Hyperelliptic d-tangential covers and $d \times d$-matrix KdV elliptic solitons

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1 Introduction

In the mid 1970's I. Krichever completed and generalized earlier work of several research teams, by developing the Theory of scalar and vector Baker-Akhiezer (B-A) functions, a powerful method to construct algebro-geometric solutions to Zakharov-Shabat type equations.
Given a compact Riemann Surface of genus $g > 0$, equipped with $d > 0$ points and the choice of a local coordinate at each of them, say $(\Gamma, \{(p_1, \lambda_1), \cdots, (p_d, \lambda_d)\})$, plus an effective non-special divisor D of degree $(g + d - 1)$, he constructed the so-called Baker-Akhiezer meromorphic vector function $\Psi_D(x, y, t)(p) : \mathbb{C}^3 \times (\Gamma \setminus \{p_i\}) \to \mathbb{P}^1$ and two differential operators

$$L_D := \partial_x^2 + U_D(x, y, t) \quad \text{and} \quad P_D := \partial_x^3 + \frac{3}{2} U_D(x, y, t)\partial_x + W_D(x, y, t)$$

with $d \times d$ matrix-valued coefficients, satisfying the system of equations:

$$\begin{cases} (\partial_y - L_D)\Psi_D = 0 \\ (\partial_t - P_D)\Psi_D = 0 \end{cases}$$

The latter system leads to the compatibility equation $[\partial_y - L_D, \partial_t - P_D] = 0$, a system of partial differential equations satisfied by the $d \times d$ matrix functions U_D and W_D (cf. [3] p. 21-22 or [1] p. 86, 2.2 & 2.3), which reduces when $d = 1$ to the scalar KP equation for U_D (see Th.2).

Moreover, the following *KdV criterion* is in order: assume there exists a meromorphic function $f : \Gamma \to \mathbb{P}^1$ with a double pole at each p_i, then, upon choosing $\frac{1}{\sqrt{f}}$ as local coordinate at each marked point, the above B-A function Ψ_D satisfies the system ([3],3.5 p. 21-22):

$$\begin{cases} (L_D - f)\Psi_D &= 0 \\ (\partial_t - P_D)\Psi_D &= 0 \end{cases}$$

It follows that P_D is the differential part of the pseudodifferential operator $L_D^{\frac{3}{2}}$ and U_D is independent of y. In particular U_D solves the simpler matrix Korteweg-deVries equation

$$(\textit{matrix KdV}) \qquad 4U_t = 3UU_x + 3U_xU + U_{xxx}$$

Later on Krichever devised an *elliptic criterion* characterizing matrix KP solutions doubly periodic in x (cf.: [4], p.289), and constructed a family of marked curves satisfying it ([1]).

In particular, any spectral data $(\Gamma,\{(p_1,\lambda_1),\cdots,(p_d,\lambda_d)\})$ satisfying both, KdV and elliptic criteria, would provide a family of algebro-geometric $d\times d$ *matrix KdV elliptic solitons*.

Let us recall that *scalar KdV elliptic solitons* have been extensively studied and its many facets are by now well understood, e.g.: they are algebro-geometric in nature, the dynamics of their poles are governed by the Calogero-Moser integrable system and concrete polynomial equations for the corresponding spectral curves are known, not to mention their deep relationship with finite-gap elliptic Schrödinger potentials. In the matrix case instead, it seems that almost none of the latter issues, not even the existence problem of *matrix KdV elliptic solitons*, has been addressed. The only exception being, to our knowledge, the construction of spectral curves associated to 2×2 *matrix KdV elliptic solitons* ([10]).

Our purpose in this article is twofold: to motivate the search for all algebro-geometric $d\times d$ *matrix KdV elliptic solitons* and to construct the first examples for any $d\geq 3$. We proceed as follows.

In Section 2 we give a quick review of the vector-valued B-A function and its application to the construction of solutions to a matrix analog of the KP equation. We then explain when and how they lead to *matrix KdV elliptic solitons*. The corresponding spectral data are then reinterpreted as (so-called *hyperelliptic d-tangential*) covers of an elliptic curve, satisfying a geometric property. Detailed proofs of all the steps being scattered throughout the litterature, we give an (almost) self-contained presentation.

In Section 3 we consider all *hyperelliptic d-tangential covers* in a suitable algebraic surface framework, prove their general properties and classify them in terms of rational irreducible curves in an anticanonical rational surface denoted S, naturally attached to X (i.e.: the *Rational characterization*).

In Section 4 at last, we prove their existence. More precisely, given $d\geq 2$ and an elliptic curve $(X,\omega_\circ):=(\mathbb{C}/\Lambda,0)$ we define infinitely many divisor classes in S and show they contain rational reducible nodal curves to which A. Tannenbaum's deformation criterion applies ([5]). It immediately follows that the corresponding *Severi Varieties* (of rational irreducible nodal curves) are non empty. We also calculate their dimensions and prove that their generic elements give rise to *hyperelliptic d-tangential covers* of arbitrarily high degree and genus. In particular we deduce the existence of infinitely many families of $d\times d$ *matrix KdV elliptic solitons*.

2 Matrix KdV elliptic solitons and hyperelliptic tangential covers

Let $\mathbb{P}^1$ denote the projective line and $(X,\omega_\circ,z)=(\mathbb{C}/\Lambda,0,z)$ the elliptic curve associated to the lattice $\Lambda\subset\mathbb{C}$, equipped with a local coordinate z at the origin $\omega_\circ\in X$. We will consider data $(\Gamma,\{(p_1,\lambda_1),\cdots,(p_d,\lambda_d)\})$ where Γ is a complex complete integral curve of positive arithmetic genus g, with given local coordinates at d smooth points. We will call Γ hyperelliptic if there exists an involution $\tau_\Gamma:\Gamma\to\Gamma$ with quotient by τ_Γ isomorphic to $\mathbb{P}^1$. The degree-2 projection $\Gamma\to\Gamma/\tau_\Gamma\cong\mathbb{P}^1$ is then ramified at so-called Weierstrass points. For any curve Γ let Γ° and $Jac\Gamma$ denote, respectively, its open subset of smooth points and generalized Jacobian. For any $p\in\Gamma^\circ$ the Abel morphism $A_p: p'\in\Gamma^\circ\mapsto\mathcal{O}_\Gamma(p'\text{-}p)\in Jac\Gamma$ is an embedding and $A_p(\Gamma^\circ)$ generates the whole Jacobian. Let us consider the exact sequence of $\mathcal{O}_\Gamma$-modules

$$0\to\mathcal{O}_\Gamma\to\mathcal{O}_\Gamma(p)\to\mathcal{O}_p(p)\to 0\ ,$$

as well as the corresponding long exact cohomology sequence

$$0 \to \mathrm{H}^{\circ}(\Gamma, \mathcal{O}_{\Gamma}) \to \mathrm{H}^{\circ}(\Gamma, \mathcal{O}_{\Gamma}(p)) \to \mathrm{H}^{\circ}(\Gamma, \mathcal{O}_{p}(p)) \to \mathrm{H}^{1}(\Gamma, \mathcal{O}_{\Gamma}) \to .. ,$$

where $\delta : \mathrm{H}^{\circ}(\Gamma, \mathcal{O}_p(p)) \to \mathrm{H}^1(\Gamma, \mathcal{O}_\Gamma)$ is the canonical coboundary morphism. The g-dimensional space $\mathrm{H}^1(\Gamma, \mathcal{O}_\Gamma)$ is canonically identified with the tangent space to $Jac\Gamma$ at its origin and the line $\mathrm{H}^{\circ}(\Gamma, \mathcal{O}_p(p))$ is generated by the class of $\delta(\frac{1}{\lambda})$, with λ any local coordinate at $p \in \Gamma^{\circ}$. Its image $\delta\big(\mathrm{H}^{\circ}(\Gamma, \mathcal{O}_p(p))\big) \subset \mathrm{H}^1(\Gamma, \mathcal{O}_\Gamma)$ is then identified with the tangent space $T_{\Gamma,p}$ to $A_p(\Gamma^{\circ})$ at $A_p(p)$.

The above identifications will lead to a geometric reinterpretation of the elliptic and the KdV criteria mentioned in the Introduction. Let us now remind the definition of the vector-valued B-A function and both criteria, proofs included.

