set of points of $\mathbb{H}^{p,q-1}$ which can be joined to o by a space-like geodesic and by $\mathcal{C}_{o,G}^>$ the set of elements g in G for which $g.o \in \mathcal{C}_o^>$.

Proposition (Proposition 4.9). *Let $o \in \mathbb{H}^{p,q-1}$ and $g \in \mathcal{C}_{o,G}^>$. Then*

$$\ell_{o,g.o} = d_{X_G}(S^o, gS^o).$$

In Corollary 7.8 we prove that given a $P_1^{p,q}$ -Anosov representation $\rho : \Gamma \rightarrow G$ and o in Ω_ρ then for all but finitely many elements γ in Γ one has $\rho(\gamma) \in \mathcal{C}_{o,G}^>$. The counterpart of Theorem A in $\mathbb{H}^{p,q-1}$ is now clear.

In order to state the corresponding geometric interpretation for Theorem B we follow Kassel & Kobayashi [28, p.151] (we refer the reader to Subsection 4.3 for precisions). Let $o \in \mathbb{H}^{p,q-1}$ and $\tau \in S^o$. These choices determine a space-like totally geodesic isometric copy $\mathbb{H}_\tau^p := (o \oplus \tau^{\perp_{p,q}}) \cap \mathbb{H}^{p,q-1}$ of \mathbb{H}^p passing through o . Let K^τ be the (maximal compact) subgroup of G stabilizing τ . As we shall see, for every g in G the point $g.o$ lies in the K^τ -orbit of a point o_g in \mathbb{H}_τ^p . The counterpart of Theorem B in $\mathbb{H}^{p,q-1}$ is provided by the following proposition.

Proposition (Proposition 4.5). *For every g in G one has*

$$\ell_{o,o_g} = d_{X_G}(\tau, gS^o).$$

1.3.1. Relationship with the works of Glorieux & Monclair and Kassel & Kobayashi. In their study of the geometry of the limit set of $\mathbb{H}^{p,q-1}$ -convex co-compact groups, Glorieux & Monclair [19] introduced an orbital counting function very much related to

$$\#\{\gamma \in \Gamma : \ell_{o, \rho(\gamma)o} \leq t\}.$$

In particular they proved that this number is finite for every $t \geq 0$ and that for all but finitely many elements γ in Γ the geodesic connecting o with $\rho(\gamma)o$ is space-like. Theorem A can then be interpreted as a precise quantitative statement related to their work. We remark however that in this paper we do not assume that ρ is $\mathbb{H}^{p,q-1}$ -convex co-compact.

On the other hand, as we shall see in Section 4 the number ℓ_{o, o_g} is related to the *polar projection* of G and therefore Theorem B addresses the problems treated by Kassel & Kobayashi in [28, Section 4].

1.4. Outline of the proof. There are three major steps in the proof of Theorems A and B.

First step. We translate the geometric quantities defining the counting functions involved in Theorems A and B into linear-algebraic quantities.

Let us be more precise. Fix $o \in \mathbb{H}^{p,q-1}$ and denote by H^o the stabilizer in G of this point. If we consider the symmetry of \mathbb{R}^d given by $J^o := \text{id}_o \oplus (-\text{id}_{o^\perp})$, we have that H^o equals the fixed point set of the involution

$$\sigma^o : g \mapsto J^o g J^o$$

of G (see Subsection 2.2.2). This identifies the tangent space at o of $\mathbb{H}^{p,q-1}$ with the subspace of $\mathfrak{so}(p, q)$ defined by $\mathfrak{q}^o := \{d\sigma^o = -1\}$. In Propositions 4.9 and 4.11 we prove that for every $g \in \mathcal{C}_{o, G}^>$ one has

$$(1.2) \quad d_{X_G}(S^o, gS^o) = \frac{1}{2} \lambda_1(J^o g J^o g^{-1}).$$

The key ingredient in the proof of equality (1.2) is the following version of the classical Cartan Decomposition of G .

Proposition (Proposition 4.8). *Let $o \in \mathbb{H}^{p,q-1}$ and $\mathfrak{b}^+ \subset \mathfrak{q}^o$ be a ray such that $\exp(\mathfrak{b}^+).o$ is space-like. Given $g \in \mathcal{C}_{o, G}^>$ there exists $h, h' \in H^o$ and a unique $X \in \mathfrak{b}^+$ such that*

$$g = h \exp(X) h'.$$

On the other hand, the linear-algebraic interpretation of the quantity $d_{X_G}(\tau, g.S^o)$ is the following: recall that K^τ is the stabilizer in G of τ and let $\|\cdot\|_\tau$ be a norm on \mathbb{R}^d invariant by K^τ . We show in Propositions 4.5 and 4.6 that for every $g \in G$ the following equality holds

$$(1.3) \quad d_{X_G}(\tau, gS^o) = \frac{1}{2} \log \|J^o g J^o g^{-1}\|_\tau.$$

Once again the proof of this equality relies on a generalization of Cartan Decomposition (see Schlichtkrull [53, Chapter 7]): every $g \in G$ can be written as

$$g = k \exp(X) h$$

for some $k \in K^\tau$, $h \in H^o$ and a unique $X \in \mathfrak{b}^+$ (see Subsection 4.3.1).

Second step. In order to simplify the exposition we assume that Γ is free of torsion and consider $\rho : \Gamma \rightarrow G$ a $P_1^{p,q}$ -Anosov representation. The key feature in the choice of a point o in Ω_ρ is that it will tell us that the proximal matrices $J^o \rho(\gamma) J^o$ and $\rho(\gamma^{-1})$ satisfy some *transversality condition* and then we can estimate the quantities (1.2) and (1.3) in terms of the highest eigenvalue of $\rho(\gamma)$.

More precisely, we will see in Proposition 3.1 that

$$(1.4) \quad \Omega_\rho = \{o \in \mathbb{H}^{p,q-1} : J^o \xi(x) \notin \eta(x) \text{ for all } x \in \partial_\infty \Gamma\}.$$

Fix $o \in \Omega_\rho$. By compactness we obtain that $J^o \xi(x)$ is uniformly far from $\eta(x)$ for every $x \in \partial_\infty \Gamma$. On the other hand if γ_+ is uniformly far from γ_- then $\xi(\gamma_+)$ (resp. $\xi(\gamma_-)$) is uniformly far to $\eta(\gamma_-)$ (resp. $\eta(\gamma_+)$). In Lemma 7.5 we apply Benoist's work [4] to

conclude that the product $J^o\rho(\gamma)J^o\rho(\gamma^{-1})$ is proximal. We obtain moreover a comparison between the quantity (1.2) (resp. (1.3)) and

$$\lambda_1(\rho(\gamma))$$

with very precise control on the error made in this comparison.

Third step. We apply Sambarino's outline [51] to our particular context². To a Hölder cocycle c on $\partial_\infty\Gamma$ the author associates a Hölder reparametrization ψ_t^c of the geodesic flow of Γ . Recall that a *Hölder cocycle* is a map $c : \Gamma \times \partial_\infty\Gamma \rightarrow \mathbb{R}$ satisfying

$$c(\gamma_0\gamma_1, x) = c(\gamma_0, \gamma_1x) + c(\gamma_1, x)$$

for every γ_0, γ_1 in Γ and $x \in \partial_\infty\Gamma$ and such that the map $c(\gamma_0, \cdot)$ is Hölder (with the same exponent for every γ_0). The cocycle c' is said to be *cohomologous* to c if there exists a Hölder continuous function $U : \partial_\infty\Gamma \rightarrow \mathbb{R}$ such that for every γ in Γ and x in $\partial_\infty\Gamma$ one has

$$c(\gamma, x) - c'(\gamma, x) = U(\gamma x) - U(x).$$

In that case ψ_t^c is conjugated to $\psi_t^{c'}$ (see [51, Section 3]). By considering a Markov coding and applying Parry & Pollicott's Prime Orbit Theorem [44], Sambarino obtains a counting theorem for the periodic orbits of ψ_t^c (see [51, Corollary 4.1]). Obviously this is a purely dynamical result, i.e. changing ψ_t^c in its conjugacy class does not affect the counting.

However, counting a quantity which is not a conjugacy invariant is more subtle: one must find a particular flow and not just *any* flow in the conjugacy class (equivalently, one must find a particular cocycle and not just *any* cocycle in the cohomology class). Indeed, we will see that the conjugacy class of the flows that we need to consider to prove Theorem A and Theorem B is the same in both cases, but only specific choices of representatives in such a class yield the respective counting results.

Let us briefly sketch the proof of Theorem A (Theorem B is proved in a similar way). Fix $o \in \Omega_\rho$ and consider

$$c_o : \Gamma \times \partial_\infty\Gamma \rightarrow \mathbb{R} : \quad c_o(\gamma, x) := \frac{1}{2} \log \left| \frac{\langle \rho(\gamma)v_x, J^o\rho(\gamma)v_x \rangle_{p,q}}{\langle v_x, J^ov_x \rangle_{p,q}} \right|$$

where $v_x \neq 0$ is any vector in $\xi(x)$ ³. This is a well-defined thanks to (1.4) and is a Hölder cocycle.

Let $\partial_\infty^2\Gamma$ be the set of pairs of distinct points in $\partial_\infty\Gamma$ and consider the action of Γ on $\partial_\infty^2\Gamma \times \mathbb{R}$ given by

$$\gamma \cdot (x, y, s) := (\gamma x, \gamma y, s - c_o(\gamma, y)).$$

We denote by $U_o\Gamma$ the quotient space. The translation flow on $\partial_\infty^2\Gamma \times \mathbb{R}$ given by

$$\psi_t(x, y, s) := (x, y, s - t)$$

descends to a flow $\psi_t = \psi_t^o$ on $U_o\Gamma$. As Sambarino shows in [51, Theorem 3.2(1)] (see also Lemma A.7) the space $U_o\Gamma$ is homeomorphic to the unit tangent bundle of Γ and the flow ψ_t is conjugated to a Hölder reparametrization of the geodesic flow of Γ . We will show (see Lemma A.7) that periodic orbits of ψ_t are parametrized by *primitive* conjugacy classes in Γ , i.e. conjugacy classes of elements which cannot be written as a power of another element. If γ is primitive, the corresponding period is given by

$$\ell_{c_o}(\gamma) := \lambda_1(\rho(\gamma)).$$

We show the following property concerning the spectral radii of a projective Anosov representation.

²The results in [51] are proven for the case on which Γ is the fundamental group of a closed negatively curved manifold. However, all the results obtained there remain valid when Γ is an arbitrary word hyperbolic group admitting an Anosov representation. This is explained in detail in Appendix A.

³When $q = 1$ this coincides with the *Busemann cocycle* of \mathbb{H}^p , i.e. $c_o(\gamma, x) = \beta_{\xi(x)}(\rho(\gamma^{-1}).o, o)$ where $\beta(\cdot, \cdot) : \partial\mathbb{H}^p \times \mathbb{H}^p \times \mathbb{H}^p \rightarrow \mathbb{R}$ is the Busemann function.

Proposition (Proposition A.2). *Let ρ be a projective Anosov representation of Γ . Then the set $\{\lambda_1(\rho(\gamma))\}_{\gamma \in \Gamma}$ spans a non discrete subgroup of \mathbb{R} .*

The flow ψ_t admits a strong Markov coding (see Subsection A.2.1), hence we can apply the techniques of the thermodynamic formalism of sub-shifts of finite type to this flow. Provided also with the previous proposition we can adapt Sambarino's ideas [51] to our context.

Denote by h the topological entropy of ψ_t . The probability of maximal entropy of ψ_t is constructed as follows: define the *Gromov product*

$$[\cdot, \cdot]_o : \partial_\infty^2 \Gamma \longrightarrow \mathbb{R} : [x, y]_o := -\frac{1}{2} \log \left| \frac{\langle v_x, J^o v_x \rangle_{p,q} \langle v_y, J^o v_y \rangle_{p,q}}{\langle v_x, v_y \rangle_{p,q} \langle v_y, v_x \rangle_{p,q}} \right|.$$

This function is well-defined by (1.4) and transversality of ξ and η . One can prove that

$$[\gamma x, \gamma y]_o - [x, y]_o = -(c_o(\gamma, x) + c_o(\gamma, y))$$

holds for every γ in Γ and $(x, y) \in \partial_\infty^2 \Gamma$. Let μ_o be a *Patterson-Sullivan probability* on $\partial_\infty \Gamma$ associated to c_o , that is, μ_o satisfies

$$\frac{d\gamma_* \mu_o}{d\mu_o}(x) = e^{-hc_o(\gamma^{-1}, x)}$$

for every $\gamma \in \Gamma$ (for the existence of such a probability see Subsection A.2.2)⁴. By [51, Theorem 3.2(2)] (see also Proposition A.12), the measure

$$e^{-h[\cdot, \cdot]_o} \mu_o \otimes \mu_o \otimes dt$$

on $\partial_\infty^2 \Gamma \times \mathbb{R}$ is Γ -invariant and induces on $U_o \Gamma$ the measure of maximal entropy of ψ_t (well-defined up to scaling).

Denote by $C_c^*(\partial_\infty^2 \Gamma)$ the dual of the space of compactly supported real continuous functions on $\partial_\infty^2 \Gamma$ equipped with the weak-star topology. For x in $\partial_\infty \Gamma$ let δ_x be the Dirac mass at x . By [51, Proposition 4.3] (see also Proposition A.13) one has

$$M e^{-ht} \sum_{\gamma \in \Gamma, \ell_{c_o}(\gamma) \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \longrightarrow e^{-h[\cdot, \cdot]_o} \mu_o \otimes \mu_o$$

on $C_c^*(\partial_\infty^2 \Gamma)$ as $t \longrightarrow \infty$ ⁵. The constant $M = M_{\rho, o} > 0$ equals the product of h with the total mass of $e^{-h[\cdot, \cdot]_o} \mu_o \otimes \mu_o \otimes dt$ on the quotient space $U_o \Gamma$.

As we show in Lemma 8.8, the number $[\gamma_-, \gamma_+]_o$ is the precise error term in the comparison between $\ell_{c_o}(\gamma)$ and $\frac{1}{2} \lambda_1(J^o \rho(\gamma) J^o \rho(\gamma^{-1}))$ provided by Benoist's Theorem 5.7. This is a key step: it tells us that is the specific cocycle c_o (and not just *any* cohomologous cocycle) that allows us to go from a purely dynamical result to the one that we want. Indeed, now it remains to adapt the proof of [51, Theorem 6.5] to obtain the following proposition from which Theorem A is directly deduced in the torsion free case.

Proposition (Proposition 8.10). *Let Γ be a torsion free word hyperbolic group, $\rho : \Gamma \longrightarrow G$ be a $P_1^{p,q}$ -Anosov representation and $o \in \Omega_\rho$. Then*

$$M e^{-ht} \sum_{\gamma \in \Gamma, \frac{1}{2} \lambda_1(J^o \rho(\gamma) J^o \rho(\gamma^{-1})) \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \longrightarrow \mu_o \otimes \mu_o$$

on $C^*(\partial_\infty \Gamma \times \partial_\infty \Gamma)$ as $t \longrightarrow \infty$.

It turns out that the previous proposition can be used to deduce Theorem A in the general case, that is, if we admit torsion elements in Γ .

Proposition (Proposition 8.12). *Let $\rho : \Gamma \longrightarrow G$ be a $P_1^{p,q}$ -Anosov representation and $o \in \Omega_\rho$. Then*

$$M e^{-ht} \sum_{\gamma \in \Gamma, \frac{1}{2} \lambda_1(J^o \rho(\gamma) J^o \rho(\gamma^{-1})) \leq t} \delta_{\rho(\gamma^{-1})o^{+p,q}} \otimes \delta_{\rho(\gamma)o} \longrightarrow \eta_*(\mu_o) \otimes \xi_*(\mu_o)$$

⁴All these objects, such as Busemann cocycles, Gromov products and Patterson-Sullivan probabilities have been studied by Glorieux & Monclair in [19] for $\mathbb{H}^{p,q-1}$ -convex co-compact groups.

⁵Recall that we have assumed for simplicity that Γ is free of torsion.

on $C^*(\mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d))$ as $t \rightarrow \infty$.

These type of distribution statements are inspired by Roblin's work [50] in the negatively curved setting.

1.5. Organization of the paper. In Section 2 we recall basic facts on the symmetric spaces X_G and $\mathbb{H}^{p,q-1}$. Section 3 is devoted to the study of end points in $\partial\mathbb{H}^{p,q-1}$ of space-like geodesics passing through our preferred point $o \in \mathbb{H}^{p,q-1}$. We give several characterizations of this set that will allow us to understand Ω_ρ in different ways, all of them used indistinctly in Sections 7, 8 and 9. In Section 4 we study the geometric quantities involved in Theorems A and B. Equalities (1.2) and (1.3) are proven respectively in Subsections 4.4.3 and 4.3.3. In Section 5 we recall Benoist's results on products of proximal matrices and Section 6 is devoted to reminders on Anosov representations. In Section 7 we define the set Ω_ρ and study the action of Γ on this set. We show in particular that the orbital counting functions involved in Theorems A and B are well-defined (Proposition 7.7 and Proposition 7.6). We also obtain some estimations for the spectral radius and operator norm of elements $J^\circ\rho(\gamma)J^\circ\rho(\gamma^{-1})$ which are of major importance (c.f. Lemma 7.5). In Section 8 (resp. Section 9) we prove Theorem A (resp. Theorem B). Finally, we include an appendix (Appendix A) on which we explain how to adapt the results of [51] to the context of arbitrary word hyperbolic groups admitting an Anosov representation.

Acknowledgements. These problems were proposed to me by Rafael Potrie and Andrés Sambarino. Without their guidance, their support and the (many) helpful discussions this work would not have been possible. I am extremely grateful for this.

The author also acknowledges Olivier Glorieux, Tal Horesh and Fanny Kassel for several enlightening discussions and comments.

2. TWO SYMMETRIC SPACES ASSOCIATED TO $\text{PSO}(p, q)$

We begin by recalling the definition and basic properties of the two symmetric spaces on which we establish our counting theorems. We fix some notations that will remain valid for the rest of the paper.

Fix two integers $p, q \geq 1$ and let $d := p + q$. We assume $d > 2$. Denote by $\mathbb{R}^{p,q}$ the vector space \mathbb{R}^d endowed with the non-degenerate quadratic form

$$\langle (x_1, \dots, x_d), (y_1, \dots, y_d) \rangle_{p,q} := \sum_{i=1}^p x_i y_i - \sum_{i=p+1}^d x_i y_i.$$

From now on we denote by $G := \text{PSO}(p, q)$ the subgroup of $\text{PSL}(d, \mathbb{R})$ consisting of elements preserving $\langle \cdot, \cdot \rangle_{p,q}$.

For a subspace π of \mathbb{R}^d we denote by $\pi^{\perp p,q}$ its orthogonal complement with respect to $\langle \cdot, \cdot \rangle_{p,q}$, i.e.

$$\pi^{\perp p,q} := \{x \in \mathbb{R}^d : \langle x, y \rangle_{p,q} = 0 \text{ for all } y \in \pi\}.$$

Let $\mathfrak{g} := \mathfrak{so}(p, q)$ be the Lie algebra of G . If \cdot^t denotes the *usual* transpose operator one has that \mathfrak{g} equals the set of matrices of the form

$$\begin{pmatrix} X_1 & X_2 \\ X_2^t & X_3 \end{pmatrix}$$

where X_1 is of size $p \times p$, X_3 is of size $q \times q$ and both are skew-symmetric with respect to \cdot^t . The *Killing form* of G is the symmetric bilinear form κ on \mathfrak{g} defined by

$$\kappa(X, Y) := \text{tr}(\text{ad}_X \circ \text{ad}_Y)$$

where $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ is the adjoint representation. It can be seen that the following equality holds:

$$\kappa(X, Y) = (d - 2)\text{tr}(XY)$$

(see Helgason [23, p.180 & p.189]).

