ERGODICITY OF PARTIALLY HYPERBOLIC DIFFEOMORPHISMS IN HYPERBOLIC 3-MANIFOLDS

SERGIO R. FENLEY AND RAFAEL POTRIE

Abstract. We study conservative partially hyperbolic diffeomorphisms in hyperbolic 3-manifolds. We show that they are always accessible and deduce as a result that every conservative C^{1+} partially hyperbolic in a hyperbolic 3-manifold must be ergodic, giving an afirmative answer to a conjecture of Hertz-Hertz-Ures in the context of hyperbolic 3-manifolds. Some of the intermediary steps are also done for general partially hyperbolic diffeomorphisms homotopic to the identity.

Keywords: Partial hyperbolicity, 3-manifold topology, foliations, ergodicity, accessibility.

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1. Introduction

The purpose of this paper is to show how some dynamical consequences of the classification of partially hyperbolic diffeomorphisms can be obtained. In particular, we will use the results of [BFFP] to establish a result about ergodicity of partially hyperbolic diffeomorphisms in hyperbolic 3-manifolds. It is also intended to be an ilustration of some of the techniques used in [BFFP] in a similar yet different context.

The description of the statistical properties of conservative dynamical systems is a central problem in dynamics. Anosov systems are well known to be stably ergodic via the Hopf argument. It took a long time before new examples of stably ergodic diffeomorphisms were shown to exist by Grayson-Pugh-Shub [GPS]: they proved this property for the time one map of the geodesic flow of a surface of constant negative curvature. A key point of the proof was to be able to extend the Hopf argument to foliations whose dimensions do not fill the ambient dimension, using the concept of *accessibility* (see section 2). This motivated Pugh and Shub to propose their celebrated conjecture about ergodicity and accessibility of partially hyperbolic systems [PS] asserting that accessible systems should be abundant among these systems and that accessibility should imply ergodicity. We refer the reader to [CHHU, W] for surveys on these topics. This paper addresses the question of establishing ergodicity of a partially hyperbolic system without need of perturbations by establishing accessibility unconditionally among conservative partially hyperbolic diffeomorphisms in certain 3-manifolds or under certain hypothesis.

The main result of this article is the following:

Theorem A. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism of class C^{1+} in a closed hyperbolic 3-manifold M which preserves a volume form.*

Then, f is a K-system with respect to volume (and in particular, it is ergodic and mixing).

The proof of this result has been reduced by [BuW, HHU], building on the arguments introduced in [GPS] as well as several new insights, to showing the following statement which is then stronger as it requires a weaker assumption than preserving volume and assuming less regularity. See subsection 2.2.

Theorem B. *Let $f : M \to M$ be a C^1-partially hyperbolic diffeomorphism of a closed hyperbolic 3-manifold, so that the non wandering set of f is all of M. Then f is accessible.*

In [HHU_2] (see also [CHHU]) this has been reduced to a problem of geometry of foliations and this is what we solve here. In the next section we introduce the setting and explain what are the main technical results we need to prove in order to establish the main theorems. Some of the intermediary results to prove Theorem B hold in the more general setting of partially hyperbolic diffeomorphisms homotopic to the identity in 3-manifolds and this is how they are proved in this paper.

We notice that in [HHU, BHHTU] it is shown that accessible partially hyperbolic diffeomorphisms are open and dense among those with one dimensional center (see [DW] for higher dimensional center, but a weaker topology). Here we treat the problem (started in [HHU_2]) of determining which manifolds or homotopy classes of partially hyperbolic diffeomorphisms are *always* accessible by topological reasons. Other results in this direction have been obtained, see [HHU_2, HU, H]. In particular, in [HRHU] it is proved that several isotopy classes of conservative partially hyperbolic diffeomorphisms of Seifert manifolds are ergodic and other results as well. Their arguments have some overlap with ours.

These results give an affirmative solution to a conjecture by Hertz-Hertz-Ures [CHHU, Conjecture 2.11] in hyperbolic 3-manifolds. To perform the proof we need to show a result about quasigeodesic pseudo-Anosov flows in 3-manifolds which may be of independent interest (see Proposition 7.2).

Notice that it makes sense to prove accessibility without any assumptions on the dynamics of f (such as being non-wandering or conservative) as done in [DW, BHHTU] for manifolds of any dimension. We provide in section 8 some results in dimension 3 that allow to treat this case too, namely, when f is leaf conjugate to an Anosov flow. Previously only in the article [HP_1, Section 6.3] accessibility in the non-conservative setting was considered in relation with the conjecture above [CHHU, Conjecture 2.11].

Being partially hyperbolic (via cone-fields) and preserving volume are conditions that are usually given a priori or can be detected by checking only finitely many iterates of f, while ergodicity (or mixing) is a chaotic property that involves the asymptotic behaviour of the system, so, this type of result allows to give precise non-perturbative information of a system by looking at finitely many iterates (c.f. [Pot]).

2. Setting and strategy

2.1. **Definitions.** Let M be a closed manifold and $f : M \to M$ a C^1-diffeomorphism. One says that f is *partially hyperbolic* if there exists a Df-invariant continuous splitting $TM = E^s \oplus E^c \oplus E^u$ into non-trivial bundles such that there is $n > 0$ so that for any unit vectors $v^\sigma \in E^\sigma(x)$ ($\sigma = s, c, u$) it follows that:

$$\|Df^n v^s\| < \min\{1, \|Df^n v^c\|\} \leq \max\{1, \|Df^n v^c\|\} < \|Df^n v^u\|.$$

Using an adapted metric, one can assume that $n = 1$. We refer the reader to [CP, HP_2] for basic properties of these diffeomorphisms.

It is well known ([HPS]) that the bundles E^s and E^u are uniquely integrable into foliations $\mathcal{W}^s$, $\mathcal{W}^u$.

Definition 2.1. (accessibility) We say that f is *accessible* if given any two points $x, y \in M$ there exists a piecewise C^1 path tangent to $E^s \cup E^u$ from x to y. More generally two points z, w in M are in the same accessibility class if there is a path as above connecting the points. This is an equivalence relation.

We refer the reader to [CHHU, W] for more on the notion of accessibility. Notice that the tangent vectors are required to be in $E^s \cup E^u$ and not on $E^s \oplus E^u$.

Recall that a diffeomorphism f is said to be *conservative* if it preserves a volume form. By Poincaré recurrence, this implies that such an f has to be *non-wandering* meaning that for every open set $U \subset M$ there exists $n > 0$ so that $f^n(U) \cap U$ is not empty. For all the results of this paper (except to deduce Theorem A from Theorem B) this weaker topological assumption (that f is non wandering) will be enough.

We will always work with M being a 3-dimensional manifold. In this setting, a hyperbolic manifold is one which can be obtained as a quotient of $\mathbb{H}^3$ by isometries of the hyperbolic metric. Contrary to what one might expect, these are well known to be quite abundant thanks to Thurston-Perelman's geometrization theorem.

For a manifold M we will denote by $\widetilde{M}$ the universal cover, and by $\pi : \widetilde{M} \to M$ the cannonical projection. Whenever an object X is lifted to $\widetilde{M}$ it will be denoted by $\widetilde{X}$.

2.2. **Ergodicity.** To prove Theorem A from Theorem B it is enough to apply the following result which is a consequence of the main result of [BuW] (in this setting it also follows from [HHU]):

Theorem 2.2. *Let $f : M \to M$ be a conservative partially hyperbolic diffeomorphism of class C^{1+} which is accessible and has center dimension equal to 1. Then, f is a K-system with respect to volume. In particular it is ergodic and mixing.*

Notice that accessibility is stable under taking finite iterates, so for proving Theorem A and B we can always pick a finite iterate of our diffeomorphism. Also, we can take finite lifts if desired as having a finite lift which is accessible implies that the original map is accessible.

We refer the reader to [Ma] for definitions and implications of being a K-system (or ergodic and mixing). We do not define these notions here, as this paper will be entirely about geometric notions pertaining to accessibility and related objects. We remark that it would be very natural to ask whether in this context every conservative partially hyperbolic diffeomorphism is Bernoulli and some indications that this might be the case can be found in [AVW]. This question is beyond the scope of this paper.

We also remark that in the setting of non-wandering diffeomorphisms, it is a classical result by Brin [Br] (see also [CP, Section 5]) that an accessible partially hyperbolic diffeomorphism which is non-wandering has a dense orbit (i.e. it is transitive).

2.3. **Accessibility.** In [HHU$_2$] (see also [CHHU]) the following result is shown based on the results of [HHU] in the specific case of dimension 3 (we remark that the results in [HHU] have their non-conservative counterparts in [BHHTU]):

Theorem 2.3. *Let M be a closed 3-manifold and $f : M \to M$ a partially hyperbolic diffeomorphism so that the non wandering set of f is all of M. Suppose that f is not accessible. Suppose that there are no tori tangent to $E^s \oplus E^u$. Then*

- *There exists an f-invariant non-empty lamination Λ^{su} with C^1-leaves tangent to $E^s \oplus E^u$, and so that Λ^{su} does not have any compact leaves.*
- *The completion of each complementary region of Λ^{su} is an I-bundle, so the lamination Λ^{su} can be extended to a foliation $\mathcal{F}$ (not necessarily f-invariant nor tangent to $E^s \oplus E^u$) without compact leaves.*
- *Finally the center bundle is uniquely integrable in the completion W of any complementary region, and in W the center foliation is made up of compact segments from one boundary component of W to the other.*

We remark here that the lamination Λ^{su} may cover the whole M in which case it would be a foliation and the second and third items become void.

Remark 2.4. In fact, under the same assumptions, the conclusions of Theorem 2.3 hold for any closed f-invariant lamination tangent to $E^s \oplus E^u$ without tori leaves. This follows directly from the proofs that we briefly summarize in the next paragraph.

The first item of Theorem 2.3 is proved in [HHU] with the hypothesis that f is conservative or non-wandering. The proof only uses that f is not accessible, an explicit proof that one only needs that f is non accessible is done in [CHHU]. The first statement of the third item is proved in Proposition 4.2 of [HHU$_2$]. It is stated for f conservative, but the proof only needs that the non wandering set of f is all of M. In fact the specific property that is used is that recurrent points are dense. The second statement of the third item is proved during the proof of Theorem 4.1 of [HHU$_2$]. Again, all that is needed is that the non-wandering set of f is all of M. This then implies the second statement in the third item.

Remark 2.5. It follows from [HHU$_3$] that if there is a torus tangent to $E^s \oplus E^u$ then it must be incompressible and the manifold M has virtually solvable fundamental group. In particular when M is a hyperbolic 3-manifold there are not tori tangent to $E^s \oplus E^u$.

Remark 2.6. For the remainder of the article, unless otherwise stated, M will have dimension 3.

2.4. **Strategy of the proof.** The proof of Theorem B (which implies Theorem A) is based on Theorem 2.3 which reduces the study to the case where there is an f-invariant lamination (which can be completed into a foliation without compact leaves) whose leaves are saturated by stable and unstable manifolds. In other words we want to show that this is not possible for M hyperbolic.

When the manifold is hyperbolic, up to taking an iterate one can assume that f is homotopic to the identity (c.f. subsection 3.1). Therefore it admits what is called a *good lift* $\widetilde{f}$ at a bounded distance from the identity and commuting with deck transformations (again see subsection 3.1). We stress that a lot of the analysis will be done in the more general setting of homeomorphisms homotopic to the identity.

The study of *taut* foliations invariant under a diffeomorphism f homotopic to the identity in general 3-manifolds was started in [BFFP]; where a general dichotomy was obtained (see subsection 3.4): Let $\widetilde{f}$ be a good lift. Then either

- there is a non empty, leaf saturated, closed set in M, and whose lift to $\widetilde{M}$ is leafwise fixed by $\widetilde{f}$;
- or, the foliation is $\mathbb{R}$-covered and uniform, and $\widetilde{f}$ acts as a translation on the leaf space of the foliation lifted to $\widetilde{M}$.

(see subsection 3.4 for more details and definitions). This works for any taut foliation and not just those associated with a partially hyperbolic diffeomorphism. These results can be extended to laminations invariant under f which is homotopic to the identity.

Then one applies the dichotomy above to the lamination Λ^{su} obtained from Theorem 2.3. One analizes each case of the dichotomy separately to get a contradiction to non accessibility of f partially hyperbolic in a hyperbolic 3-manifold. The case where some of the leaves are fixed by $\widetilde{f}$ requires a general result of [BFFP] stating that f cannot have contractible periodic points, and this then allows to perform arguments similar to those dealing with the *doubly invariant case* of [BFFP] which we do in section 6. This case can be dealt with for general partially hyperbolic diffeomorphisms of 3-manifolds that are homotopic to the identity, and that is what is done here.

The full translation case (dealt with in section 7) is specific to hyperbolic manifolds as it uses the existence of a regulating pseudo-Anosov flow (see subsection 3.2) to show that the partially hyperbolic diffeomorphism needs to be leaf conjugate to a topological Anosov flow. The ideas to show this are similar to the ones appearing in the classification of dynamically coherent partially hyperbolic diffeomorphisms in hyperbolic 3-manifolds (this is done in subsection 7.4). Then one uses some well known properties of topological Anosov flows (see subsection 3.3) and some properties of regulating pseudo-Anosov flows (in particular, we mention Proposition 7.2 which may be of independent interest) to get a contradiction (c.f. subsection 7.1). One much easier subcase of the translation case also works for general partially hyperbolic diffeomorphisms homotopic to the identity when $\pi_1(M)$ is not virtually solvable (c.f. Remark 2.5).

3. Preliminaries and reductions

3.1. **Mostow rigidity and good lifts.** Let $f : M \to M$ be a homeomorphism of a closed hyperbolic 3-manifold. Mostow rigidity [Mo], implies that any homeomorphism of a closed hyperbolic manifold (in fact in any dimension ≥ 3) is homotopic to an isometry. In any closed manifold an isometry has an iterate close to the identity. This implies that in a hyperbolic manifold, every homeomorphism has an iterate homotopic to the identity.

Definition 3.1. (good lift) Let f be a homeomorphism of a closed manifold. We say that a lift $\widetilde{f} : \widetilde{M} \to \widetilde{M}$ is a *good lift* of f, if $\widetilde{f}$ is at bounded distance from the identity (i.e. there exists $K > 0$ so that $d(x, \widetilde{f}(x)) < K$ for all $x \in \widetilde{M}$), and $\widetilde{f}$ commutes with all deck transformations.