Definition 1 (*the vector Baker-Akhiezer function, cf. [3]*)

Given any spectral data $(\Gamma, \{(p_1, \lambda_1), \cdots, (p_d, \lambda_d)\})$ *and non-special degree* $d + g$ - 1 *effective divisor* D *on* Γ°*, such that* D *and* $\{p_i\}$ *have disjoint supports, the corresponding meromorphic vector B-A function* $\Psi_D : \mathbb{C}^3 \times (\Gamma \setminus \{p_i\}) \to \mathbb{P}^1$ *is uniquely characterized by the following properties*:

1. *for any* $(x, y, t) \in \mathbb{C}^3$ *its divisor of poles is bounded by* D;
2. *in a neighbourhood of each* p_i *it has an essential singularity of the following type*

$$\Psi_D(x, y, t)(\lambda_i) = e^{\frac{x}{\lambda_i} + \frac{y}{\lambda_i^2} + \frac{t}{\lambda_i^3}} \left(\vec{e}_i + \vec{\xi_1^i}(x, y, t)\lambda_i + \vec{\xi_2^i}(x, y, t)\lambda_i^2 + O(\lambda_i^3)\right) ,$$

where $\vec{e}_i \in \mathbb{C}^d$ *is the vector having a* 1 *at the i-th place and* 0 *everywhere else.*

Theorem 2

Having fixed $\Psi_D(x, y, t)$ *as above, let* V *and* T *denote the* $d \times d$ *matrix-valued functions with i-th column equal, respectively, to* $\vec{\xi_1^i}(x, y, t)$ *and* $\vec{\xi_2^i}(x, y, t)$*, i.e.:* $V = (\vec{\xi_1^1}, \cdots, \vec{\xi_1^d})$ *and* $T = (\vec{\xi_2^1}, \cdots, \vec{\xi_2^d})$*. Set also* $U :=$ -$2V_x$ *and* $W := 3V_x.V$ - $3V_{xx}$ - $3T_x$*. Then, the differential operators* $L := \partial_x^2 + U$ *and* $P := \partial_x^3 + \frac{3}{2}U\partial_x + W$ *satisfy the system of equations*

$$\begin{cases} (\partial_y - L)\Psi_D = 0 \\ (\partial_t - P)\Psi_D = 0 \end{cases}$$

and the compatibility equation $[\partial_y - L, \partial_t - P] = 0$*. They imply that* U *and* W *satisfy*

$$\begin{cases} W = -\frac{3}{2}(V_y + V_{xx}) \\ \\ U_t - W_y = U_{xxx} + \frac{3}{2}U.U_x + [W, U] - W_{xx} \end{cases}$$

as well as the following matrix analog of the KP equation

$$(matrix\, KP) \qquad 3U_{yy} = \big\{U_t - U_{xxx} - 6U.U_x + 4[U, W]\big\}_x .$$

Proof. The functions $(\partial_y - L)\Psi_D$ and $(\partial_t - P)\Psi_D$ are meromorphic outside $\{p_1, \cdots, p_d\}$, have pole divisors bounded by D and (it can be checked that) have essential singularites at each marked point p_i with local development $e^{\frac{x}{\lambda_i} + \frac{y}{\lambda_i^2} + \frac{t}{\lambda_i^3}} O(\lambda_i)$. The uniqueness of the B-A function implies the vanishing of $(\partial_y - L)\Psi_D$ and $(\partial_t - P)\Psi_D$ as asserted. In particular the coefficient of λ_i in the local development of $(\partial_y - L)\Psi_D$ at each marked point should also vanish, giving the relation:

$$2V_x.V = 2T_x + V_{xx} - V_y .$$

Replacing in $W = 3V_x.V - 3V_{xx} - 3T_x$ gives at last $W = -\frac{3}{2}(V_y + V_{xx})$.

On the other hand, the commutator $[\partial_y - L, \partial_t - P]$ is a first order differential in ∂_x such that $[\partial_y - L, \partial_t - P]\Psi_D = 0$. Hence $[\partial_y - L, \partial_t - P] = 0$, which is equivalent to the two equations

$$4W_x = 3(U_y + U_{xx}) \quad \text{and} \quad U_t - W_y = U_{xxx} + \frac{3}{2}U.U_x + [W,U] - W_{xx}.$$

Differentiating the latter one with respect to x and replacing W_{xy} and W_{xxx} in terms of derivatives of U, we get that the couple (U, W) satisfies (matrix KP). ■

Proposition 3 (*KdV criterion*, [*3*](3.5), *p.* 22)

Let $(\Gamma, \{(p_i, \lambda_i)\}, D, U(x,y,t))$ be as above and assume there exists a meromorphic function $f : \Gamma \to \mathbb{P}^1$ with a double pole at each p_i and such that $f = \frac{1}{\lambda_i^2} + O(\lambda_i^2)$. Then the $d \times d$ matrix function U is independent of y and solves the following matrix analog of the KdV equation:

$$(matrix\, KdV) \qquad\qquad 4U_t = U_{xxx} + 3U.U_x + 3U_x.U\,.$$

Proof. The function e^{yf} is holomorphic outside $\{p_1, \cdots, p_d\}$ and equal, over a neighbourhood of each p_i, to $e^{\frac{y}{\lambda_i{}^2}}(1 + O(\lambda_i^2))$. Hence $\Psi_D(x,y,t) = e^{yf}\Psi_D(x,0,t)$, implying the functions $\{V, U, W\}$ defined above are independent of y. Recalling that $W = -\frac{3}{2}(V_y + V_{xx})$ and $U = -2V_x$ and replacing in the latter equality we get

$$4W = 3U_x \quad \text{and} \quad U_t = U_{xxx} + \frac{3}{2}U.U_x + [W,U] - W_{xx}$$

from which we deduce immediately that U satifies (matrix KdV). ■

Proposition 4 (*Elliptic criterion*; *taken partly from* [*4*] *Assertion*, *p.* 289)

Let $(X, \omega_\circ) = (\mathbb{C}/\Lambda, 0)$ be the elliptic curve associated to the lattice of periods $\Lambda \subset \mathbb{C}$ and z the canonical coordinate at its origin. Let also $(\Gamma, \{(p_i, \lambda_i)\}, D, \Psi_D)$ be as above and (U, W) the corresponding solution of (matrix KP). Assume there exist:

1. *a d-marked projection $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to X$ such that $\{p_1, \cdots, p_d\} \subset \pi^{-1}(\omega_\circ)$;*
2. *a morphism $\kappa : \Gamma \to \mathbb{P}^1$, without poles outside $\pi^{-1}(\omega_\circ)$ and such that:*
   - *(a) $\kappa + \pi^*(\frac{1}{z})$ has simple poles at $\{p_1, \cdots, p_d\}$ and is holomorphic elsewhere;*
   - *(b) $\kappa + \pi^*(\frac{1}{z}) = \frac{1}{\lambda_i}$ over a neighbourhood of each p_i.*

Then, U and W are Λ-periodic in x. More precisely $U(x,y,t) = \sum_{j=1}^{n} A_j(y,t)\wp(x - x_j(y,t))$, where $\sum_j A_j$ is a constant diagonal matrix, and for generic $(y,t) \in \mathbb{C}^2$ we have $n = deg\,\pi$, $A_j^2 = -2A_j$, $rkA_j = 1$ and $TrA_j = -2$.

Proof. Let $\zeta(z) : \mathbb{C} \to \mathbb{P}^1$ denote the ζ-Weierstrass function. It is odd, Λ-additive and has a local development $\zeta(z) = \frac{1}{z} + O(z^3)$ at the origin. Given a $\mathbb{Z}$-basis $(2\omega_1, 2\omega_2)$ of Λ, let $(\eta_1, \eta_2) \in \mathbb{C}^2$ denote the corresponding couple of quasi-periods, i.e.: for any $\omega = m_1 2\omega_1 + m_2 2\omega_2 \in \Lambda$ and $z \notin \Lambda$, $\zeta(z + \omega) = \zeta(z) + \eta$ with $\eta = m_1\eta_1 + m_2\eta_2$. Thus, Legendre's relation $\eta_1 2\omega_2 - \eta_2 2\omega_1 = 2\pi i$ implies that the map

$$\varphi_\omega : \Gamma \setminus \{p_i\} \to \mathbb{C}, \qquad p \mapsto e^{\left(\omega(\kappa(p) + \zeta(z)) - \eta z\right)}$$

with $z - \pi(p) \in \Lambda$, is well defined, holomorphic and has the following essential singularity at p_i :

$$\varphi_\omega(\lambda_i) = e^{\frac{\omega}{\lambda_i} - \eta z} = e^{\frac{\omega}{\lambda_i} - \eta a_i \lambda_i + O(\lambda_i^2)} = e^{\frac{\omega}{\lambda_i}}(1 - \eta a_i \lambda_i + O(\lambda_i^2)),$$

where $\pi^*(z) = a_i \lambda_i + O(\lambda_i^2)$.

