PALINDROMIC GRAPHS

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Abstract. If $A(G)$ is the adjacent matrix of graph G and $\chi(G;\lambda) = \det(\lambda I - A(G)) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ is its characteristic polynomial, the graph G is *palindromic* if $\chi(G;\lambda)$ is, i.e, if $a_i = a_{n-i}$ for all i. The family of palindromic trees is found, as well as general efficient method to construct palindromic graphs.

1. Introduction

Despite the extensive research in the field of spectral graph theory, as far as we known the characterization of those graphs with palindromic and antipalindromic characteristic polynomial has passed unnoticed. Maybe because, like in [2] those works that give the characteristic polynomials of the first graphs, does not list the first of its coefficients, since $a_0 = 1$ for any graph. In this work we completely close the problem for trees and show that the same construction works for the general case, though it is not exhaustive.

2. Definitions

Given a graph $G = (V, E)$ with order $n = |V|$, its adjacent matrix is the matrix $A(G) \in \mathbb{R}^{n\times n}$ such that

$$(A(G))_{ij} = \begin{cases} 1 & ij \in E, \\ 0 & ij \notin E. \end{cases}$$

The characteristic polynomial of G is $\chi(G;\lambda) = \det(\lambda I - A(G)) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n$. We say that G is *palindromic* if $\chi(G;\lambda)$, is *self-reciprocal* or *palindromic*, i.e. if $a_i = a_{n-i}$ for $i = 0, \ldots n$. Analogously, We say that G is *antipalindromic* if $\chi(G;\lambda)$, is antipalindromic, i.e. if $a_i = -a_{n-i}$ for $i = 0, \ldots, n$.

Let us give some further notation:

- P_n is the path with n vertices.
- $\mathcal{T}_n$ will denote the set of non-isomorphic trees of order n.
- $\mathcal{T}$ will denote the set of all non-isomorphic trees, i.e., $\mathcal{T} = \cup_{i=1}^{\infty}\mathcal{T}_n$.
- $L(G)$ will denote the set of leaves of graph G.

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- If $w \notin V(G)$, and $v \in V(G)$, then $G +_v w$ will denote the graph obtained from G by pending a vertex w from vertex v, i.e.

$$G +_v w = (V(G) \cup \{w\}, E(G) \cup \{vw\})$$

- If $x, y \notin V(G)$ and $v \in V(G)$, then $G +_v xy = (G +_v x) +_x y$ is the graph obtained from G by pending vertices x, y in a path from vertex v.
- Let $H(G)$ be the *hairness* of graph G made by adding one leave for each of its vertices, i.e.

$$H(G) = (\{V(G) \times \{0,1\}\}, \{\{(v,0)(w,0)\} : vw \in E(G)\} \cup \{(v,0)(v,1) : v \in V(G)\}).$$

or if $V(G) = \{v_1, \ldots, v_n\}$, then $H(G) = G +_{v_1} v_1' +_{v_2} v_2' +_{v_3} \cdots +_{v_n} v_n'$.

We define the following family of non isomorphic trees as $\mathcal{F} = \cup_{i=1}^{\infty} F_i$ recursively as:

- $\mathcal{F}_1 = \{P_2\}$, $\mathcal{F}_2 = \{P_4\}$,
- $\mathcal{F}_n = \{T +_v xy : v \notin L(T), x, y \notin T, T \in \mathcal{F}_{n-1}\}$/isomorphism.

Next, we prove that $\mathcal{F}$ is in fact the family of palindromic and antipalindromic non isomorphic trees.

Theorem 1. *The family $\mathcal{F}$ verifies the following properties:*

(1) *The map H restricted to $\mathcal{T}_n$ is a bijection from $\mathcal{T}_n$ to $\mathcal{F}_n$.*
(2) $|\mathcal{T}_n| = |\mathcal{F}_n|$.
(3) *The characteristic polynomial of the graphs in $\mathcal{F}_n$ are palindromic if n is even and antipalindromic if n is odd.*

Proof. In order to prove (1), we notice that by definition, $\mathcal{F}_n \subset T_{2n}$ and H is injective. Therefore, we only need to prove that $H(T) \in \mathcal{F}_n$ for any $T \in \mathcal{T}_n$. We apply induction on n: $H(\mathcal{T}_1) = H(\{K_1\}) = \{P_2\} = \mathcal{F}_2$. Let $T \in T_n$ with $n > 1$, and $v \in L(T)$ one of its leaves pending from, say, vertex w. Then, by inductive hypothesis, $H(T - v) = F \in \mathcal{F}_{n-1}$ with $w \notin L(F)$. By definition of $\mathcal{F}_n$, we have that $F' = F +_w xy \in \mathcal{F}_n$ with $x, y \notin F$. But clearly F' and $H(T)$ are isomorphic, thus $H(T) \in F_n$, as we wanted to prove. Clearly (2) follows from (1).