Remark 3.2. In fact it is easy to prove that the first condition is implied by the second one (see [BFFP]). We state the definition of good lifts this way, to emphasize that both properties are used throughout the analysis.

Lemma 3.3. *If $f : M \to M$ is homotopic to identity, then a lift to $\widetilde{M}$ of a homotopy from the identity to f in M provides a good lift of f.*

In conclusion:

Proposition 3.4. *Let f be a diffeomorphism of a closed hyperbolic 3-manifold M. Then, there is an iterate of f which admits a good lift to $\widetilde{M}$.*

Notice that in hyperbolic M the good lift is unique (but we will not use this fact).

3.2. **Foliations without compact leaves.** A foliation $\mathcal{F}$ on a closed 3-manifold M will mean a continuous two dimensional foliation with leaves of class C^1 and tangent to a continuous distribution (foliations of class $C^{0,1+}$ according to [CC]). We will work with foliations without compact leaves. In particular thanks to results of Novikov, Palmeira and others (see [Ca$_3$, CC]) it follows that the fundamental group of each leaf injects in $\pi_1(M)$ and therefore every leaf lifts to a plane in the universal cover $\widetilde{M}$ which necessarily is diffeomorphic to $\mathbb{R}^3$. We denote by $\widetilde{\mathcal{F}}$ the foliation lifted to $\widetilde{M}$.

For such a foliation, there cannot be a nullhomotopic closed curve transverse to the foliation, and this implies that the leaf space

$$\mathcal{L}_{\widetilde{\mathcal{F}}} \ := \ \widetilde{M}/_{\widetilde{\mathcal{F}}}$$

is a one-dimensional simply connected manifold (possibly non-Hausdorff). If the leaf space is Hausdorff, then it is homeomorphic to $\mathbb{R}$ and in this case we say the foliation $\mathcal{F}$ is $\mathbb{R}$-*covered.*

We also consider the following geometric condition: suppose that given any two leaves $L, L' \in \widetilde{\mathcal{F}}$ the Hausdorff distance between the leaves is bounded (by a bound that obviously depends on L and L'). In this case we say that the foliation is *uniform* [Th$_2$]. We notice that the distance one needs to consider in $\widetilde{M}$ is relevant as this is a geometric condition, but since M is compact, any distance which is $\pi_1(M)$-equivariant will work (in particular, when one works with M hyperbolic, one can consider the standard hyperbolic metric on $\widetilde{M} = \mathbb{H}^3$).

If $\mathcal{F}$ is a (transversally orientable) foliation on a closed 3-manifold M and $\Phi_t : M \to M$ is a flow on M, we say that Φ is *regulating* for $\mathcal{F}$ if the orbits of Φ are transverse to $\mathcal{F}$ and when lifted to the universal cover, it holds that for every $x \in \widetilde{M}$ and $L \in \widetilde{\mathcal{F}}$ it follows that there exists $t \in \mathbb{R}$ such that $\widetilde{\Phi}_t(x) \in L$. (Notice that under the assumption that $\mathcal{F}$ has no closed leaves, this implies that the orbit of x cannot intersect L more than once by Novikov's theorem [CC]. Here $\widetilde{\Phi}$ is the flow Φ lifted to $\widetilde{M}$.)

If $\mathcal{F}$ admits a regulating flow, it follows that $\mathcal{F}$ is $\mathbb{R}$-covered. In hyperbolic manifolds, one has the following very interesting strong converse to this proved by Thurston [Th$_2$] (see also [Ca, Fen$_3$]):

Theorem 3.5 ([Th$_2$, Ca, Fen$_3$]). *Given an $\mathbb{R}$-covered uniform (transversally orientable) foliation $\mathcal{F}$ on a closed hyperbolic manifold M, there exists a regulating pseudo-Anosov flow $\Phi_{\mathcal{F}}$ to $\mathcal{F}$.*

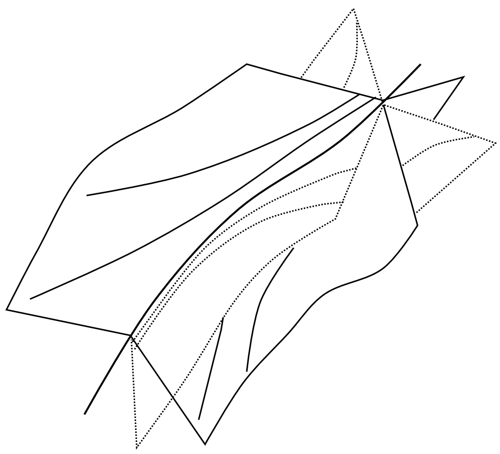

FIGURE 1. The local figure close to a p-prong with $p = 3$.

We recall that a flow $\Phi : M \to M$ with C^1 flow lines is said to be *pseudo-Anosov* if there are 2-dimensional, possibly singular, foliations, $\mathcal{G}^s$ (stable), $\mathcal{G}^u$ (unstable) which are flow invariant which verify that:

- In a leaf of $\mathcal{G}^s$ all orbits are forward asymptotic, and backwards orbits diverge from each other; the analogous opposite statement for leaves of $\mathcal{G}^u$;
- Singularities, if any, are of p-prong type, with $p \geq 3$. This means that they are periodic orbits, finitely many, and locally the stable leaf is a p-prong in the plane times an interval (see figure 1).

3.3. **Discretized Anosov flows.** A diffeomorphism $g : M \to M$ is said to be a *discretized Anosov flow* if there exists a g-invariant 1-dimensional continuous foliation $\mathcal{F}^c$ which supports a topological Anosov flow $\{\Phi_t : M \to M\}$ (that is, $\Phi_t(x)$ is always in the same $\mathcal{F}^c$ leaf as x for any x in M and any real t); and such that there exists a continuous function $\tau : M \to \mathbb{R}_{>0}$ so that $g(x) = \Phi_{\tau(x)}(x)$.

Recall that a *topological Anosov flow* is a pseudo-Anosov flow which has no singular periodic orbits.

Remark 3.6. A partially hyperbolic diffeomorphism which is a discretized Anosov flow will be necessarily dynamically coherent and the center leaves will correspond to the orbits of the flow (see [BFFP$_2$]).

We will use the following result from topological Anosov flows:

Theorem 3.7 ([Fen$_5$, Fen$_7$]). *Let Φ be a topological Anosov flow on a 3-manifold M which is regulating for a uniform $\mathbb{R}$-covered foliation $\mathcal{F}$ in M. Then M has virtually solvable fundamental group, and Φ_t is orbit equivalent to a suspension flow of a linear automorphism of $\mathbb{T}^2$.*

It is interesting to compare the case of suspensions (which yield solvable fundamental group of the 3-manifold) with the thorough study of accessibility classes in such manifolds performed in [H].

3.4. **Dichotomies for foliations and laminations.** Here we collect some of the results that are needed from [BFFP] as well as some properties that we will use.

We will use the following result:

Proposition 3.8 (Dichotomy for lifted foliations [BFFP]). *Let $h : M \to M$ be a homeomorphism of a 3-manifold which is homotopic to the identity. Let $\mathcal{F}$ be a foliation without compact leaves in M preserved by h. Let $\widetilde{h}$ be a good lift of h. Suppose that there is a leaf L of $\widetilde{\mathcal{F}}$ such that $\widetilde{h}(L) \neq L$. Then there are two possibilities:*

(1) The foliation $\mathcal{F}$ is $\mathbb{R}$-covered and uniform and $\widetilde{h}$ acts as a translation on the leaf space of $\widetilde{\mathcal{F}}$.

(2) The leaves L and $\widetilde{h}(L)$ bound a region in $\widetilde{M}$ with closure U, and the foliation $\widetilde{\mathcal{F}}$ has leaf space homeomorphic to a closed interval in U. In addition if $V = \bigcup_{n\in\mathbb{Z}} \widetilde{h}^n(U)$ then each leaf of ∂V is fixed by $\widetilde{h}$ and the set V is precisely invariant (meaning that if $\gamma \in \pi_1(M)$ verifies that $\gamma V \cap V \neq \emptyset$ then $\gamma V = V$).

For an account on complementary regions to laminations, we refer the reader to [CC, Section 5.2].

Lemma 3.9. *Let $f : M \to M$ be a diffeomorphism which preserves a lamination Λ with C^1 leaves, and such that each completion of a complementary region of Λ is an I-bundle. Then, there exists a homeomorphism $h : M \to M$ homotopic to f which coincides with f in Λ, and there exists a h-invariant foliation $\mathcal{F}$ which extends Λ.*

Proof. Since the completion of complementary regions are I-bundles, one can extend Λ to a foliation $\mathcal{F}_0$ so that in the closure of each complementary region of Λ the foliation $\mathcal{F}_0$ is a product foliation. The completion being an I-bundle means that the completion is diffeomorphic to $F \times [0,1]$ where F is surface. Extend the foliation so that the leaves in $F \times [0,1]$ are $F \times \{t\}$. In fact this can be done so that $\mathcal{F}$ is a $C^{0,1+}$ foliation [CC].

Choose a one dimensional foliation η with C^1 leaves, transverse to $\mathcal{F}_0$, and so that in each closure of a complementary component of Λ the foliation η provides an I-bundle structure. This is standard, see [CC], but we provide the main ideas: choose a one dimensional foliation $\mathcal{G}$ with C^1 leaves transverse to $\mathcal{F}_0$. Let U be a complementary region and V its completion. If V is compact, it is obvious how to do this. Otherwise $V = K \cup B$ where K is compact and in B the distance between the upper and lower boundary is very small. In particular if this distance is very small, then every leaf of $\mathcal{G}$ in B goes from the lower boundary of B to the upper boundary of B. Then $K = C \times [0,1]$ where C is a compact surface. In addition in $\partial C \times [0,1]$ one can assume the one dimensional foliation $\mathcal{G}$ is vertical. Since $\mathcal{F}$ is a product two dimensional foliation in $C \times [0,1]$ it is easy to prove that every leaf of $\mathcal{G}$ in $C \times [0,1]$ is a compact segment from $C \times \{0\}$ to $C \times \{1\}$.

We will now define $\mathcal{F}$ and the map h. Let A be a complementary region.

First consider the case that the orbit of A is infinite, in other words A is not periodic. Then for each n define a foliation $\mathcal{F}$ in $f^n(A)$ as $f^n(\mathcal{F}_0|A)$. Let $h = f$ in the f orbit of A. Clearly this satisfies the conditions of the Lemma in these regions.

The other case is that A is periodic, let n be the smallest positive integer so that $f^n(A) = A$. A priori there could be infinitely many such regions. For each complemetary region B, then B union its two boundary leaves F_0, F_1 is an I-bundle with a product foliation. We consider the case that F_0 is non compact, the other case being simpler. Then any such $C = B \cup F_0 \cup F_1$ is the union of a compact core and non compact parts [CC]. By choosing the non compact parts very thin, we can ensure that in such a non compact part of C, then $f(\mathcal{F}_0)$ is transverse the I-fibers in the particular I-bundle. Finally only finitely many complementary regions have a compact part with thickness bigger than a given $\delta > 0$. Hence except for these finitely many compact regions in these finitely many regions then $f(\mathcal{F}_0)$ is transverse to the I-fibers.

Now consider a specific periodic A with period $n > 0$. Let the boundary leaves be F_0, F_1. In each $0 \leq i < n$ define $\mathcal{F}$ in $f^i(A)$ to be the image of $\mathcal{F}_0|A$ by f^i. Now we define h. For each $0 \leq i < n-1$ (here $i \leq n-1$ and not $i \leq n$), let h to be equal to f in these complementary regions. Now consider $f^n(A) = A$. The foliation $\mathcal{F}$ in A is already defined to be equal to $\mathcal{F}_0$ here. Now we will define h in $f^{n-1}(A)$. By hypothesis $f(\mathcal{F})$ is transverse to the I-fibration in the non compact part of $f^{n-1}(A \cup F_0 \cup F_1)$. So we can just homotope f to a map h along the I-fibers in the non compact part of $A \cup F_0 \cup F_1$, so that the new map h sends $\mathcal{F}|f^{n-1}(A)$ to $\mathcal{F}|A$, restricted to the non compact part. In fact we can do this so that f^n fixes every leaf in the non compact part. The remaining set in A is compact, where the foliations are products. Here again f can be homotoped to h satisfying that h sends $\mathcal{F}|f^{n-1}(A)$ to $\mathcal{F}|A$.

This finishes the proof of the Lemma. □

As a consequence, we obtain:

Corollary 3.10. *Suppose that f is a diffeomorphism homotopic to the identity. Suppose that f preserves a lamination Λ with C^1 leaves, and that each completion of a complementary region of Λ is an I-bundle. Then, any good lift $\widetilde{f}$ of f verifies that*

- *either there is a compact sublamination Λ' of Λ, whose leaves are fixed by $\widetilde{f}$ when lifted to $\widetilde{M}$, or*
- *let $\mathcal{F}$ be a foliation which extends Λ as in Lemma 3.9. Then $\mathcal{F}$ is $\mathbb{R}$-covered and uniform, and $\widetilde{f}$ acts as a translation on the leaf space of $\widetilde{\Lambda}$ as a subset of the leaf space of $\widetilde{\mathcal{F}}$. We sometimes refer to this as $\widetilde{f}$ acts as a translation on $\widetilde{\Lambda}$.*

Proof. First we apply Lemma 3.9 to obtain $\mathcal{F}$ and h. At this point, the only property that needs to be proved is that if $\widetilde{h}$ is a good lift of h, and $\widetilde{h}$ fixes a leaf of $\widetilde{\mathcal{F}}$ then $\widetilde{f}$ fixes a leaf of $\widetilde{\Lambda}$. This is because we also proved in [BFFP] that the set of leaves of $\widetilde{\Lambda}$ fixed by $\widetilde{h}$ is a closed subset of leaves, and hence projects to a sublamination of Λ in M.

Suppose that there is a leaf L of $\widetilde{\mathcal{F}}$ that is fixed by $\widetilde{h}$. If L is in $\widetilde{\Lambda}$, then $\widetilde{f}$ also fixes L and we are done.