It follows from its uniqueness property, that Ψ_D is Λ-multiplicative, meaning

$$\Psi_D(x+\omega, y, t)(p) = \varphi_\omega(p)\Psi_D(x, y, t)(p) \quad \text{for any} \quad (\omega, x, y, t, p) \in \Lambda \times \mathbb{C}^3 \times (\Gamma \setminus \{p_i\}).$$

It also implies that $\vec{\xi}_1^i(x+\omega, y, t) = \vec{\xi}_1^i(x, y, t) - \eta a_i \vec{e}_i$, hence $V(x+\omega, y, t) = V(x, y, t) - \eta Diag(a_i)$, where $Diag(a_i)$ denotes the diagonal matrix with coefficient a_i at the i-th place $(1 \le i \le d)$. Hence $U = -2V_x$ and $W = -\frac{3}{2}(V_y + V_{xx})$ are Λ-periodic in x. Recall that X acts on the compactified Jacobian $W(\Gamma)$ (equal to $Jac^{g-1}\Gamma$ if Γ is smooth) as follows: $(\mathcal{L}, \alpha) \in W(\Gamma) \times X \mapsto \mathcal{L}(\pi^*(\alpha - \omega_\circ))$. In particular the orbit of $\mathcal{L} := \mathcal{O}_\Gamma(D - \sum_i p_i)$ intersects the theta divisor $\Theta \subset W(\Gamma)$ at the set of poles of $U(x, 0, 0)$. For a generic divisor D the latter orbit is transverse to Θ and intersects it at exactly $n = deg(\pi)$ points ([7] App. A-2). Hence, $U(x, y, t)$ also has n poles for generic $(y, t) \in \mathbb{C}^2$, say $\{x_j(y, t)\}$. Plugging its local development into (matrix KP) we immediately get that $U(x, y, t) = \sum_{j=1}^n A_j(y, t)\wp(x - x_j(y, t)) + A_\circ(y, t)$ with $A_j^2 + 2A_j = (O)$ for any $1 \le j \le n$. On the other hand, its integral over a ω-cycle is equal to

$$\int_0^\omega U dx = -2\big(V(x+\omega) - V(x)\big) = 2\eta Diag(a_i), \text{ as well as to } \int_0^\omega U dx = -\eta(\sum_j A_j) + \omega A_\circ.$$

It follows that $\sum_{j=1}^n A_j = -2Diag(a_i)$ and $A_\circ = (O)$. At last we remark that $\sum_{i=1}^d a_i = deg(\pi)$ because

$$\frac{1}{z}\sum_{i=1}^d a_i + O(1) = Tr_\pi(\kappa + \pi^*(\zeta(z)) = Tr_\pi(\kappa) + Tr_\pi\big(\pi^*(\zeta(z))\big) = \frac{deg(\pi)}{z} + O(1).$$

Hence $Tr \sum_{j=1}^n A_j = -2\sum_{i=1}^d a_i = -2deg(\pi)$, which coupled with the equalities $\{A_j^2 = -2A_j\}$ mentioned above imply that $TrA_j = -2rkA_j$ and $2n = 2deg(\pi) = -\sum_{j=1}^n TrA_j = 2\sum_{j=1}^n rkA_j$. The latter can only happen if $rkA_j = 1$ for each $1 \le j \le n$ as asserted. ■

Remark 1

1. *It should be stressed that for a spectral data $(\Gamma, \{(p_i, \lambda_i)\})$ to satisfy both criteria, hence provide $d \times d$-matrix KdV elliptic solitons, the local coordinates $\{\lambda_1, \cdots, \lambda_d\}$ must match the functions $\kappa + \pi^*(\frac{1}{z})$ and f. This will be indeed the case if $\kappa + \pi^*(\frac{1}{z}) = \frac{1}{\lambda_i} + O(\lambda_i)$ and $f = \frac{1}{\lambda_i^2} + O(\lambda_i^2)$ at each marked point p_i.*

2. *The above property is not easy to deal with. We overcome this difficulty at the cost of restraining our framework. Namely, we will only consider d-marked curves $(\Gamma, \{p_i\})$ such that Γ is hyperelliptic and p_i a Weierstrass point for each i.*

Definition 5

Let $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to X$ be a projection marked at d smooth points of the fibre $\pi^{-1}(\omega_\circ)$.

1. *We will call π a d-tangential cover if and only if it satisfies the following conditions:*

   (a) $d(\pi^*)(T_{X,\omega_\circ}) \subset \sum_{i=1}^d T_{\Gamma,p_i} \subset \mathrm{H}^1(\Gamma, \mathcal{O}_\Gamma)$;

(b) $d(\pi^*)(T_{X,\omega_\circ}) \nsubseteq \sum_{i\neq j} T_{\Gamma,p_i}$ *for each* $1 \leq j \leq d$.

2. *If in addition* Γ *is a hyperelliptic curve and any* p_i *a Weierstrass point, we will say* π *is hyperelliptic* d*-tangential. In the latter case there exists a unique involution* $\tau_\Gamma : \Gamma \to \Gamma$ *fixing the marked points* $\{p_i\}$ *and with quotient curve isomorphic to* $\mathbb{P}^1$.

Definition 6

Let the marked projections $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to X$ *and* $\underline{\pi} : (\underline{\Gamma}, \{r_1, \cdots, r_s\}) \to X$ *be hyperelliptic* d*-tangential and hyperelliptic* s*-tangential respectively. We will say that* π *dominates* $\underline{\pi}$*, if and only if there exists a non-trivial morphism* $\varphi : \Gamma \to \underline{\Gamma}$ *such that* $\pi = \underline{\pi} \circ \varphi$ *and* $\varphi^{-1}(\{r_1, \cdots, r_s\}) = \{p_1, \cdots, p_d\}$*. Conversely,* π *is called*:

1. *indecomposable if it does not dominate any hyperelliptic tangential cover of smaller degree*;
2. *minimal if it does not dominate any other hyperelliptic* d*-tangential cover of same degree.*

Theorem 7 (d*-tangency criterion cf.* : [*8*]-1.8*, see also* [*10*]-1.8)

A d*-marked cover* $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to X$ *is* d*-tangential if and only if* $\mathrm{h}^0(\Gamma, \mathcal{O}_\Gamma(\sum_i p_i)) = 1$, $\{p_i\} \subset \pi^{-1}(\omega_\circ)$ *and there exists a morphism* $\kappa : \Gamma \to \mathbb{P}^1$*, called henceforth* d*-tangential, such that*:

1. κ *is holomorphic outside* $\pi^{-1}(\omega_\circ)$;
2. *over a neighbourhood of* $\pi^{-1}(\omega_\circ)$*, the divisor of poles of* $\kappa + \pi^*(\frac{1}{z})$ *is equal to* $\sum_i p_i$.

Lemma 8

Let $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to X$ *be a hyperelliptic* d*-tangential cover. Then*:

1. *there exists a unique* d*-tangential function* $\kappa : \Gamma \to \mathbb{P}^1$ *satisfying* $\kappa \circ \tau_\Gamma = -\kappa$;
2. *there exists a projection* $f : \Gamma \to \mathbb{P}^1$ *with pole divisor* $(f)_\infty = \sum_i 2p_i$ *and a* d*-uple of local coordinates* $(\lambda_1, \cdots, \lambda_d)$ *at* $(p_1, \cdots, p_d)$*, such that over a neighbourhood of each* p_i *we have*:

$$f = \frac{1}{\lambda_i^2} \quad \textit{and} \quad \kappa + \pi^*(\tfrac{1}{z}) = \frac{1}{\lambda_i} + O(\lambda_i).$$

Proof.