2.1. The Riemannian symmetric space X_G . A *Cartan involution* on G is an involutive automorphism $\tau : G \rightarrow G$ such that the bilinear form

$$(X, Y) \mapsto -\kappa(X, d\tau(Y))$$

is positive definite. The fixed point set K^τ of such an involution τ is a maximal compact subgroup of G (see Knapp [31, Theorem 6.31]). The *Riemannian symmetric space* of G is the set X_G of all Cartan involutions in G . It is equipped with a natural action of G which is transitive (c.f. [31, Corollary 6.19]) and the stabilizer of τ is K^τ , thus

$$G/K^\tau \cong X_G.$$

Remark 2.1. The space X_G can be identified with the space of q -dimensional negative definite subspaces of \mathbb{R}^d . Explicitly, to a q -dimensional negative definite subspace π one associates the Cartan involution of G determined by the inner product of \mathbb{R}^d which equals $-\langle \cdot, \cdot \rangle_{p,q}$ (resp. $\langle \cdot, \cdot \rangle_{p,q}$) on π (resp. $\pi^{\perp_{p,q}}$) and for which π and $\pi^{\perp_{p,q}}$ are orthogonal. \diamond

The choice of a point τ in X_G determines a *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{p}^\tau \oplus \mathfrak{k}^\tau$$

where $\mathfrak{p}^\tau := \{d\tau = -1\}$ and $\mathfrak{k}^\tau := \{d\tau = 1\}$. The group K^τ is tangent to \mathfrak{k}^τ and one has a G -equivariant identification

$$(2.1) \quad \mathfrak{p}^\tau \cong T_\tau X_G$$

given by $X \mapsto \left. \frac{d}{dt} \right|_0 \exp(tX) \cdot \tau$ (see [23, Theorem 3.3 of Ch. IV]).

Example 2.2. Consider the involution in G defined by $\tau(g) := (g^{-1})^t$. One sees that $\tau \in X_G$ and \mathfrak{p}^τ (resp. \mathfrak{k}^τ) is the set of symmetric matrices (resp. skew-symmetric matrices) in $\mathfrak{so}(p, q)$. Moreover K^τ is the subgroup $\text{PS}(\text{O}(p) \times \text{O}(q))$. \diamond

The Killing form is positive definite (resp. negative definite) on \mathfrak{p}^τ (resp. \mathfrak{k}^τ), hence the identification (2.1) defines a G -invariant Riemannian metric on X_G by restricting the Killing form to tangent spaces. It is well-known (see [23, Theorem 4.2 of Ch. IV]) that X_G equipped with this metric is a symmetric space which is non-positively curved. For reasons that will become clear in the sequel (c.f. Remark 2.3), we change the normalization of the metric and define

$$(2.2) \quad d_{X_G}(\tau, \exp(X) \cdot \tau) := \left(\frac{1}{2(d-2)} \kappa(X, X) \right)^{\frac{1}{2}}$$

for all $X \in \mathfrak{p}^\tau$.

2.2. The pseudo-Riemannian symmetric space $\mathbb{H}^{p,q-1}$. We now define the pseudo-Riemannian hyperbolic space and recall some basic facts on its geometry.

2.2.1. Definitions. Let

$$\hat{\mathbb{H}}^{p,q-1} := \{\hat{o} \in \mathbb{R}^{p,q} : \langle \hat{o}, \hat{o} \rangle_{p,q} = -1\}$$

endowed with the restriction of the form $\langle \cdot, \cdot \rangle_{p,q}$ to tangent spaces. This metric induces on

$$\mathbb{H}^{p,q-1} := \{o = [\hat{o}] \in \mathbb{P}(\mathbb{R}^{p,q}) : \langle \hat{o}, \hat{o} \rangle_{p,q} < 0\}$$

a pseudo-Riemannian structure invariant under the projective action of G . This space is called the *pseudo-Riemannian hyperbolic space of signature $(p, q-1)$* . The *boundary* of $\mathbb{H}^{p,q-1}$ is the space of *isotropic lines* defined by

$$\partial \mathbb{H}^{p,q-1} := \{\xi = [\hat{\xi}] \in \mathbb{P}(\mathbb{R}^{p,q}) : \langle \hat{\xi}, \hat{\xi} \rangle_{p,q} = 0\}.$$

It is also equipped with the natural (transitive) action of G . If we denote by $P_1^{p,q}$ the (parabolic) subgroup of G stabilizing an isotropic line, then

$$\partial\mathbb{H}^{p,q-1} \cong G/P_1^{p,q}.$$

2.2.2. Structure of symmetric space. The action of $G = \text{PSO}(p, q)$ on $\mathbb{H}^{p,q-1}$ is transitive, hence $\mathbb{H}^{p,q-1} \cong G/H^o$ where H^o is the stabilizer in G of $o \in \mathbb{H}^{p,q-1}$. For instance, when $o = [0, \dots, 0, 1] \in \mathbb{H}^{p,q-1}$ one has

$$H^o = \left\{ \begin{bmatrix} \hat{g} & 0 \\ 0 & 1 \end{bmatrix} \in G : \hat{g} \in \text{O}(p, q-1) \right\}.$$

Fix any $o \in \mathbb{H}^{p,q-1}$. Since o and $o^{\perp_{p,q}}$ are transverse we can consider the matrix

$$J^o := \text{id}_o \oplus (-\text{id}_{o^{\perp_{p,q}}}).$$

It follows that $H^o = \text{Fix}(\sigma^o)$ where σ^o is the involution of G defined by

$$(2.3) \quad \sigma^o(g) := J^o g J^o.$$

Thus $\mathbb{H}^{p,q-1} \cong G/H^o$ is a symmetric space of G .

Remark 2.3. Let $o \in \mathbb{H}^{p,q-1}$ and $\mathfrak{q}^o := \{d\sigma^o = -1\}$. There exists a G -equivariant identification

$$\mathfrak{q}^o \cong T_o\mathbb{H}^{p,q-1}$$

given by $X \mapsto \left. \frac{d}{dt} \right|_0 \exp(tX).o$. We denote by $\langle \cdot, \cdot \rangle_o$ the pull-back of the $(p, q-1)$ -form on $T_o\mathbb{H}^{p,q-1}$ under this map and, for $X \in \mathfrak{q}^o$, we set $|X|_o := \langle X, X \rangle_o$ ⁶.

Recall that κ is the Killing form of $\mathfrak{so}(p, q)$. From explicit computations one sees that

$$(2.4) \quad |X|_o = \frac{1}{2(d-2)} \kappa(X, X)$$

for all $X \in \mathfrak{q}^o$. The choice of normalization made in Subsection 2.1 is now justified. ◇

The following remark will be useful in the sequel. It tell us that the stabilizer H^o contains sufficiently many isometries of $T_o\mathbb{H}^{p,q-1}$.

Remark 2.4. Let $o \in \mathbb{H}^{p,q-1}$. Then the action of the connected component of H^o containing the identity is conjugated to the action of $\text{SO}(p, q-1)$ on $\mathbb{R}^{p,q-1}$. ◇

2.2.3. Geodesics in $\mathbb{H}^{p,q-1}$. Geodesics of $\mathbb{H}^{p,q-1}$ are the intersections of straight lines of $\mathbb{P}(\mathbb{R}^{p,q})$ with $\mathbb{H}^{p,q-1}$. They are divided in three types:

- *Space-like geodesics:* Those associated to 2-dimensional subspaces of \mathbb{R}^d on which $\langle \cdot, \cdot \rangle_{p,q}$ has signature $(1, 1)$. Hence, they have positive speed and meet the boundary $\partial\mathbb{H}^{p,q-1}$ in two distinct points.
- *Time-like geodesics:* Those associated to 2-dimensional subspaces of \mathbb{R}^d on which $\langle \cdot, \cdot \rangle_{p,q}$ has signature $(0, 2)$. Hence, they have negative speed and do not meet the boundary (they are closed).
- *Light-like geodesics:* Those associated to 2-dimensional subspaces of \mathbb{R}^d on which $\langle \cdot, \cdot \rangle_{p,q}$ has signature $(0, 1)$, that is, degenerates but has a negative eigenvalue. They have zero speed and meet the boundary $\partial\mathbb{H}^{p,q-1}$ in a single point.

For a point $o \in \mathbb{H}^{p,q-1}$ we denote by \mathcal{C}_o (resp. $\mathcal{C}_o^>$) the set of points of $\mathbb{H}^{p,q-1}$ which can be joined with o by a light-like (resp. space-like) geodesic. Its closure in $\mathbb{P}(\mathbb{R}^{p,q})$ is denoted by $\overline{\mathcal{C}}_o$ (resp. $\overline{\mathcal{C}}_o^>$).

⁶This number could be negative or zero for $X \neq 0$.

2.2.4. Light-cones. The following lemma is proved by Glorieux & Monclair in [19, Lemma 2.2]. We include a proof for completeness.

Lemma 2.5. *Let $o \in \mathbb{H}^{p,q-1}$. Then $\overline{\mathcal{C}}_o \cap \partial\mathbb{H}^{p,q-1} = o^{\perp_{p,q}} \cap \partial\mathbb{H}^{p,q-1}$.*

Proof. If $q = 1$ the statement holds since in that case $\overline{\mathcal{C}}_o = \emptyset = o^{\perp_{p,q}} \cap \partial\mathbb{H}^{p,q-1}$.

Assume $q > 1$ and let $\mathbb{P}(\pi)$ be the projectivized of a 2-dimensional subspace π of \mathbb{R}^d of signature $(0, 1)$ containing o . Then $\mathbb{P}(\pi) \cap \partial\mathbb{H}^{p,q-1}$ coincides with the (only) isotropic line ξ in $\mathbb{P}(\pi)$, which is easily seen to be orthogonal to o . The reverse inclusion is proved in a similar way. \square

2.2.5. Lengths of space-like geodesics. For a point o' in $\mathcal{C}_o^>$ we denote by $\ell_{o,o'}$ the length of the geodesic segment connecting o with o' . For instance the geodesic

$$(2.5) \quad s \mapsto [\sinh(s), 0, \dots, 0, \cosh(s)] \in \mathbb{H}^{p,q-1}$$

is parametrized by arc-length.

2.2.6. Space-like copies of \mathbb{H}^p . We now observe that $\mathbb{H}^{p,q-1}$ contains many space-like isometric copies of \mathbb{H}^p (and this will give us a geometric interpretation of Theorem B in $\mathbb{H}^{p,q-1}$). Indeed, let π be a $(p+1)$ -dimensional subspace of \mathbb{R}^d of signature $(p, 1)$. Then $\mathbb{P}(\pi) \cap \mathbb{H}^{p,q-1}$ identifies with

$$\{o = [\hat{o}] \in \mathbb{P}(\mathbb{R}^{p,1}) \mid \langle \hat{o}, \hat{o} \rangle_{p,1} < 0\}.$$

It follows that $\mathbb{P}(\pi) \cap \mathbb{H}^{p,q-1}$ is a totally geodesic isometric copy of \mathbb{H}^p inside $\mathbb{H}^{p,q-1}$. Moreover this sub-manifold is space-like, in the sense that any of its tangent vectors has positive norm.

3. END POINTS OF SPACE-LIKE GEODESICS

Let o be a point in $\mathbb{H}^{p,q-1}$. The goal of this section is to give several characterizations of the set of end points (in $\partial\mathbb{H}^{p,q-1}$) of space-like geodesics passing through o . All these characterizations will be used in sections 7, 8 and 9.

Recall that

$$J^o = \text{id}_o \oplus (-\text{id}_{o^{\perp_{p,q}}})$$

is the matrix defining the involution σ^o of Subsection 2.2.2. Note that J^o preserves the form $\langle \cdot, \cdot \rangle_{p,q}$ and in particular acts on $\partial\mathbb{H}^{p,q-1}$. Set

$$\mathcal{O}^o := \{\xi \in \partial\mathbb{H}^{p,q-1} : J^o\xi \neq \xi\}.$$

Recall from Subsection 2.2.3 that \mathcal{C}_o denotes the light-cone of o .

Proposition 3.1. *Let $o \in \mathbb{H}^{p,q-1}$. Then the following holds:*

$$\mathcal{O}^o = \{\xi \in \partial\mathbb{H}^{p,q-1} : J^o\xi \notin \xi^{\perp_{p,q}}\} = \partial\mathbb{H}^{p,q-1} \setminus o^{\perp_{p,q}} = \partial\mathbb{H}^{p,q-1} \setminus \overline{\mathcal{C}}_o.$$

It follows that, unless $q = 1$, the set \mathcal{O}^o is not the whole boundary of $\mathbb{H}^{p,q-1}$.

Proof of Proposition 3.1. From Lemma 2.5 we know that $\overline{\mathcal{C}}_o \cap \partial\mathbb{H}^{p,q-1} = o^{\perp_{p,q}} \cap \partial\mathbb{H}^{p,q-1}$. Hence we only have to prove that

$$\mathcal{O}^o \subset \{\xi \in \partial\mathbb{H}^{p,q-1} : J^o\xi \notin \xi^{\perp_{p,q}}\} \subset \partial\mathbb{H}^{p,q-1} \setminus o^{\perp_{p,q}} \subset \mathcal{O}^o.$$

This follows from definitions. \square

4. GENERALIZED CARTAN DECOMPOSITIONS

The goal of this section is to define two generalized Cartan projections and to give geometric and linear-algebraic interpretations of both of them. In Subsection 4.1 we establish some notations and in Subsection 4.2 we define the totally geodesic sub-manifold $S^o \subset X_G$ and discuss some of its properties. Section 4.3 is devoted to the study of the well-known *polar projection* of G . A direct link between this projection and Theorem B is established. Finally in Subsection 4.4 we propose a generalized Cartan decomposition for elements of G satisfying some property with respect to the choice of a particular H^o . We provide a link between this decomposition and Theorem A.

4.1. Notations. The standard reference for this subsection is Schlichtkrull [53, Chapter 7].

Let $o \in \mathbb{H}^{p,q-1}$ and $H^o = \text{Fix}(\sigma^o)$ be its stabilizer in G (c.f. Subsection 2.2.2). Let \mathfrak{h}^o be the Lie algebra of fixed points of $d\sigma^o$ and $\mathfrak{q}^o := \{d\sigma^o = -1\}$. One has the following decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{so}(p, q)$ of G :

$$\mathfrak{g} = \mathfrak{h}^o \oplus \mathfrak{q}^o.$$

Moreover, this decomposition is orthogonal with respect to the Killing form of \mathfrak{g} .

Take a Cartan involution τ commuting with σ^o : such involutions always exist and two of them differ by conjugation by an element in H^o (see Matsuki [38, Lemma 4]). Let $K^\tau := \text{Fix}(\tau)$, which is a maximal compact subgroup of G . Let \mathfrak{p}^τ and \mathfrak{k}^τ be the subspaces defined in Subsection 2.1. From the commutation of σ^o and τ the following holds:

$$\mathfrak{g} = (\mathfrak{p}^\tau \cap \mathfrak{q}^o) \oplus (\mathfrak{p}^\tau \cap \mathfrak{h}^o) \oplus (\mathfrak{k}^\tau \cap \mathfrak{q}^o) \oplus (\mathfrak{k}^\tau \cap \mathfrak{h}^o).$$

Let $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ be a (necessarily abelian) maximal subalgebra: two of them differ by conjugation by an element in $K^\tau \cap H^o$. We choose a closed Weyl chamber \mathfrak{b}^+ in \mathfrak{b} , corresponding to a positive system of restricted roots of \mathfrak{b} in $\mathfrak{g}^{\sigma^o \tau} := (\mathfrak{p}^\tau \cap \mathfrak{q}^o) \oplus (\mathfrak{k}^\tau \cap \mathfrak{h}^o)$.

Example 4.1. Let $o = [0, \dots, 0, 1]$. Recall from Subsection 2.2.2 that H^o is the upper left corner embedding of $O(p, q-1)$ in G and the involution σ^o is obtained by conjugation by $J^o = \text{diag}(-1, \dots, -1, 1)$. One sees that \mathfrak{h}^o equals the upper left corner embedding of $\mathfrak{so}(p, q-1)$ in $\mathfrak{so}(p, q)$ and that

$$\mathfrak{q}^o = \left\{ \begin{pmatrix} 0 & 0 & Y_1 \\ 0 & 0 & Y_2 \\ Y_1^t & -Y_2^t & 0 \end{pmatrix} : Y_1 \in M(p \times 1, \mathbb{R}), Y_2 \in M((q-1) \times 1, \mathbb{R}) \right\}.$$

Let τ be the Cartan involution of Example 2.2. One observes that τ commutes with σ^o and

$$\mathfrak{p}^\tau \cap \mathfrak{q}^o = \{X \in \mathfrak{q}^o : Y_2 = 0\} \quad \mathfrak{k}^\tau \cap \mathfrak{q}^o = \{X \in \mathfrak{q}^o : Y_1 = 0\}.$$

Pick \mathfrak{b} to be the subset of $\mathfrak{p}^\tau \cap \mathfrak{q}^o$ of matrices with Y_1 of the form

$$\begin{pmatrix} s \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for some $s \in \mathbb{R}$: this is a maximal subalgebra of $\mathfrak{p}^\tau \cap \mathfrak{q}^o$. A Weyl chamber \mathfrak{b}^+ is defined by the inequality $s \geq 0$. ◇

The following remark will be used repeatedly in the sequel.

Remark 4.2. Let $o \in \mathbb{H}^{p,q-1}$ and τ a Cartan involution commuting with σ^o . Consider a K^τ -invariant norm $\|\cdot\|_\tau$ on \mathbb{R}^d . Then J^o preserves $\|\cdot\|_\tau$. Indeed, this is obvious for the choices of Example 4.1 and follows in general by conjugating by an element g in G which takes $[0, \dots, 0, 1]$ to o . ◇

4.2. **The sub-manifold S^o .** Fix $o \in \mathbb{H}^{p,q-1}$ and define

$$S^o := \{\tau \in X_G : \tau\sigma^o = \sigma^o\tau\}.$$

Remark 4.3. Recall from Remark 2.1 that X_G can be identified with the space of q -dimensional negative definite subspaces of \mathbb{R}^d . Under this identification S^o correspond to the set of subspaces that contain the line o . By considering the $\langle \cdot, \cdot \rangle_{p,q}$ -orthogonal complement we see that S^o parametrizes the space of totally geodesic space-like copies of \mathbb{H}^p inside $\mathbb{H}^{p,q-1}$ passing through o (c.f. Subsection 2.2.6). ◇

Using the fact that two elements of S^o differ by conjugation by an element in H^o one observes that for any $\tau \in S^o$ the following holds

$$S^o = H^o.\tau.$$

The group H^o has several (but finite) connected components. One can see that these components differ by right translation by elements in H^o that fix some $\tau \in S^o$. Hence S^o is connected and

$$S^o = \exp(\mathfrak{p}^\tau \cap \mathfrak{h}^o).\tau.$$

It follows that $S^o \subset X_G$ is a totally geodesic sub-manifold of X_G and $T_\tau S^o \cong \mathfrak{p}^\tau \cap \mathfrak{h}^o$ (see [23, Theorem 7.2 of Ch. IV]).

4.3. **$K \exp(\mathfrak{b}^+)H$ -decomposition.** We now define the polar projection of G and give geometric and linear-algebraic interpretations of it. Fix $o \in \mathbb{H}^{p,q-1}$, $\tau \in S^o$ and $\mathfrak{b}^+ \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ as in Subsection 4.1. Recall that $|\cdot|_o$ is the form on \mathfrak{q}^o defined in Remark 2.3, which is positive definite on $\mathfrak{p}^\tau \cap \mathfrak{q}^o$.

4.3.1. **Definition.** By Schlichtkrull [53, Proposition 7.1.3] the following decomposition of G holds:

$$(4.1) \quad G = K^\tau \exp(\mathfrak{b}^+)H^o$$

where the $\exp(\mathfrak{b}^+)$ -component is uniquely determined and one can define

$$(4.2) \quad b^\tau : G \longrightarrow \mathfrak{b}^+$$

by taking the log of this component⁷. This is a continuous map called the *polar projection* of G associated to the choice of τ and \mathfrak{b}^+ . It generalizes the usual Cartan projection of G .

Remark 4.4. Note that b^τ is not proper (unless $q = 1$). However it descends to a map $\mathbb{H}^{p,q-1} \cong G/H^o \longrightarrow \mathfrak{b}^+$ which, by definition, is proper. ◇

4.3.2. **Geometric interpretation.** The geometric interpretation of the polar projection b^τ in $\mathbb{H}^{p,q-1}$ follows Kassel & Kobayashi [28, p.151]. The geometric interpretation in the Riemannian symmetric space X_G is inspired by the work of Oh & Shah [43] for the case $p = 1$ and $q = 3$.