Otherwise $\pi(L)$ is in a complementary region W of Λ. The completion of W is a product foliated I-bundle. Since $\widetilde{f}$ preserves $\widetilde{W}$ and $\widetilde{f}$ preserves $\widetilde{\Lambda}$, it now follows that $\widetilde{f}$ fixes each of the two boundary leaves of $\widetilde{W}$. This finishes the proof of the corollary. □

Remark 3.11. In the first case, where there is a compact lamination Λ' of leaves whose lifts are fixed by $\widetilde{f}$, it follows that there is a uniform bound on the displacement of points inside the leaves by $\widetilde{f}$. This is because $\widetilde{f}$ is bounded distance from the identity and the lamination Λ' is compact. See [BFFP, Section 4].

3.5. **Dynamical coherence and incoherence.** A partially hyperbolic diffeomorphism is *dynamically coherent* if there are foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ that are f-invariant, and everywhere tangent to $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$. These are called the center stable and center unstable foliations. In that case the intersection of these two foliations is a one dimensional foliation tangent to E^c, called the center foliation. But this condition is not always satisfied, there are several recent examples which are not dynamically coherent [HHU$_4$, BGHP]. See also [HP$_2$, Section 4] for more information and context.

However, under very general orientability conditions there are "generalized" foliations tangent to E^{cs} and E^{cu}:

Theorem 3.12 (Burago-Ivanov [BI])**.** *Let f be a partially hyperbolic diffeomorphism of a 3-manifold M so that the bundles E^s, E^c, E^u are orientable and Df preserves these orientations. Then, there are collections $\mathcal{F}^{cs}_{bran}$ and $\mathcal{F}^{cu}_{bran}$*

of complete immersed surfaces tangent respectively to E^{cs} and E^{cu} satisfying the following properties:

- *every point $x \in M$ belongs to at least one surface (called* leaf*) of $\mathcal{F}^{cs}_{bran}$ (resp. $\mathcal{F}^{cu}_{bran}$),*
- *the collection is f-invariant: if $L \in \mathcal{F}^{cs}_{bran}$ (resp. $L \in \mathcal{F}^{cu}_{bran}$) then $f(L) \in \mathcal{F}^{cs}_{bran}$ (resp. $\mathcal{F}^{cu}_{bran}$),*
- *different leaves of $\mathcal{F}^{cs}_{bran}$ (resp. $\mathcal{F}^{cu}_{bran}$) do not topologically cross,*
- *if $x_n \to x$ and L_n are leaves of $\mathcal{F}^{cs}_{bran}$ (resp. $\mathcal{F}^{cu}_{bran}$) containing x_n then up to subsequence $L_n \to L$ for a leaf L of $\mathcal{F}^{cs}_{bran}$ (resp. $L \in \mathcal{F}^{cu}_{bran}$).*

Moreover, for every $\varepsilon > 0$ there are approximating foliations $\mathcal{F}^{cs}_{\varepsilon}$ and $\mathcal{F}^{cu}_{\varepsilon}$ tangent to subspaces making angle less than ε with E^{cs} (resp. E^{cu}). In addition there are continuous maps $h^{cs}_{\varepsilon} : M \to M$ and $h^{cu}_{\varepsilon} : M \to M$ at C^0-distance less than ε from identity so that h^{cs}_{ε} (resp. h^{cu}_{ε}) maps an arbitrary leaf of of $\mathcal{F}^{cs}_{\varepsilon}$ (resp. $\mathcal{F}^{cu}_{\varepsilon}$) into a leaf of $\mathcal{F}^{cs}_{bran}$ (resp. $\mathcal{F}^{cu}_{bran}$), by a local diffeomorphism.

The collections of surfaces $\mathcal{F}^{cs}_{bran}, \mathcal{F}^{cu}_{bran}$ are called the center stable and center unstable branching foliations. The individual surfaces are called the leaves. If say $\mathcal{F}^{cs}_{bran}$ does not form a foliation, then h^{cs}_{ε} is not a local diffeomorphims for any $\varepsilon > 0$: there are open sets where it collapses leaves transversely. When h^{cs}_{ε} is restricted to an arbitrary leaf of $\mathcal{F}^{cs}_{bran}$, then h^{cs} is a local diffeomorphism. Even then it may not be a global diffeomorphism because leaves of $\mathcal{F}^{cs}_{bran}$ may self intersect, forming branching locus.

When lifted to the universal cover the foliations $\widetilde{\mathcal{F}}^{cs}_{bran}, \widetilde{\mathcal{F}}^{cu}_{bran}$ have "leaf spaces" which are one dimensional, simply connected, but possibly non Hausdorff, just as in the case of foliations. Since a point does not determine the leaf it is on, this is not immedidate as in the case of actual foliations. But one can approximate $\mathcal{F}^{cs}_{bran}$ by actual foliations and the above follows, using such an approximation. We refer the reader to [BFFP, Section 9.1] for a detailed treatment. A good lift acts on $\widetilde{M}$ and hence acts as homeomorphisms on the leaf spaces of $\widetilde{\mathcal{F}}^{cs}_{bran}$ and $\widetilde{\mathcal{F}}^{cu}_{bran}$.

Remark 3.13. One can always take a finite lift g of an iterate of f so that g satisfies the condition of Theorem 3.12, and hence g admits branching foliations as in Theorem 3.12.

3.6. **Classification results.** We need the following classification result from [BFFP] for hyperbolic manifolds[1].

Theorem 3.14 ([BFFP])**.** *Let $f : M \to M$ be a partially hyperbolic diffeomorphism of a hyperbolic 3-manifold M and let $\widetilde{f}$ be a good lift of f. Suppose that f preserves branching foliations $\mathcal{F}^{cs}_{bran}, \mathcal{F}^{cu}_{bran}$. Then either*

(1) f is a discretized Anosov flow, and in particular f is dynamically coherent;

(2) or f is not dynamically coherent and $\widetilde{f}$ is a double translation*. That is, $\widetilde{f}$ is a translation on each of the leaf spaces of $\widetilde{\mathcal{F}}^{cs}_{bran}$ and $\widetilde{\mathcal{F}}^{cu}_{bran}$.*

[1]More general results exist for general partially hyperbolic diffeomorphisms homotopic to the identity, we refer the reader to [BFFP] for such statements including some stronger results even in the hyperbolic manifold case.

By a translation on the leaf space of $\widetilde{\mathcal{F}}^{cs}_{bran}$ we mean the following: the foliation $\mathcal{F}^{cs}_{bran}$ is approximated arbitrarily well by a Reebless foliation. This implies that one can also define the leaf space of the branched foliation $\widetilde{\mathcal{F}}^{cs}_{bran}$ and that it is a simply connected one manifold, possibly non Hausdorff. If it is Hausdorff, then it is homeomorphic to $\mathbb{R}$. So $\widetilde{f}$ acting as a translation on the leaf space of $\widetilde{\mathcal{F}}^{cs}_{bran}$, means that $\widetilde{f}$ does not fix any leaf of $\widetilde{\mathcal{F}}^{cs}_{bran}$, this leaf space is homeomorphic to $\mathbb{R}$, and $\widetilde{f}$ acts as a translation on $\mathbb{R}$.

We remark that there are no known examples of partially hyperbolic diffeomorphisms of the form (2) in M hyperbolic. On the other hand it is shown in [BFFP] that such hypothetical examples should share several features with the actual recent examples discovered in [BGHP]. We stress that this refers to a hypothetical possibility in hyperbolic 3-manifolds in [BFFP], and to actual examples in some Seifert manifolds $M = T^1 S$ in [BGHP].

We also mention the following dynamical consequence that will be useful for this paper.

Theorem 3.15 ([BFFP])**.** *Let $f : M \to M$ be a partially hyperbolic diffeomorphism homotopic to identity in M so that $\pi_1(M)$ is not virtually solvable. Suppose that there are f-invariant branching foliations $\mathcal{F}^{cs}_{bran}, \mathcal{F}^{cu}_{bran}$. Suppose that either $\mathcal{F}^{cs}_{bran}$ or $\mathcal{F}^{cu}_{bran}$ is f-minimal. Let $\widetilde{f}$ be a good lift of f to $\widetilde{M}$. Then $\widetilde{f}$ has no periodic points.*

This result does need M hyperbolic. On the other hand, when M is hyperbolic one does not need to assume that the branching foliations exist, or are f-minimal as this can be achieved by a finite cover (see [BFFP]). Now we define the unknown terms: A fixed point of $\widetilde{f}^k$ where $\widetilde{f}$ is a good lift of f is called a *contractible periodic point.* The property of say the foliation $\mathcal{F}^{cs}_{bran}$ being *f-minimal* means the following: M is the only non empty, closed, $\mathcal{F}^{cs}_{bran}$ saturated set that is f-invariant.

4. Proof of Theorem B

Consider a partially hyperbolic diffeomorphism $f : M \to M$ where M is a closed hyperbolic 3-manifold. The goal is to prove that f is accessible. Since the strong stable and unstable foliations are the same for iterates, it is no loss of generality to assume that f is homotopic to identity and admits a good lift $\widetilde{f}$ to $\widetilde{M}$ (c.f. Proposition 3.4). We will assume that f is non-wandering (recall that if f is non-wandering, so are its iterates[2]).

For most of the analysis we will not need to assume that M is hyperbolic, just that f admits a good lift $\widetilde{f}$ to $\widetilde{M}$ and that $\pi_1(M)$ is not virtually solvable. We will state explicitely the place where we use that M is hyperbolic.

We will proceed by contradiction assuming that f is not accessible and appeal to Theorem 2.3 that provides an f-invariant lamination Λ^{su} whose leaves are tangent to $E^s \oplus E^u$. Moreover, this lamination has no closed leaves (because $\pi_1(M)$ is not virtually solvable, c.f. Remark 2.5) and can be completed to a foliation $\mathcal{F}$ so that the completion U of any complementary region of Λ^{su}

[2]This is because for a homeomorphism of a compact metric space whose non-wandering set is all the space, the set of recurrent points is dense.

is a product foliated I-bundle. In fact, from Theorem 2.3 we know that the open set $U \cong F \times (0,1)$ (the completion of U being the addition of $F \times \{0\}$ and $F \times \{1\}$) and:

(1) the foliation $\mathcal{F}$ in U is a product foliation, and,
(2) the center bundle in U is uniquely integrable and the center one dimensional foliation is also a product foliation in U.

This is the only place where the fact that f is non-wandering will be used. (We do not really need in (2) the center to be uniquely integrable in those regions, just that center curves join both sides of the complementary region, but it is helpful for the presentation of the arguments.)

We denote by $\widetilde{\Lambda}^{su}$ and $\widetilde{\mathcal{F}}$ the lifts of these objects to $\widetilde{M}$. The lifted lamination $\widetilde{\Lambda}^{su}$ is invariant under $\widetilde{f}$. In addition there is h homotopic to f so that $h = f$ in Λ and h preserves $\mathcal{F}$. Let $\widetilde{h}$ be a corresponding good lift of h to $\widetilde{M}$.

We can apply Corollary 3.10 to the lamination $\widetilde{\Lambda}^{su}$. This separates the study into two cases, which we will deal with separately.

Case I – There is a leaf of $\widetilde{\Lambda}^{su}$ which is fixed by the good lift $\widetilde{f}$.

By Corollary 3.10 this is equivalent to $\widetilde{h}$ fixing a leaf of $\widetilde{\mathcal{F}}$. Here we will prove the following:

Proposition 4.1. *Consider a good lift $\widetilde{f}$ of a partially hyperbolic diffeomorphism $f : M \to M$, homotopic to the identity, so that $\pi_1(M)$ is not virtually solvable. Then $\widetilde{f}$ cannot leave invariant a leaf of $\widetilde{\Lambda}^{su}$.*

The proof of this proposition is deferred to section 6. This will prove that Case I cannot happen. We mention here that there is a simpler proof of Proposition 4.1 communicated to us by Andy Hammerlindl [HRHU] using an old result from Mendes [Men]. Both proofs work without any assumption on the topology of M other than having that $\pi_1(M)$ is not solvable. We present our proof here in order to have a complete proof of Theorem B. In addition our proof exemplifies, in a simplified context, some of the technical results of [BFFP] (which are simplified by the fact that leaves are subfoliated by strong stables and unstables while in [BFFP] they are foliated by strong stables and have some branching center foliation which is harder to deal with).

Case II – $\widetilde{f}$ does not fix any leaf of $\widetilde{\Lambda}^{su}$.

The analysis of this case will be divided in two subcases that will be proven in section 7. The fact that it can be divided into these subcases needs M to be hyperbolic so that Theorem 3.14 applies. Even if the dichotomy holds (without assuming that M is hyperbolic), we need M to be hyperbolic to treat one of the cases (the double translation).

Up to a finite cover and iterate, we may assume that f leaves invariant branching foliations $\mathcal{F}^{cs}_{bran}, \mathcal{F}^{cu}_{bran}$. By Theorem 3.14, one considers the action of $\widetilde{f}$ on the leaf spaces of $\widetilde{\mathcal{F}}^{cs}_{bran}, \widetilde{\mathcal{F}}^{cu}_{bran}$: either f is a discretized Anosov flow ($\widetilde{f}$ fixes every leaf of both $\widetilde{\mathcal{F}}^{cs}_{bran}, \widetilde{\mathcal{F}}^{cu}_{bran}$), or $\widetilde{f}$ acts as a translation on both leaf spaces, called *double translation*. Recall that $\widetilde{f}$ does not fix a leaf of $\widetilde{\Lambda}^{su}$ if and only if $\widetilde{h}$ acts as a translation on the leaf space of $\widetilde{\mathcal{F}}$. Notice that in this particular case there are three foliations: $\mathcal{F}^{cs}_{bran}, \mathcal{F}^{cu}_{bran}$ and $\mathcal{F}$. There are actions on the leaf spaces of each of these three foliations. Here we will prove:

Proposition 4.2. *Suppose that $\pi_1(M)$ is not virtually solvable. If $\mathcal{F}$ is $\mathbb{R}$-covered and uniform and $\widetilde{h}$ acts as a translation on the leaf space of $\widetilde{\mathcal{F}}$ then f cannot be a discretized Anosov flow.*

As stated, this proposition only requires the manifold not to have (virtually) solvable fundamental group (see [H] for a study of this case). The next result however really uses that the manifold is hyperbolic.