1. Let $\kappa : \Gamma \to \mathbb{P}^1$ be the unique d-tangential function up to an additive constant. One can first check that $-\tau_\Gamma^*(\kappa)$ is also d-tangential and has same principal parts as κ at $\{p_i\}$. Hence $\kappa + \tau_\Gamma^*(\kappa)$ is equal to a constant, say $c \in \mathbb{C}$. It follows that $\kappa + \frac{c}{2}$ is τ_Γ-anti-invariant.
2. Let w_i denote the τ_Γ-anti-invariant local coordinate at p_i such that $\frac{1}{w_i} := \kappa + \pi^*(\frac{1}{z})$ and f_i the meromorphic function with a double pole at p_i such that $f_i = \frac{1}{w_i^2} + O(w_i^2)$. Then $f = \sum_{i=1}^d f_i$ has a double pole at each p_i, with same polar part $\frac{1}{w_i^2}$, and there exists a unique local coordinate λ_i such that $\frac{1}{\lambda_i}$ is a square-root of f (i.e.: $\frac{1}{\lambda_i^2} = f = \frac{1}{w_i^2} + O(1)$) and $\frac{1}{\lambda_i} = \frac{1}{w_i} + O(w_i)$. Hence we also have $\kappa + \pi^*(\frac{1}{z}) = \frac{1}{w_i} = \frac{1}{\lambda_i} + O(\lambda_i)$. ∎

Proposition 9

Let $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to X$ *be a* d*-tangential cover equipped with a* d*-tangential function* κ *and a local coordinate* λ_i *at each* p_i *such that* $\kappa + \pi^*(\frac{1}{z}) = \frac{1}{\lambda_i} + O(\lambda_i)$*. Then, the corresponding* $d \times d$ *matrix KP solutions are* Λ*-periodic in* x*.*

Proof. The *d-tangency criterion* implies the *elliptic* one, hence the result. ■

Theorem 10

Given a hyperelliptic d-tangential cover $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to X$ *of genus g, equipped with its unique data* $(\kappa, (f_i), f, (\lambda_i))$ *as in the latter Lemma, we let* $A_\circ$ *and* $B_\circ$ *denote the constant diagonal* $d \times d$ *matrices with i-th coefficient equal to* a_i*, such that* $\pi^*(z) = a_i\lambda_i + O(\lambda_i^2)$*, and* $b_i := \sum_{j\neq i} f_j(p_i)$*. Then, to any non-special effective divisor D of* Γ *Krichever's construction associates a* $d \times d$ *matrix KdV elliptic soliton of the form:*

$$U(x,t) = \sum_{i=1}^{n} A_i(t)\wp(x - x_i(t)) + B_\circ,$$

where $n = deg\,\pi$*,* $A_i^2 = -2A_i$*,* $rkA_i = 1$*,* $TrA_i = -2$ *and* $\sum_i A_i = -2A_\circ$*.*

Proof. We let again w_i denote the local coordinate at p_i such that $\frac{1}{w_i} = \kappa + \pi^*(\zeta(z)$, $f_i : \Gamma \to \mathbb{P}^1$ the unique meromorphic function with local development $f_i(w_i) = \frac{1}{w_i^2} + O(w_i^2)$ and $\frac{1}{\lambda_i}$ the square root of $f := \sum_{j=1}^{d} f_j$. In particular we have $\frac{1}{\lambda_i^2} = f = \frac{1}{w_i^2} + b_i + O(w_i^2)$, $\frac{1}{w_i} = \frac{1}{\lambda_i} - \frac{1}{2}b_i\lambda_i + O(\lambda_i^2)$ and $z = a_i\lambda_i + O(\lambda_i^2)$ with $\sum_i a_i = n := deg\,\pi$ as already proven.

The latter equalities imply that $\varphi_\omega(p)$ has the following essential singularity at each point p_i :

$$\varphi_\omega(\lambda_i) = e^{\frac{\omega}{\lambda_i} - (\frac{\omega}{2}b_i + \eta a_i)\lambda_i + O(\lambda_i^2)} = e^{\frac{\omega}{\lambda_i}}(1 - (\frac{\omega}{2}b_i + \eta a_i)\lambda_i + O(\lambda_i^2)).$$

It follows from its uniqueness property, that Ψ_D is Λ-multiplicative, meaning

$$\Psi_D(x+\omega, y, t)(p) = \varphi_\omega(p)\Psi_D(x,y,t)(p) \quad \text{for any} \quad (\omega, x, y, t, p) \in \Lambda \times \mathbb{C}^3 \times (\Gamma \setminus \{p_i\}).$$

It also implies that $\vec{\xi}_1^{\,i}(x+\omega, y, t) = \vec{\xi}_1^{\,i}(x,y,t) - (\eta a_i + \frac{\omega}{2}b_i)\vec{e}_i$, hence $V(x+\omega, y, t) = V(x,y,t) - \eta A_\circ - \frac{\omega}{2}B_\circ$.

For any $\omega = m_1 2\omega_1 + m_2 2\omega_2 \in \Lambda$ the integral over a ω-cycle of the corresponding KP elliptic soliton $U(x,y,t) = -2V_x$ is equal to

$$\int_0^\omega U dx = -2\big(V(x+\omega) - V(x)\big) = 2\eta A_\circ + \omega B_\circ, \text{ where } \eta = m_1\eta_1 + m_2\eta_2.$$

On the other hand, it follows from the KdV criterion and the latter Lemma that U is a matrix KdV elliptic soliton $U(x,t) = \sum_{i=1}^{n} A_i(t)\wp(x - x_i(t)) + B(t)$ (for some matrix $B(t)$ and coefficients $\{A_i\}$ satisfying $A_i^2 = -2A_i$ and $TrA_i = -2rkA_i$ as in the d-tangential case).

Hence $\int_0^\omega U dx = -\eta(\sum_i A_i) + \omega B$. Comparing both values gives $\sum_i A_i = -2A_\circ$ and $B = B_\circ$ as asserted, and $-2n = Tr(\sum_i A_i) = -2\sum_i rkA_i$. The latter equalities imply at last that $rkA_i = 1$ for each i. ■

3 Hyperelliptic tangential covers through algebraic surfaces

Let z denote the canonical coordinate of $X = \mathbb{C}/\Lambda$ at its origin $\omega_\circ$, and let $\overline{U}$ denote an open neighbourhood of $\omega_\circ$ and $U := X \setminus \{\omega_\circ\}$. We start constructing a ruled surface $\pi_S : S \to X$ through which any d-tangential cover factors. We also define a suitable blowing up $e : S^\perp \to S$, which we prove is a degree-2 cover of a rational surface $\varphi : S^\perp \to \widetilde{S}$. The latter give a natural framework for studying all (and constructing some) hyperelliptic d-tangential covers. We recall hereafter their definition and main properties (cf.: [6]; see also [10]).

Proposition 11

Let $\pi_S : S \to X$ denote the ruled surface obtained by glueing the fibers of $\mathbb{P}^1 \times U$ and $\mathbb{P}^1 \times \overline{U}$ over $U \cap \overline{U}$ by means of a translation as follows:

for any $z \in U \cap \overline{U}$ we identify $(T, z) \in \mathbb{P}^1 \times U$ with $(\overline{T}\text{-}\frac{1}{z}, z) \in \mathbb{P}^1 \times \overline{U}$.