In order to prove (3), let $F \in \mathcal{F}$, and $\chi(F;\lambda)$ its characteristic polinomial. First of all, notice that a polynomial $p(\lambda)$ of degree n is palindromic (resp. antipalindromic), iff $p(\lambda) = \lambda^n p(1/\lambda)$ (respectively $p(\lambda) = -\lambda^n p(1/\lambda)$). Thus, we want to prove that

$$\chi(F;\lambda) = (-1)^n \lambda^{2n} \chi(F;1/\lambda) \quad F \in \mathcal{F}_n. \tag{1}$$

We proceed by induction. For $n = 1$, $\mathcal{F}_1 = \{K_2\}$ and $\chi(K_2;\lambda) = \lambda^2 - 1$ which is antipalindromic. Let $F \in \mathcal{F}_n$, and $F' \in \mathcal{F}_{n-1}$ such that $F = F' +_v xy$, by a trivial result (additional results 2a in [1]):

$$\chi(F;\lambda) = \lambda\chi(F' +_v x;\lambda) - \chi(F';\lambda).$$

By the same computation

$$\chi(F' +_v x; \lambda) = \lambda\chi(F'; \lambda) - \chi(F' - v; \lambda),$$

where $F' - v$ is the disjoint union of K_1 and some trees $F_1, \dots, F_k \in \mathcal{F}$ of orders $2n_1, \dots, 2n_k$ since, by (1) F' has a leave pending from v, and the trees F_i have a leave pending from each of its vertices. Therefore

$$\chi(F' - v; \lambda) = \lambda\chi(F_1; \lambda)\chi(F_2; \lambda) \dots \chi(F_k; \lambda).$$

Summing up the equalities we obtain:

$$\chi(F; \lambda) = (\lambda^2 - 1)\chi(F'; \lambda) - \lambda^2\chi(F_1; \lambda) \dots \chi(F_k; \lambda).$$

Now, with this formulation of $\chi(F; \lambda)$ we can prove that it verifies (1). Indeed,

$$\begin{aligned}\lambda^{2n}\chi(F; 1/\lambda) &= -(\lambda^2 - 1)\lambda^{2n-2}\chi(F'; 1/\lambda) - \lambda^2\lambda^{2n_1}\chi(F_1; 1/\lambda) \dots \lambda^{2n_k}\chi(F_k; 1/\lambda) \\ &= -(\lambda^2 - 1)(-1)^{n-1}\chi(F'; \lambda) - \lambda^2(-1)^{n_1}\chi(F_1; \lambda) \dots (-1)^{n_k}\chi(F_k; \lambda)\end{aligned}$$

where the last equality follows from the strong induction hypothesis. Finally, since $2n_1 + \dots + 2n_k = 2n - 4$, then $n_1 + \dots + n_k = n - 2$ and there is an even (respectively odd) number of odd n_i's if n is even (respectively odd), so the last polynomial is

$$(-1)^n[(\lambda^2 - 1)\chi(F'; \lambda) - \lambda^2\chi(F_1; \lambda) \dots \chi(F_k; \lambda)]$$

which is $\chi(F; \lambda)$ if n is even and $-\chi(F; \lambda)$ if n is odd, as we wanted to prove. □

Now we will prove that the only palindromic and antipalindromic trees are those in $\mathcal{F}$. Let us first remember that the coefficient a_i of the characteristic polynomial of a tree is, in absolute value, the number of matching of cardinality i of the tree, i.e., the number of i-sets of non adjacent edges (see, for instance, Proposition 7.3 in [1]).