Proposition 4.3. *Suppose that M is a hyperbolic 3-manifold. If f is a double translation on $\widetilde{\mathcal{F}}^{cs}_{bran}, \widetilde{\mathcal{F}}^{cu}_{bran}$ then $\widetilde{h}$ cannot act as a translation on $\widetilde{\mathcal{F}}$.*

This treats all possibilities and therefore these propositions complete the proof of Theorem B (and as a consequence proves Theorem A, see subsection 2.2).

5. Minimality of Λ^{su}, Gromov hyperbolicity, and contractible fixed points

Before we begin the proof of Theorem B, we obtain some general results that will be useful later. The main point of this section is to show that leaves of Λ^{su} are Gromov hyperbolic so that Candel's uniformisation theorem [Can] applies. We remark that this property is quite direct in hyperbolic 3-manifolds as they are atoroidal (see e.g. [Ca_3, Chapter 7] or the proof of Lemma 5.6 below), so the reader can skip this section if it is only interested in the hyperbolic 3-manifold case. This section also shows Lemma 5.7 which allows to consider finite lifts and remain being non-wandering to apply Theorem 3.15. This is also not needed in the case where M is a hyperbolic 3-manifold where Theorem 3.15 holds without any further assumption.

5.1. A general result about foliations with transverse invariant measures.

The following is a general result about minimal foliations in 3-manifolds. We have not found this statement in the literature, so we give a proof; but it is likely to be known by the experts. Raul Ures has informed us that he also has a proof of some parts of this result [Ur]. We refer the reader to [CC, Chapters 11 and 12 Book I] for background on transverse invariant measures and growth of leaves. We will also use some standard results from foliations theory in dimension 3 that can be found e.g. in [CC, Chapter 9 Book 2].

Theorem 5.1. *Let $\mathcal{F}$ be a minimal transversally orientable codimension one foliation in a closed 3-manifold M admitting a holonomy invariant transverse measure ν. Then, all leaves are pairwise homeomorphic. Moreover, the foliation is $\mathbb{R}$-covered and uniform and one of the following options holds for $\mathcal{F}$.*

- *If there is a planar leaf in $\mathcal{F}$, then M is the 3-dimensional torus;*
- *If there is a leaf which is an annulus or Möbius band, then M is a nil-manifold;*
- *Otherwise in the universal cover $\widetilde{M}$ all the leaves are uniformly Gromov hyperbolic.*

If the transverse orientability hypothesis is not fulfilled, one can obtain it via taking a double cover (which will not affect the existence of a holonomy invariant transverse measure). Minimality of the foliation after a double lift is not immediate, we explain how to obtain this along the way.

We divide the proof into several lemmas.

First, notice that minimality implies that there are no compact leaves, in particular no Reeb components. In addition the foliation does not have sphere or projective plane leaves. Therefore one can apply Novikov's and Palmeira's (see [CC]) theorems to show that $\widetilde{M}$ is diffeomorphic to $\mathbb{R}^3$ and leaves of $\tilde{\mathcal{F}}$ are properly embedded planes in $\widetilde{M}$.

Also, thanks to [Ca_2, Lemma 3.3] one can without loss of generality assume that leaves of $\mathcal{F}$ are smoothly immersed and with immersions varying continuously in the C^∞ topology. This is obtained after a global isotopy of the original foliation which does not affect minimality, the existence of a holonomy invariant transverse measure nor the topological type of the foliation. If one picks a Riemannian structure on M one can consider the unit vector field orthogonal to $T\mathcal{F}$. This vector field is not necessarily C^1, but can be approximated arbitrarilily close by a smooth vector field. Pick one such smooth vector field, and consider the one dimensional foliation τ obtained by integrating this vector field.

Cover M by finitely many charts of $\mathcal{F}$ which are also foliated charts for the one dimensional foliation τ. In each chart any transverse arc from the "bottom" leaf of $\mathcal{F}$ to the "top" leaf of $\mathcal{F}$ has exactly the same measure under ν, by the holonomy invariance. In addition this measure is not zero, because of the minimality of $\mathcal{F}$. The support of the measure has to intersect the interior of this chart. Hence there is a minimal $a > 0$ measure of transverse arcs for any of the elements of this finite family of charts and also a maximum $b > 0$.

Let $\widetilde{\mathcal{F}}$, $\widetilde{\tau}$ the lifts to $\widetilde{M}$. We first establish the fact that the foliation is $\mathbb{R}$-covered. The proof will help to show that all leaves are homeomorphic in a double cover.

Lemma 5.2. *The foliation $\mathcal{F}$ is $\mathbb{R}$-covered.*

Proof. Start with a leaf L of $\widetilde{\mathcal{F}}$ and fix a point x in L. Choose a near enough leaf E of $\widetilde{\mathcal{F}}$ so that the leaf of $\widetilde{\tau}$ through x intersects E. Let $\widetilde{\tau}(y)$ be the leaf of $\widetilde{\tau}$ through y. Let

$$D = \{y \in L, \text{ so that } \widetilde{\tau}(y) \cap E \neq \emptyset\}; \quad \text{let } \varphi : D \to E, \ \varphi(y) = \widetilde{\tau}(y) \cap E.$$

Then we claim that $D = L$ and the length of the $\widetilde{\tau}$ segments from L to E have bounded length. Clearly D is open. Suppose that y_i are in D converging to y in L. Consider the segments v_i of the foliation $\widetilde{\tau}$ from y_i to $\varphi(y_i)$. If these segments have bounded length, then they are all in a compact set in $\widetilde{M}$. Then by the local product structure of the foliation τ lifted to $\widetilde{M}$, it follows that, up to subsequence, the segments converge to a segment of the foliation $\widetilde{\tau}$ from y to E. Hence y is in D.

Suppose on the other hand that the length of v_i converges to infinity. By the holonomy invariance of the measure ν, it follows that they all have the same measure ($\widetilde{\nu}$, the lift of ν to $\widetilde{M}$). Projecting to M these are segments of τ of length going to infinity. At most b length of these segments can be contained in one of finitely many foliated boxes. Hence the sum of the measures of these segments is going to infinity, contradiction.

Therefore the length of v_i is bounded. It now follows that D is both open and closed, and so $D = L$. In addition the length of the $\widetilde{\tau}$ segments from L to E is bounded. We stress this fact

($*$) every $\widetilde{\tau}$ leaf intersecting L also intersects E, and the length of $\widetilde{\tau}$ segments from L to E is bounded.

This implies that if W is the region of $\widetilde{M}$ bounded by L, E (including L, E), then the foliation $\widetilde{\mathcal{F}}$ restricted to W has leaf space homeomorphic to a closed interval.

The same holds for images of W under deck transformations, that is $\gamma(W)$ where γ is in $\pi_1(M)$. Hence if $\gamma(W)$ and $\beta(W)$ intersect the foliation $\widetilde{\mathcal{F}}$ has leaf space a closed interval in the union of these two sets. Since the foliation $\mathcal{F}$ is minimal, it follows that the union of deck translates of W cover M.

This shows that $\mathcal{F}$ is $\mathbb{R}$-covered. □

The property ($*$) obtained in the proof is important to obtain:

Lemma 5.3. *Suppose that $\mathcal{F}$ is transversely orientable. Then the leaves of $\mathcal{F}$ are pairwise homeomorphic.*

Here we shall use the transverse orientability of $\mathcal{F}$. The possible necessity of this is demonstrated by the following foliation: start with the product foliation of $S^2 \times S^1$ with transverse measure given by the S^1 measure. Let η be a free involution of S^2 and take the quotient of $S^2 \times S^1$ by the involution $\eta'(p,t) = (\eta(p), 1-t)$ there t is mod one. The quotient has a foliation with spheres and two projective planes, and a holonomy invariant measure. Not all leaves are homeomorphic to each other. Here the foliation is not minimal. We do not know whether this behavior can occur with minimal foliations. If one does not have transverse orientability, one can always lift to a double cover to get it (minimality of the lifted foliation can be tricky, we explain this at the end).

Proof. Let L, E as in the proof of the previous lemma. Let γ in the stabilizer of L, in other words, γ is in $\pi_1(\pi(L))$. Fix a basepoint x in L. Consider the $\widetilde{\tau}$ segment v from x to $z = \varphi(x)$. Choose a path α in L from x to $\gamma(x)$. Pushing this path along the $\widetilde{\tau}$ foliation, this produces a path α_L from z to another point w in E. We can push the whole path because of fact ($*$) above. On the other hand, the image of v under γ is a segment of $\widetilde{\tau}$, starting in $\gamma(x)$ and with same $\widetilde{\nu}$ length as v. But the $\widetilde{\tau}$ segment from $\gamma(x)$ to E also has this same length. Since no non degenerate segment in $\widetilde{\nu}$ has zero length, because of the holonomy invariance of $\widetilde{\nu}$; it follows that $\gamma(v)$ has to end in E. In other words

$$\gamma\varphi(x) \quad = \quad \varphi(\gamma(x)).$$

In fact this works for any x in L. It now follows that $\pi(L)$ is homeomorphic to $\pi(E)$, and this is true for any leaf of $\widetilde{\mathcal{F}}$ in between L and E. Since the leaves of $\mathcal{F}$ are dense the result follows. □

We first prove minimality in a double cover.

Lemma 5.4. *Let $\mathcal{G}$ be a transversely orientable double cover of $\mathcal{F}$ in a double cover M_2 of M. Then $\mathcal{G}$ is a minimal foliation too.*

Proof. Let ν_2 be the lift of ν to M_2, which is a holonomy invariant transverse measure for $\mathcal{G}$.

We must prove that $\mathcal{G}$ is a minimal foliation. Suppose not. Let V be a leaf of $\mathcal{G}$ which is not dense. Suppose first that V contains a segment of the transverse foliation. Let η be a maximal such segment, which is compact. It has to be compact, because the $\mathcal{G}$ saturation of a transverse segment of big enough ν_2 measure is all of M_2. Let W be a leaf through one endpoint of η. Since W is not compact it limits somewhere so that are points in M_2 with infinitely many returns of W. This contradicts the maximality of η. It follows that V is dense, so $\mathcal{G}$ is minimal. This proves the Lemma. □

Now we can prove the theorem.

Proof of Theorem 5.1. Now we prove the trichotomy in the statement of Theorem 5.1. Lift to a double cover M_2 so that the lift $\mathcal{G}$ of $\mathcal{F}$ is transversely orientable. By the previous lemma, $\mathcal{G}$ is minimal, and also the leaves of $\mathcal{G}$ are pairwise homeomorphic.

Case 1 – Suppose that $\mathcal{G}$ has a plane leaf.

Then all the leaves of $\mathcal{G}$ are planes. Then leaves of $\mathcal{G}$ cover those of $\mathcal{F}$ at most two to one, so the fundamental group of leaves of $\mathcal{F}$ is a subgroup of $\mathbf{Z}_2$. If the fundamental group is not trivial, then the leaf is the projective plane, which is disallowed by minimality of $\mathcal{F}$. It follows that all the leaves of $\mathcal{F}$ are also planes. Then $M = \mathbb{T}^3$, by a result of Rosenberg [Ros].

Case 2 – Suppose that $\mathcal{G}$ has an annulus or a Möbius band leaf.

We first rule out a Möbius band leaf of $\mathcal{G}$. Suppose that E is a Möbius band leaf. Let α be a simple closed curve in R so that it has a small neighborhood U which is homeomorphic to a compact Möbius band. Then $E - U$ is an open annulus. Now take a small transversal neighborhood of U in M foliated by the transversal foliation. Since there is a holonomy invariant transverse measure, this neighborhood is product foliated by $\mathcal{G}$. In particular all the local leaves of $\mathcal{G}$ in this neighborhood are compact Möbius bands. There are infinitely many returns of $E - U$ to the fixed transversal neighborhood of U, because E is dense. The local leaves of E intersected with this neighborhood are local leaves of $\mathcal{G}$ which are compact Möbius bands. This contradicts that $E - U$ is an open annulus.

This shows that there are no Möbius band leaves of $\mathcal{G}$ and all leaves of $\mathcal{G}$ are annuli.

Let V be a leaf of $\mathcal{G}$. Fix a simple not null homotopic closed curve δ in V through a point x. Then δ generates the fundamental group of V. Since ν is holonomy invariant and V is dense, the curve δ lifts to a nearby closed curve β through y in V. In addition is the boundary of an embedded annulus A in M_2 made up of very small segments of the fixed transverse foliation. In addition since V is an annulus, then $\beta \cup \delta$ also bound a unique annulus B contained in

V. We can cut and paste, choosing the first such intersection of B with the interior of A, so that B does not intersect the interior of A. For any $\epsilon > 0$, we can also choose B very big in V so that B is ϵ dense in M. In particular we can choose B so that it intersects every flow line of the transverse flow.

The union $A \cup B$ is a torus, transverse to the flow along B. We can slightly adjust it along A so that it is transverse to the flow. The resulting torus T is transverse to the flow and it intersects every orbit of the transverse flow. In other words the torus is a cross section of the flow. Hence the manifold fibers over the circle with fiber a torus.

Suppose first that M is a solv manifold. It is proved in appendix B of [HP$_1$] that $\mathcal{G}$ is weakly equivalent to either a stable or unstable foliation of a suspension Anosov flow or a fibration by tori. In the first case $\mathcal{G}$ has to have planar leaves, in the second case $\mathcal{G}$ has to have tori leaves. Any of these is disallwed by the conditions here. So we conclude that M cannot be a solv manifold. This implies that M is a nil manifold, which could be $\mathbb{T}^3$.

This finishes the analysis of Case 2.

Case 3 $-$ No leaves of $\mathcal{G}$ are planes, annuli or Möbius bands.

In particular since there are no compact leaves, it follows that the leaves of $\mathcal{G}$ cannot be conformally elliptic or parabolic and they are all conformally hyperbolic. Each one is separately uniformized with a metric of constant sectional curvature -1.

In this case we use the results of Candel in [Can]. In section 4.2 of [Can] he proves that if all leaves of $\mathcal{G}$ uniformize to being hyperbolic, then the uniformization is continuous and one can choose a metric in M so that each leaf of $\mathcal{G}$ has a metric of constant negative curvature.

It now follows that the leaves of $\widetilde{\mathcal{G}}$ (which are the same as the leaves of $\widetilde{\mathcal{F}}$) are uniformly Gromov hyperbolic in any metric. This proves Case 3.