We immediately deduce that:

1. *the infinity sections $\omega \in U \mapsto (\infty, \omega) \in \mathbb{P}^1 \times U$ and $\overline{\omega} \in \overline{U} \mapsto (\infty, \overline{\omega}) \in \mathbb{P}^1 \times \overline{U}$ get glued together defining a particular one, denoted by $C_\circ \subset S$;*
2. *the canonical symmetry $[\text{-}1] : X \to X$ fixes its origin $\omega_\circ$ as well as the three other half-periods $\{\omega_1, \omega_2, \omega_3\}$, and lifts to an involution $\tau : S \to S$ fixing $C_\circ$, with two fixed points over each ω_i: one in $C_\circ$ denoted by s_i and the other one denoted by r_i.*

Consider the blowing up $e : S^\perp \to S$ of $\{s_i, r_i\}$, the eight fixed points of τ, its lift $\tau^\perp : S^\perp \to S^\perp$ to an involution fixing the corresponding exceptional divisors $\{s_i^\perp, r_i^\perp\}$, and the strict transform $C_\circ^\perp \subset S^\perp$ of $C_\circ$. Then:

1. *the quotient surface $\widetilde{S} := S^\perp/\tau^\perp$ is birational to $\mathbb{P}^1 \times (X/[\text{-}1]) = \mathbb{P}^1 \times \mathbb{P}^1$ and the canonical degree-2 projection $\varphi : S^\perp \to \widetilde{S}$ is ramified along $\{s_i^\perp, r_i^\perp\}$. Hence $\widetilde{S}$ is a smooth rational surface with canonical divisor $\widetilde{K}$ satisfying $\varphi^*(\widetilde{K}) = e^*(\text{-}2C_\circ)$ ([6]);*
2. *the eight curves $\{\widetilde{s}_i := \varphi(s_i^\perp), \widetilde{r}_i := \varphi(r_i^\perp)\}$ and $\widetilde{C}_\circ := \varphi(C_\circ^\perp)$ in $\widetilde{S}$ are isomorphic to $\mathbb{P}^1$, have self-intersection -2 and $\text{-}\widetilde{K} = 2\widetilde{C}_\circ + \sum_i \widetilde{s}_i$. In particular $\text{-}\widetilde{K}$ is nef, hence $\widetilde{S}$ is an anticanonical rational surface (cf. : [2]).*

Proposition 12 *Let $K_{\mathcal{S}}$ denote the canonical divisor of S, $S_\circ := \pi_S^{\text{-}1}(\omega_\circ)$ and $\kappa_{\mathcal{S}} : \mathcal{S} \to \mathbb{P}^1$ the pullback of the first projection $T : \mathbb{P}^1 \times X \to \mathbb{P}^1$. Then:*

1. *$K_{\mathcal{S}}$ is linearly equivalent to $\text{-}2C_\circ$;*
2. *$C_\circ$ has zero self-intersection;*
3. *the divisor of poles of $\kappa_{\mathcal{S}}$ is equal to $C_\circ + S_\circ$;*
4. *the restriction of $\kappa_{\mathcal{S}} + \pi_{\mathcal{S}}^*(\frac{1}{z})$ to $\mathbb{P}^1 \times \overline{U}$ has a simple pole along $C_\circ$.*

Proof.

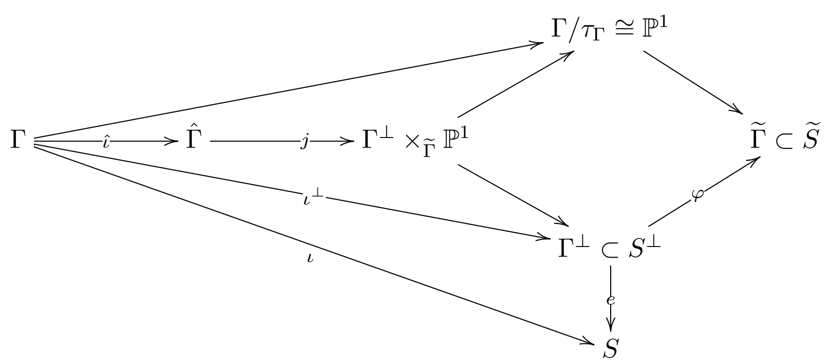
1. The wedge products $dT \wedge dz$ (on $\mathbb{P}^1 \times U$) and $d\overline{T} \wedge dz$ (on $\mathbb{P}^1 \times \overline{U}$) get glued over $U \cap \overline{U}$, defining a meromorphic differential with divisor class $K_{\mathcal{S}} = \text{-}2C_\circ$ as announced.
2. $C_\circ$ has genus equal to $1 = 1 + \frac{1}{2}C_\circ.(C_\circ + K_{\mathcal{S}}) = 1 + \frac{1}{2}C_\circ.(C_\circ\text{-}2C_\circ)$. Hence $C_\circ.C_\circ = 0$.
3. $\kappa_{\mathcal{S}}$ is represented over the open subsets $\mathbb{P}^1 \times U$ and $\mathbb{P}^1 \times \overline{U}$ by T and $\overline{T}\text{-}\frac{1}{z}$, respectively. Hence, its divisor of poles is equal to $C_\circ + S_\circ$.
4. It also follow that $\kappa_{\mathcal{S}} + \pi_{\mathcal{S}}^*(\frac{1}{z})$ is represented by $\overline{T}$ over $\mathbb{P}^1 \times \overline{U}$ and has a simple pole along $C_\circ$ as asserted. ∎

Proposition 13 *Let* $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to (X, \omega_\circ)$ *be a hyperelliptic* d*-tangential cover of degree* n *and* $\tau_\Gamma : \Gamma \to \Gamma$ *the corresponding hyperelliptic involution. Then* π *factors through a unique morphism* $\iota^\perp : \Gamma \to S^\perp$ *satisfying the following properties, where* m *denotes the degree of* $\iota^\perp$ *over its image* $\Gamma^\perp := \iota^\perp(\Gamma) \subset S^\perp$ *and* $(\gamma_i) := (\Gamma^\perp.r_i^\perp)$ *:*

1. $d^\perp := \frac{d}{m}$ *and* $n^\perp := \frac{n}{m}$ *are positive integers,* $(\iota^\perp)^*(s_\circ^\perp) = \{p_1, \cdots, p_d\}$ *and* $\iota^\perp \circ \tau_\Gamma = \tau^\perp \circ \iota^\perp$*;*
2. $\Gamma^\perp := \iota^\perp(\Gamma) \subset S^\perp$ *is a* $\tau^\perp$*-invariant curve and* $\widetilde{\Gamma} := \varphi(\Gamma^\perp) \subset \widetilde{S}$ *a rational irreducible curve*;
3. $\Gamma^\perp$ *belongs to the linear system* $|e^*(n^\perp C_\circ + d^\perp S_\circ)\text{-}\, d^\perp s_\circ^\perp\text{-}\sum_i \gamma_i r_i^\perp|$.

Moreover, π *dominates a unique minimal hyperelliptic* $d^\perp$*-tangential cover.*

Proof. The three items have already been worked out for $d = 1$ in [6] 4.3, 4.4 & 4.5, and can be proven along the same lines for any $d \geq 2$. We will only explain the last issue. The above morphism $\iota^\perp : \Gamma \to S^\perp$ factors through $\Gamma^\perp \times_{\widetilde{\Gamma}} \mathbb{P}^1$, the fiber product of the degree-2 projection $\varphi : \Gamma^\perp \to \widetilde{\Gamma}$ and the normalization map $\mathbb{P}^1 \to \widetilde{\Gamma}$. We let $j : \hat{\Gamma} \to \Gamma^\perp \times_{\widetilde{\Gamma}} \mathbb{P}^1$ denote the minimal birational morphism smooth over the image of $\{p_1, \ldots, p_d\}$, and lift $\iota^\perp$ to $\hat{\iota} : \Gamma \to \hat{\Gamma}$. The composed morphism $\iota := e \circ \iota^\perp : \Gamma \to S$ satisfies $\iota^*(C_\circ) = \sum_i p_i$ (cf.: [8] Prop. 2.5 p.533) and factors via $\hat{\iota}$.