Let us begin with the interpretation in the pseudo-Riemannian setting. Let $o \in \mathbb{H}^{p,q-1}$, $\tau \in S^o$. As we already discussed (c.f. Remark 4.3), this determines a totally geodesic space-like copy \mathbb{H}_τ^p of \mathbb{H}^p inside $\mathbb{H}^{p,q-1}$ passing through o . From explicit computations one can show that

$$\mathbb{H}_\tau^p = \exp(\mathfrak{p}^\tau \cap \mathfrak{q}^o).o.$$

⁷We will be interested in the sequel in the growth of $|b^\tau(\cdot)|_o$. Since two different choices of \mathfrak{b} differ by conjugation by an element in $H^o \cap K^\tau$, the function $|b^\tau(\cdot)|_o$ remains unchanged under this ambiguity hence we do not emphasize the dependence on the choice of \mathfrak{b} (and \mathfrak{b}^+) in the notations.

Let $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ be a maximal subalgebra. This determines the geodesic $\exp(\mathfrak{b}).o \subset \mathbb{H}_\tau^p$ and the choice of \mathfrak{b}^+ corresponds to the choice of one of the geodesic rays in $\exp(\mathfrak{b}).o$ starting from o . Equation (4.1) tell us that for every g in G the point $g.o$ lies in the K^τ -orbit of $o_g := \exp(b^\tau(g)).o \in \exp(\mathfrak{b}^+).o$. The geometric interpretation of the polar projection is now clear: the number $|b^\tau(g)|_o^{\frac{1}{2}}$ equals the length of the geodesic segment connecting o with o_g (see Figure 1).

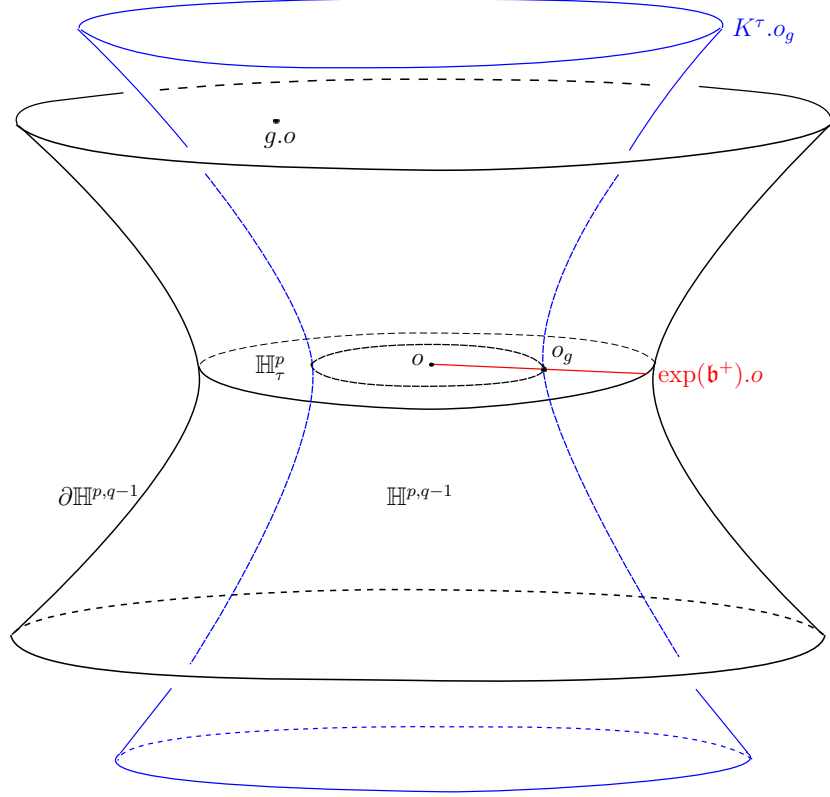


FIGURE 1. Geometric interpretation of polar projection.

We now turn our attention to the Riemannian symmetric space X_G .

Proposition 4.5. *Let b^τ be a polar projection associated to τ . Then for every g in G one has*

$$|b^\tau(g)|_o^{\frac{1}{2}} = d_{X_G}(g^{-1}.\tau, S^o).$$

Proof. The function $g \mapsto d_{X_G}(g^{-1}.\tau, S^o)$ is K^τ -invariant on the left and H^o -invariant on the right, hence it suffices to check that the equality on the statement holds for $g = \exp(X)$ with $X \in \mathfrak{b}^+$.

Since X_G is non-positively curved, there exists a unique geodesic trough $\exp(-X).\tau$ which is orthogonal to $S^o = \exp(\mathfrak{p}^\tau \cap \mathfrak{h}^o).\tau$. This geodesic is $\exp(\mathfrak{b}).\tau$ and meets S^o in τ , hence

$$d_{X_G}(\exp(-X).\tau, S^o) = d_{X_G}(\exp(-X).\tau, \tau).$$

Thanks to Remark 2.3 and equation (2.2) the proof is complete. \square

4.3.3. Linear-algebraic interpretation. Let $\|\cdot\|_\tau$ be the operator norm induced by the homothety class of norms in \mathbb{R}^d preserved by K^τ . Recall from Subsection 2.2.2 that J° is the matrix defining the involution σ° : commutes with elements of H° and for $X \in \mathfrak{b}$ satisfies $J^\circ \exp(X) = \exp(-X)J^\circ$.

Proposition 4.6. *For all g in G one has*

$$|b^\tau(g)|_\delta^{\frac{1}{2}} = \frac{1}{2} \log \|J^\circ g J^\circ g^{-1}\|_\tau.$$

Proof. We prove the proposition for the particular choices of Example 4.1, the general case follows from this one by conjugating by appropriate elements in G .

By Remark 4.2 the matrix J° preserves $\|\cdot\|_\tau$ thus

$$\frac{1}{2} \log \|J^\circ g J^\circ g^{-1}\|_\tau = \frac{1}{2} \log \|g J^\circ g^{-1}\|_\tau.$$

The map $g \mapsto \frac{1}{2} \log \|g J^\circ g^{-1}\|_\tau$ is K^τ -invariant on the left and H° -invariant on the right, hence it remains to check that the equality of the statement holds on $\exp(\mathfrak{b}^+)$. Let $X \in \mathfrak{b}^+$, that is,

$$X = \begin{pmatrix} & & s \\ & \cdot \cdot & \\ s & & \end{pmatrix}$$

for some $s \geq 0$. We have $|X|_\delta^{\frac{1}{2}} = s = \frac{1}{2} \log \|\exp(X) J^\circ \exp(-X)\|_\tau$. \square

4.4. $H \exp(\mathfrak{b}^+) H$ -decomposition. The structure of this subsection is the same as that of Subsection 4.3. Fix $o \in \mathbb{H}^{p,q-1}$. For elements g in G such that the geodesic connecting o and $g.o$ is space-like we define a decomposition in the same spirit of the usual Cartan decomposition of G . In order to do that, we fix a ray $\mathfrak{b}^+ \subset \mathfrak{q}^\circ$ on which the form $|\cdot|_o$ of Remark 2.3 is positive definite.

Let us begin with a remark.

Remark 4.7. There exists $\tau \in S^\circ$ such that $\mathfrak{b}^+ \subset \mathfrak{p}^\tau \cap \mathfrak{q}^\circ$. Indeed, by Remark 2.4 the action of H° on space-like geodesic rays in $T_o \mathbb{H}^{p,q-1}$ is transitive. Fix $\mathfrak{b}_1^+ \subset \mathfrak{p}^{\tau_1} \cap \mathfrak{q}^\circ$ for some $\tau_1 \in S^\circ$. There exists then $h \in H^\circ$ such that $h^{-1} \mathfrak{b}_1^+ h = \mathfrak{b}^+$ and thus

$$\mathfrak{b}^+ \subset h \mathfrak{p}^{\tau_1} h^{-1} \cap h \mathfrak{q}^\circ h^{-1}.$$

We have $\mathfrak{b}^+ \subset \mathfrak{p}^\tau \cap \mathfrak{q}^\circ$ where $\tau := h.\tau_1 \in S^\circ$. \diamond

4.4.1. Definition. Recall from Subsection 2.2.3 the definition of the set $\mathcal{C}_o^>$ and set

$$\mathcal{C}_{o,G}^> := \{g \in G : g.o \in \mathcal{C}_o^>\}.$$

Proposition 4.8. *Let $o \in \mathbb{H}^{p,q-1}$ and g be an element in $\mathcal{C}_{o,G}^>$. Then*

$$g = h \exp(X) h'$$

for some $h, h' \in H^\circ$ and a unique $X \in \mathfrak{b}^+$.

It is clear that this decomposition of g can only hold when $g \in \mathcal{C}_{o,G}^>$.

Proof of Proposition 4.8. Take h in H° such that $h^{-1} g.o \in \exp(\mathfrak{b}^+).o$. There exists then $X \in \mathfrak{b}^+$ and $h' \in H^\circ$ such that $h^{-1} g = \exp(X) h'$. Note that X is unique since it is determined by the length of the geodesic segment connecting o with $g.o$. \square

We define the map

$$(4.3) \quad b^o : \mathcal{C}_{o,G}^> \longrightarrow \mathfrak{b}^+ : g \mapsto b^o(g)$$

where $g = h \exp(b^o(g)) h'$ for some $h, h' \in H^\circ$.

Note that b^o descends to the quotient $\mathcal{C}_o^>$ but this map is not proper (compare with Remark 4.4).

4.4.2. Geometric interpretation. Recall that for $g \in \mathcal{C}_{o,G}^>$ we denote by $\ell_{o,g,o}$ the length of the (space-like) geodesic segment connecting o with $g.o$.

Proposition 4.9. *Let $o \in \mathbb{H}^{p,q-1}$ and g be an element in $\mathcal{C}_{o,G}^>$. Then*

$$\ell_{o,g,o} = |b^o(g)|_o^{\frac{1}{2}} = d_{X_G}(S^o, g.S^o).$$

Proof. The first equality was already discussed in the proof of Proposition 4.8. For the second one write $g = h \exp(b^o(g))h'$. Since $S^o = H^o.\tau$ we have

$$d_{X_G}(S^o, h \exp(b^o(g))h'.S^o) = d_{X_G}(H^o.\tau, \exp(b^o(g))H^o.\tau).$$

Set $X := b^o(g)$. If $X = 0$ there is nothing to prove, so assume $X \neq 0$. In that case $H^o.\tau$ is disjoint from $\exp(X)H^o.\tau$. Take $\tau' \in H^o.\tau$ and $\tau'' \in \exp(X)H^o.\tau$.

Claim 4.10. The following holds $d_{X_G}(\tau', \tau'') \geq d_{X_G}(\tau, \exp(X).\tau)$.

Proof of Claim 4.10. Let $\beta_1 \subset H^o.\tau$ (resp. $\beta_2 \subset \exp(X)H^o.\tau$) be the unit-speed geodesic connecting $\beta_1(0) = \tau$ (resp. $\beta_2(0) = \exp(X).\tau$) with τ' (resp. τ''). Then β_1 and β_2 are disjoint and from the fact that X_G is non-positively curved it follows that the map

$$(t, s) \mapsto d_{X_G}(\beta_1(t), \beta_2(s))$$

is smooth (see Petersen [46, p.129]). Moreover, since $\exp(\mathfrak{b}).\tau$ is orthogonal both to $H^o.\tau$ and $\exp(X)H^o.\tau$ we conclude that the differential at $(0, 0)$ of this map is zero.

Take $t_0 > 0$ such that $\beta_1(t_0) = \tau'$ and a positive a such that the geodesic $t \mapsto \beta_2(at)$ equals τ'' in t_0 . By Busemann [12, Theorem 3.6] the map

$$t \mapsto d_{X_G}(\beta_1(t), \beta_2(at))$$

is convex. Since it has a critical point in 0 the proof of the claim is finished. \square

Thanks to Remark 2.3 and equation (2.2) the proof of Proposition 4.9 is now complete. \square

4.4.3. Linear-algebraic interpretation. Recall that $J^o = (J^o)^{-1}$ is the matrix defining the involution σ^o (c.f. Subsection 2.2.2). We denote by $\lambda_1(g)$ the logarithm of the modulus of the top eigenvalue of the element g of G .

Proposition 4.11. *Let $o \in \mathbb{H}^{p,q-1}$ and g be an element in $\mathcal{C}_{o,G}^>$. Then*

$$|b^o(g)|_o^{\frac{1}{2}} = \frac{1}{2} \lambda_1(J^o g J^o g^{-1}).$$

Proof. It suffices to prove the proposition for the choices of o and \mathfrak{b}^+ of Example 4.1, the general case follows after conjugation by correct elements of G . Write $g = h \exp(b^o(g))h'$ with

$$b^o(g) = \begin{pmatrix} & & s \\ & \cdot & \\ s & & \end{pmatrix}$$

for some $s \geq 0$. We have $|b^o(g)|_o^{\frac{1}{2}} = s$. On the other hand, J^o commutes with elements of H^o and thus the number $\frac{1}{2} \lambda_1(J^o g J^o g^{-1})$ equals to

$$\frac{1}{2} \lambda_1(J^o h \exp(b^o(g))J^o \exp(b^o(g))^{-1} h^{-1}) = \frac{1}{2} \lambda_1(J^o \exp(b^o(g))J^o \exp(b^o(g))^{-1}).$$

Since $b^o(g) \in \mathfrak{q}^o$ we have $J^o \exp(b^o(g))^{-1} = \exp(b^o(g))J^o$ and the proof is complete. \square

5. PROXIMALITY

In this section we recall basic facts on product of proximal matrices, the main one being Benoist's Theorem 5.7. All these facts will be used in sections 6 to 9. The results presented here are well-known but we provide proofs for those which are not explicitly stated in the literature (the reader familiarized with these concepts may skip this section). Standard references for this section are the works of Benoist [2, 3, 4].

5.1. Notations and basic definitions. We begin with some notations.

A norm $\|\cdot\|$ on \mathbb{R}^d will be fixed from now on and through the whole the section. For $\xi_1, \xi_2 \in \mathbb{P}(\mathbb{R}^d)$ define the distance

$$d(\xi_1, \xi_2) := \inf\{\|v_{\xi_1} - v_{\xi_2}\| : v_{\xi_i} \in \xi_i \text{ and } \|v_{\xi_i}\| = 1 \text{ for all } i = 1, 2\}.$$

Let $\text{Gr}_{d-1}(\mathbb{R}^d)$ be the Grassmannian of $(d-1)$ -dimensional subspaces of \mathbb{R}^d . For $\eta_1, \eta_2 \in \text{Gr}_{d-1}(\mathbb{R}^d)$ we denote by $d(\xi_1, \eta_1)$ the distance from ξ_1 to the compact set $\mathbb{P}(\eta_1) \subset \mathbb{P}(\mathbb{R}^d)$ and by $d(\eta_1, \eta_2)$ the Hausdorff distance between $\mathbb{P}(\eta_1)$ and $\mathbb{P}(\eta_2)$. Given a positive ε we set

$$b_\varepsilon(\xi_1) := \{\xi \in \mathbb{P}(\mathbb{R}^d) : d(\xi_1, \xi) < \varepsilon\}$$

and

$$B_\varepsilon(\eta_1) := \{\xi \in \mathbb{P}(\mathbb{R}^d) : d(\xi, \eta_1) \geq \varepsilon\}.$$

On the other hand, let

$$\mathbb{P}^{(2)} := \{(\theta, v) \in \mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d) : v \notin \ker \theta\}$$

and

$$\mathbb{P}^{(4)} := \{(\theta, v, \phi, u) \in \mathbb{P}^{(2)} \times \mathbb{P}^{(2)} : v \notin \ker \phi \text{ and } u \notin \ker \theta\}.$$

Observe that

$$(5.1) \quad \mathcal{G}_{\|\cdot\|} = \mathcal{G} : \mathbb{P}^{(2)} \longrightarrow \mathbb{R} : \mathcal{G}(\theta, v) := \log \frac{|\theta(v)|}{\|\theta\| \|v\|}$$

is well-defined (c.f. Sambarino [51, p.472]). Similarly the following map is well-defined

$$(5.2) \quad \mathbb{B} : \mathbb{P}^{(4)} \longrightarrow \mathbb{R} : \mathbb{B}(\theta, v, \phi, u) := \log \left| \frac{\theta(u) \phi(v)}{\theta(v) \phi(u)} \right|$$

and is called de *cross-ratio* of (θ, v, ϕ, u) ⁸. Both \mathcal{G} and \mathbb{B} are continuous.

Remark 5.1. There exists a G -equivariant identification $\mathbb{P}((\mathbb{R}^d)^*) \longrightarrow \text{Gr}_{d-1}(\mathbb{R}^d)$ given by

$$\theta \mapsto \ker \theta$$

where the action of G on the left side is given by $g.\theta := \theta \circ g^{-1}$. We will sometimes abuse of notations and think \mathcal{G} and \mathbb{B} to be defined by elements of $\text{Gr}_{d-1}(\mathbb{R}^d)$ instead of $\mathbb{P}((\mathbb{R}^d)^*)$.

◇

5.2. Product of proximal matrices. Given g in $\text{End}(\mathbb{R}^d) \setminus \{0\}$ we denote by

$$\lambda_1(g) \geq \dots \geq \lambda_d(g)$$

the logarithm of the moduli of the eigenvalues of g , repeated with multiplicity (we use the convention $\log 0 = -\infty$). The matrix g is said to be *proximal* in $\mathbb{P}(\mathbb{R}^d)$ if $\lambda_1(g)$ is simple. In that case we let g_+ (resp. g_-) to be the attractive fixed line (resp. repelling fixed hyperplane) of g in $\mathbb{P}(\mathbb{R}^d)$. Note that if g is non invertible then g_- contains the kernel of g .

⁸Usually $e^{\mathbb{B}}$ is called the cross-ratio.

We now define a quantified version of proximality. The definition that we propose is (slightly) weaker to the one given by Benoist in [2, 3, 4]. We provide proofs of the basic facts established in those works when necessary.

Definition 5.2. Let $0 < \varepsilon \leq r$ and $g \in \text{End}(\mathbb{R}^d) \setminus \{0\}$ be a proximal matrix. The matrix g is called (r, ε) -proximal if $d(g_+, g_-) \geq 2r$ and $g(B_\varepsilon(g_-)) \subset b_\varepsilon(g_+)$.

Lemma 5.3 (Benoist [2, Corollaire 6.3]). *Let $0 < \varepsilon \leq r$. There exists a constant $c_{r,\varepsilon} > 0$ such that for every (r, ε) -proximal matrix g one has*

$$\log \|g\| - c_{r,\varepsilon} \leq \lambda_1(g) \leq \log \|g\|.$$

□

The following criterion of (r, ε) -proximality will be very useful in the sequel.

Lemma 5.4 (Benoist [2, Lemme 6.2]). *Let g be an element in $\text{End}(\mathbb{R}^d) \setminus \{0\}$, $\eta \in \text{Gr}_{d-1}(\mathbb{R}^d)$, $\xi \in \mathbb{P}(\mathbb{R}^d)$ and $0 < \varepsilon \leq r$. If $d(\xi, \eta) \geq 6r$ and $g(B_\varepsilon(\eta)) \subset b_\varepsilon(\xi)$ then g is $(2r, 2\varepsilon)$ -proximal with $d(g_+, \xi) \leq \varepsilon$ and $d(g_-, \eta) \leq \varepsilon$.*

Proof. Consider the Hilbert distance on the convex set $B_\varepsilon(\eta)$ (see [5]). The condition $g(B_\varepsilon(\eta)) \subset b_\varepsilon(\xi)$ implies that g is contracting for this metric and thus has a unique fixed point in $B_\varepsilon(\eta)$, which belongs in fact to $b_\varepsilon(\xi)$. The proof now finishes as in [2, Lemme 6.2].

□

Corollary 5.5 (Benoist [4, Lemme 1.4]). *Let $0 < \varepsilon \leq r$. If g_1 and g_2 are (r, ε) -proximal and satisfy*

$$d(g_{1+}, g_{2-}) \geq 6r \text{ and } d(g_{2+}, g_{1-}) \geq 6r$$

then $g_1 g_2$ is $(2r, 2\varepsilon)$ -proximal.