This finishes the proof of Theorem 5.1. □

5.2. **The partially hyperbolic case.**

Lemma 5.5. *Suppose that f is a partially hyperbolic diffeomorphism in M which is not accessible and such that the non-wandering set of f is all of M. Suppose further that there are no compact surfaces tangent to $E^s \oplus E^u$. Then the lamination Λ^{su} is minimal.*

Proof. Suppose on the contrary that Λ^{su} is not minimal, and let Λ be a minimal sublamination of Λ^{su}. Consider

$$L_i \ := \ f^i(\Lambda), \ i \in \mathbb{Z}.$$

Then L_i is contained in Λ^{su}. Since Λ is minimal then for any i, j either $L_i = L_j$ or they are disjoint. Let E be the closure in M of the union of the L_i. Then $E \subset \Lambda^{su}$ is a sublamination and E is f-invariant.

We recall that to obtain the results of [HHU$_2$] it is not necessary to consider the full lamination Λ^{su} (c.f. Remark 2.4). However it is necessary to consider an f-invariant (sub)lamination. This is why we consider the set E as above.

Let U be the metric completion of a complementary region of Λ. Recall that we can think of the interior of U as a subset of M. If $\Lambda = M$ there is nothing to prove. Otherwise there is a non empty such component U, which is non compact, since there are no compact leaves in Λ^{su}. Then U has an octopus decomposition (see [CC, Proposition 5.2.14]):

$$U \quad = \quad K \cup D,$$

where K is compact, D is non compact and D is an I-bundle over a non compact surface. If for some i, the set $f^i(\Lambda)$ intersects the interior of D (this is a subset of M), then $f^i(\Lambda)$ intersects all the I-fibers in such a component of D. Hence $f^i(D)$ (with a fixed i), limits on Λ in M, as Λ is minimal. This contradicts that Λ, $f^i(\Lambda)$ are disjoint and compact.

Notice that $f^i(\Lambda)$ is distinct from Λ as it intersects the interior of U. So we now obtain that $f^i(\Lambda) \cap U$ is contained in the compact set K. It follows that the intersection of E with the interior of U is contained in K, and hence this intersection does not intersect the interior of D. Because E is f-invariant, we can now apply Theorem 2.3 (recall Remark 2.4). It follows that the completions of the complementary regions of E are I-bundles. But there is a complementary region A containing the interior of a component of D as above. This implies that $f^i(\Lambda)$ cannot intersect A, hence cannot intersect the interior of U. Since this applies to any complementary component of Λ, it follows that $f^i(\Lambda)$ cannot intersect any complementary region of Λ, hence $f^i(\Lambda) = \Lambda$.

The conclusion is Λ is f invariant.

Now, again applying Theorem 2.3, we obtain that the completion of the complementary regions of Λ are I-bundles. In the same as above we prove that Λ^{su} cannot intersect the interior of any complementary region of Λ.

Therefore $\Lambda^{su} = \Lambda$, and as a consequence Λ^{su} is minimal as claimed. □

We stress that this has no assumption on M, or homotopic assumptions on f. We only have the dynamical hypothesis (non-wandering of f is all of M) on f. Lemma 5.5 allows us to use Theorem 5.1 to obtain:

Proposition 5.6. *Let $\Lambda \subset M$ be a minimal lamination without compact leaves and so that the complementary regions of Λ are I-bundles. Assume that M is a irreducible 3-manifold whose fundamental group is not (virtually) solvable. Then, leaves of Λ are uniformly Gromov hyperbolic.*

Proof. Suppose that leaves of Λ are not uniformly Gromov hyperbolic. By Candel's theorem [Can] there is a holonomy invariant transverse measure ν to Λ.

The support of ν is a sublamination of Λ. By hypothesis, this lamination is minimal, so the support of ν is all of Λ.

The complementary regions to Λ have completions that are I-bundles. Hence one can blow down these complementary regions so that Λ blows down to a foliation $\mathcal{H}$. Since Λ is minimal, then $\mathcal{H}$ is minimal. The holonomy invariant transverse measure μ blows down to a holonomy invariant transverse measure to $\mathcal{H}$. The support is all of M. Now one can apply Theorem 5.1 and the fact that $\pi_1(M)$ is not (virtually) solvable to conclude. □

As a consequence, leaves of Λ^{su} are uniformly Gromov hyperbolic.

We will also need the following easy fact.

Lemma 5.7. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism homotopic to the identity and such that the non-wandering set of f is all of M. Then f has no contractible periodic points.*

Proof. Up to taking a finite lift g of an iterate of f we can assume that the bundles of g are orientable and its orientation is preserved by Df. This allows us to apply Theorem 3.12 to get branching foliations $\mathcal{F}^{cs}_{bran}, \mathcal{F}^{cu}_{bran}$, for the map g.

Assume, say that $\mathcal{F}^{cs}_{bran}$ is not minimal. Then a minimal g-invariant set Λ of g is a proper repeller (because it is transversally unstable). This implies that g cannot be non-wandering (see [CP, Section 1.1.2]).

We will show that this would force f to have wandering points which contradicts the hypothesis: For simplicity we will assume that f is homotopic to identity, but the proof can be adapted to be more general.

So let $g : M_1 \to M_1$ be a finite lift of an iterate of f which is also homotopic to the identity (obtained by lifting the homotopy). For simplicity assume that M_1 is a normal cover.

Since $\Omega(f) = M$, the set of recurrent points of f is dense in M. This is because for a basis of the topology $\{U_n\}_n$ we have that the sets $A_n = \{x \in U_n \ : \ \exists k > 1 \ , \ f^k(x) \in U_n\} \cup \overline{U_n}^c$ is open and dense and a point in $\bigcap_n A_n$ must be recurrent. Since recurrent points are non-wandering for iterates of f we can assume that g is a lift of f and not of an iterate.

Let x be the lift to M_1 of a recurrent point. We want to show that x is non-wandering for g. Since the non-wandering set is closed and the lifts of recurrent points of f is dense we get $\Omega(g) = M_1$.

As x is the lift of a recurrent point, there are $n_j \to \infty$ so that $g^{n_j} \to \gamma x$, where γ is some deck transformation from $M_1 \to M$. Since $M_1 \to M$ is a finite cover there is some $k > 0$ so that $\gamma^k = id$. In addition g commutes with γ, since g is homotopic to the identity.

Now take a neighborhood U of x, Let U_{k-1} be an open set around $\gamma^{k-1}x$ so that for some large m_{k-1} one has that

$$g^{m_{k-1}}(U_{k-1}) \ \subset \ U$$

This exists because $g^{n_j}(\gamma^{k-1}(x))$ converges to x.

Similarly, inductively construct U_i neighborhood of $\gamma^i x$ and m_i so that $g^{m_i}(U_i) \subset U_{i+1}$. Once one has constructed U_1, it follows that for some m_0 one has that $g^{m_0}(x)$ is in U_1 and then

$$g^{m_0+m_1+\ldots m_{k-1}}(x) \ \in \ U$$

as desired. This shows that if $\Omega(f) = M$, then g is non wandering.

Now, we can apply Theorem 3.15 to g which implies that f cannot have contractible periodic points either, since one such point would give rise to a contractible periodic point for g. □

6. Case I – Fixed leaves

This section will be devoted to the proof of Proposition 4.1.

Assumption in section 6 – The hypothesis in this section are those of Proposition 4.1. In particular, no need to assume that M is hyperbolic. We will assume that there is a leaf $\tilde{\Lambda}^{su}$ which is fixed by a good lift $\widetilde{f}$ of f to $\widetilde{M}$.

We start by showing that leaves of Λ^{su} whose lifts are fixed by $\widetilde{f}$ all have cyclic fundamental group, and there is at least one which is not a plane.

Suppose that C is a leaf of Λ^{su} which has a lift L to $\widetilde{M}$ which is fixed by $\widetilde{f}$. The goal to obtain Proposition 4.1 is to show that this assumption on the existence of C leads to a contradiction. Since $\widetilde{f}$ commutes with deck transformations, then $\widetilde{f}$ fixes any lift of C to $\widetilde{M}$. Hence this is a property of C and not of the particular lift.

The proof of the following lemma will make use of the theory of *axes* for free actions on leaf spaces of foliations, we refer the reader to [Fen$_4$] or [RS] for a general account or [BFFP, Section 4.6] for a more direct account for use in a similar context.

Lemma 6.1. *Let* $L \in \widetilde{\Lambda}^{su}$ *which is fixed by* $\widetilde{f}$. *Then,* $\mathrm{Stab}_L := \{\gamma \in \pi_1(M) : \gamma L = L\}$ *is cyclic.*

Proof. Suppose that L is fixed by $\widetilde{f}$. In this case we proceed as in the proof of Proposition 5.4 of [BFFP]: By Lemma 5.7 the map $\widetilde{f}$ has no periodic points. It follows that $\widetilde{f}$ does not fix any stable leaf in L: otherwise if s is such a leaf, since $\widetilde{f}$ is a contraction in the stable leaves, there would an $\widetilde{f}$ fixed point in s. Hence $\widetilde{f}$ acts freely on the stable leaf space of L, i.e. $\mathcal{L}^s_L = L/_{\widetilde{\mathcal{W}^s}}$. Therefore, there is an axis Ax^s of $\widetilde{f}$ acting on $\mathcal{L}^s_L$. Similarly $\widetilde{f}$ acts freely on the unstable leaf space $\mathcal{L}^u_L$ and has an axis Ax^u.

The axis is the set of leaves s so that $\widetilde{f}(s)$ separates s from $\widetilde{f}^2(s)$ in L. In [Fen$_4$] it is shown that if $\widetilde{f}$ acts freely on the leaf space, then the axis is non empty, and the axis is either a line or a $\mathbb{Z}$-union of disjoint intervals in $\mathcal{L}^s_L$ (see also [BFFP, Section 4.6]). In the second case

$$\mathrm{Ax}^s \; = \; \bigcup_{i \in \mathbb{Z}} I_i \; = \; \bigcup_{i \in \mathbb{Z}} [x_i, y_i]$$

Here $[x_i, y_i]$ are closed intervals in the leaf space $\mathcal{L}^s_L$ and y_i is non separated from x_{i+1} in $\mathcal{L}^s_L$.

In addition, there are no closed stable leaves in M, hence any deck transformation fixing L must act freely on $\mathcal{L}^s_L$ too. As deck transformations commute with $\widetilde{f}$ it follows that any deck transformation fixing L must have the same axis as $\widetilde{f}$ (see [Fen$_4$, Lemma 3.11]).

This implies that deck transformations all act without fixed points in the single set Ax^s. If Ax^s is a line, then by Hölder's theorem (see [BFFP, Proposition 4.16]) the group must be abelian. In other words, the fundamental group of the leaf must be abelian. If the axis is an infinite union of intervals, then the fundamental group of $\pi(L)$ acts on this collection without any fixed points, that is $\pi_1(\pi(L))$ acts freely on $\mathbb{Z}$. Again this implies that $\pi_1(\pi(L))$ is abelian. Since there are no closed leaves of $\mathcal{F}$ one obtains that Stab_L is cyclic. $\square$

Lemma 6.2. *Suppose that $\widetilde{f}$ fixes a leaf of $\widetilde{\Lambda}^{su}$. Then there exists at least one leaf L of $\widetilde{\Lambda}^{su}$ which is fixed by $\widetilde{f}$ and has stabilizer* $\mathrm{Stab}_L := \{\gamma \in \pi_1(M) : \gamma L = L\}$ *which is isomorphic to $\mathbb{Z}$.*

Proof. Assume by way of contradiction that all leaves C of Λ^{su} which have a lift L to $\widetilde{M}$ fixed by $\widetilde{f}$ are planes (in other words if L is such a lift, then Stab_L is trivial, or equivalently $\pi_1(C)$ is trivial). Since only $\mathbb{T}^3$ admits a foliation by planes (see [Ros]), one has to have one leaf C of $\mathcal{F}$ with $\pi_1(C)$ not trivial, and $\pi_1(C)$ contains $\gamma \in \pi_1(M) - id$. Since complementary regions to Λ^{su} are I-bundles, it follows that there is a leaf $B \in \Lambda^{su}$, so that γ is in $\pi_1(B)$. Lift B to a leaf $\hat{B}$ of $\widetilde{\Lambda}^{su}$, so that $\hat{B}$ is fixed by γ. By assumption in the beginning of this paragraph, $\hat{B}$ cannot be fixed by $\widetilde{f}$. By Proposition 3.8 the completion U of the region between $\hat{B}$ and $\widetilde{f}(\hat{B})$ is an I-bundle. In addition since $\widetilde{f}$ has fixed leaves, then $V = \bigcup_{n\in\mathbb{Z}} \widetilde{f}^n(U)$ is not all of $\widetilde{M}$, and all leaves in ∂V are fixed by $\widetilde{f}$. Here $\hat{B}$ is in $\widetilde{\Lambda}^{su}$, and each leaf in ∂V is accumulated by $\widetilde{f}^n(\hat{B})$ with $n \to \infty$ or $n \to -\infty$. Hence any leaf in ∂V is in $\widetilde{\Lambda}^{su}$. Since each such leaf L is fixed by $\widetilde{f}$, then $\pi(L)$ it must be a plane, by the assumption in the beginning of this paragraph.

Again, by Proposition 3.8 the set V is precisely invariant, hence $\pi(V)$ is an open, $\mathcal{F}$ saturated set, which is not all of M and whose leaves in the boundary are all in Λ^{su} and are all planes. One can do the octopus decomposition of the completion W of $\pi(V)$ (see [CC, Proposition 5.2.14]). The decomposition is $W = K \cup D$ (not unique), where K is compact, and D is an I-bundle and very thin (meaning that local product structure boxes are not completely contained in D so that center curves go from side to side). It follows from the fact that boundary leaves of $\pi(V)$ are planes, that $\pi(V)$ has to be an I-bundle, that is, a disk times an open interval. (This uses that M is irreducible.)

But $\pi(V)$ contains $\pi(\hat{B})$ which does not have trivial fundamental group, contradiction. This completes the proof of the lemma. □

Proof of Proposition 4.1. Suppose that there is a leaf of $\widetilde{\Lambda}^{su}$ that is fixed by $\widetilde{f}$. We proceed as in [BFFP, Section 6] to get a contradiction. By the previous lemma, there is a leaf L of $\widetilde{\Lambda}^{su}$, which is fixed by both $\widetilde{f}$ and $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$. We will work in L. We just sketch the main arguments and refer the reader to [BFFP] for full details. We remark that several arguments are simpler in this setting because we know precisely how the dynamics of both foliations look like (in contraposition to [BFFP] where there is a center foliation for which the dynamics of $\widetilde{f}$ is not determined, that is, $\widetilde{f}$ could be contracting, expanding, or neither along the center foliation, depending on the particular point).