Hence $\hat{\iota}$ is étale over $\hat{\iota}(\{p_1, \cdots, p_d\})$, which must consist of $d^\perp$ smooth points $\{r_1, \cdots, r_{d^\perp}\}$ at which the degree-2 cover $\hat{\Gamma} \to \Gamma^\perp \times_{\widetilde{\Gamma}} \mathbb{P}^1 \to \mathbb{P}^1$ is ramified. In other words, $(\hat{\Gamma}, \{r_1, \cdots, r_{d^\perp}\})$ is a hyperelliptic curve marked at $d^\perp$ smooth Weierstrass points and $\hat{\iota}^*\big(\sum_{j=1}^{d^\perp} r_j\big) = \sum_{i=1}^{d} p_i$. Hence $\mathrm{h}^0\big(\hat{\Gamma}, \mathcal{O}_{\hat{\Gamma}}(\sum_j r_j)\big) \leq \mathrm{h}^0\big(\Gamma, \mathcal{O}_\Gamma(\sum_i p_i)\big)$ and π factors as $\hat{\pi} \circ \hat{\iota}$, where $\hat{\pi} : \hat{\Gamma} \to X$ is the natural projection. On the other hand, π being a hyperelliptic d-tangential cover, $\mathrm{h}^0\big(\Gamma, \mathcal{O}_\Gamma(\sum_i p_i)\big) = 1$, hence $\mathrm{h}^0\big(\hat{\Gamma}, \mathcal{O}_{\hat{\Gamma}}(\sum_j r_j)\big) = 1$. Consider at last the rational projection $\kappa_S : S \to \mathbb{P}^1$ defined in the preceding Proposition. We easily check that its pull-back $\hat{\kappa} : \hat{\Gamma} \to \mathbb{P}^1$ to $\hat{\Gamma}$ is a well defined morphism and satisfies the *tangency criterion.* Hence $\hat{\pi} : (\hat{\Gamma}, \{r_1, \cdots, r_{d^\perp}\}) \to X$ is a $d^\perp$-hyperelliptic tangential cover dominated by π. ■

Definition 14

The intersection multiplicity vector $\gamma \in \mathbb{N}^4$, $\gamma_i := \Gamma^\perp.r^\perp$ $(i = 0, \cdots, 3)$*, canonically associated to the hyperelliptic* d*-tangential cover* $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to (X, \omega_\circ)$ *will be called the type of* π*. Given any* $\alpha, \beta \in \mathbb{N}^4$ *we will denote hereafter*

$$\alpha^{(1)} := \sum_i \alpha_i, \quad \alpha^{(2)} := \sum_i \alpha_i^2, \quad \alpha.\beta := \sum_i \alpha_i\beta_i \quad \textit{and} \quad \alpha.r^\perp := \sum_i \alpha_i r_i^\perp.$$

Corollary 15

Let $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to (X, \omega_\circ)$ *be a hyperelliptic d-tangential cover of degree n, arithmetic genus g and type* γ *factoring through* $\iota^\perp : \Gamma \to \Gamma^\perp \subset S^\perp$ *as above. Then,* π *is indecomposable if and only if deg*$(\iota^\perp : \Gamma \to \Gamma^\perp) = 1$. *In the latter case* π *dominates a unique minimal hyperelliptic d-tangential cover and* (n, d, γ) *satisfies* $\gamma_\circ + d \equiv \gamma_j \equiv n (\bmod 2)$ *for each* $j \neq 0$, *as well as*:

$$2g + 2 - d \leq \gamma^{(1)}, \qquad (\gamma^{(1)})^2 \leq 4\gamma^{(2)} - \tfrac{3}{2}(1 - (-1)^d) \quad and \quad \gamma^{(2)} \leq 2nd - d^2 - 4d + 7 + (-1)^d.$$

It follows at last that

$$(2g + 2 - d)^2 \leq 8nd - 4d^2 - 8d + 16 - \tfrac{3}{2}(1 - (-1)^d),$$

and an obvious upper bound for g in terms solely of d and n.

Proof. For any $0 \leq i \leq 3$ the strict transform $S_i^\perp \subset S^\perp$ of $S_i := \pi_S^{-1}(\omega_i) \subset S$ is $\tau^\perp$-invariant, projects with degree two onto $\widetilde{S}_i := \varphi(S_i^\perp) \subset \widetilde{S}$ and $e^*(S_i) = S_i^\perp + s_i^\perp + r_i^\perp$ is numerically equivalent to $e^*(S_\circ) = S_\circ^\perp + s_\circ^\perp + r_\circ^\perp$. Hence $\varphi_*(S_i^\perp) = 2\widetilde{S}_i$ and $2\widetilde{S}_i + \widetilde{s}_i + \widetilde{r}_i$ is numerically equivalent to $2\widetilde{S}_\circ + \widetilde{s}_\circ + \widetilde{r}_\circ$. Taking into account that $\widetilde{\Gamma}$ does not intersect $\bigcup_{j=1}^{3} \widetilde{s}_j$ we get for any $j = 1, 2, 3$:

$$\begin{cases} n = \Gamma^\perp.e^*(S_\circ) = \widetilde{\Gamma}.(2\widetilde{S}_\circ + \widetilde{s}_\circ + \widetilde{r}_\circ) = 2\widetilde{\Gamma}.\widetilde{S}_\circ + d + \gamma_\circ \\ \quad n = \Gamma^\perp.e^*(S_j) = \widetilde{\Gamma}.(2\widetilde{S}_j + \widetilde{s}_j + \widetilde{r}_j) = 2\widetilde{\Gamma}.\widetilde{S}_j + \gamma_j \end{cases}$$

Hence $n \equiv d + \gamma_\circ \equiv \gamma_j (\bmod 2)$ as asserted.

Recall also that $\pi : \Gamma \to X$ factors through $\hat{\Gamma} \to \Gamma^\perp \times_{\widetilde{\Gamma}} \mathbb{P}^1$. On the other hand, applying the adjoint formula we immediately calculate the arithmetic geni of $\Gamma^\perp$ and $\widetilde{\Gamma}$, namely:

$$p_a(\Gamma^\perp) = \frac{1}{2}(2nd - d^2 - d - \gamma^{(2)} + \gamma^{(1)} + 2) \quad and \quad p_a(\widetilde{\Gamma}) = \frac{1}{4}(2nd - d^2 - 2d - \gamma^{(2)} + 4).$$

Hence

$$g \leq p_a(\Gamma^\perp \times_{\widetilde{\Gamma}} \mathbb{P}^1) = p_a(\Gamma^\perp) - 2p_a(\widetilde{\Gamma}) = \frac{1}{2}(d + \gamma^{(1)} - 2) \quad and \quad 0 \leq 4p_a(\widetilde{\Gamma}) = 2nd - d^2 - 2d - \gamma^{(2)} + 4,$$

which settles the first and third inequalities. As for the middle one, it comes as a direct application of the equality

$$(\gamma^{(1)})^2 = \sum_i \gamma_i^2 + \sum_{i<j} 2\gamma_i\gamma_j = 4\sum_i \gamma_i^2 - \sum_{i<j} (\gamma_i - \gamma_j)^2,$$

after taking account of the congruences $\gamma_\circ + d \equiv \gamma_j (\bmod 2)$ for each $j \neq 0$. ■

Remark 2 *The previous results tell us that any indecomposable hyperelliptic d-tangential cover can be uniquely realized as a degree-2 cover of a rational irreducible curve in* $\widetilde{S}$. *On the other hand, we can start from suitable rational irreducible curves in* $\widetilde{S}$ *and go backwards as explained hereafter. We first state and prove the following basic Lemma for which we did not find a proper reference in the litterature.*

Lemma 16

Let $(\Gamma, \{p_1, \cdots, p_d\})$ be a hyperelliptic curve of positive arithmetic genus g marked at d smooth Weierstrass points, for some $0 < d \leq g$. Then $\mathrm{h}^0(\Gamma, \mathcal{O}_\Gamma(\sum_i p_i)) = 1$. In other words the tangent subspace $\sum_{i=1}^d T_{\Gamma,p_i} \subset \mathrm{H}^1(\Gamma, \mathcal{O}_\Gamma)$ has dimension d.

Proof. Assume, on the contrary, that there exists a morphism $f : \Gamma \to \mathbb{P}^1$ satisfying $(f)_\infty = \sum_{i=1}^d p_i$ and let the affine curve $\{y^2 = \Pi_{j=1}^{2g+1}(x - \alpha_j)\} \subset \mathbb{C}^2$ represent $\Gamma \setminus \{p_d\}$. We identify its *infinity point* with p_d and $\big((\alpha_1, 0), \cdots, (\alpha_{d-1}, 0)\big)$ with $(p_1, \cdots, p_{d-1})$. In particular α_i must be a simple root of $P(x) := \Pi_{j=1}^{2g+1}(x - \alpha_j)$ for each $1 \leq i < d$. We can also, and will, assume f is τ_Γ-anti-invariant. It follows that $f = y\frac{A(x)}{B(x)}$ with A and B two coprime polynomials such that:

1. $deg(B) = g + deg(A) \geq g$, because f has a simple pole at the *infinity point* p_d;
2. α_i is a simple root of $B(x)$ for each $1 \leq i < d$, because f has a simple pole at p_i.