□

Let g_1 and g_2 be two matrices as in Corollary 5.5. The goal now is to state a theorem (Theorem 5.7) which provides a comparison between the top eigenvalue (and operator norm) of $g_1 g_2$ in terms of the highest eigenvalues of g_1 and g_2 and the maps \mathcal{G} and \mathbb{B} . These estimates will play a key roll in the proofs of Theorems A and B.

Lemma 5.6. *Fix $r > 0$ and $\delta > 0$. For every ε small enough, the following property is satisfied: for every pair of (r, ε) -proximal elements g_1 and g_2 such that*

$$d(g_{1+}, g_{2-}) \geq 6r \text{ and } d(g_{2+}, g_{1-}) \geq 6r$$

one has

$$|\mathcal{G}(g_{2-}, g_{1+}) - \mathcal{G}((g_1 g_2)_-, (g_1 g_2)_+)| < \delta.$$

Proof. For every $0 < \varepsilon \leq r$, consider the compact set $C_{r,\varepsilon}$ of pairs (g_1, g_2) of norm-one (r, ε) -proximal matrices in $\text{End}(\mathbb{R}^d) \setminus \{0\}$ satisfying

$$d(g_{1+}, g_{2-}) \geq 6r \text{ and } d(g_{2+}, g_{1-}) \geq 6r.$$

The function

$$(g_1, g_2) \mapsto |\mathcal{G}(g_{2-}, g_{1+}) - \mathcal{G}((g_1 g_2)_-, (g_1 g_2)_+)|$$

is continuous and equals zero on $C_r := \bigcap_{\varepsilon > 0} C_{r,\varepsilon} \subset \text{End}(\mathbb{R}^d) \setminus \{0\}$.

□

Theorem 5.7 (Benoist [4, Lemme 1.4]). *Fix $r > 0$ and $\delta > 0$. Then for every ε small enough, the following properties are satisfied: for every pair of (r, ε) -proximal elements g_1 and g_2 such that*

$$d(g_{1+}, g_{2-}) \geq 6r \text{ and } d(g_{2+}, g_{1-}) \geq 6r$$

one has:

(1) *The number*

$$\left| \lambda_1(g_1 g_2) - (\lambda_1(g_1) + \lambda_1(g_2)) - \mathbb{B}(g_{1-}, g_{1+}, g_{2-}, g_{2+}) \right|$$

is less than δ .

(2) *The number*

$$\left| \log \|g_1 g_2\| - (\lambda_1(g_1) + \lambda_1(g_2)) - \mathbb{B}(g_{1-}, g_{1+}, g_{2-}, g_{2+}) + \mathcal{G}(g_{2-}, g_{1+}) \right|$$

is less than δ .

Proof. (1) See [4, Lemme 1.4].

(2) Let ε be as in (1). For every g_1 and g_2 as in the statement Corollary 5.5 implies that $g_1 g_2$ is $(2r, 2\varepsilon)$ -proximal. Applying Sambarino [51, Lemma 5.6] and taking ε smaller if necessary we have

$$|\log \|g_1 g_2\| - \lambda_1(g_1 g_2) + \mathcal{G}((g_1 g_2)_-, (g_1 g_2)_+)| < \delta.$$

Lemma 5.6 finishes the proof. \square

Remark 5.8. There exist compact sets $C \subset \mathbb{P}^{(4)}$ and $C' \subset \mathbb{P}^{(2)}$ such that for every pair of matrices g_1 and g_2 as in the statement of Theorem 5.7 one has

$$(g_{1-}, g_{1+}, g_{2-}, g_{2+}) \in C \text{ and } (g_{2-}, g_{1+}) \in C'.$$

It follows that the quantities

$$\mathbb{B}(g_{1-}, g_{1+}, g_{2-}, g_{2+}) \text{ and } \mathcal{G}(g_{2-}, g_{1+})$$

are bounded. \diamond

6. PROJECTIVE ANOSOV REPRESENTATIONS

Anosov representations were introduced by Labourie [32] for surface groups and extended by Guichard & Wienhard [21] to word hyperbolic groups. In this section we recall the definition of (projective) Anosov representations and some well-known facts concerning (r, ε) -proximality of matrices in the image of such a representation. The reader familiarized with this concept can skip this section.

6.1. Singular values. The most useful characterization of Anosov representations for our purposes is the one given in terms of *singular values*. We begin by recalling this notion and we fix also some notations that we will use thorough the paper.

Let τ be a q -dimensional subspace of \mathbb{R}^d which is negative definite for $\langle \cdot, \cdot \rangle_{p,q}$. Consider $\langle \cdot, \cdot \rangle_\tau$ to be the inner product of \mathbb{R}^d that coincides with $-\langle \cdot, \cdot \rangle_{p,q}$ (resp. $\langle \cdot, \cdot \rangle_{p,q}$) on τ (resp. $\tau^\perp_{p,q}$) and for which τ and $\tau^\perp_{p,q}$ are orthogonal. Given g in $\mathrm{PSL}(d, \mathbb{R})$, we let $g^{*\tau}$ to be the adjoint operator with respect to $\langle \cdot, \cdot \rangle_\tau$. Set

$$a_1^\tau(g) \geq \dots \geq a_d^\tau(g)$$

to be the logarithm of the eigenvalues of $\sqrt{g^{*\tau} g}$ counted with multiplicity. These are called the τ -singular values of g . Geometrically, these numbers represent the (logarithm of the) lengths of the semi axes of the ellipsoid which is the image by g of the unit sphere

$$\mathbb{S}_\tau^{d-1} := \{x \in \mathbb{R}^d : \langle x, x \rangle_\tau = 1\}.$$

Let $i = 1, \dots, d-1$. Given an element g in $\mathrm{PSL}(d, \mathbb{R})$ such that $a_i^\tau(g) > a_{i+1}^\tau(g)$ we denote by $U_i(g)$ the i -dimensional subspace of \mathbb{R}^d spanned by the i biggest axes of $g(\mathbb{S}_\tau^{d-1})$. We also set

$$S_{d-i}(g) := U_{d-i}(g^{-1}).$$

Remark 6.1. Let $\varepsilon > 0$. It follows from Singular Value Decomposition (see Horn & Johnson [24, Section 7.3 of Chapter 7]), that there exists $L > 0$ such that for every g in $\mathrm{PSL}(d, \mathbb{R})$ satisfying $a_1^\tau(g) - a_2^\tau(g) > L$ one has

$$g(B_\varepsilon(S_{d-1}(g))) \subset b_\varepsilon(U_1(g)),$$

where $B_\varepsilon(S_{d-1}(g))$ and $b_\varepsilon(U_1(g))$ are defined as in Subsection 5.1.

◇

6.2. The definition of projective Anosov representations. A lot of work has been done in order to simplify the original definition of Anosov representations, here we follow mainly the work of Bochi & Potrie & Sambarino [6] (see also Guichard & Guéritaud & Kassel & Wienhard [22] or Kapovich & Leeb & Porti [25]).

Fix τ as in the previous subsection and let Γ be a finitely generated group. Consider a finite symmetric generating set S of Γ and take $|\cdot| = |\cdot|_S$ to be the associated word length: for γ in Γ , it is the minimum number required to write γ as a product of elements of S . Let $\rho : \Gamma \rightarrow \mathrm{PSL}(d, \mathbb{R})$ be a representation. We say that ρ is *projective Anosov* if there exist positive constants C and α such that for all $\gamma \in \Gamma$ one has⁹

$$(6.1) \quad a_1^\tau(\rho(\gamma)) - a_2^\tau(\rho(\gamma)) \geq \alpha|\gamma| - C.$$

By Kapovich & Leeb & Porti [27, Theorem 1.4] (see also [6, Section 3]), condition (6.1) implies that Γ is word hyperbolic¹⁰. Let $\partial_\infty\Gamma$ be its Gromov boundary and $\Gamma_{\mathbb{H}}$ be the set of infinite order elements of Γ . Every γ in $\Gamma_{\mathbb{H}}$ has exactly two fixed points in $\partial_\infty\Gamma$: the attractive one denoted by γ_+ and the repelling one denoted by γ_- . The dynamics of γ on $\partial_\infty\Gamma$ is of type *north-south*.

Fix $\rho : \Gamma \rightarrow \mathrm{PSL}(d, \mathbb{R})$ a projective Anosov representation and recall that $\mathrm{Gr}_{d-1}(\mathbb{R}^d)$ denotes the Grassmannian of $(d-1)$ -dimensional subspaces of \mathbb{R}^d . By [6, 22, 25] we know that there exist continuous equivariant maps

$$\xi : \partial_\infty\Gamma \rightarrow \mathbb{P}(\mathbb{R}^d) \text{ and } \eta : \partial_\infty\Gamma \rightarrow \mathrm{Gr}_{d-1}(\mathbb{R}^d)$$

which are *transverse*, i.e. for every $x \neq y$ in $\partial_\infty\Gamma$ one has

$$(6.2) \quad \xi(x) \oplus \eta(y) = \mathbb{R}^d.$$

In particular both ξ and η are homeomorphisms onto their images. In fact, these homeomorphisms are Hölder (see Bridgeman & Canary & Labourie & Sambarino [11, Lemma 2.5]).

One can see that condition (6.1) implies that for every γ in $\Gamma_{\mathbb{H}}$ the matrix $\rho(\gamma)$ is proximal. Equivariance implies then that

$$\xi(\gamma_+) = \rho(\gamma)_+ \text{ and } \eta(\gamma_+) = \rho(\gamma^{-1})_-.$$

In fact, these equalities characterize ξ and η .

We denote by $\Lambda_{\rho(\Gamma)} \subset \mathbb{P}(\mathbb{R}^d)$ the image of ξ , which is called the *limit set* of $\rho(\Gamma)$: it is the closure of the set of attractive fixed points in $\mathbb{P}(\mathbb{R}^d)$ of proximal elements in $\rho(\Gamma)$. The image of η is called the *dual limit set* of $\rho(\Gamma)$.

Here is another characterization of the limit sets which is very useful. An explicit reference is [22, Theorem 5.3] (it can also be deduced from [6, Subsection 3.4]).

Let $d = d_\tau$ be the distance on $\mathbb{P}(\mathbb{R}^d)$ associated to $\langle \cdot, \cdot \rangle_\tau$ (c.f. Subsection 5.1).

Proposition 6.2. *Let $\rho : \Gamma \rightarrow \mathrm{PSL}(d, \mathbb{R})$ be a projective Anosov representation. Then $\xi(\partial_\infty\Gamma)$ (resp. $\eta(\partial_\infty\Gamma)$) equals the set of accumulation points of sequences $\{U_1(\rho(\gamma_n))\}_n$ (resp. $\{S_{d-1}(\rho(\gamma_n))\}_n$) where $\gamma_n \rightarrow \infty$. Moreover, given a positive ε there exists $L > 0$ such that for every γ in $\Gamma_{\mathbb{H}}$ with $|\gamma| > L$ one has*

$$d(U_1(\rho(\gamma)), \rho(\gamma)_+) < \varepsilon \text{ and } d(S_{d-1}(\rho(\gamma)), \rho(\gamma)_-) < \varepsilon.$$

□

We are interested in projective Anosov representations whose image is contained in $G = \mathrm{PSO}(p, q)$. The following remark is then important for our purposes.

⁹For the equivalence between this definition and the one given in the Introduction see [6, 22, 25].

¹⁰We refer the reader to the book of Ghys & de la Harpe [18] for definitions and standard facts on word hyperbolic groups.

Remark 6.3. Let $\rho : \Gamma \rightarrow \mathrm{PSL}(d, \mathbb{R})$ be a projective Anosov representation. If $\rho(\Gamma)$ is contained in G we say that ρ is $P_1^{p,q}$ -Anosov (recall that $P_1^{p,q}$ denotes the (parabolic) subgroup of G stabilizing an isotropic line). In this case, the image of ξ is contained in $\partial\mathbb{H}^{p,q-1}$ and the dual map η equals $\xi^{\perp_{p,q}}$.

◇

6.3. Proximity properties. We keep the notations from the previous subsection. The following lemma will be useful in the next section.

Lemma 6.4 (Sambarino [51, Lemma 5.7]). *Let $\rho : \Gamma \rightarrow \mathrm{PSL}(d, \mathbb{R})$ be a projective Anosov representation and $0 < \varepsilon \leq r$. Then*

$$\#\{\gamma \in \Gamma_{\mathbb{H}} : d(\rho(\gamma)_+, \rho(\gamma)_-) \geq 2r \text{ and } \rho(\gamma) \text{ is not } (r, \varepsilon)\text{-proximal}\} < \infty.$$

Proof. Consider a sequence $\gamma_n \rightarrow \infty$ in $\Gamma_{\mathbb{H}}$ such that $d(\rho(\gamma_n)_+, \rho(\gamma_n)_-) \geq 2r$ for all n . By Proposition 6.2 for every n big enough the following holds

$$b_{\frac{\varepsilon}{2}}(U_1(\rho(\gamma_n))) \subset b_{\varepsilon}(\rho(\gamma_n)_+)$$

and

$$B_{\varepsilon}(\rho(\gamma_n)_-) \subset B_{\frac{\varepsilon}{2}}(S_{d-1}(\rho(\gamma_n))).$$

By Remark 6.1 and equation (6.1) the condition $\rho(\gamma_n)(B_{\varepsilon}(\rho(\gamma_n)_-)) \subset b_{\varepsilon}(\rho(\gamma_n)_+)$ is satisfied for sufficiently large n .

□

7. THE SET Ω_{ρ}

Fix a $P_1^{p,q}$ -Anosov representation $\rho : \Gamma \rightarrow G$ as in Section 6 and consider the set

$$\Omega_{\rho} := \{o \in \mathbb{H}^{p,q-1} : J^o \xi(x) \notin \eta(x) \text{ for all } x \in \partial_{\infty} \Gamma\}$$

where J^o is the matrix defined in Subsection 2.2.2. The goal of this section is to study the dynamics and the geometry of orbits $\gamma \mapsto \rho(\gamma).o$ for points o in Ω_{ρ} . More precisely, in Subsection 7.1 we show that the action of Γ on Ω_{ρ} is properly discontinuous (this is a well-known fact). In Subsection 7.2 we prove that, for some $0 < \varepsilon \leq r$, the matrix $J^o \rho(\gamma) J^o \rho(\gamma^{-1})$ is (r, ε) -proximal for *typical* γ in Γ and obtain some estimates for its highest eigenvalue and operator norm. These results are of major importance and will be used repeatedly in the sequel. Subsection 7.3 is devoted to the study of the orbital counting functions of Theorems A and B: we show that they are finite. Finally, in Subsection 7.4 we prove that for *typical* elements in Γ the geodesic connecting $o \in \Omega_{\rho}$ with $\rho(\gamma).o$ is space-like and we establish a weak version of the triangle inequality for lengths of geodesics that will be useful in the next section.

Before we start, let us discuss some examples for which Ω_{ρ} is non empty. From Proposition 3.1 we know that the following alternative description of Ω_{ρ} holds

$$\Omega_{\rho} = \{o = [\hat{o}] \in \mathbb{H}^{p,q-1} : \langle \hat{o}, \hat{\xi} \rangle_{p,q} \neq 0 \text{ for all } \xi = [\hat{\xi}] \in \Lambda_{\rho(\Gamma)}\}.$$

We have the following important example.

Example 7.1. • Let Γ be the fundamental group of a convex co-compact hyperbolic manifold of dimension $m \geq 2$ and $\iota_0 : \Gamma \rightarrow \mathrm{SO}(m, 1)$ be the holonomy representation. Fix $p \geq m$ and $q \geq 2$. Consider the embedding $\mathbb{R}^{m,1} \hookrightarrow \mathbb{R}^{p,q}$ given by

$$\mathbb{R}^{m,1} \cong \mathrm{span}\{e_{p-m+1}, \dots, e_{p+1}\},$$

where e_i denotes the vector of \mathbb{R}^d with all entries equal to zero except for the i -th entry which is equal to one. This induces a projection $j : \mathrm{SO}(m, 1) \rightarrow G$ and a representation $\rho_0 : \Gamma \rightarrow G$ defined by

$$\rho_0 := j \circ \iota_0.$$

Thus ρ_0 is $P_1^{p,q}$ -Anosov, because ι_0 is $P_1^{m,1}$ -Anosov. The set Ω_{ρ_0} is non empty: every point $o \in \mathbb{H}^{p,q-1}$ for which the plane

$$\text{span}\{o, e_{p+2}, \dots, e_d\}$$

has signature $(0, q)$ belongs to Ω_{ρ_0} . Since the condition of being Anosov is open in the space of all representations of Γ into G and the limit map ξ varies continuously with the representation (see Guichard & Wienhard [21, Theorem 5.13]), we obtain that if ρ is a small deformation of ρ_0 then Ω_ρ is non empty.

- The previous example generalizes to a large class of representations introduced by Danciger & Guéritaud & Kassel in [14, 15] called $\mathbb{H}^{p,q-1}$ -convex co-compact¹¹. Let $\Gamma < G$ be a $\mathbb{H}^{p,q-1}$ -convex co-compact group and $\rho : \Gamma \rightarrow G$ be the inclusion representation, which is $P_1^{p,q}$ -Anosov as proved in [15, Theorem 1.25]. Let Ω be a non empty Γ -invariant properly convex open subset of $\mathbb{H}^{p,q-1}$. By [15, Proposition 4.5] Ω is contained in Ω_ρ .

Note however that there exist examples of $P_1^{p,q}$ -Anosov representations ρ whose image is not convex co-compact but satisfy $\Omega_\rho \neq \emptyset$ (c.f. [14, Examples 5.2 & 5.3]).

◇

7.1. Dynamics on Ω_ρ . Observe that Ω_ρ is Γ -invariant. The goal of this subsection is to show that the action of Γ on Ω_ρ is properly discontinuous. This is a particular case of well-known results due to Guichard & Wienhard [21] and Kapovich & Leeb & Porti [26]. We provide a short proof for completeness.

Recall from Subsection 5.1 the definition of the sets $b_\varepsilon(\cdot)$ and $B_\varepsilon(\cdot)$ for a given $\varepsilon > 0$ and from Subsection 6.1 the definition of $U_1(\cdot)$ and $S_{d-1}(\cdot)$.

Proposition 7.2 ([21, 26]). *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation. Then the action of Γ on Ω_ρ is properly discontinuous, i.e. for every compact set $C \subset \Omega_\rho$ one has*

$$\#\{\gamma \in \Gamma : \rho(\gamma).C \cap C \neq \emptyset\} < \infty.$$

Moreover, for any o in Ω_ρ the set of accumulation points of $\rho(\Gamma).o$ in $\mathbb{H}^{p,q-1} \cup \partial\mathbb{H}^{p,q-1}$ coincides with the limit set $\Lambda_{\rho(\Gamma)}$.

Proof. Let $C \subset \Omega_\rho$ be a compact set and fix a distance on $\mathbb{P}(\mathbb{R}^d)$ as in Section 5. Take a positive ε such that

$$C \cap \bigcup_{x \in \partial_\infty \Gamma} b_\varepsilon(\xi(x)) = \emptyset \text{ and } C \subset \bigcap_{x \in \partial_\infty \Gamma} B_\varepsilon(\eta(x)).$$

By Proposition 6.2, Remark 6.1 and equation (6.1) we know that, up to a finite subset of elements γ in Γ , the following holds:

$$b_{\frac{\varepsilon}{2}}(U_1(\rho(\gamma))) \subset \bigcup_{x \in \partial_\infty \Gamma} b_\varepsilon(\xi(x)),$$

$$\bigcap_{x \in \partial_\infty \Gamma} B_\varepsilon(\eta(x)) \subset B_{\frac{\varepsilon}{2}}(S_{d-1}(\rho(\gamma)))$$

and

$$\rho(\gamma)(B_{\frac{\varepsilon}{2}}(S_{d-1}(\rho(\gamma)))) \subset b_{\frac{\varepsilon}{2}}(U_1(\rho(\gamma))).$$

For these γ we have then that $\rho(\gamma).C$ is contained in the ε -neighbourhood of $\Lambda_{\rho(\Gamma)}$ and thus is disjoint from C .