The leaf L has stable and unstable one dimensional foliations, on which both $\widetilde{f}$ and γ act.

- As shown in the proof of Lemma 6.1, the maps $\widetilde{f}$ and γ act freely on both $\mathcal{L}^s_L$ and $\mathcal{L}^u_L$. Since $\widetilde{f}$ and γ commute they both share the same axis Ax^s in $\mathcal{L}^s_L$, and Ax^u in $\mathcal{L}^u_L$. Each of these axes is either a line or a $\mathbb{Z}$-union of intervals.
- An unstable leaf in L cannot intersect a stable leaf s in Ax^s and its image $\gamma(s)$. Otherwise a graph transform argument (as in e.g. [BFFP,

Lemma 4.24]) would produce a closed unstable leaf in $\pi(L)$, which is impossible.

- This allows us to find a stable leaf s_1 which is fixed by $h := \gamma^n \circ \widetilde{f}^m$ where $m > 0$, see [BFFP, Proposition 6.3 and Lemma 6.4]. This map g is also partially hyperbolic and with uniform constants. It also follows that $s_2 = \gamma^{-n} s_1 = \widetilde{f}^m s_1$ is at bounded distance from s_1 in L, because $\widetilde{f}$ moves points a bounded distance inside L (Remark 3.11).
- The next step is to show that leaves of $\widetilde{\mathcal{F}}$ are Gromov hyperbolic. Here there is a substantial difference with [BFFP]. There we prove Gromov hyperbolicity for the leaves of $\mathcal{F}^{cs}_{bran}, \mathcal{F}^{cu}_{bran}$ in Lemma 5.7 of [BFFP]. A fundamental property that is used in [BFFP] is that f acts in an expanding way transverse to $\mathcal{F}^{cs}_{bran}$. In our situation, transverse to the Λ^{su} lamination is the center direction. A priori this could be contracting, expanding or neither under f. Hence this step necessitates a different proof from what is done in [BFFP]. This was done in Lemma 5.6. Then by Candel's result [Can], M admits a metric that makes all leaves of $\mathcal{F}$ of constant negative curvature.

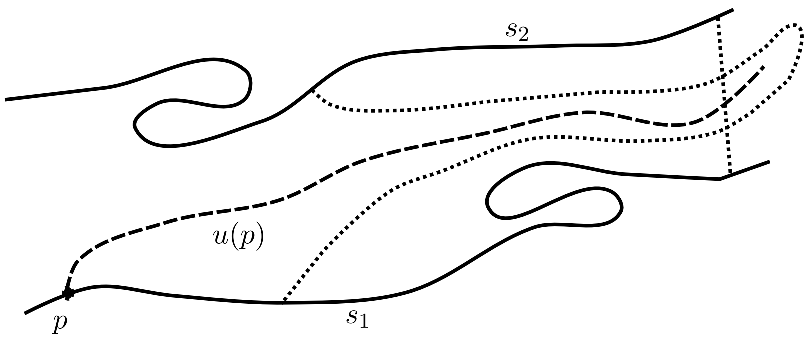

FIGURE 2. Such a configuration gives a contradiction with the fact that there are points in $s_1 \setminus \{p\}$ which are mapped very far away, because h cannot map curves of bounded length to arbitrarily long curves.

- Using a comparison with hyperbolic isometries, it is possible to show that there are points in s_1 which are mapped arbitrarily far apart from themselves by h in each ray of $s_1 \setminus \{p\}$ (where p is the fixed point in s_1 for h). This is [BFFP, Lemma 6.10].
- As in the proof of [BFFP, Proposition 6.3], one can find an unstable leaf u_1 in L, fixed by h and separating s_1 from s_2 in L. Using the previous remark and the fact that the band between s_1 and s_2 in L has bounded width, this forces the unstable u_1 to be coarsely contracting under h, as is done in [BFFP, Proposition 6.3] (see figure 2). This is a contradiction because $m > 0$. This finishes the proof of Proposition 4.1.

□

7. Case II – Translation case on $\widetilde{\mathcal{F}}$

Here we assume that $\widetilde{f}$ does not fix any leaf of $\widetilde{\Lambda}^{su}$, or equivalently that $\widetilde{h}$ does not fix any leaf of $\widetilde{\mathcal{F}}$.

We apply Proposition 3.8 to $\mathcal{F}$ and h, and we obtain that $\mathcal{F}$ is $\mathbb{R}$-covered and uniform. We fix a parametrization of the leaf space of $\widetilde{\mathcal{F}}$ as the reals $\mathbb{R}$,

$$\mathcal{F} \ = \ \{L_t \mid t \in \mathbb{R}\},$$

with $\widetilde{h}$ acting as an increasing homeomorphism.

7.1. **Discretized Anosov flow case.** In this subsection we prove Proposition 4.2. The goal is to prove that f cannot be a discretized Anosov flow. The proof is done under more general assumptions:

Assumption in subsection 7.1 – We assume that $\pi_1(M)$ is not virtually solvable, and therefore, there are no tori tangent to $E^s \oplus E^u$ (c.f. Remark 2.5).

Assume by way of contradiction that f is a discretized Anosov flow. In particular f is dynamically coherent and there exists a (topological) Anosov flow Φ and a positive continuous function $\tau : M \to \mathbb{R}_{>0}$ so that $f(x) = \Phi_{\tau(x)}(x)$ and center leaves of f are the orbits of Φ. In particular since M is compact, there are $a, b > 0$ so that $a < \tau(x) < b$. Notice that one could apply Theorem 3.7 if one showed that the flow Φ_t is regulating. This can be done, but we chose to give a more direct proof (which uses similar ideas to the proof of Theorem 3.7).

Since Φ_t is an Anosov flow, [Fen$_2$, Corollary E] shows that either Φ_t is topologically conjugate to a suspension Anosov flow or there are periodic orbits which are freely homotopic. In fact in the second case there are periodic orbits η_1 and η_2 of Φ_t which are freely homotopic but with different orientation (see [Fen$_2$, Corollary 4.5] or [BaFe, Section 2]). In other words the following happens: lift η_i coherently to $\widetilde{\eta}_i$ orbits of $\widetilde{\Phi}$. Let γ be a deck transformation associated with η_1 in the flow forward direction, hence associated with η_2 in the backwards direction, so γ preserves both $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$. Choose points x_i in $\widetilde{\eta}_i$. Then

- $\gamma \widetilde{x}_1 = \widetilde{\Phi}_{t_1}(\widetilde{x}_1)$ with $t_1 > 0$ and,
- $\gamma \widetilde{x}_2 = \widetilde{\Phi}_{t_2}(\widetilde{x}_2)$ with $t_2 < 0$.

Now we will consider $\widetilde{f}^n(\widetilde{x}_1)$ as $n \to \infty$. Since f is a discretized Anosov flow, then $\widetilde{f}^n(\widetilde{x}_1)$ is in $\widetilde{\Phi}_{\mathbb{R}}(\widetilde{x}_1)$, and it is equal to $\widetilde{\Phi}_{t_n}(\widetilde{x}_1)$ where $t_n > an$, since each application of $\widetilde{f}$ moves at least $a > 0$ time forward along the orbit. But the flow line $\widetilde{\eta}_1$ is a bounded distance from $\widetilde{\eta}_2$ (because η_1, η_2 are freely homotopic). Therefore $\widetilde{\Phi}_{t_n}(\widetilde{x}_1)$ is a bounded distance from $\widetilde{\Phi}_{r_n}(\widetilde{x}_2)$, and here $r_n \to -\infty$ as $n \to \infty$. This is because η_1 is freely homotopic to the inverse of η_2. But the points $\widetilde{f}^i(\widetilde{x}_2), i < 0$ are evenly spaced along the negative ray $\widetilde{\Phi}_{t<0}(\widetilde{x}_2)$. This is because $\tau(x) < b$ for all x. It follows that $\widetilde{f}^n(\widetilde{x}_1)$ is a bounded distance from $\widetilde{f}^{i_n}(\widetilde{x}_2)$, where $i_n \to -\infty$ as $n \to \infty$.

This will give a contradiction, the idea is that since Φ is transverse to $\mathcal{F}$ orbits must move all in the *same direction with respect to* $\mathcal{F}$. We carry the details below.

At this point we will only use the maps $f, \widetilde{f}$ and the foliations $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. First we need more information about the foliation $\mathcal{F}$. Here $\mathcal{F}$ is $\mathbb{R}$-covered and does not have any compact leaves. Proposition 2.6 of [Fen$_3$] shows that $\mathcal{F}$ can be collapsed to a minimal foliation $\mathcal{E}$ which is minimal and which is obtained from $\mathcal{F}$ by collapsing at most a countably many foliated I-bundles of $\mathcal{F}$ to single leaves. This collapsing is homotopic to the identity and can be lifted to $\widetilde{M}$: the distances are distorted a bounded additive amount.

Now $\mathcal{E}$ is $\mathbb{R}$-covered, uniform and minimal. Theorem 2.7 of [Th$_2$] shows that $\mathcal{F}$ is a *slithering* of M around $\mathbb{S}^1$. Slithering means that there is a fibration $\nu : \widetilde{M} \to \mathbb{S}^1$ and the deck transformations of $\widetilde{M} \to M$ induce bundle automorphisms. In other words the pre images of ν form a foliation in $\widetilde{M}$ that is $\pi_1(M)$ invariant, and induces a foliation in M, which in this case is $\mathcal{E}$. The slithering comes with a coarsely defined distance [Th$_2$] between leaves E, E' of $\widetilde{\mathcal{E}}$: Parametrize the circle $\mathbb{S}^1$ as $[0,1]$ with 0 identified to 1. The universal cover of the circle is $\mathbb{R}$, where the generator deck transformation is a translation by 1. Now pick one lift $\widetilde{\nu} : \nu \to \mathbb{R}$ of ν. Given E, E' leaves of $\widetilde{\mathcal{E}}$, define the rough distance $z(E,E')$ as the absolute value of $\widetilde{\nu}(E) - \widetilde{\nu}(E')$. In [Th$_2$] Thurston is more careful defining this "pseudo-distance" to be an integer, but there is a bounded error between the definitions. Then Theorem 2.6 of [Th$_2$] shows that the distance between any two points $x \in E$ and $y \in E'$ is bounded below by a constant times $z(E,E')$.

Now we are ready to finish the proof of Proposition 4.2. Suppose that $\widetilde{x}_i$ is in L_i, where L_i are leaves of $\widetilde{\mathcal{F}}$. Then $\widetilde{f}^n(\widetilde{x}_1)$ is in L_{u_n}, with $u_n \to \infty$. We proved above that $\widetilde{f}^{i_n}(\widetilde{x}_2) \in L_{v_n}$, with $v_n \to -\infty$, both with $n \to \infty$. In addition the distance from $\widetilde{f}^n(\widetilde{x}_1)$ to $\widetilde{f}^{i_n}(\widetilde{x}_2)$ is uniformly bounded in n. Since leaves of $\widetilde{\mathcal{F}}$ are a uniformly bounded distance from leaves in $\widetilde{\mathcal{E}}$, the same holds for the projection of the points to leaves of $\widetilde{\mathcal{E}}$. But the previous paragraph shows that the distance between any points in L_{v_n} and L_{u_n} is converging to infinity because $z(L_{v_n}, L_{u_n})$ is converging to infinity. This is a contradiction.

This shows that this case cannot happen. This finishes the proof of Proposition 4.2.

7.2. **Pseudo-Anosov flows transverse to $\mathbb{R}$-covered foliations.** To analyze the case that $\widetilde{f}$ does not fix any leaf of $\widetilde{\Lambda}^{su}$ and is a double translation (in the sense of Theorem 3.14) we will need some properties of pseudo-Anosov flows. (This case will be called *triple translation*, c.f. subsection 7.4.)

Each of the 3 foliations will come equipped with a transverse pseudo-Anosov flow and to pursue our arguments we need to be able to compare such flows. This is the purpose of this section. We remark that the results proved in this subsection are completely general and can be of independent interest.

Let $\mathcal{G}$ be a foliation or a branching foliation (such as $\mathcal{F}^{cs}_{bran}, \mathcal{F}^{cu}_{bran}$) in a hyperbolic 3-manifold M. Let g be a homeomorphism homotopic to the identity preserving $\mathcal{G}$. If $\mathcal{G}$ is a branching foliation, then we assume that it is approximated arbitrarily close by a Reebless foliation, as in the case of the branching foliations $\mathcal{F}^{cs}_{bran}, \mathcal{F}^{cu}_{bran}$ of a partially hyperbolic diffeomorphism. As explained in [BFFP, Section 9.1], one can talk about the leaf space of $\widetilde{\mathcal{G}}$. It is the same as the leaf space of the approximating Reebless foliation. Suppose that this leaf space is $\mathbb{R}$ and that $\mathcal{G}$ is uniform and transversely orientable. Here uniform has the same definition as that for regular (unbranched) foliations. Then, by Theorem 3.5 there is a pseudo-Anosov flow Φ_t transverse to $\mathcal{G}$ and regulating for it (see also [BFFP, Section 12]). This only works if M is hyperbolic.

First we define a very weak form of free homotopy for periodic orbits of flows.

Definition 7.1. (freely homotopic orbits) Let Φ_1, Φ_2 be two flows in a closed 3-manifold. We say that periodic orbits α_1 of Φ_1 and α_2 of Φ_2 are *freely homotopic,* if there are $i, j \in \mathbb{Z}$ so that α_1^i is freely homotopic to α_2^j as oriented closed curves.

In other words, since i, j may have different signs, we do not care about powers or orientations in this definition.