Having assumed $d - 1 < g \leq deg(B)$ forces the existence of another root $\beta \notin \{\alpha_1, \cdots, \alpha_{d-1}\}$ of multiplicity $m \geq 1$ of $B(x)$. Notice that $f^2 = P(x)\frac{A(x)^2}{B(x)^2}$ has a double pole at each p_i but no pole at $(\beta, 0)$, implying $P(x)$ vanishes at β with multiplicity $l \geq 2m \geq 2$. In particular $(\beta, 0)$ is a singular point of Γ.

Let $\nu : \Gamma_\nu \to \Gamma$ denote the normalization map and $f_\nu : \Gamma_\nu \to \mathbb{P}^1$ the pull-back of $f : \Gamma \to \mathbb{P}^1$.

We have three possibilities for its values at $f_\nu^{-1}\big(\{(\beta, 0)\}\big) \subset \Gamma_\nu$:

1. $l = 2m$ and f_ν has non-zero opposite values at the two points in $f_\nu^{-1}\big(\{(\beta, 0)\}\big)$;
2. $l = 2m + 2k$ is even and f_ν has a zero of order $k > 0$ at the two points in $f_\nu^{-1}\big(\{(\beta, 0)\}\big)$;
3. $l = 2m + 2k + 1$ is odd and f_ν has a zero of order $l - 2m$ at the unique point in $f_\nu^{-1}\big(\{(\beta, 0)\}\big)$.

The three of them contradict f_ν being the pull-back of a well defined morphism $f : \Gamma \to \mathbb{P}^1$:

1. $l = 2m$: f_ν should have the same value at both points;
2. $l = 2m + 2k$: f_ν should have a zero of order $\geq m + k > k$ at both points;
3. $l = 2m+2k+1$: f_ν should vanish to order $\geq l > l - 2m$ at the unique point in $f_\nu^{-1}\big(\{(\beta, 0)\}\big)$. ∎

Proposition 17 (*Rational characterization of hyperelliptic tangential covers*)

Let $\widetilde{\Gamma} \subset \widetilde{S}$ be a rational irreducible curve satisfying the following properties:

1. *$\widetilde{\Gamma}$ does not intersect $\widetilde{C}_\circ \cup \bigcup_{j=1}^3 \widetilde{s}_j$ but $d := \widetilde{\Gamma}.\widetilde{s}_\circ > 0$;*
2. *$\#(\widetilde{\Gamma} \cap \widetilde{s}_\circ) = \widetilde{\Gamma}.\widetilde{s}_\circ$, i.e.: $\widetilde{\Gamma}$ is smooth and transverse to $\widetilde{s}_\circ$ at each point of their intersection*;
3. *$d + 2 \leq \sum_i \widetilde{\Gamma}.\widetilde{r}_i$.*

Let $\Gamma^\perp$ denote its inverse image in $S^\perp$ and $\hat{\Gamma}$ the fiber product of the projection $\Gamma^\perp \to \widetilde{\Gamma}$ and normalization morphism $\mathbb{P}^1 \to \widetilde{\Gamma}$, marked at the d smooth pre-images $\{\hat{p}_1, \cdots, \hat{p}_d\}$ of $\widetilde{\Gamma} \cap \widetilde{s}_\circ$. Then $\hat{\pi} : (\hat{\Gamma}, \{\hat{p}_1, \cdots, \hat{p}_d\}) \to \Gamma^\perp \to X$ is a minimal, indecomposable, hyperelliptic d-tangential cover of degree $n := \widetilde{\Gamma}.(2\widetilde{S}_\circ + \widetilde{s}_\circ + \widetilde{r}_\circ)$, type $\gamma := (\widetilde{\Gamma}.\widetilde{r}_i)$ and arithmetic genus $g := \frac{1}{2}(d - 2 + \gamma^{(1)})$.

Proof. The rational curves $\widetilde{\Gamma} = \widetilde{S}_\circ$ and $\widetilde{\Gamma} = \widetilde{r}_\circ$ do not satisfy properties 3 and 1, respectively. Hence $\widetilde{\Gamma} \neq \widetilde{S}_\circ, \widetilde{r}_\circ$ and $n = \widetilde{\Gamma}.(2\widetilde{S}_\circ + \widetilde{s}_\circ + \widetilde{r}_\circ) \geq d > 0$. We then calculate the arithmetic geni $p_a(\hat{\Gamma})$ and $p_a(\Gamma^\perp)$ via the adjunction formula, obtaining $p_a(\hat{\Gamma}) = p_a(\Gamma^\perp) - 2p_a(\widetilde{\Gamma}) = \frac{1}{2}(d - 2 + \gamma^{(1)})$. Notice that the degree-2 projection $\hat{\Gamma} \to \mathbb{P}^1$ is ramified and smooth at the d pre-images of $\widetilde{\Gamma} \cap \widetilde{s}_\circ$ because $\varphi : S^\perp \to \widetilde{S}$ is ramified along $\widetilde{s}_\circ$ and $\widetilde{\Gamma}$ is transverse to $\widetilde{s}_\circ$. Moreover, property 3 is equivalent to $g := p_a(\hat{\Gamma}) \geq d$ and most importantly, $\hat{\Gamma}$ is irreducible: otherwise it would break as a sum of two copies of $\mathbb{P}^1$ projecting onto the elliptic curve X. Contradiction! In particular the hyperelliptic curve $\hat{\Gamma}$, marked at the d pre-images $\{\hat{p}_1, \cdots, \hat{p}_d\} \subset \hat{\Gamma}$ of $\widetilde{\Gamma} \cap \widetilde{s}_\circ$, satisfies the preceding Lemma, i.e.: each $\hat{p}_i$ is a Weierstrass point and $\mathrm{h}^0\big(\hat{\Gamma}, \mathcal{O}_{\hat{\Gamma}}(\sum_i \hat{p}_i)\big) = 1$.

On the other hand, recall the rational map $\kappa_S : \mathbb{P}^1 \times \overline{U} \subset S \to \mathbb{P}^1$, $(\overline{T}, z) \mapsto \overline{T}$, and let $\hat{\kappa} : \hat{\Gamma} \to \mathbb{P}^1$ and $\hat{\pi} : \hat{\Gamma} \to X$ denote the pull-backs of $\kappa_S \circ e : \Gamma^\perp \to \mathbb{P}^1$ and $\pi_S \circ e : \Gamma^\perp \to X$, respectively. The latter are well defined morphisms and we have proved at the beginning of this section that $\kappa_S + \pi_S^*(\frac{1}{z})$ has a simple pole along $C_\circ$. Thus $\hat{\kappa}$ satisfies conditions 1. & 2. of the d-tangency criterion. In other words, $\hat{\pi} : (\hat{\Gamma}, \{\hat{p}_1, \cdots, \hat{p}_d\}) \to X$ is a minimal, indecomposable, hyperelliptic d-tangential cover as asserted. ■

4 From Severi Varieties to hyperelliptic tangential covers

Given $(d, \gamma, n) \in \mathbb{N}^* \times \mathbb{N}^4 \times \mathbb{N}^*$ satisfying $\gamma_\circ + d \equiv \gamma_j \equiv n (\mathrm{mod}\, 2)$ for each $j = 1, 2, 3$, there is a unique linear equivalence class $\widetilde{\Lambda}$ on $\widetilde{S}$ such that $\varphi^*(\widetilde{\Lambda}) = e^*(nC_\circ + dS_\circ) - ds_\circ^\perp - \gamma.r^\perp$.
Let $V_\circ|\widetilde{\Lambda}| \subset |\widetilde{\Lambda}|$ denote the locally closed (so-called Severi) subvariety of rational irreducible nodal curves. The above *Rational characterization* reduces the construction of examples of indecomposable hyperelliptic d-tangential covers to that of corresponding Severi varieties. More precisely, the former are parameterized by the open subset of curves in $V_\circ|\widetilde{\Lambda}|$ transverse to $\widetilde{s}_\circ$.

This program has indeed been achieved for $d = 1, 2$ as recalled hereafter.