¹¹These are inclusion representations induced by taking an infinite discrete subgroup $\Gamma < G$ which preserves some properly convex non empty open set $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ whose boundary is strictly convex and of class C^1 . One requires that Γ preserves some *distinguished* non empty convex subset of Ω on which the action is co-compact (see [14, 15] for precisions).

We have shown that the action of Γ on Ω_ρ is properly discontinuous and that for any point o in Ω_ρ the accumulation points of $\rho(\Gamma).o$ belong to $\Lambda_{\rho(\Gamma)}$. Conversely, the Γ -orbit of any point in $\Lambda_{\rho(\Gamma)}$ is dense in the limit set and now the proof is complete. \square

7.2. Proximity of $J^o\rho(\gamma)J^o\rho(\gamma^{-1})$. We now study the matrices $J^o\rho(\gamma)J^o\rho(\gamma^{-1})$ for a point o in Ω_ρ .

Recall that for a point $\tau \in K^\tau$ we denote by K^τ its stabilizer in G and by $\|\cdot\|_\tau$ any norm on \mathbb{R}^d invariant by K^τ . Following Subsection 5.1 we denote by d_τ the induced distance on $\mathbb{P}(\mathbb{R}^d)$. The next lemma is a direct consequence of Proposition 6.2, transversality condition (6.2) and the definition of Ω_ρ .

Lemma 7.3. *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation, $o \in \Omega_\rho$ and $\tau \in S^o$. There exists a positive constant D such that*

$$\#\{\gamma \in \Gamma : d_\tau(J^oU_1(\rho(\gamma)), S_{d-1}(\rho(\gamma^{-1}))) < D\} < \infty.$$

\square

Lemma 7.4. *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation, $o \in \Omega_\rho$ and $\tau \in S^o$. Then there exist $0 < \varepsilon \leq r$ such that, up to a finite subset of elements $\gamma \in \Gamma$, the matrix $J^o\rho(\gamma)J^o\rho(\gamma^{-1})$ is (r, ε) -proximal.*

Proof. We apply a *ping-pong* argument together with Benoist's criterion of proximality (Lemma 5.4). By Lemma 7.3 we can take a positive constant r such that for all but finitely many $\gamma \in \Gamma$ one has

$$(7.1) \quad d_\tau(J^oU_1(\rho(\gamma)), S_{d-1}(\rho(\gamma^{-1}))) \geq 6r.$$

Without loss of generality assume that (7.1) is satisfied for every $\gamma \in \Gamma$. Take $0 < \varepsilon \leq r$ such that for every $\gamma \in \Gamma$ one has

$$b_\varepsilon(J^oU_1(\rho(\gamma))) \subset B_\varepsilon(S_{d-1}(\rho(\gamma^{-1}))).$$

By Remark 4.2 the matrix J^o preserves d_τ thus

$$J^o(b_\varepsilon(U_1(\rho(\gamma)))) \subset B_\varepsilon(S_{d-1}(\rho(\gamma^{-1})))$$

holds for every γ in Γ . Remark 6.1 and equation (6.1) imply that

$$\rho(\gamma^{-1})(B_\varepsilon(S_{d-1}(\rho(\gamma^{-1})))) \subset b_\varepsilon(U_1(\rho(\gamma^{-1})))$$

for all but finitely many elements γ in Γ . It follows that $J^o\rho(\gamma^{-1})(B_\varepsilon(S_{d-1}(\rho(\gamma^{-1})))) \subset B_\varepsilon(S_{d-1}(\rho(\gamma)))$ and applying $\rho(\gamma)$ we obtain

$$\rho(\gamma)J^o\rho(\gamma^{-1})(B_\varepsilon(S_{d-1}(\rho(\gamma^{-1})))) \subset b_\varepsilon(U_1(\rho(\gamma))).$$

Then

$$J^o\rho(\gamma)J^o\rho(\gamma^{-1})(B_\varepsilon(S_{d-1}(\rho(\gamma^{-1})))) \subset b_\varepsilon(J^o(U_1(\rho(\gamma)))).$$

By (7.1) and Lemma 5.4 the proof is finished. \square

The following is a strengthenin of Lemma 7.4 (it will only be used in the proof of Propositions 8.10 and 9.8). Recall from Subsection 5.1 the definitions of \mathcal{S}_τ and \mathbb{B} and recall also that Γ_H denotes the set of infinite order elements in Γ .

Lemma 7.5. *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation, $o \in \Omega_\rho$ and $\tau \in S^o$. Fix any $\delta > 0$ and A and B two compact disjoint sets in $\partial_\infty\Gamma$. Then there exists $0 < \varepsilon \leq r$ such that for all but finitely many elements $\gamma \in \Gamma_H$ with $\gamma_- \in A$ and $\gamma_+ \in B$ the following holds:*

- (1) *The matrices $J^o\rho(\gamma)J^o$ and $\rho(\gamma^{-1})$ are (r, ε) -proximal.*

- (2) $d_\tau(J^o\rho(\gamma)_+, \rho(\gamma^{-1})_-) \geq 6r$ and $d_\tau(\rho(\gamma^{-1})_+, J^o\rho(\gamma)_-) \geq 6r$.
(3) $d_\tau((J^o\rho(\gamma)J^o)_+, \rho(\gamma^{-1})_-) \geq 6r$ and $d_\tau(\rho(\gamma^{-1})_+, (J^o\rho(\gamma)J^o)_-) \geq 6r$.
(4) The number

$$\frac{1}{2}\lambda_1(J^o\rho(\gamma)J^o\rho(\gamma^{-1})) - \lambda_1(\rho(\gamma))$$

is at distance at most δ from

$$\frac{1}{2}\mathbb{B}(J^o\rho(\gamma)_-, J^o\rho(\gamma)_+, \rho(\gamma^{-1})_-, \rho(\gamma^{-1})_+).$$

- (5) The number

$$\frac{1}{2}\log\|J^o\rho(\gamma)J^o\rho(\gamma^{-1})\|_\tau - \lambda_1(\rho(\gamma))$$

is at distance at most δ from

$$\frac{1}{2}\mathbb{B}(J^o\rho(\gamma)_-, J^o\rho(\gamma)_+, \rho(\gamma^{-1})_-, \rho(\gamma^{-1})_+) - \frac{1}{2}\mathcal{G}_\tau(\rho(\gamma^{-1})_-, J^o\rho(\gamma)_+).$$

Proof. By transversality condition (6.2) there exists $r > 0$ such that

$$(7.2) \quad d_\tau(\xi(x), \eta(y)) \geq 2r \text{ and } d_\tau(\xi(y), \eta(x)) \geq 2r$$

for all $(x, y) \in A \times B$. Further, since $o \in \Omega_\rho$ we may assume

$$(7.3) \quad d_\tau(J^o\xi(x), \eta(x)) \geq 6r$$

for all $x \in \partial_\infty\Gamma$.

Given these $r > 0$ and $2\delta > 0$, we consider $\varepsilon > 0$ as in Benoist's Theorem 5.7.

By Lemma 6.4 we know that elements satisfying $d_\tau(\rho(\gamma)_+, \rho(\gamma)_-) \geq 2r$ are (r, ε) -proximal, up to a finite set in Γ_H . Thanks to (7.2), for all but finitely many $\gamma \in \Gamma_H$ with $\gamma_- \in A$ and $\gamma_+ \in B$ one has that $\rho(\gamma^{\pm 1})$ is (r, ε) -proximal. Moreover, since $J^o = (J^o)^{-1}$ preserves $\|\cdot\|_\tau$ we have that $J^o\rho(\gamma)J^o$ is (r, ε) -proximal with $(J^o\rho(\gamma)J^o)_\pm = J^o\rho(\gamma)_\pm$. In fact, by (7.3) we have

$$d_\tau(J^o\rho(\gamma)_+, \rho(\gamma^{-1})_-) \geq 6r \text{ and } d_\tau(\rho(\gamma^{-1})_+, J^o\rho(\gamma)_-) \geq 6r.$$

Theorem 5.7 together with the fact that $\lambda(\rho(\gamma^{-1}))$ equals $\lambda(\rho(\gamma))$ for all γ finish the proof. \square

7.3. The orbital counting functions of Theorems A and B. In this section we prove that the orbital counting functions involved in Theorems A and B are well-defined.

We keep the notations from the previous subsection.

Proposition 7.6. *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation, $o \in \Omega_\rho$ and $\tau \in S^o$. Then for every $t \geq 0$ one has*

$$\#\{\gamma \in \Gamma : \frac{1}{2}\log\|J^o\rho(\gamma)J^o\rho(\gamma^{-1})\|_\tau \leq t\} < \infty.$$

Proof. Let $t \geq 0$ and recall from Proposition 4.6 that

$$\frac{1}{2}\log\|J^o\rho(\gamma)J^o\rho(\gamma^{-1})\|_\tau = |b^\tau(\rho(\gamma))|_o^{\frac{1}{2}}$$

where b^τ is the polar projection defined in Subsection 4.3 and $|\cdot|_o$ is the form of Remark 2.3. We know by Remark 4.4 that b^τ defines a proper map in $\mathbb{H}^{p,q-1} \cong G/H^o$ hence

$$C := \{o' \in \mathbb{H}^{p,q-1} : |b^\tau(o')|_o \leq t^2\}$$

is compact. By Proposition 7.2 for all but finitely many γ in Γ we have that $\rho(\gamma).o$ does not belong to C . \square

Combining Lemma 7.4, Lemma 5.3 and the previous proposition we find the following result.

Proposition 7.7. *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation and $o \in \Omega_\rho$. Then for every $t \geq 0$ one has*

$$\#\{\gamma \in \Gamma : \frac{1}{2}\lambda_1(J^o\rho(\gamma)J^o\rho(\gamma^{-1})) \leq t\} < \infty.$$

□

7.4. Geometric properties of orbits on Ω_ρ . In this section we present two results concerning the geometry of orbits in Ω_ρ . The first one (Corollary 7.8) is a corollary of Proposition 7.2. Together with Propositions 4.9, 4.11 and 7.7 it recovers the fact that the orbital counting function of Glorieux & Monclair [19] is well-defined. The second one (Proposition 7.9) can be interpreted as a weak triangle inequality and it is inspired on [19, Theorem 3.5]. It will only be used in the proof of Proposition 8.10.

Let $o \in \Omega_\rho$. Recall the notations from Subsection 2.2.3. Given an open set $W \subset \partial\mathbb{H}^{p,q-1}$ disjoint with $\overline{\mathcal{C}}_o \cap \partial\mathbb{H}^{p,q-1}$ we denote by $\mathcal{C}_o^{>W}$ the subset of $\mathcal{C}_o^{>}$ consisting of points o' such that the (space-like) geodesic ray connecting o with o' has its end point in W .

Corollary 7.8. *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation, a point $o \in \Omega_\rho$ and $W \subset \partial\mathbb{H}^{p,q-1}$ an open set containing $\Lambda_{\rho(\Gamma)}$ with closure disjoint from $\overline{\mathcal{C}}_o \cap \partial\mathbb{H}^{p,q-1}$. Then up to a finite subset of elements γ in Γ one has $\rho(\gamma).o \in \mathcal{C}_o^{>W}$. In particular the geodesic joining o with $\rho(\gamma).o$ is space-like.*

Proof. Let C be the closure of $\mathbb{H}^{p,q-1} \setminus \mathcal{C}_o^{>W}$ in $\mathbb{H}^{p,q-1} \cup \partial\mathbb{H}^{p,q-1}$. Note that C is compact and by Proposition 7.2 does not contain accumulation points of $\rho(\Gamma).o$, hence $\rho(\Gamma).o \cap C$ is finite. Since $\gamma \mapsto \rho(\gamma).o$ is proper the proof is complete. □

Recall that $|\cdot|$ denotes the word length on Γ associated to a finite symmetric generating set.

Proposition 7.9. *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation and $o \in \Omega_\rho$. There exists a constant $L > 0$ such that for every $f \in \Gamma$ there exists $D_f > 0$ with the following property: for every $\gamma \in \Gamma$ with $|\gamma| > L$ one has*

$$\frac{1}{2}\lambda_1(J^o\rho(f)\rho(\gamma)J^o\rho(\gamma^{-1})\rho(f^{-1})) \leq D_f + \frac{1}{2}\lambda_1(J^o\rho(\gamma)J^o\rho(\gamma^{-1})).$$

Proof. Fix $\tau \in S^o$ and take $0 < \varepsilon \leq r$ as in Lemma 7.4. Let $L > 0$ such that for every $\gamma \in \Gamma$ with $|\gamma| > L$ the matrix $J^o\rho(\gamma)J^o\rho(\gamma^{-1})$ is (r, ε) -proximal. Fix $f \in \Gamma$ and let γ be an element in Γ with $|\gamma| > L$. We have

$$\frac{1}{2}\lambda_1(J^o\rho(f)\rho(\gamma)J^o\rho(\gamma^{-1})\rho(f^{-1})) \leq \frac{1}{2}\log \|J^o\rho(f)\rho(\gamma)J^o\rho(\gamma^{-1})\rho(f^{-1})\|_\tau.$$

By Remark 4.2 the right side number equals $\frac{1}{2}\log \|\rho(f)\rho(\gamma)J^o\rho(\gamma^{-1})\rho(f^{-1})\|_\tau$ which is less or equal than

$$D'_f + \frac{1}{2}\log \|J^o\rho(\gamma)J^o\rho(\gamma^{-1})\|_\tau$$

where $D'_f := \frac{1}{2}\log \|\rho(f)\|_\tau + \frac{1}{2}\log \|\rho(f^{-1})\|_\tau$. Since $J^o\rho(\gamma)J^o\rho(\gamma^{-1})$ is (r, ε) -proximal, we conclude by applying Lemma 5.3. □

8. DISTRIBUTION OF THE ORBIT OF o WITH RESPECT TO b^o

In this section we prove Theorem A. As we already said the proof relies on Sambarino's work [51]. In that paper the author deals with the case on which Γ is the fundamental group of a negatively curved closed manifold and study reparametrizations of its geodesic flow. However all the results presented in [51] generalize to word hyperbolic groups admitting an Anosov representation: as shown by Bridgeman & Canary & Labourie & Sambarino in [11, Sections 3 to 5] the Gromov geodesic flow of Γ admits a Markov

coding, hence all the results of [51] can be adapted to this more general setting (details are presented in Appendix A).

The section is structured as follows. In Subsection 8.1 we define a Hölder cocycle on $\partial_\infty\Gamma$ and its corresponding flow as in [51]. In Subsection 8.2 we study the associated Gromov product. In Subsection 8.3 we use this object to deduce a distribution result of fixed points in $\partial_\infty\Gamma$ of infinite order elements with respect b^ρ (Proposition 8.10). This proves Theorem A in the torsion free case. In Subsection 8.4 we use Proposition 8.10 to deduce a distribution result for the orbit of o in $\mathbb{H}^{p,q-1}$ which allows us to include torsion in the statement and then to prove Theorem A in the general case (Proposition 8.12).

Trough this section we fix $\rho : \Gamma \rightarrow G$ a $P_1^{p,q}$ -Anosov representation as in Section 6 and a point o in Ω_ρ (c.f. Section 7).

8.1. The cocycle c_o . We first define a flow which is a Hölder reparametrization of the Gromov geodesic flow of Γ . In order to do this, observe that by definition of Ω_ρ and equivariance of the curves ξ and η the following map is well-defined.

Definition 8.1. Let

$$c_o : \Gamma \times \partial_\infty\Gamma \rightarrow \mathbb{R} : c_o(\gamma, x) := \frac{1}{2} \log \left| \frac{\theta_x(\rho(\gamma^{-1})J^\rho\rho(\gamma)v_x)}{\theta_x(J^\rho v_x)} \right|,$$

where $\theta_x : \mathbb{R}^d \rightarrow \mathbb{R}$ is a non-zero linear functional whose kernel equals $\eta(x)$ and $v_x \neq 0$ belongs to $\xi(x)$.

A geometric interpretation of the map c_o is provided by the following remark. We point out however that this characterization will not be used in the sequel.

Remark 8.2. One can prove that for every $\gamma \in \Gamma$ and $x \in \partial_\infty\Gamma$ one has

$$c_o(\gamma, x) = \beta_{\xi(x)}(\rho(\gamma^{-1})o, o)$$

where $\beta(\cdot, \cdot)$ is the pseudo-Riemannian Busemann function defined by Glorieux & Monclair [19, Definition 3.8].

◇

Recall that a *Hölder cocycle* is a function $c : \Gamma \times \partial_\infty\Gamma \rightarrow \mathbb{R}$ satisfying that for every γ_0, γ_1 in Γ and $x \in \partial_\infty\Gamma$ one has

$$c(\gamma_0\gamma_1, x) = c(\gamma_0, \gamma_1x) + c(\gamma_1, x)$$

and such that the map $c(\gamma_0, \cdot)$ is Hölder (with the same exponent for every γ_0). The *period* of $\gamma \in \Gamma_{\mathbb{H}}$ is defined by $\ell_c(\gamma) := c(\gamma, \gamma_+)$ where γ_+ is the attractor of γ in $\partial_\infty\Gamma$ (recall that $\Gamma_{\mathbb{H}}$ denotes the set of infinite order elements in Γ).

Lemma 8.3. *The map c_o is a Hölder cocycle. The period of $\gamma \in \Gamma_{\mathbb{H}}$ is given by*

$$\ell_{c_o}(\gamma) = \lambda_1(\rho(\gamma)) > 0.$$

Proof. A direct computation shows that c_o is a Hölder cocycle.

On the other hand let $\gamma \in \Gamma_{\mathbb{H}}$ and fix a particular choice of a linear functional θ_{γ_+} . Since $\lambda_1(\rho(\gamma)) = \lambda_1(\rho(\gamma^{-1}))$ one sees that $\theta_{\gamma_+} \circ (\pm\rho(\gamma^{-1}))$ coincides with $e^{\lambda_1(\rho(\gamma))}\theta_{\gamma_+}$ up to a sign (here $\pm\rho(\gamma^{-1})$ denotes some lift of $\rho(\gamma^{-1})$ to $\text{SO}(p, q)$). The proof is now complete.

□

Following Sambarino [51] we set $\partial_\infty^2\Gamma := \{(x, y) \in \partial_\infty\Gamma \times \partial_\infty\Gamma : x \neq y\}$ and consider the flow on $\partial_\infty^2\Gamma \times \mathbb{R}$ defined by

$$(8.1) \quad \psi_t(x, y, s) := (x, y, s - t).$$

The group Γ acts on $\partial_\infty^2\Gamma \times \mathbb{R}$ by

$$(8.2) \quad \gamma.(x, y, s) := (\gamma x, \gamma y, s - c_o(\gamma, y)).$$

This action is proper and co-compact and the quotient space $U_o\Gamma$ is homeomorphic to the unit tangent bundle of Γ . The flow ψ_t descends to a flow on $U_o\Gamma$, still denoted ψ_t , which is a Hölder reparametrization of the Gromov geodesic flow of Γ . This is the analogue of [51, Theorem 3.2(1)] (see Lemma A.7).

For an element γ in Γ we denote by $[\gamma]$ its conjugacy class and we say that it is *primitive* if cannot be written as a positive power of another element in Γ . Periodic orbits of ψ_t are in one-to-one correspondence with conjugacy classes of primitive elements in Γ . If $[\gamma]$ is such a conjugacy class, the period of the corresponding periodic orbit is

$$\ell_{c_o}(\gamma) = \lambda_1(\rho(\gamma))$$

(see Fact A.1 and Lemma A.7). The topological entropy of ψ_t coincides with the *entropy* of ρ defined by Bridgeman & Canary & Labourie & Sambarino [11]:

$$h_{\text{top}}(\psi_t) = h_\rho := \limsup_{t \rightarrow \infty} \frac{\log \#\{[\gamma] \in [\Gamma] \text{ primitive} : \lambda_1(\rho(\gamma)) \leq t\}}{t}.$$

It is positive and finite (c.f. Fact A.3) and will be denoted by h from now on.