Proposition 7.2. *Let Φ_i, $i = 1, 2$ be pseudo-Anosov flows transverse and regulating to a foliation or a branched foliation $\mathcal{G}_i$ in the same hyperbolic 3-manifold M. (The foliations $\mathcal{G}_i$ may be different from each other.) Suppose that every periodic orbit of Φ_1 is freely homotopic to a periodic orbit of Φ_2. Then Φ_1 is topologically orbit equivalent to Φ_2, by an equivalence that is homotopic to the identity.*

Proof. Since M is hyperbolic, then Φ_i is transitive [Mosh], and the union of periodic orbits is dense. Since the flows are regulating for $\mathcal{G}_i$, it follows that the leaf space of $\widetilde{\mathcal{G}}_i$ is homeomorphic to $\mathbb{R}$ for $i = 1, 2$. By [Fen$_3$] if there are periodic orbits of say Φ_2 that are freely homotopic to each other, then there are η_i with η_1 freely homotopic to η_2^{-1}. Let γ be a deck transformation representing η_1, and hence η_2^{-1}. The first implies that γ acts increasing on the leaf space of $\widetilde{\mathcal{G}}_2$ and the second implies that γ acts decreasingly there, a contradiction. Therefore there is at most one periodic orbit of Φ_2 in each conjugacy class in $\pi_1(M)$. Similarly for Φ_1.

Since M is hyperbolic, and Φ_1 is regulating for a foliation it follows that Φ_1 is *quasigeodesic* [Th$_2$, Fen$_6$]. This means that orbits are rectifiable and when lifted to the universal cover length along a given any orbit of $\widetilde{\Phi}_1$ is uniformly efficient up to a bounded multiplicative distortion in measuring distance in $\widetilde{M}$. In other words if δ is such an orbit of $\widetilde{\Phi}_1$, there is $K > 0$ so that for any x, y in δ then

$$l_\delta(x, y) \quad \leq \quad K d(x, y) + K$$

where $l_\delta(x, y)$ is the length along δ from x to y. In addition there is a uniform bound see Lemma 3.10 of [Ca$_3$]. In other words there is a single K so that the inequality above works for all orbits. Using properties of quasigeodesics in hyperbolic manifolds, this implies that any orbit of $\widetilde{\Phi}_1$ is a uniformly bounded distance from the corresponding geodesic of $\mathbb{H}^3$ with the same ideal points as this orbit (see e.g. [Th$_1$]). The same holds for Φ_2.

Let $\mathcal{O}_i$ be the orbit space of the flow $\widetilde{\Phi}_i$. In [FM] it is proved that $\mathcal{O}_i$ is homeomorphic to the plane $\mathbb{R}^2$. We will produce a homeomorphism $\tau : \mathcal{O}_1 \to \mathcal{O}_2$ that is $\pi_1(M)$ equivariant. This will produce the topological equivalence.

For any point p in $\mathcal{O}_1$ which is a lift of a periodic orbit γ_1 of Φ_1, then γ_1 is freely homotopic to a periodic orbit γ_2 of Φ_2. We remark that since M is hyperbolic, there is essentially one free homotopy between the corresponding powers of γ_1 and γ_2 $-$ any two free homotopies are homotopic. Lift this homotopy to $\widetilde{M}$ so that the first orbit lifts to the orbit in $\mathcal{O}_1$ corresponding to p. The second orbit lifts to an orbit of $\widetilde{\Phi}_2$ and it corresponds to q in $\mathcal{O}_2$. The second lift is uniquely determined by the uniqueness of free homotopies above.

Now let δ be any orbit of $\widetilde{\Phi}_1$. Since the periodic orbits of Φ_1 are dense, there are orbits δ_n of $\widetilde{\Phi}_1$, which are lifts of periodic orbits and which converge to δ. Let β_n be the corresponding lifts of periodic orbits of Φ_2. Notice that any orbit of $\widetilde{\Phi}_i$ is a uniformly bounded distance from the corresponding geodesic with the same endpoints at the sphere at infinity.. The orbits δ_n intersect a fixed compact set in $\widetilde{M}$. Hence the same is true for the corresponding geodesics by the above property. Since the orbits of Φ_2 also satisfy this property, it follows that the orbits β_n intersect a fixed compact set in $\widetilde{M}$. So there is a subsequence β_{n_j} which converges to an orbit β of $\widetilde{\Phi}_2$.

Since all orbits of $\widetilde{\Phi}_1$ (or $\widetilde{\Phi}_2$) are quasigeodesics with uniform constants, then the ideal points of the quasigeodesics β_{n_j} also converge to the ideal points of β (c.f. [Th$_1$]). In addition since Φ_i is transverse and regulating for a foliation $\mathcal{G}_i$, it follows that no two orbits of $\widetilde{\Phi}_1$ (or $\widetilde{\Phi}_2$) can have the same forward and backwards ideal point (notice that for this we need the flow to be pseudo-Anosov and not just quasi-geodesic, see [Fen$_6$] for more details). In particular the orbit β of $\widetilde{\Phi}_2$ is uniquely determined by its ideal points. In particular for any subsequence β_{n_k} of β_n which converges, it has to converge to β. In other words, since any such sequence has a convergent subsequence, then the whole sequence β_n converges to β. Furthermore, now for any sequence α_n of lifts of periodic orbits of Φ_1 converging to δ, the corresponding orbits ϵ_n of $\widetilde{\Phi}_2$ converge to β $-$ again this is because of the continuity property of the ideal points. It follows that β is uniquely defined.

This defines a map from $\mathcal{O}_1$ to $\mathcal{O}_2$. Again because of the uniformity of the constants of quasigeodesic behavior, this map is continuous. In addition, by the property that ideal points determine the orbit, this map is a bijection onto its image, which is homeomorphic to $\mathbb{R}^2$. The image is also $\pi_1(M)$ invariant. Since Φ_2 is transitive, there is a dense orbit, so this now implies that the image is $\mathcal{O}_2$.

Using the theory of classifying spaces Haefliger [Hae] proved that there is a homotopy equivalence of M sending orbits of Φ_1 to orbits of Φ_2. Subsequently Ghys [Gh] and later Barbot [Ba$_2$] upgraded this to a homeomorphism that sends orbits to orbits[3]. Up to reversing the flow direction, this homeomorphism also preserves orientation along orbits.

Since this homeomorphism sends periodic orbits to closed curves freely homotopic to themselves, it follows that this homeomorphism acts trivially on $\pi_1(M)$. Since M is aspherical, this implies that the homeomorphism is homotopic to the identity [He]. □

7.3. **Lefschetz fixed points.** Recall the Leftschetz formula for fixed points. We refer to the monograph by Franks [Fr, Chapter 5] or [BFFP, Section 12.1]. for details and more generality. Here we only work in the context we shall use.

Let P be a topological plane, $g : P \to P$ be a orientation preserving continuous map and $D \subset P$ a compact disk such that $\mathrm{Fix}(g) \subset \mathrm{int}(D)$.

The Lefschetz index $\mathrm{Ind}(g)$ of g in P is defined to be the intersection number of the graph of g with the diagonal in $P \times P$.

We will use the following facts about the Lefschetz index (see [KH, Section 8.6] for more details):

- If all fixed points of g are isolated then $\mathrm{Ind}(g)$ is the sum of all the indices of each fixed point. Moreover, a (topologically) hyperbolic fixed saddle point is -1 and the index of a p-prong fixed point ($p \geq 3$) is $1-p$.
- If $g, h : P \to P$ are orientation preserving maps for which there is $R > 0$ so that $d(g(x), h(x)) < R$ and there is a disk $D \subset P$ so that for every $x \notin D$ one has that $d(g(x), x) > 2R$ then $\mathrm{Ind}(g) = \mathrm{Ind}(h)$.

7.4. **Triple translation.** This section will use the setting of Proposition 4.3 which is what is left to prove to complete the proof of Theorem B. Recall that we can work up to finite covers and iterates.

Assumption in subsection 7.4 The diffeomorphism f is a non-accessible partially hyperbolic diffeomorphism of a hyperbolic 3-manifold M and f is a double translation (c.f. Theorem 3.14). Moreover, a good lift $\widetilde{f}$ of f acts as a translation in the lamination $\widetilde{\Lambda}^{su}$ (which therefore completes to an $\mathbb{R}$-covered foliation, see Corollary 3.10).

The assumption above implies that the diffeomorphism f preserves branching foliations $\mathcal{F}^{cs}_{bran}, \mathcal{F}^{cu}_{bran}$ and $\widetilde{f}$ acts as a translation on both leaf spaces of $\widetilde{\mathcal{F}}^{cs}_{bran}, \widetilde{\mathcal{F}}^{cu}_{bran}$. Here $\mathcal{F}^{cs}_{bran}, \mathcal{F}^{cu}_{bran}$ are branching foliations, but are approximated arbitrarily closely by actual foliations (as in Theorem 3.12).

Recall the foliation $\mathcal{F}$ which is an extension of the lamination Λ^{su}. In addition in this case there is a homeomorphism h homotopic to f so that h preserves $\mathcal{F}$. There is a good lift $\widetilde{h}$ of h to $\widetilde{M}$, so that $\widetilde{h}$ also acts as a translation on the leaf space of $\widetilde{\mathcal{F}}$, which is also $\mathbb{R}$. Recall that $h = f$ when restricted to Λ^{su}, and likewise for corresponding lifts to $\widetilde{M}$. Recall that f does not a priori preserve $\mathcal{F}$ in the complement of Λ^{su} (Corollary 3.10). This case is then called *triple translation*. Up to a finite cover and iterate if necessary, we may assume that $\mathcal{F}$ is transversely orientable. By Theorem 3.5 there is a

[3] This argument seems to have appeared several times in several places. See [HP_2, Section 8] for further references.

pseudo-Anosov flow Φ_{su} which is transverse to $\mathcal{F}$ and which is regulating for $\mathcal{F}$. This means that every orbit of $\widetilde{\Phi}_{su}$ intersects every leaf of $\widetilde{\mathcal{F}}$.

We know that f is a double translation meaning that $\widetilde{f}$ also translates $\widetilde{\mathcal{F}}^{cs}_{bran}$ and $\widetilde{\mathcal{F}}^{cu}_{bran}$ (which are therefore also $\mathbb{R}$-covered and uniform).

Denote by Φ_{cs} and Φ_{cu} the corresponding regulating pseudo-Anosov flows transverse to $\mathcal{F}^{cs}_{bran}, \mathcal{F}^{cu}_{bran}$ respectively. In particular, there are 3 pseudo-Anosov flows: Φ_{cs}, Φ_{cu} and Φ_{su}. We now state [BFFP, Proposition 12.6]:

Proposition 7.3. *For each $\gamma \in \pi_1(M)$ associated to a periodic orbit of Φ_{cs} (resp. Φ_{cu}) there exists $x \in \widetilde{M}$ which verifies that $\gamma^i \circ \widetilde{f}^k(x) = x$ for some $i, k \in \mathbb{Z} \setminus \{0\}$.*

See also the research announcement [BFFP$_2$].

In particular since $\widetilde{f}$ acts freely on the leaf space of $\widetilde{\mathcal{F}}^{cs}_{bran}$, then both i, k in Proposition 7.3 are non zero. To prove this result, in [BFFP] we use the transverse pseudo-Anosov flow to make an index argument, but we also use crucially the fact that we have information of the transverse dynamics to (say) $\mathcal{F}^{cs}_{bran}$: transversely we have the unstable foliation $\mathcal{W}^u$ and f expands unstable length. This is the reason why we cannot apply this result directly to Λ^{su} (even if Λ^{su} were a true foliation): transverse to Λ^{su} we have the center bundle E^c. But f can contract, expand or leave invariant E^c lengths depending on the particular point.

We will need some properties relating the lifted flow $\widetilde{\Phi}_{su}$ with the map $\widetilde{h}$. Recall that $\widetilde{h}$ is an extension of $\widetilde{f}|_{\widetilde{\Lambda}^{su}}$ to $\widetilde{M}$, and $\widetilde{h}$ preserves $\widetilde{\mathcal{F}}$. Let L be a leaf of $\widetilde{\mathcal{F}}$, which could be a leaf of $\widetilde{\Lambda}^{su}$ or not. Then $\widetilde{h}$ induces a map from L to $\widetilde{h}(L)$. This maps a point x to a point a bounded distance from it in $\widetilde{M}$, because $\widetilde{h}$ is a good lift of h.

We define another map $\mu : L \to \widetilde{h}(L)$: for each x in L, consider the $\widetilde{\Phi}_{su}$ flow line and define $\mu(x)$ to the intersection of this flow line with $\widetilde{h}(L)$, that is:

$$x \in L \;\mapsto\; \mu(x) \;:=\; \widetilde{\Phi}_{su}(x) \cap \widetilde{h}(L).$$

Since Φ_{su} is regulating for $\mathcal{F}$, there is always such an intersection and this is unique so this map is well defined. This map μ depends on L, but for notational simplicity we will omit this dependence. In addition since $\mathcal{F}$ is uniform, then the distance in $\widetilde{M}$ from x to $\mu(x)$ is uniformly bounded in $\widetilde{M}$, see [Fen$_3$]. Because $\mathcal{F}$ is $\mathbb{R}$-covered this implies that the distance in $\widetilde{h}(L)$ from $\widetilde{h}(x)$ to $\mu(x)$ is uniformly bounded above [Fen$_3$].

Another property we need is the following (see [BFFP, Section 13]):

Lemma 7.4. *For every $L \in \widetilde{\mathcal{F}}$, R a positive number, and $\eta \in \pi_1(M) \setminus \{\mathrm{id}\}$ such that $\eta \circ \mu(L) = L$, there exists a compact set $K \subset L$ such that if $y \notin K$ then $d_L(\eta \circ \mu(y), y) > R$.*

This is just a property about regulating pseudo-Anosov flows. In a nutshell, the exponential expansion and contraction of the foliations of the pseudo-Anosov flows and the fact that one has some control on the geometry of the intersection of the stable and unstable laminations of the pseudo-Anosov flow with the transverse foliation allow one to show that outside a compact set, points must move arbitrarily a lot. It is illustrating to check this for different lifts of a pseudo-Anosov homeomorphism of a surface (which would be the case of the suspension flow). Details are carried in [BFFP, Section 13] separating the case where η is associated to a periodic orbit of Φ_{su} and the case where it is not.

We can now prove one main property we need.

Proposition 7.5. *(equivalent flows) The flows Φ_{cs} and Φ_{su} are topologically equivalent, by an equivalence homotopic to the identity. The same holds for Φ_{cu} and Φ_{su}.*

Proof. We show the result for Φ_{cs} and Φ_{su}. Let α be a periodic orbit of Φ_{cs} represented by a deck transformation γ. By Proposition 7.3 there is $x \in \widetilde{M}$ with $\gamma^i \circ \widetilde{f}^k(x) = x$, and $k > 0$. We want to show that γ^i is associated to a periodic orbit of Φ_{su} so that we can apply Proposition 7.2.