Theorem 2. *The trees in $\mathcal{F}$ are the only palindromic and an antipalindromic trees.*

Proof. By *reductio ad absurdum* let T be a (anti)palindromic tree not in $\mathcal{F}$. So, T has a perfect matching $M = \{v_1v_{-1}, \dots, v_nv_{-n}\}$, i.e. $v_iv_{-i} \in E(T)$ and $V(T) = V_+ \cup V_-$ with $V_+ = \{v_1, \dots, v_n\}$ and $V_- = \{v_{-1}, \dots, v_{-n}\}$.

We will prove that the number of matchings of cardinality $n - 1$ is greater than the number of edges of T, so $|a_2| > |a_{n-2}|$.

First, let us prove that $|a_2| \geq |a_{n-2}|$, by showing a different $(n-1)$-matching for each edge of the graph. Indeed, for each edge e in M the set $M - e$ is a $(n-1)$-matching. On the other hand, if e is an edge not in M, then it is adjacent with two edges f and g in M, so $M - f - g + e$ is a $(n-1)$-matching, as well. Clearly, the matchings obtained in these two ways are different.

Now, we will prove the strictness of the inequality, under the hypothesis that $T \notin \mathcal{F}$. Indeed, under such hypothesis we have two possibilities: either there is an edge with its ends in V_+ and V_- different from those in M, or we have edges between vertices in V_+ and edges between vertices in V_-. In the first case where, without loss of generality, we can suppose that v_1v_{-2} is such an edge of T while in the second case, we can suppose that both v_1v_2 and v_2v_3 are edges of T. In the first case, let v_i be a vertex adjacent with v_2 with $|i| > 2$, then the set $M - v_1v_{-1} + v_1v_{-2} - v_{-2}v_2 + v_2v_i - v_iv_{-i}$ is a $(n-1)$-matching different than those constructed before. In the second case, we consider the set $M - v_1v_{-1} - v_2v_{-2} - v_3v_{-3} + v_1v_2 + v_{-2}v_{-3}$ is a $(n-1)$-matching different than those constructed before as well, thus the inequality is strict.

□

3. A Generalization with a "symplectic" approach

The equation (1) is valid for general graph and we can prove it by using a similar approach to the theory of symplectic matrices.

Theorem 3. *Given a graph G of order n, then $H(G)$ is palindromic if n is even and antipalindromic if n is odd.*

Proof. As we notice before, we need to prove (1), i.e.

$$\chi(H(G);\lambda) = (-1)^n\lambda^{2n}\chi(H(G);1/\lambda);$$

We will proceed like in Appendix A of [3]. First of all notice that an adjacent matrix A of $H(G)$ could be

$$R = A(H(G)) = \begin{pmatrix} A(G) & I \\ I & 0 \end{pmatrix},$$

with a proper sort of the vertices. Clearly, matrix R is invertible, and it has determinant equal to $(-1)^n$ (thus R is not symplectic when n is odd), since we can transform matrix R into the matrix

$$\begin{pmatrix} I & A(G) \\ 0 & I \end{pmatrix},$$

by n column transpositions. Besides, matrix R verifies the "quasisymplectic" equation $RJR = -J$ where J is the *canonical matrix*

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Moreover, the inverse of R is $R^{-1} = -JRJ$, since $RJRJ = -JJ = I$. Let us now compute $\chi(R;\lambda)$:

$$\begin{aligned}\chi(R;\lambda) &= \det(\lambda I - R) = \lambda^{2n}\det(I - \lambda^{-1}R) = \lambda^{2n}\det(R)\det(R^{-1} - \lambda^{-1}I)\\ &= \lambda^{2n}(-1)^n\det(-JRJ - \lambda^{-1}(-J)J) = \lambda^{2n}(-1)^n\det(-J)\det(R - \lambda^{-1}I)\det(J)\\ &= \lambda^{2n}(-1)^n\chi(R;1/\lambda),\end{aligned}$$

as we wanted to prove. □

4. Conclusions

Our characterization of palindromic trees, allows its recognition in linear time, in contrast with the $O(n^{2.3})$ algorithm to compute the characteristic polynomial based on the Coppersmith-Winograd algorithm.

Unfortunately, despite of the trees case, applying the map H to a graph is not the only way to obtain (anti)palindromic graphs. So we leave open the problem of characterizes palindromic graph in general.

References

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