Remark 8.4. One can prove that if we *push* all this construction by the limit map $\xi : \partial_\infty\Gamma \rightarrow \Lambda_{\rho(\Gamma)}$ we recover, up to reversing time, the geodesic flow defined in [19, Subsection 6.1] for $\mathbb{H}^{p,q-1}$ -convex co-compact groups. This remark will not be used in the sequel.

◇

8.2. Dual cocycle and Gromov product. We now introduce an object which is useful to describe the probability of maximal entropy of ψ_t and the distribution of fixed points in the boundary of elements γ in Γ_H .

Remark 8.5. The cocycle c_o is dual to itself, i.e. $\ell_{c_o}(\gamma) = \ell_{c_o}(\gamma^{-1})$ for every $\gamma \in \Gamma_H$. Indeed, this follows from Lemma 8.3 and the fact that $\lambda_1(g) = \lambda_1(g^{-1})$ for all g in G .

◇

Thanks to transversality condition (6.2) and the fact that o belongs to Ω_ρ the following map is well-defined.

Definition 8.6. Let

$$[\cdot, \cdot]_o : \partial_\infty^2\Gamma \rightarrow \mathbb{R} : [x, y]_o := -\frac{1}{2} \log \left| \frac{\theta_x(J^o v_x) \theta_y(J^o v_y)}{\theta_x(v_y) \theta_y(v_x)} \right|,$$

where θ_x (resp. θ_y) is a non-zero linear functional whose kernel is $\eta(x)$ (resp. $\eta(y)$) and v_x (resp. v_y) is a non-zero vector in $\xi(x)$ (resp. $\xi(y)$)¹².

Lemma 8.7. *The map $[\cdot, \cdot]_o$ is a Gromov product for the pair $\{c_o, c_o\}$, that is, for every $\gamma \in \Gamma$ and every $(x, y) \in \partial_\infty^2\Gamma$ one has*

$$[\gamma x, \gamma y]_o - [x, y]_o = -(c_o(\gamma, x) + c_o(\gamma, y)).$$

Proof. Direct computation. □

Recall from Section 5 that \mathbb{B} denotes the cross-ratio between two lines and two hyperplanes transverse to those lines.

Lemma 8.8. *Let γ be an element of Γ_H . Then*

$$[\gamma_-, \gamma_+]_o = -\frac{1}{2} \mathbb{B}(J^o \rho(\gamma)_-, J^o \rho(\gamma)_+, \rho(\gamma^{-1})_-, \rho(\gamma^{-1})_+).$$

Proof. From Section 6 we know that $\rho(\gamma^{\pm 1})$ is proximal and that the following holds:

$$\rho(\gamma)_+ = \xi(\gamma_+), \quad \rho(\gamma^{-1})_+ = \xi(\gamma_-), \quad \rho(\gamma)_- = \eta(\gamma_-), \quad \rho(\gamma^{-1})_- = \eta(\gamma_+).$$

Since $J^o = (J^o)^{-1}$ the matrix $J^o \rho(\gamma) J^o$ is proximal and

¹²This map coincides, up to a sign, with the Gromov product of [19, Subsection 3.5].

$$(J^\circ \rho(\gamma) J^\circ)_+ = J^\circ \xi(\gamma_+)$$

and

$$(J^\circ \rho(\gamma) J^\circ)_- = J^\circ \eta(\gamma_-).$$

The proof is now a direct computation. \square

8.3. Distribution of attractors & repellers with respect to b° . As we said in the previous subsection, the Gromov product $[\cdot, \cdot]_o$ is useful to describe the probability of maximal entropy of ψ_t . Indeed recall that $h = h_{\text{top}}(\psi_t)$ and let μ_o be a *Patterson-Sullivan probability* on $\partial_\infty \Gamma$ associated to c_o , i.e. μ_o satisfies

$$\frac{d\gamma_* \mu_o}{d\mu_o}(x) = e^{-hc_o(\gamma^{-1}, x)}$$

for every $\gamma \in \Gamma$ (such a probability exists, see Subsection A.2.2). By Lemma 8.7 the measure

$$(8.3) \quad e^{-h[\cdot, \cdot]_o} \mu_o \otimes \mu_o \otimes dt$$

on $\partial_\infty^2 \Gamma \times \mathbb{R}$ is Γ -invariant and induces on the quotient a ψ_t -invariant measure. By Sambarino [51, Theorem 3.2(2)] this measure is, up to scaling, the probability of maximal entropy of ψ_t (see Proposition A.12).

Sambarino also obtains a result on distribution of fixed points in $\partial_\infty \Gamma$, which we now state. For a metric space X we denote by $C_c^*(X)$ the dual of the space of compactly supported continuous real functions on X equipped with the weak-star topology. If x is a point in X , let $\delta_x \in C_c^*(X)$ be the Dirac mass at x .

Proposition 8.9 (Sambarino [51, Proposition 4.3]¹³). *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation and $o \in \Omega_\rho$. There exists a constant $M = M_{\rho, o} > 0$ such that*

$$M e^{-ht} \sum_{\gamma \in \Gamma_{\mathbb{H}, \ell_{c_o}(\gamma)} \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \rightarrow e^{-h[\cdot, \cdot]_o} \mu_o \otimes \mu_o$$

as $t \rightarrow \infty$ on $C_c^*(\partial_\infty^2 \Gamma)$. \square

From Proposition 8.9 we deduce Proposition 8.10 which directly implies Theorem A in the torsion free case. We go from a conjugacy invariant result to one that is not. In order to do that we eliminate the Gromov product from the convergence in Proposition 8.9: this is achieved by an application of Lemma 7.5 (item (4)) and Lemma 8.8.

The proof follows almost line by line the proof of [51, Theorem 6.5].

Proposition 8.10. *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation and $o \in \Omega_\rho$. There exists a constant $M > 0$ such that*

$$M e^{-ht} \sum_{\gamma \in \Gamma_{\mathbb{H}, \frac{1}{2}\lambda_1(J^\circ \rho(\gamma) J^\circ \rho(\gamma^{-1}))} \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \rightarrow \mu_o \otimes \mu_o$$

as $t \rightarrow \infty$ on $C^*(\partial_\infty \Gamma \times \partial_\infty \Gamma)$.

Proof. Set

$$\theta_t := M e^{-ht} \sum_{\gamma \in \Gamma_{\mathbb{H}, \frac{1}{2}\lambda_1(J^\circ \rho(\gamma) J^\circ \rho(\gamma^{-1}))} \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+}.$$

¹³For a proof of this result in our context see Proposition A.13.

We first proof the statement outside the diagonal. Let $\delta > 0$ and $A, B \subset \partial_\infty \Gamma$ disjoint open sets. Consider an element $\gamma \in \Gamma_{\mathbb{H}}$ such that $\gamma_- \in A$ and $\gamma_+ \in B$ and let $s := [\gamma_-, \gamma_+]_o$. By taking A and B smaller we may assume

$$(8.4) \quad |[x, y]_o - s| < \delta$$

for all $(x, y) \in A \times B$. We assume further that A and B have disjoint closures.

Let $\tau \in S^o$. By Lemma 7.5 there exists $0 < \varepsilon \leq r$ such that for all but finitely many elements $\gamma \in \Gamma_{\mathbb{H}}$ with $(\gamma_-, \gamma_+) \in A \times B$ the following holds:

$$\left| \frac{1}{2} \lambda_1(J^o \rho(\gamma) J^o \rho(\gamma^{-1})) - \lambda_1(\rho(\gamma)) - \frac{1}{2} \mathbb{B}(J^o \rho(\gamma)_-, J^o \rho(\gamma)_+, \rho(\gamma^{-1})_-, \rho(\gamma^{-1})_+) \right| < \delta.$$

Applying Lemma 8.3 and Lemma 8.8 we conclude that

$$\left| \frac{1}{2} \lambda_1(J^o \rho(\gamma) J^o \rho(\gamma^{-1})) - \ell_{c_o}(\gamma) + [\gamma_-, \gamma_+]_o \right| < \delta.$$

By (8.4) it follows that

$$\ell_{c_o}(\gamma) - s - 2\delta < \frac{1}{2} \lambda_1(J^o \rho(\gamma) J^o \rho(\gamma^{-1})) < \ell_{c_o}(\gamma) - s + 2\delta$$

for all but finite $\gamma \in \Gamma_{\mathbb{H}}$ such that $\gamma_- \in A$ and $\gamma_+ \in B$. From now on, the proof of the convergence

$$\theta_t(A \times B) \longrightarrow \mu_o(A) \mu_o(B)$$

follows line by line the proof of Sambarino's Theorem [51, Theorem 6.5].

It remains to prove the convergence in the diagonal, but once again, the proof is the same as the one of given in [51, Theorem 6.5]. For completeness we briefly sketch it.

Since μ_o has no atoms (see Lemma A.10), for every γ in Γ the diagonal has $\mu_o \otimes \gamma_* \mu_o$ -measure equal to zero. We fix two elements $\gamma_0, \gamma_1 \in \Gamma_{\mathbb{H}}$ with no common fixed point in $\partial_\infty \Gamma$ and let $\varepsilon_0 > 0$. There exists a finite open covering \mathcal{U} of $\partial_\infty \Gamma$ such that for $i = 0, 1$ one has

$$\sum_{U \in \mathcal{U}} \mu_o(U) \mu_o(\gamma_i^{-1}(U)) < \varepsilon_0.$$

We can assume that for every $U \in \mathcal{U}$ there exists $i \in \{0, 1\}$ such that $\gamma_i^{-1}(\bar{U})$ is disjoint from \bar{U} . There exists an open covering \mathcal{V} of $\partial_\infty \Gamma$ with the following properties:

- (1) $\sum_{V \in \mathcal{V}} \mu_o(V) \mu_o(\gamma_i^{-1}(V)) < \varepsilon_0$ for $i = 0, 1$.
- (2) The closure of every element in \mathcal{U} is contained in a unique element of \mathcal{V} and if $\gamma_i^{-1}(\bar{U})$ is disjoint from \bar{U} the same holds for this element in \mathcal{V} .
- (3) Suppose that $\gamma_i^{-1}(\bar{U}) \cap \bar{U} = \emptyset$ and let $V \in \mathcal{V}$ be the unique element such that $\bar{U} \subset V$. Then up to a finite subset of elements γ with $\gamma_\pm \in U$ one has $(\gamma_i^{-1} \gamma)_- \in V$ and $(\gamma_i^{-1} \gamma)_+ \in \gamma_i^{-1}(V)$.

Set $D := \max_{i=0,1} \{D_{\gamma_i^{-1}}\}$ where $D_{\gamma_i^{-1}}$ is the constant given by Proposition 7.9 and take $U \in \mathcal{U}$ as in (3). By Proposition 7.9 we have

$$\begin{aligned} \theta_t(U \times U) &\leq M e^{-ht} \sum_{\gamma \in \Gamma_{\mathbb{H}}, \frac{1}{2} \lambda_1(J^o \rho(\gamma) J^o \rho(\gamma^{-1})) \leq t+D} \delta_{\gamma_-}(V) \delta_{\gamma_+}(\gamma_i^{-1}(V)) \\ &\quad + M e^{-ht} \#F \end{aligned}$$

where F is a finite set independent of t . Since $V \times \gamma_i^{-1}(V)$ is far from the diagonal the right side converges to

$$e^D \mu_o(V) \mu_o(\gamma_i^{-1}(V))$$

as $t \rightarrow \infty$. Adding up in $U \in \mathcal{U}$ we conclude

$$\limsup_{t \rightarrow \infty} \sum_{U \in \mathcal{U}} \theta_t(U \times U) \leq 2e^D \varepsilon_0.$$

Hence $\theta_t(\{(x, x) : x \in \partial_\infty \Gamma\})$ converges to zero and since the diagonal has measure zero for $\mu_o \otimes \mu_o$ the proof is finished. \square

8.4. Proof of Theorem A. We now prove Theorem A.

The following is a corollary of Proposition 8.10. Recall that if $f : X \rightarrow Y$ is a map and m is a measure on X then $f_*(m)$ denotes the measure on Y defined by $A \mapsto m(f^{-1}(A))$.

Corollary 8.11. *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation and $o \in \Omega_\rho$. There exists a constant $M = M_{\rho,o} > 0$ such that*

$$Me^{-ht} \sum_{\gamma \in \Gamma_{\mathbb{H}}, \frac{1}{2}\lambda_1(J^\circ \rho(\gamma)J^\circ \rho(\gamma^{-1})) \leq t} \delta_{\rho(\gamma^{-1})o^{\perp p,q}} \otimes \delta_{\rho(\gamma)o} \rightarrow \eta_*(\mu_o) \otimes \xi_*(\mu_o)$$

on $C^*(\mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d))$ as $t \rightarrow \infty$.¹⁴

Proof. Set

$$\nu_t^{\mathbb{H}} := Me^{-ht} \sum_{\gamma \in \Gamma_{\mathbb{H}}, \frac{1}{2}\lambda_1(J^\circ \rho(\gamma)J^\circ \rho(\gamma^{-1})) \leq t} \delta_{\rho(\gamma^{-1})o^{\perp p,q}} \otimes \delta_{\rho(\gamma)o}$$

and take θ_t the measure defined in the proof of Proposition 8.10. We know that

$$(\eta, \xi)_*(\theta_t) \rightarrow \eta_*(\mu_o) \otimes \xi_*(\mu_o).$$

Hence we only have to show

$$(8.5) \quad \nu_t^{\mathbb{H}} - (\eta, \xi)_*(\theta_t) \rightarrow 0.$$

Let $\tau \in S^o$ and take a small positive δ . By Proposition 6.2 and the proof of Proposition 7.2 we know that for all but finite γ in $\Gamma_{\mathbb{H}}$ one has

$$d(\rho(\gamma)o, \rho(\gamma)_+) < \delta \text{ and } d(\rho(\gamma^{-1})o, \rho(\gamma^{-1})_+) < \delta.$$

By taking $\cdot^{\perp p,q}$ we can assume further that $d(\rho(\gamma^{-1})o^{\perp p,q}, \rho(\gamma)_-) < \delta$. Now the proof of (8.5) follows from evaluation on continuous functions of $\mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d)$. \square

We now include torsion elements to the previous statement and finish the proof of Theorem A.

Proposition 8.12. *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation and $o \in \Omega_\rho$. There exists a constant $M = M_{\rho,o} > 0$ such that*

$$Me^{-ht} \sum_{\gamma \in \Gamma, \frac{1}{2}\lambda_1(J^\circ \rho(\gamma)J^\circ \rho(\gamma^{-1})) \leq t} \delta_{\rho(\gamma^{-1})o^{\perp p,q}} \otimes \delta_{\rho(\gamma)o} \rightarrow \eta_*(\mu_o) \otimes \xi_*(\mu_o)$$

on $C^*(\mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d))$ as $t \rightarrow \infty$.

Proof. The structure of the proof is the same as that of Proposition 8.10, that is, we first prove the statement outside the diagonal and deduce from that the statement on the diagonal. Here by *diagonal* we mean the set

$$\Delta := \{(\theta, v) \in \mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d) : \theta(v) = 0\}.$$

Let

$$\nu_t := Me^{-ht} \sum_{\gamma \in \Gamma, \frac{1}{2}\lambda_1(J^\circ \rho(\gamma)J^\circ \rho(\gamma^{-1})) \leq t} \delta_{\rho(\gamma^{-1})o^{\perp p,q}} \otimes \delta_{\rho(\gamma)o}$$

¹⁴Recall from Section 5 that there exists an equivariant identification between $\mathbb{P}((\mathbb{R}^d)^*)$ and $\text{Gr}_{d-1}(\mathbb{R}^d)$.

and take ν_t^{H} as in the proof of Corollary 8.11.

Consider first a continuous function f on $\mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d)$ whose support $\text{supp}(f)$ is disjoint from Δ .

Claim 8.13. The following holds

$$\#\{\gamma \in \Gamma : (\rho(\gamma^{-1})o^{\perp p,q}, \rho(\gamma)o) \in \text{supp}(f) \text{ and } \gamma \notin \Gamma_{\text{H}}\} < \infty.$$

Proof of Claim 8.13. Fix $\tau \in S^o$ and take a positive D such that for every $(\theta, v) \in \text{supp}(f)$ one has $d(\theta, v) > D$. As we saw in the proof of Proposition 7.2, the distances

$$d(\rho(\gamma)o, U_1(\rho(\gamma))) \text{ and } d(\rho(\gamma^{-1})o^{\perp p,q}, S_{d-1}(\rho(\gamma)))$$

converge to zero as $\gamma \rightarrow \infty$. We conclude that, up to finitely many elements γ in Γ with $(\rho(\gamma^{-1})o^{\perp p,q}, \rho(\gamma)o) \in \text{supp}(f)$, one has

$$d(U_1(\rho(\gamma)), S_{d-1}(\rho(\gamma))) > D.$$

Now apply equation (6.1), Remark 6.1 and Benoist's Lemma 5.4 to conclude that for $|\gamma|$ large enough the matrix $\rho(\gamma)$ is proximal. \square

From Claim 8.13 we conclude that

$$\lim_{t \rightarrow \infty} \nu_t(f) = \lim_{t \rightarrow \infty} \nu_t^{\text{H}}(f)$$

which by Corollary 8.11 equals $(\eta_*(\mu_o) \otimes \xi_*(\mu_o))(f)$.

It remains to prove the convergence on the diagonal. It suffices to prove that for every positive ε_0 there exists an open covering $\{U^* \times U\}$ of Δ such that

$$\limsup_{t \rightarrow \infty} \nu_t \left(\bigcup (U^* \times U) \right) \leq \varepsilon_0.$$

The proof is the same as in Proposition 8.10. Namely, take two elements γ_0, γ_1 in Γ_{H} with no common fixed point in $\partial_{\infty}\Gamma$ and a coverings $\mathcal{U} = \{U^* \times U\}$ and $\mathcal{V} = \{V^* \times V\}$ of Δ by open sets with the following properties:

- (1) For every $U^* \times U$ in \mathcal{U} there exists $i = 0, 1$ such that $\rho(\gamma_i^{-1})(\overline{U})$ is transverse to $\overline{U^*}$.
- (2) $\sum_{V^* \times V \in \mathcal{V}} (\eta_*(\mu_o) \otimes \xi_*(\mu_o))(V^* \times \rho(\gamma_i^{-1})(V)) < \varepsilon_0$ for $i = 0, 1$.
- (3) The closure of every element in \mathcal{U} is contained in a unique element of \mathcal{V} and if $\rho(\gamma_i^{-1})(\overline{U})$ is transverse to $\overline{U^*}$ the same holds for this element in \mathcal{V} .
- (4) Suppose that $\rho(\gamma_i^{-1})(\overline{U})$ is transverse to $\overline{U^*}$ and let $V^* \times V \in \mathcal{V}$ be the unique element such that $\overline{U} \subset V$ and $\overline{U^*} \subset V^*$. Then up to a finite subset of elements γ with $(\rho(\gamma^{-1})o^{\perp p,q}, \rho(\gamma)o) \in U^* \times U$ one has

$$(\rho((\gamma_i^{-1}\gamma)^{-1})o^{\perp p,q}, \rho(\gamma_i^{-1}\gamma)o) \in V^* \times \rho(\gamma_i^{-1})(V).$$

Provided with this construction, the proof finishes in the same way as that of Proposition 8.10. \square

9. DISTRIBUTION OF THE ORBIT OF o WITH RESPECT TO b^{τ}

We now prove Theorem B. The main lines of the proof are the same to those of the proof of Theorem A, we just have to pick a (slightly) different flow ψ_t .

From now on we fix a $P_1^{p,q}$ -Anosov representation $\rho : \Gamma \rightarrow G$ as in Section 6, a point o in Ω_{ρ} (c.f. Section 7) and $\tau \in S^o$ where S^o is the sub-manifold defined in Subsection 4.2.

9.1. The cocycle c_τ . Let $\|\cdot\|_\tau$ be the class of norms in \mathbb{R}^d preserved by K^τ , the compact group of fixed points of τ in G .

Definition 9.1. Let

$$c_\tau : \Gamma \times \partial_\infty \Gamma \longrightarrow \mathbb{R} : c_\tau(\gamma, x) := \frac{1}{2} \log \left(\frac{\|\rho(\gamma) \cdot \theta_x\|_\tau \|\rho(\gamma) \cdot v_x\|_\tau}{\|\theta_x\|_\tau \|v_x\|_\tau} \right)$$

where $\theta_x : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a non-zero linear functional whose kernel equals $\eta(x)$ and $v_x \neq 0$ belongs to $\xi(x)$.