First we want to show that x can be chosen to be in $\widetilde{\Lambda}^{su}$. Recall from Theorem 2.3 that the completion of any complementary region of Λ^{su} is an I-bundle, the center bundle is uniquely integrable in this completion, and any center segment in this completion must connect the two boundary components in $\widetilde{\Lambda}^{su}$. Suppose then that x is not in $\widetilde{\Lambda}^{su}$. Then x is in a center segment which connects two boundary leaves of $\widetilde{\Lambda}^{su}$ and this center segment is fixed by $\gamma^i \circ \widetilde{f}^k$. In particular if y is an endpoint of the center segment through x connecting two boundary leaves of $\widetilde{\Lambda}^{su}$ then $\gamma^i \circ \widetilde{f}^k(y) = y$. So we can assume that x is in $\widetilde{\Lambda}^{su}$.

Let L be the leaf of $\widetilde{\Lambda}^{su}$ containing x. As above consider the maps $\mu, \widetilde{f}^k$ from L to $\widetilde{f}^k(L) = \widetilde{h}^k(L)$. Then we have maps $\gamma^i \circ \mu$, $\gamma^i \circ \widetilde{f}^k$ from L to itself. They are a bounded distance $R > 0$ from each other.

Moreover, we know that $\gamma^i \circ \widetilde{f}^k$ has at least one fixed point (since $\gamma^i \circ \widetilde{f}^k(x) = x$) and it has Lefschetz index equal to -1 since it is a hyperbolic fixed point (because the action of $\gamma^i \circ \widetilde{f}^k$ in L is the composition of a isometry with a diffeomorphism with two hyperbolic bundles, so x is a hyperbolic saddle, c.f. subsection 7.3).

By Lemma 7.4 we know that the maps $\gamma^i \circ \widetilde{f}^k$ and $\gamma^i \circ \mu$ move points more than $2R$ outside a compact set, so we are in the setting of subsection 7.3. In particular $\gamma^i \circ \widetilde{f}^k$ and $\gamma^i \circ \mu$ have the same Lefschetz index. Then one can deduce that $\gamma^i \circ \mu$ has a fixed point of negative index.

But the map μ is flow along $\widetilde{\Phi}_{su}$ from L to $\widetilde{f}^k(L)$. Therefore a fixed point of $\gamma^i \circ \mu$ produces a periodic orbit of Φ_{su} which is represented by a power of γ.

Therefore any periodic orbit of Φ_{cs} is freely homotopic to a periodic orbit of Φ_{su}. By Proposition 7.2 this implies that Φ_{cs} is topologically equivalent to Φ_{su} by a topological equivalence homotopic to the identity.

This finishes the proof of the proposition. □

Now we show the following:

Proposition 7.6. *Let $\gamma \in \pi_1(M)$ associated to a periodic orbit of Φ_{su} so that there exists $x \in \widetilde{M}$ which belongs to a leaf $L \in \widetilde{\Lambda}^{cs}$ so that $\gamma^i \circ \widetilde{f}^k(x) = x$.*

Then it follows that x is the unique fixed point of $\gamma^i \circ \widetilde{f}^k$ in L and that the orbit γ is associated to a regular periodic orbit of Φ_{su} (which has index -1).

Proof. The proof is very similar to the proof that one cannot have a "mixed" behavior for a hyperbolic partially hyperbolic diffeomorphism in a hyperbolic 3-manifold M. This is done in [BFFP, Section 14].

In that case one had a center foliation and a stable foliation in such a leaf L and the center foliation may be branched. The analysis here is much simpler because we have in L a stable and an unstable foliation. In particular positive powers of f are known to expand length along unstable leaves, as opposed to the unknown actual behavior of f along center leaves. We explain the main steps.

As done in the previous propositions let μ be the $\widetilde{\Phi}_{su}$ flow along from L to $\widetilde{f}^k(L)$. Then $\gamma^i \circ \widetilde{f}^k$ has non zero Lefschetz index in L, and so does $\gamma^i \circ \mu$. Therefore $\gamma^i \circ \mu$ has a unique fixed point z in L. Up to taking powers, suppose that γ leaves invariant all prongs of the orbit through z. If z is a p-prong orbit, then the Lefschetz index $\gamma^i \circ \mu$ is equal to $1-p$, and hence so is the Lefschetz index of $\gamma^i \circ \widetilde{f}^k$ (c.f. subsection 7.3).

Our goal is to prove that $p = 2$. The leaf L is quasi-isometric to the hyperbolic plane [Can], and hence can be canonically compactified with an ideal circle ∂L. In particular $\gamma^i \circ \mu$ has exactly $2p$ fixed points in ∂L. It follows that the action of $\gamma^i \circ \widetilde{f}^k(p)$ has exactly $2p$ fixed points in ∂L, which are alternatively attracting (the set $P = \{P_1, \ldots, P_p\}$) and repelling (the set $N = \{N_1, \ldots, N_p\}$). This is because $\gamma^i \circ \mu$ and $\gamma^i \circ \widetilde{f}^k$ act in exactly the same way on ∂L, as they are maps a bounded distance from each other in L and then use Lemma 7.4 (see figure 3).

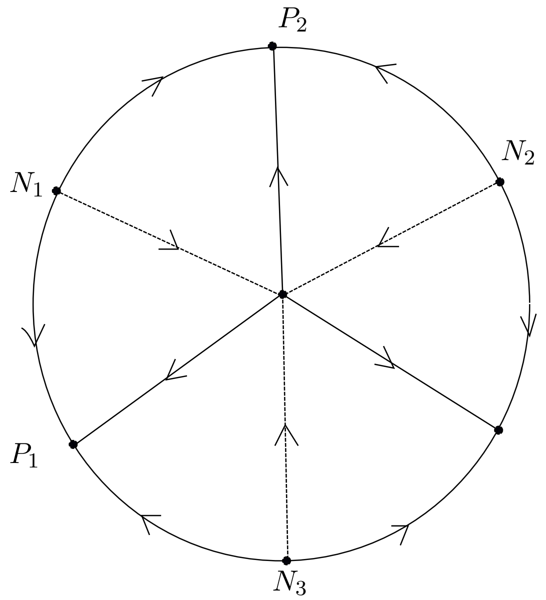

FIGURE 3. The dynamics of $\gamma^i \circ \mu$ in L. By Lemma 7.4 the dynamics of $\gamma^i \circ \widetilde{f}^k$ is very similar near the boundary.

In order to do prove that $p = 2$, we obtain several facts, whose proof is done in detail in [BFFP, Theorem 14.1]:

- First, if x is a fixed point of $\gamma^i \circ \widetilde{f}^k$ in L and $s(x)$ is a stable ray ending in x, then $s(x)$ can only limit in a single point w in N.
- An unstable ray through x can only limit in a single point w in P.
- Since P, N are disjoint the ideal points of periodic stable rays and periodic unstable rays are disjoint.
- Hence the two rays of $s(x)$ limit to distinct points, same for unstable leaves.
- Because the total index is $1 - p$, there are exactly $p - 1$ fixed points in L.

If x_1, x_2 are distinct fixed points of $\gamma^i \circ \widetilde{f}^k$, then no ray of $s(x_1)$ can share an ideal point with a ray of $s(x_2)$. This is because if two periodic rays share an ideal point it is possible to either construct a periodic unstable leaf limiting in the repelling point in N or get a configuration as in figure 4 which gives a contradiction as one is able to find a tangency between stables and unstables.

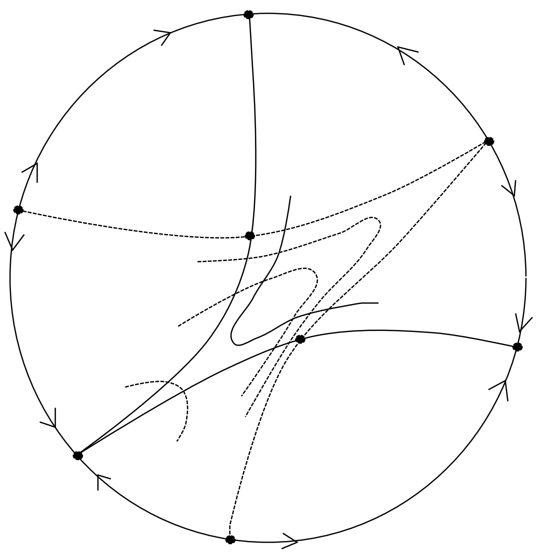

FIGURE 4. If two stable rays limit in the same point either the unstables do not cross from one side to the other and one obtains a periodic unstable limiting in a repelling point or one gets the configuration in the drawing that forces some tangency.

Therefore one gets for each fixed point x in L, there are four fixed points in ∂L. These collections of 4 points are pairwise disjoint. Hence the total is $4(p - 1)$. Since this number is equal to $2p$, it follows that $p = 2(p - 1)$ or that $p = 2$.

This finishes the proof of the proposition. □

7.5. **End of the proof of Proposition 4.3.** In this section we will assume that M is a hyperbolic 3-manifold.

Suppose first that Φ_{cs} does not have singular orbits. Then Φ_{cs} is a topological Anosov flow. In that case, Theorem 3.7 implies that $\pi_1(M)$ is solvable. But this contradicts that M is hyperbolic.

Suppose now that the flow Φ_{cs} has singular orbits. Let α be a p-prong singular orbit, with $p \geq 3$. Let γ be the deck transformation associated with α. By Proposition 7.3 there is x in $\widetilde{M}$ with $\gamma^i \circ \widetilde{f}^k(x) = x$, and $k > 0$. In the proof of Proposition 7.5 we showed that one may assume that x is in $\widetilde{\Lambda}^{su}$. Let L be the leaf of $\widetilde{\Lambda}^{su}$ containing x. By Proposition 7.6, there is a periodic orbit of Φ_{su} associated with γ and it is non singular. But Proposition 7.5 shows that the flows Φ_{cs} and Φ_{su} are topologically equivalent, by an equivalence homotopic to the identity. This would imply that the orbit of Φ_{su} is singular. This is a contradiction with Proposition 7.6 which states that the orbit of Φ_{su} in question is a regular orbit.

This contradiction shows that the assumption that there is a lamination Λ^{su} is impossible. It follows that f is accessible. This finishes the proof of Proposition 4.3 and therefore of Theorem B.

8. Accessibility without conservative behavior

In this section we obtain a result in the direction of having something like Theorem 2.3 without assuming that f is non-wandering. As we explained, the only place in the proofs of accessibility that uses that f is non-wandering is to be able to use the second and third conclusions Theorem 2.3. One other context where accesssibility has been established without use of conservativity is [HP_1, Section 6.3] for partially hyperbolic diffeomorphisms in nilmanifolds which are not the 3-torus. (In the generic setting, the non-conservative case has also been considered in some references, e.g. [DW, BHHTU].)

Here we show:

Theorem 8.1. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism of a closed 3-manifold M whose fundamental group is not virtually solvable. Suppose that f is a discretized Anosov flow. Then f is accessible.*

Proof. Suppose not, then there is Λ^{su} given by the first conclusion of Theorem 2.3. That only uses that f is not accessible. Notice that in this case we cannot say anything a priori about the complementary regions of Λ^{su} since f is not necessarily non-wandering. The goal will be to show that these complementary regions are I-bundles in order to be able to apply the results of the previous sections.

Let Φ be a topological Anosov flow so that $f(x) = \Phi_{\tau(x)}(x)$ where $\tau(x) > 0$. Let $\widetilde{f}$ be the lift preserving flow lines of $\widetilde{\Phi}$, so $\widetilde{f}(x) = \widetilde{\Phi}_{\tau(\pi(x))}(x)$.

Consider a stable leaf s of f lifted to $\widetilde{M}$. Then all points in s converge together under positive iteration of $\widetilde{f}$. It follows that s is contained in a (weak) stable leaf L of $\widetilde{\Phi}$.

Any center leaf c in L is an orbit of $\widetilde{\Phi}$. We claim that that s intersects c. To see this, take a point $x \in s$ and call its center leaf c_x. It follows that there exists $n > 0$ so that $\widetilde{f}^n(x)$ is at distance less than ε from c, because they are in the same weak stable leaf of $\widetilde{\Phi}$. By the local product structure, it follows that the stable leaf $s(\widetilde{f}^n(x))$ intersects c. Since $\widetilde{f}^{-n}(s(\widetilde{f}^n(x))) = s$ and c is $\widetilde{f}$-invariant it follows that s intersects c. Therefore, stable leaves are global sections of the center leaves inside each center-stable leaf. These facts are already implicit in [BoW] (see in particular [BoW, Lemma 3.16]). Likewise for the center-unstable leaves.

We now consider the orbit space $\mathcal{O}$ of the lifted flow $\widetilde{\Phi}$. This orbit space is homeomorphic to $\mathbb{R}^2$ (see [Fen$_1$, Ba$_1$]). Take any leaf E of $\widetilde{\Lambda}^{su}$. Then E is transverse to $\widetilde{\Phi}$, as it is transverse to the center bundle. Hence the projection of E to $\mathcal{O}$ is locally injective. The previous paragraph implies that E projects to $\mathcal{O}$ as a stable and unstable saturated set. This implies that E is a global section of the projection $\widetilde{M} \to \mathcal{O}$, in other words a global section for the flow $\widetilde{\Phi}$.

It now follows that the completion of any complementary region of $\widetilde{\Lambda}^{su}$ is an I-bundle. In addition the center/flow foliation is a product in the completion of this complementary region. Therefore the same is true when projecting to M.

This recovers the second and third conclusions of Theorem 2.3. One can now proceed as in sections 6 and 7.1 to get a contradiction and show that f is accessible. $\square$

Notice that this result is strictly larger than the previous as it has been shown in [BG] that there are examples of partially hyperbolic diffeomorphisms which are discretized Anosov flows and which have proper attractors and therefore cannot be non-wandering.

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Florida State University, Tallahassee, FL 32306
E-mail address: fenley@math.fsu.edu

Centro de Matemática, Universidad de la República, Uruguay
E-mail address: rpotrie@cmat.edu.uy
URL: http://www.cmat.edu.uy/~rpotrie/