Remark 9.2. One can prove that for every $\gamma \in \Gamma$ and $x \in \partial_\infty \Gamma$ one has

$$c_\tau(\gamma, x) = \log \frac{\|\rho(\gamma) \cdot v_x\|_\tau}{\|v_x\|_\tau},$$

that is, c_τ coincides with the map $\beta_1(\cdot, \cdot)$ defined by Sambarino in [51, Section 5]. This remark will not be used in the sequel. ◇

Lemma 9.3. *The function c_τ is a Hölder cocycle. The period of γ in Γ_H is given by*

$$\ell_{c_\tau}(\gamma) = \lambda_1(\rho(\gamma)) > 0.$$

Proof. Follows from direct computations. □

The quotient space of $\partial_\infty^2 \Gamma \times \mathbb{R}$ by the action of Γ induced by c_τ will be denoted by $U_\tau \Gamma$.

9.2. Dual cocycle and Gromov product. Recall that c_o is the cocycle defined in Section 8.

Remark 9.4. The cocycle c_τ is dual to c_o , i.e. $\ell_{c_o}(\gamma) = \ell_{c_\tau}(\gamma^{-1})$ for every $\gamma \in \Gamma_H$. Indeed, this follows from Lemmas 9.3 and 8.3 and the fact that $\lambda_1(g) = \lambda_1(g^{-1})$ for all g in G . ◇

Recall that $\partial_\infty^2 \Gamma := \{(x, y) \in \partial_\infty \Gamma \times \partial_\infty \Gamma : x \neq y\}$. Since $o \in \Omega_\rho$ and the curves ξ and η are transverse the following map is well-defined.

Definition 9.5. Let

$$[\cdot, \cdot]_\tau : \partial_\infty^2 \Gamma \longrightarrow \mathbb{R} : [x, y]_\tau := \frac{1}{2} \log \left| \frac{\theta_y(v_x) \theta_x(v_y)}{\theta_x(J^o v_x) \|\theta_y\|_\tau \|v_y\|_\tau} \right|.$$

The proof of the following lemma is a straightforward computation.

Lemma 9.6. *The map $[\cdot, \cdot]_\tau$ is a Gromov product for the pair $\{c_o, c_\tau\}$, that is, for every $\gamma \in \Gamma$ and every $(x, y) \in \partial_\infty^2 \Gamma$ one has*

$$[\gamma x, \gamma y]_\tau - [x, y]_\tau = -(c_o(\gamma, x) + c_\tau(\gamma, y)).$$

□

Let \mathcal{G}_τ be the map defined in Section 5, associated to the choice of the norm $\|\cdot\|_\tau$.

Lemma 9.7. *Let γ be an element of Γ_H . Then*

$$[\gamma_-, \gamma_+]_\tau = -\frac{1}{2} \mathbb{B}(J^o \rho(\gamma)_-, J^o \rho(\gamma)_+, \rho(\gamma^{-1})_-, \rho(\gamma^{-1})_+) + \frac{1}{2} \mathcal{G}_\tau(\rho(\gamma^{-1})_-, J^o \rho(\gamma)_+).$$

Proof. Recall the definition of $[\cdot, \cdot]_o$ from Subsection 8.2. One has

$$[\gamma_-, \gamma_+]_\tau = [\gamma_-, \gamma_+]_o + \frac{1}{2} \log \frac{|\theta_{\gamma_+}(J^o v_{\gamma_+})|}{\|\theta_{\gamma_+}\|_\tau \|v_{\gamma_+}\|_\tau}.$$

The proof then follows from Lemma 8.8 and the fact that J^o preserves the norm $\|\cdot\|_\tau$ (c.f. Remark 4.2). □

9.3. Distribution of attractors & repellers with respect to b^τ . Consider μ_τ a Patterson-Sullivan probability on $\partial_\infty\Gamma$ associated to c_τ (c.f. Subsection A.2.2) and recall that μ_o is the one associated to c_o (c.f. Subsection 8.4). The analogue of Proposition 8.9 is available for the flow $U_\tau\Gamma$. The limiting measure can be written in this case as¹⁵

$$e^{-h[\cdot,\cdot]_\tau} \mu_o \otimes \mu_\tau.$$

From this fact we deduce the following proposition.

Proposition 9.8. *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation, a point $o \in \Omega_\rho$ and $\tau \in S^o$. There exists a constant $M' = M'_{\rho,\tau} > 0$ such that*

$$M' e^{-ht} \sum_{\gamma \in \Gamma_{\mathbb{H}, \frac{1}{2}} \log \|J^\circ \rho(\gamma) J^\circ \rho(\gamma^{-1})\|_\tau \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \rightarrow \mu_o \otimes \mu_\tau$$

as $t \rightarrow \infty$ on $C^*(\partial_\infty\Gamma \times \partial_\infty\Gamma)$.

Proof. The proof is the same that the one given in Proposition 8.10 adapted to the pair $\{c_o, c_\tau\}$ and the Gromov product $[\cdot, \cdot]_\tau$: apply item (5) of Lemma 7.5 and Lemma 9.7. \square

9.4. Proof of Theorem B. The following proposition, which implies Theorem B, can be proved in the same way as Proposition 8.12.

Proposition 9.9. *Let $\rho : \Gamma \rightarrow G$ be a $P_1^{p,q}$ -Anosov representation, a point $o \in \Omega_\rho$ and $\tau \in S^o$. There exists a constant $M' = M'_{\rho,\tau} > 0$ such that*

$$M' e^{-ht} \sum_{\gamma \in \Gamma, \frac{1}{2} \|J^\circ \rho(\gamma) J^\circ \rho(\gamma^{-1})\|_\tau \leq t} \delta_{\rho(\gamma^{-1})o^{+p,q}} \otimes \delta_{\rho(\gamma)o} \rightarrow \eta_*(\mu_o) \otimes \xi_*(\mu_\tau)$$

on $C^*(\mathbb{P}((\mathbb{R}^d)^*) \times \mathbb{P}(\mathbb{R}^d))$ as $t \rightarrow \infty$. \square

APPENDIX A. DISTRIBUTION OF PERIODIC ORBITS OF $U_o\Gamma$ AND $U_\tau\Gamma$

The goal of this appendix is to describe the distribution of periodic orbits of the flows $U_o\Gamma$ and $U_\tau\Gamma$ defined in Sections 8 and 9 (Proposition A.13 and Remark A.14). For the case on which Γ is the fundamental group of a closed negatively curved manifold, this result is covered by [51, Proposition 4.3]. We explain how to adapt Sambarino's proof to obtain the desired results for word hyperbolic groups admitting an Anosov representation.

In [51] Sambarino considers the geodesic flow of the manifold and explains how Hölder cocycles give rise to reparametrizations ψ_t of such flow. He uses Patterson-Sullivan theory to describe the probability of maximal entropy of ψ_t and apply the thermodynamic formalism of suspensions of sub-shifts of finite type to obtain a spatial distribution result for periodic orbits of ψ_t . Here we benefit from the fact that a projective Anosov representation ρ is given and use the *geodesic flow* of ρ introduced by Bridgeman & Canary & Labourie & Sambarino in [11] as a reference flow. This is a canonical flow associated to a projective Anosov representation and we show that it is Hölder conjugate to the flows $U_o\Gamma$ and $U_\tau\Gamma$. Since the techniques of the thermodynamic formalism are available for the geodesic flow of the representation (see [11, 13]), the adaptations needed in our context are straightforward.

The appendix is structured as follows. In Subsection A.1 we recall the definition of the geodesic flow of a representation and its main properties. We are interested in two descriptions of its probability of maximal entropy (Facts A.3 and A.6). In Subsection A.2 we translate these results to the flows $U_o\Gamma$ and $U_\tau\Gamma$.

A.1. The geodesic flow $U_\rho\Gamma$. We fix from now on a projective Anosov representation $\rho : \Gamma \rightarrow G$.

¹⁵For a proof, see Remark A.14.

A.1.1. Definition and the metric Anosov property. The standard reference for this subsection is [11]. Identify $\mathbb{P}((\mathbb{R}^d)^*)$ with $\text{Gr}_{d-1}(\mathbb{R}^d)$ as in Section 5. Given $(x, y) \in \partial_\infty^2 \Gamma$ let

$$M(x, y) := \{(\theta, v) \in \eta(x) \times \xi(y) : \theta(v) = 1\} / \sim$$

where $(\theta, v) \sim (-\theta, -v)$. Consider the line bundle over $\partial_\infty^2 \Gamma$ defined by

$$F_\rho := \{(x, y, \theta, v) : (x, y) \in \partial_\infty^2 \Gamma \text{ and } (\theta, v) \in M(x, y)\}.$$

Fact A.1 (Bridgeman & Canary & Labourie & Sambarino [11, Sections 4 & 5]). The following holds:

- The group Γ acts naturally on F_ρ and this action is proper and co-compact. The quotient space is denoted by $U_\rho \Gamma$ and is Hölder homeomorphic to the unit tangent bundle of Γ .
- The flow ϕ_t on F_ρ defined by

$$\phi_t(x, y, \theta, v) := (x, y, e^{-t}\theta, e^t v)$$

descends to a flow on $U_\rho \Gamma$, still denoted by ϕ_t , and called the *geodesic flow* of ρ . The geodesic flow of ρ is conjugated to a Hölder reparametrization of the Gromov geodesic flow of Γ (see Mineyev [39]).

- Periodic orbits of ϕ_t are in one-to-one correspondence with conjugacy classes of primitive elements γ in Γ , that is, elements which cannot be written as a power of another element. The corresponding period is $\lambda_1(\rho(\gamma))$.
- The geodesic flow ϕ_t is a transitive *metric Anosov flow*. Very informally, this means that there exists laminations W^{ss} , W^{uu} , W^{cs} and W^{cu} of $U_\rho \Gamma$, called respectively *strong stable lamination*, *strong unstable lamination*, *central stable lamination* and *central unstable lamination*, defining a *local product structure* and with the property that W^{ss} (resp. W^{uu}) is exponentially contracted by the flow (resp. the inverse flow). For precise definitions see [11, Subsection 3.2].

Explicitly, for a point $Z_0 = (x_0, y_0, \theta_0, v_0)$ in $U_\rho \Gamma$ the strong stable and strong unstable leaves through Z_0 are given by:

$$W^{ss}(Z_0) = \{(x, y_0, \theta, v_0) \in U_\rho \Gamma : \theta \in \eta(x), \theta(v_0) = 1\}$$

and

$$W^{uu}(Z_0) = \{(x_0, y, \theta_0, v) \in U_\rho \Gamma : v \in \xi(y), \theta_0(v) = 1\}.$$

The central stable and central unstable leaves are given by:

$$W^{cs}(Z_0) = \{(x, y_0, \theta, v) \in U_\rho \Gamma : \theta \in \eta(x), v \in \xi(y_0), \theta(v) = 1\}$$

and

$$W^{cu}(Z_0) = \{(x_0, y, \theta, v) \in U_\rho \Gamma : v \in \xi(y), \theta \in \eta(x_0), \theta(v) = 1\}.$$

◇

A.1.2. Entropy and distribution of periodic orbits. Recall that a flow is said to be *topologically weakly-mixing* if all the periods of its periodic orbits are not multiple of a common constant.

Proposition A.2. *The geodesic flow of ρ is topologically weakly-mixing.*

Before proving Proposition A.2 let us state the main result of this subsection. Indeed, the following fact is a consequence of the existence of a strong Markov coding for ϕ_t (see [11, 13]) together with the weak-mixing property. For Axiom A flows it was originally proved by Bowen [8] (the counting result is due to Parry & Pollicott [44]). In order to obtain it in our more general context, we need to apply Pollicott's work [47, Subsection 3.5].

Fact A.3. The following holds:

- The topological entropy of ϕ_t is positive and finite. It is given by

$$h = h_\rho := \limsup_{t \rightarrow \infty} \frac{\log \#\{[\gamma] \in [\Gamma] \text{ primitive} : \lambda_1(\rho(\gamma)) \leq t\}}{t}.$$

- As $t \rightarrow \infty$, one has

$$hte^{-ht} \#\{[\gamma] \in [\Gamma] \text{ primitive} : \lambda_1(\rho(\gamma)) \leq t\} \rightarrow 1.$$

- There exists a unique probability $m = m_\rho$ of maximal entropy for ϕ_t , called the *Bowen-Margulis probability*.
- Periodic orbits become equidistributed with respect to m : if $\text{Leb}_{[\gamma]}$ denotes the Lebesgue measure of length $\lambda_1(\rho(\gamma))$ supported on the periodic orbit $[\gamma]$, then

$$hte^{-ht} \sum \frac{1}{\lambda_1(\rho(\gamma))} \text{Leb}_{[\gamma]} \rightarrow m$$

in the weak-star topology as $t \rightarrow \infty$. Here the sum is taken over all primitive $[\gamma]$ such that $\lambda_1(\rho(\gamma)) \leq t$.

◇

We finish this subsection with an elementary proof of Proposition A.2 inspired by the work of Benoist [4].

Proof of Proposition A.2. Suppose by contradiction that ϕ_t is not topologically weakly-mixing. By Fact A.1 this implies that there exists a constant $a > 0$ such that the group spanned by the set $\{\lambda_1(\rho(\gamma))\}_{\gamma \in \Gamma}$ is contained in $a\mathbb{Z}$.

Set

$$\partial_\infty^4 \Gamma := \{(x_1, x_2, x_3, x_4) \in (\partial_\infty \Gamma)^4 : (x_i, x_j) \in \partial_\infty^2 \Gamma \text{ for all } i \neq j\}.$$

Since $\{(\gamma_-, \gamma_+)\}_{\gamma \in \Gamma_{\text{H}}}$ is dense in $\partial_\infty^2 \Gamma$ (see Gromov [20, Corollary 8.2.G]), Benoist's Theorem 5.7 implies that

$$(A.1) \quad \{\mathbb{B}(\eta(x'), \xi(y'), \eta(x), \xi(y)) : (x', y', x, y) \in \partial_\infty^4 \Gamma\} \subset a\mathbb{Z}$$

where \mathbb{B} is the cross-ratio defined in Section 5.

Fix three different points x', y' and y in $\partial_\infty \Gamma$. Transversality condition (6.2) and the definition of the cross-ratio implies the following: for every $x \in \partial_\infty \Gamma$ such that $(x', y', x, y) \in \partial_\infty^4 \Gamma$ there exists a neighbourhood V of x and a point $\xi_{x,y,y'}$ in the projective line $\xi(y) \oplus \xi(y')$ such that

$$(A.2) \quad \eta(\tilde{x}) \cap (\xi(y) \oplus \xi(y')) = \{\xi_{x,y,y'}\}$$

holds for every $\tilde{x} \in V$.

We then have the following.

Claim A.4. The limit set $\Lambda_{\rho(\Gamma)}$ is not contained in $\xi(y) \oplus \xi(y')$.

Proof of Claim A.4. Suppose by contradiction that $\Lambda_{\rho(\Gamma)} \subset \xi(y) \oplus \xi(y')$. Transversality condition (6.2) implies that for every $x \in \partial_\infty \Gamma$ different of y' and y one has

$$\eta(x) \cap (\xi(y) \oplus \xi(y')) = \{\xi(x)\}.$$

Then by Equation (A.2) the map ξ is not injective and this is a contradiction.

□

Because of Claim A.4 we can take y'' in $\partial_\infty \Gamma$ such that $\xi(y'')$ does not belong to $\xi(y) \oplus \xi(y')$. We can assume further that $y'' \neq x'$.

By (A.1) we have again the following: for every $x \notin \{x', y, y', y''\}$ there exists a neighbourhood V of x and a point $\xi_{x,y,y''}$ in the projective line $\xi(y) \oplus \xi(y'')$ such that

$$\eta(\tilde{x}) \cap (\xi(y) \oplus \xi(y'')) = \{\xi_{x,y,y''}\}$$

holds for every $\tilde{x} \in V$.

As in Claim A.4 we conclude that $\Lambda_{\rho(\Gamma)}$ cannot be contained in $\xi(y) \oplus \xi(y') \oplus \xi(y'')$ and now an inductive argument yields the desired contradiction. \square

A.1.3. The invariant measure of the strong stable lamination. As shown by Margulis [37], for Anosov flows there exists an invariant measure of the strong stable lamination which is exponentially contracted by the flow. In our context this measure is also available. Indeed by [11, 13] the flow ϕ_t admits a strong Markov coding and, as explained by Bowen & Marcus in [10, Section 4], this implies the existence of such a measure. As we shall see in Fact A.6, the importance for us of this measure relies on the fact that describes the probability measure of maximal entropy of ϕ_t in a different way than the one provided by Fact A.3.

The statement that we need is the following (for precisions see [10]).

Fact A.5. For every $Z_0 \in U_{\rho}\Gamma$ and every *plaque* of central unstable leaf $W_{\text{loc}}^{cu}(Z_0)$ there exists a positive and finite Borel measure $\nu_{\text{loc}}^{cu}(Z_0)$ on $W_{\text{loc}}^{cu}(Z_0)$ such that:

- The family $\{\nu_{\text{loc}}^{cu}(Z_0)\}_{Z_0 \in U_{\rho}\Gamma}$ is invariant under strong stable holonomy.
- There exists a real number $h^u \geq 0$ such that for every t and every $Z_0 \in U_{\rho}\Gamma$ one has¹⁶

$$(\phi_t)_*(\nu_{\text{loc}}^{cu}(Z_0)) = e^{-h^u t} \nu_{\text{loc}}^{cu}(\phi_t(Z_0)).$$

\diamond

A.1.4. The Bowen-Margulis probability. By reversing time and desintegrating along flow lines, Fact A.5 yields a family of measures $\{\nu_{\text{loc}}^{ss}(Z_0)\}$ on strong stable plaques which is expanded by the flow. In the case of Anosov flows, Margulis [37] first showed how the families $\{\nu_{\text{loc}}^{cu}(Z_0)\}$ and $\{\nu_{\text{loc}}^{ss}(Z_0)\}$ with the above properties combine to produce a ϕ_t -invariant finite Borel measure ν on the whole space. This measure coincides, up to scaling, with the Bowen-Margulis probability of the flow.

The statement that we need in our context is the following. Once again, this is a standard fact and the reader is referred for instance to Katok & Hasselblatt's book [30, Section 5 of Chapter 20] for a proof in the case of Anosov flows. With obvious adaptations the same proof works in our setting.

Fact A.6. Suppose that one has a family of measures $\{\nu_{\text{loc}}^{ss}(Z_0)\}_{Z_0 \in U_{\rho}\Gamma}$ on the strong stable plaques with the following properties:

- There exists a real number $h^s \geq 0$ such that for every t and every $Z_0 \in U_{\rho}\Gamma$ one has

$$(\phi_t)_*(\nu_{\text{loc}}^{ss}(Z_0)) = e^{h^s t} \nu_{\text{loc}}^{ss}(\phi_t(Z_0)).$$

- For every $Z_0 \in U_{\rho}\Gamma$ and every open set A contained in a neighbourhood of Z_0 with local product structure the map

$$W_{\text{loc}}^{cu}(Z_0) \longrightarrow \mathbb{R} : Z \mapsto \nu_{\text{loc}}^{ss}(Z)(A \cap W_{\text{loc}}^{ss}(Z))$$

is upper semi continuous.

Consider the family $\{\nu_{\text{loc}}^{cu}(Z_0)\}_{Z_0 \in U_{\rho}\Gamma}$ provided by Fact A.5. Then the following holds:

- If $A \subset U_{\rho}\Gamma$ is an open set contained in a neighbourhood of $Z_0 \in U_{\rho}\Gamma$ with local product structure, set

$$\nu(A) := \int_{Z \in W_{\text{loc}}^{cu}(Z_0)} \nu_{\text{loc}}^{ss}(Z)(A \cap W_{\text{loc}}^{ss}(Z)) d\nu_{\text{loc}}^{cu}(Z_0)(Z).$$

Then this measure extends to a finite Borel measure ν on $U_{\rho}\Gamma$ such that for every $t \in \mathbb{R}$ the following holds:

¹⁶For a map $f : X \longrightarrow Y$ and a measure ν on X we denote by $f_*(\nu)$ the measure on Y given by $A \mapsto \nu(f^{-1}(A))$.

$$(\phi_t)_* \nu = e^{(h^s - h^u)t} \nu.$$

In particular $h^s = h^u$ and ν is ϕ_t -invariant.

- The number h^u equals the topological entropy h of the flow and the probability proportional to ν is the Bowen-Margulis probability of ϕ_t .

◇

A.2. The flows $U_o\Gamma$ and $U_\tau\Gamma$. In this subsection we study the quotient spaces of $\partial_\infty^2\Gamma \times \mathbb{R}$ by the actions defined by the cocycles c_o and c_τ of Subsections 8.1 and 9.1. We exhibit explicit equivariant homeomorphisms

$$F_\rho \longrightarrow \partial_\infty^2\Gamma \times \mathbb{R}$$

which conjugate the respective flows (Subsection A.2.1). We then use Fact A.5 to show the existence of *Patterson-Sullivan probabilities* associated to c_o and c_τ (Subsection A.2.2) and explain how these measures describe the Bowen-Margulis probabilities of the flows $U_o\Gamma$ and $U_\tau\Gamma$ (Proposition A.12 and Remark A.14). Finally, we establish the analogue of [51, Proposition 4.3] for our particular cocycles (Proposition A.13 and Remark A.14).

A.2.1. Explicit conjugations between $U_\rho\Gamma$, $U_o\Gamma$ and $U_\tau\Gamma$. Recall that $U_o\Gamma$ is the quotient space of $\partial_\infty^2\Gamma \times \mathbb{R}$ by the action (8.2) and that $\psi_t = \psi_t^o$ is the flow on $U_o\Gamma$ induced by the translation flow (8.1).

The following lemma implies in particular that the action of Γ on $\partial_\infty^2\Gamma \times \mathbb{R}$ via c_o is proper and co-compact and that $U_o\Gamma$ is homeomorphic to the unit tangent bundle of Γ .

Lemma A.7. *There exists a Hölder homeomorphism $U_\rho\Gamma \longrightarrow U_o\Gamma$ which conjugates the flows ϕ_t and ψ_t . Further, for every point $(x_0, y_0, t_0) \in U_o\Gamma$ the central unstable and strong stable leaves through (x_0, y_0, t_0) are given by*

$$W^{cu}(x_0, y_0, t_0) = \{(x_0, y, t) \in U_o\Gamma : y \in \partial_\infty\Gamma \setminus \Gamma x_0 \text{ and } t \in \mathbb{R}\}$$

and

$$W^{ss}(x_0, y_0, t_0) = \{(x, y_0, t_0) \in U_o\Gamma : x \in \partial_\infty\Gamma \setminus \Gamma y_0\}.$$

Proof. Consider the map $F_\rho \longrightarrow \partial_\infty^2\Gamma \times \mathbb{R}$ defined by

$$(x, y, \theta, v) \mapsto (x, y, -\frac{1}{2} \log |\langle v, J^o v \rangle_{p,q}|),$$

which is easily seen to be Hölder continuous, injective and equivariant. Moreover one can prove that it is proper and surjective, hence a homeomorphism.

The statement involving the flows and the laminations is straightforward (c.f. Fact A.1).

□

We now turn our attention to the flow $U_\tau\Gamma$ (c.f. Subsection 9.1). An analogue of Lemma A.7 is also available. In fact, the analogue holds because of the following remark.

Remark A.8. The cocycles c_o and c_τ are cohomologous. Indeed, this follows from the fact that c_o and c_τ have the same periods and a theorem due to Livsic [35]. Explicitly, let

$$U : \partial_\infty\Gamma \longrightarrow \mathbb{R} : U(x) := \frac{1}{2} \log \frac{\|v_x\|_\tau \|\theta_x\|_\tau}{|\theta_x(J^o v_x)|}$$

where $\theta_x : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a non-zero linear functional whose kernel equals $\eta(x)$ and $v_x \neq 0$ belongs to $\xi(x)$. Then for every γ in Γ and x in $\partial_\infty\Gamma$ one has

$$c_\tau(\gamma, x) - c_o(\gamma, x) = U(\gamma x) - U(x) .$$

◇

A.2.2. Patterson-Sullivan probabilities for c_o and c_τ . The goal of this subsection is to show the existence of a *Patterson-Sullivan probability of dimension h^u* for the cocycle c_o , that is, a probability measure μ_o on $\partial_\infty\Gamma$ such that

$$(A.3) \quad \frac{d\gamma_*\mu_o}{d\mu_o}(x) = e^{-h^u c_o(\gamma^{-1}, x)}$$

holds for every $\gamma \in \Gamma$ ¹⁷. We will see in the next subsection that in fact one has $h^u = h$. The existence of a Patterson-Sullivan probability μ_τ for c_τ follows directly from this one by Remark A.8.

When Γ is the fundamental group of a closed negatively curved manifold, the existence (and uniqueness) of such probability is proved by Ledrappier [33]. When $\rho(\Gamma)$ is Zariski dense one can apply explicitly the work of Quint [48] and for the case of $\mathbb{H}^{p,q-1}$ -convex co-compact groups we find also the construction presented by Glorieux & Monclair [19].

Even though Patterson's method [45] works correctly in our setting and produces directly a Patterson-Sullivan probability of dimension h , we choose a shorter approach. Applying Fact A.5 and Lemma A.7 we find a transverse measure $\{\nu_{\text{loc}}^{cu}(u_0)\}_{u_0 \in U_o\Gamma}$ of the strong stable lamination of $\psi_t : U_o\Gamma \curvearrowright$ which has the property of being contracted by the flow. Lifting this measure to $\partial_\infty^2\Gamma \times \mathbb{R}$ yields a probability μ_o on $\partial_\infty\Gamma$ satisfying (A.3). Indeed, for closed negatively curved manifolds and the Busemann cocycle this procedure is explained for instance by Babillot in [1, Subsection 7.1]. With obvious adaptations the procedure equally applies in our setting.

Remark A.9. Recall that $h^u \geq 0$. Equation (A.3) shows in fact that h^u is positive. Otherwise the probability μ_o is Γ -invariant but one can see that this is not possible for a non elementary word hyperbolic group. ◇

We finish this subsection by showing that μ_o has no atoms (this property is needed in the proof of Proposition 8.10). The proof presented here is an adaptation of [19, Proposition 4.3].

Lemma A.10. *The measure μ_o has no atoms.*

Proof. Suppose that there exists an atom $y \in \partial_\infty\Gamma$ for μ_o . Since h^u is positive the point y cannot be fixed by an element of Γ , hence

$$(A.4) \quad 1 = \mu_o(\partial_\infty\Gamma) \geq \sum_{\gamma \in \Gamma} e^{-h^u c_o(\gamma^{-1}, y)} \mu_o(y).$$

Claim A.11. There exists a sequence $\gamma_n \rightarrow \infty$ such that $c_o(\gamma_n^{-1}, y) \rightarrow -\infty$.

Proof of Claim A.11. Let x be a point in $\partial_\infty\Gamma$ different from y and $\|\cdot\|$ be any norm on \mathbb{R}^d . Take a sequence $\gamma_n \rightarrow \infty$ such that

$$(\gamma_n)_+ \rightarrow y \text{ and } (\gamma_n)_- \rightarrow x.$$

Up to taking a subsequence we may suppose that γ_n converges uniformly to y on compact sets of $\partial_\infty\Gamma \setminus \{x\}$ (c.f. Bowditch [7, Lemma 2.11]). Let $B(x) \subset \partial_\infty\Gamma$ be the complement of a small neighbourhood of x in $\partial_\infty\Gamma$ and $b(y) \subset B(x)$ be a small neighbourhood of y . Then we can suppose that $\gamma_n(B(x)) \subset b(y)$ holds for every n .

By Proposition 6.2 there exists $\varepsilon > 0$ such that for all n one has

$$\xi(B(x)) \subset B_\varepsilon(S_{d-1}(\rho(\gamma_n))),$$

where $S_{d-1}(\rho(\gamma_n))$ is as in Subsection 6.1 and $B_\varepsilon(\cdot)$ is as in Section 5.1. Take a positive c with the following property: for every n and every vector v in $B_\varepsilon(S_{d-1}(\rho(\gamma_n)))$ one has

¹⁷Recall that h^u is the constant of Fact A.5.

$$\|\rho(\gamma_n)v\| \geq c\|\rho(\gamma_n)\|\|v\|.$$

Let $v \neq 0$ be a vector in $\xi(y)$. We have that $\rho(\gamma_n^{-1})v$ belongs to $B_\varepsilon(S_{d-1}(\rho(\gamma_n)))$ hence

$$\rho(\gamma_n^{-1})v \longrightarrow 0$$

as $n \longrightarrow \infty$. The divergence $c_o(\gamma_n^{-1}, y) \longrightarrow -\infty$ follows. \square

A combination of (A.4) and Claim A.11 yields the desired contradiction. \square

A.2.3. The Bowen-Margulis probability for $U_o\Gamma$ and $U_\tau\Gamma$. Recall that $[\cdot, \cdot]_o$ is the Gromov product of the pair $\{c_o, c_o\}$ defined in Subsection 8.2. The following result is the analogue of [51, Theorem 3.2(2)].

Proposition A.12 (Sambarino [51, Theorem 3.2]). *The number h^u equals the topological entropy h of ψ_t and the measure*

$$e^{-h[\cdot, \cdot]_o} \mu_o \otimes \mu_o \otimes dt$$

induces a measure on the quotient space $U_o\Gamma$ proportional to the Bowen-Margulis probability of ψ_t .

Proof. From explicit computations one can show that $e^{-h^u[\cdot, \cdot]_o} \mu_o \otimes \mu_o \otimes dt$ equals the product of measures ν_{loc}^{cu} and ν_{loc}^{ss} as in Fact A.6. \square

We now state the desired result of this appendix: the analogue of [51, Proposition 4.3]. Provided with Proposition A.12, the same proof applies in our setting.

Proposition A.13 (Sambarino [51, Proposition 4.3]). *There exists a positive M such that*

$$Me^{-ht} \sum_{\gamma \in \Gamma_H, \ell_{c_o}(\gamma) \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \longrightarrow e^{-h[\cdot, \cdot]_o} \mu_o \otimes \mu_o$$

as $t \longrightarrow \infty$ on $C_c^(\partial_\infty^2 \Gamma)$.* \square

For the flow $U_\tau\Gamma$ we obtain analogue results.

Remark A.14. Let $[\cdot, \cdot]_\tau$ be the Gromov product of the pair $\{c_o, c_\tau\}$ defined in Subsection 9.2. The same arguments of Proposition A.12 and Proposition A.13 apply to obtain that

$$e^{-h[\cdot, \cdot]_\tau} \mu_o \otimes \mu_\tau \otimes dt$$

induces the Bowen-Margulis probability of $U_\tau\Gamma$ and that there exists a positive M' such that

$$M'e^{-ht} \sum_{\gamma \in \Gamma_H, \ell_{c_\tau}(\gamma) \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \longrightarrow e^{-h[\cdot, \cdot]_\tau} \mu_o \otimes \mu_\tau$$

as $t \longrightarrow \infty$ on $C_c^(\partial_\infty^2 \Gamma)$.* \diamond

REFERENCES

- [1] Babillot, M.: *Points entieres et groupes discrets: de l'analyse aux systèmes dynamiques*. Panoramas et synthèses **13** (2002), p. 1-119.
- [2] Benoist, Y.: *Actions propres sur les espaces homogènes réductifs*. Ann. of Math. **144** (1996), p. 315-347.
- [3] Benoist, Y.: *Propriétés asymptotiques des groupes linéaires*. Geom. Funct. Anal. **7** (1997), p. 1-47.
- [4] Benoist, Y.: *Propriétés asymptotiques des groupes linéaires II*. Advanced Studies Pure Math. **26** (2000), p. 33-48.
- [5] Benoist, Y.: *Convexes divisibles I*. Tata Inst. Fund. Res. Stud. Math. **17** (2004), p. 339-374.
- [6] Bochi, J., Potrie, R., Sambarino, A. : *Anosov representations and dominated splittings*. Preprint, arXiv:1605.01742 [math.GR].

- [7] Bowditch, B.: *Convergence groups and configuration spaces in Geometric group theory down under*. Walter de Gruyter (1999), Berlin.
- [8] Bowen, R.: *Periodic orbits of hyperbolic flows*. Amer. J. Math. **94** (1972), p. 1-30.
- [9] Bowen, R.: *Symbolic dynamics for hyperbolic flows*. Amer. J. Math. **95** (1973), p. 429-460.
- [10] Bowen, R., Marcus, B.: *Unique ergodicity for horocycle foliations*. Israel J. Math. **26(1)** (1977), p. 43-67.
- [11] Bridgeman, M., Canary, R., Labourie, F., Sambarino, A.: *The pressure metric for Anosov representations*. Geom. Funct. Anal. **25** (2015), p. 1089-1179.
- [12] Busemann, H.: *Spaces with non-positive curvature*. Acta Math. **80** (1948), p. 259-310.
- [13] Constantine, D., Lafont, J.-F., Thompson, D.: *Strong symbolic dynamics for geodesic flow on CAT(-1) spaces and other metric Anosov flows*. Preprint, arXiv:1808.04395 [math.DS].
- [14] Danciger, J., Guéritaud, F., Kassel, F.: *Convex cocompactness in pseudo-Riemannian hyperbolic spaces*. Geom. Dedicata **192** (2018), volume Special Geometries: A Celebration of Bill Goldman's 60th Birthday, p. 87-126.
- [15] Danciger, J., Guéritaud, F., Kassel, F.: *Convex cocompact actions in real projective geometry*. Preprint, arXiv:1704.08711 [math.GT].
- [16] Duke, W., Rudnick, Z., Sarnak, P.: *Density of integer points on affine homogeneous varieties*. Duke Math. Jour **71(1)** (1993), p. 143-179.
- [17] Eskin, A., McMullen, C.: *Mixing, counting and equidistribution in Lie groups*. Duke Math. Jour **71(1)** (1993), p. 181-209.
- [18] Ghys, E., de la Harpe, P.: *Sur les groupes hyperboliques d'après Mikhael Gromov*. Progress in Mathematics **83** (1990). Springer Science+Business Media, LLC, New York.
- [19] Glorieux, O., Monclair, D.: *Critical exponent and Hausdorff dimension for quasi-Fuchsian AdS manifolds*. Preprint, arXiv:1606.05512 [math.DG].
- [20] Gromov, M.: *Hyperbolic groups in Essays in group theory*. Springer-Verlag (1987), New York.
- [21] Guichard, O., Wienhard, A.: *Anosov representations: domains of discontinuity and applications*. Invent. Math. **190** (2012), p. 357-438.
- [22] Guichard, O., Guéritaud, F., Kassel, F., Wienhard, A.: *Anosov representations and proper actions*. Geom. Topol. **21** (2017), p. 485-584.
- [23] Helgason, S.: *Differential geometry, Lie groups, and symmetric spaces*. Academic Press (1978), San Diego.
- [24] Horn, R., Johnson, C.: *Matrix analysis*. Cambridge University Press (1985), New York.
- [25] Kapovich, M., Leeb, B., Porti, J.: *Anosov subgroups: Dynamical and geometric characterizations*. European Journal of Mathematics **3** (2017), p. 808-898.
- [26] Kapovich, M., Leeb, B., Porti, J.: *Dynamics on flag manifolds: domains of proper discontinuity and cocompactness*. Geom. Topol. **22** (2018), p. 157-234.
- [27] Kapovich, M., Leeb, B., Porti, J.: *A Morse Lemma for quasigeodesics in symmetric spaces and euclidean buildings*. Preprint, arXiv:1411.4176 [math.GR].
- [28] Kassel, F., Kobayashi, T.: *Poincaré series for non-Riemannian locally symmetric spaces*. Adv. Math. **287** (2016), p. 123-236.
- [29] Kassel, F.: *Geometric structures and representations of discrete groups*. Proc. Int. Cong. of Math. **1** (2018), p. 1113-1150.
- [30] Katok, A., Hasselblatt, B.: *Introduction to the modern theory of dynamical systems*. Cambridge University Press (1995), New York.
- [31] Knapp, A.: *Lie groups beyond an introduction*. Progress in Mathematics **140** (1996). Springer Science+Business Media, LLC, Boston.
- [32] Labourie, F.: *Anosov flows, surface groups and curves in projective space*. Invent. Math. **165** (2006), p. 51-114.
- [33] Ledrappier, F.: *Structure au bord des variétés à courbure négative*. In Séminaire de Théorie spectrale et géométrie de Grenoble **13** (1994-1995), p. 97-122.
- [34] Lee, M., Oh, H.: *Effective circle count for Apollonian packings and closed horospheres*. Geom. Funct. Anal. **23** (2013), p. 580-621.
- [35] Livsic, A.N.: *Cohomology of dynamical systems*. Math. USSR Izvestija **6** (1972), p. 1278-1301.
- [36] Margulis, G.: *Applications of ergodic theory to the investigation of manifolds with negative curvature*. Functional Anal. Appl. **3** (1969), p. 335-336.
- [37] Margulis, G.: *Certain measures associated with U-flows on compact manifolds*. Functional Anal. Appl. **4** (1969), p. 55-67.
- [38] Matsuki, T.: *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*. J. Math. Soc. Japan **31(2)** (1979), p. 331-357.
- [39] Mineyev, I.: *Flows and joins of metric spaces*. Geom. Topol. **9** (2005), p. 403-482.
- [40] Mohammadi, A., Oh, H.: *Matrix coefficients, counting and primes for orbits of geometrically finite groups*. Journal of the EMS **17** (2015), p. 837-897.
- [41] Oh, H., Shah, N.: *The asymptotic distribution of circles in the orbits of Kleinian groups*. Inven. Math. **187** (2012), p. 1-35.

- [42] Oh, H., Shah, N.: *Equidistribution and counting for orbits of geometrically finite hyperbolic groups*. Journal of the AMS **26** (2013), p. 511–562.
- [43] Oh, H., Shah, N.: *Counting visible circles on the sphere and Kleinian groups*. Proceedings of the conference on "Geometry, Topology and Dynamics in negative curvature", LMS series **425** (2016), p. 272-288.
- [44] Parry, W., Pollicott, M.: *An analogue of the prime number theorem and closed orbits of Axiom A flows*. Ann. of Math. **118** (1983), p. 573-591.
- [45] Patterson, S. J.: *The limit set of a fuchsian group*. Acta Math. **136** (1976), p. 241-273.
- [46] Petersen, P.: *Riemannian geometry*. Graduate Texts in Mathematics **171** (1998). Springer Science+Business Media, LLC, New York.
- [47] Pollicott, M.: *Symbolic dynamics for Smale flows*. American Journal of Mathematics **109(1)** (1987), p. 183-200.
- [48] Quint, J.-F.: *Mesures de Patterson-Sullivan en rang supérieur*. Geom. Funct. Anal. **12** (2002), p. 776-809.
- [49] Quint, J.-F.: *Groupes de Schottky et comptage*. Ann. Inst. Fourier **55** (2005), p. 373-429.
- [50] Roblin, T.: *Ergodicité et équidistribution en courbure négative*. Mémoires de la SMF **95** (2003).
- [51] Sambarino, A.: *Quantitative properties of convex representations*. Comment. Math. Helv. **89** (2014), p. 443-488.
- [52] Sambarino, A.: *The orbital counting problem for hyperconvex representations*. Ann. Inst. Fourier **65(4)** (2015), p. 1755–1797.
- [53] Schlichtkrull, H.: *Hyperfunctions and harmonic analysis on symmetric spaces*. Progress in Mathematics **49** (1984). Birkhäuser-Verlag, Boston-Basel-Stuttgart.
- [54] Wienhard, A.: *An invitation to higher Teichmüller theory*. Proc. Int. Cong. of Math. **1** (2018), p. 1007–1034.

(León Carvajales)

CENTRO DE MATEMÁTICA

UNIVERSIDAD DE LA REPÚBLICA

IGUÁ 4225, 11400, MONTEVIDEO, URUGUAY.

E-mail address: lcarvajales@cmat.edu.uy