Theorem 18 (*the case $d = 1$; cf. : [6] 4.5, 4.9*)

For any $(\mu, n) \in \mathbb{N}^4 \times \mathbb{N}^*$ such that $2n+1 = \mu^{(2)}$ and $n \equiv 1 + \mu_\circ \equiv \mu_j (\mathrm{mod}\, 2)$ there is a unique exceptional curve $\widetilde{\Gamma}_\mu \subset \widetilde{S}$ of the first kind, such that $\Gamma_\mu^\perp := \varphi^*(\widetilde{\Gamma}_\mu) \in |e^*(nC_\circ + S_\circ) - s_\circ^\perp - \mu.r^\perp|$. Moreover:

1. *the natural projection $\pi_S \circ e : \Gamma_\mu^\perp \to X$, marked at the unique point in $\Gamma_\mu^\perp \cap s_\circ^\perp$, is the unique hyperelliptic 1-tangential cover of degree n and type μ;*

2. *any other hyperelliptic 1-tangential cover of type μ has degree $n + 2l$ for some $l > 0$, and there are a finite number of them.*

Proposition 19
Given any $(\mu, n) \in \mathbb{N}^4 \times \mathbb{N}^*$ such that $\mu_\circ + 1 \equiv \mu_j (\mathrm{mod}\, 2)$ and $n = \mu^{(2)} + 3 + \mu.(2, 0, 2, 0)$ there exists a unique linear equivalence class $\widetilde{\Lambda}_{2,\mu}$ on $\widetilde{S}$ satisfying

$$\varphi^*(\widetilde{\Lambda}_{2,\mu}) = e^*(nC_\circ + 2S_\circ) - 2s_\circ^\perp - (2\mu + (2, 0, 2, 0)).r^\perp \qquad .$$

Moreover:

1. *$\widetilde{\Lambda}_{2,\mu}$ is nef and has zero arithmetic genus and zero self-intersection*;

2. *$|\widetilde{\Lambda}_{2,\mu}|$ is a pencil with generic element transverse to $\widetilde{s}_\circ$ and isomorphic to $\mathbb{P}^1$.*

Proof. The reduced divisors $\widetilde{\Gamma}_\mu + \widetilde{\Gamma}_{\mu+(2,0,2,0)}$ and $\widetilde{\Gamma}_{\mu+(2,0,0,0)} + \widetilde{\Gamma}_{\mu+(0,0,2,0)}$ have zero self-intersection, are transverse to $\widetilde{s}_\circ$ and belong to $|\widetilde{\Lambda}_{2,\mu}|$. Hence they generate a pencil with irreducible generic element of arithmetic genus 0 and transverse to $\widetilde{s}_\circ$. Moreover, the latter pencil has a unique element going through any point of $\widetilde{S}$, implying it has to be equal to $|\widetilde{\Lambda}_{2,\mu}|$. ■

Corollary 20 (*the case $d = 2$; cf. : [10]*)

There exists a 1-dimensional family of smooth hyperelliptic 2-tangential covers of arithmetic genus $g := \mu^{(1)} + 2$, degree $n := \mu^{(2)} + 3 + \mu.(2,0,2,0)$ and type $2\mu + (2,0,2,0)$.

Proof. According to the above proposition, the generic element $\widetilde{\Gamma} \in |\widetilde{\Lambda}_{2,\mu}|$ satisfies the *Rational characterization of hyperelliptic tangential covers.* ■

In order to generalize the latter results to any $d \geq 2$ we make the following definitions.

Definition 21
Given $d \geq 2$ and $\mu \in \mathbb{N}^4$ such that $\mu_\circ + 1 \equiv \mu_j (\mathrm{mod}\, 2)$, we let $\widetilde{\Lambda}_{d,\mu}$ denote the unique linear equivalence class on $\widetilde{S}$ such that

$$\varphi^*(\widetilde{\Lambda}_{d,\mu}) = e^*(nC_\circ + dS_\circ) - ds_\circ^\perp - (d\mu + \alpha).r^\perp$$

with $n = \frac{d}{2}(\mu^{(2)} + 3) + \mu.\alpha$ and, either $\alpha := (d, d-2, 1, 1)$ if d is odd or $\alpha := (d, d-2, 2, 0)$ if d is even.
At last, we let $\widetilde{L}_{2,\mu}$ and $\widetilde{M}_{2,\mu}$ denote the unique linear equivalence classes on $\widetilde{S}$ such that

$$\varphi^*(\widetilde{L}_{2,\mu}) = e^*(lC_\circ + 2S_\circ) - 2s_\circ^\perp - \big(2\mu + (2,2,0,0)\big).r^\perp, \quad \textit{with} \quad l = \mu^{(2)} + 3 + \mu.(2,2,0,0)$$

and

$$\varphi^*(\widetilde{M}_{2,\mu}) = e^*(mC_\circ + 2S_\circ) - 2s_\circ^\perp - \big(2\mu + (2,0,0,2)\big).r^\perp, \quad \textit{with} \quad m = \mu^{(2)} + 3 + \mu.(2,0,0,2).$$

Lemma 22
For any $d \geq 2$ and $\mu \in \mathbb{N}^4$ as above $\widetilde{\Lambda}_{d,\mu}$ is nef and $\widetilde{\Lambda}_{d,\mu}.\widetilde{K} = -d$. Moreover:

1. *$\widetilde{\Lambda}_{d,\mu}$ has arithmetic genus $p_a(\widetilde{\Lambda}_{d,\mu}) = [\frac{d-1}{2}]$ and self-intersection $\widetilde{\Lambda}_{d,\mu}.\widetilde{\Lambda}_{d,\mu} = d - 2 + 2[\frac{d-1}{2}]$;*
2. *the linear system $|\widetilde{\Lambda}_{d,\mu}|$ is basepoint free and $dim(|\widetilde{\Lambda}_{d,\mu}|) = d - 1 + [\frac{d-1}{2}]$;*
3. *the generic elements of the pencils $|\widetilde{L}_{2,\mu}|$, $|\widetilde{\Lambda}_{2,\mu}|$ and $|\widetilde{M}_{2,\mu}|$have zero self-intersection, are transverse to $\widetilde{s}_\circ$ and intersect one another at two distinct points;*
4. *for any $h \geq 0$ the classes $\widetilde{\Lambda}_{2h+2,\mu}$ and $\widetilde{\Lambda}_{2h+3,\mu}$ are equal to $\widetilde{\Lambda}_{2,\mu} + h\widetilde{L}_{2,\mu}$ and $\widetilde{\Lambda}_{3,\mu} + h\widetilde{L}_{2,\mu}$ respectively;*
5. *there exists a reduced nodal curve $\widetilde{D}$ in $|\widetilde{\Lambda}_{d,\mu}|$ such that each one of its irreducible components has negative intersection with the canonical divisor $\widetilde{K}$.*

Sketch of proof.
1) & 2) The rational surface $\widetilde{S}$ being anticanonical, they both follow from [2], Th. III-1, p.1197.
3) & 4) Straightforward verification (use Bertini for the last issue of 3).!
5) If $d = 2h + 2$ is even, we take any smooth curve $\widetilde{C} \in |\widetilde{\Lambda}_{2,\mu}|$ and add h distinct generic fibers of $|\widetilde{L}_{2,\mu}|$ transverse to $\widetilde{C}$. For $d = 3$, instead, there are several possibilities to take into account.

1. If $\mu_1 \neq 0$ (respectively $\mu_2 \neq 0$ or $\mu_3 \neq 0$) we take $\widetilde{D}$ equal to:

$$\widetilde{\Gamma}_{\mu+(1,\text{-}1,1,1)} + \widetilde{L}_{2,\mu} \text{ (respectively: } \widetilde{\Gamma}_{\mu+(1,1,\text{-}1,1)} + \widetilde{\Lambda}_{2,\mu} \quad \text{or} \quad \widetilde{\Gamma}_{\mu+(1,1,1,\text{-}1)} + \widetilde{M}_{2,\mu}).$$

2. When $\mu_1 = \mu_2 = \mu_3 = 0$ (the trickiest case) it would be enough to find an irreducible curve $\widetilde{C} \in |\widetilde{\Gamma}_{\mu+(1,1,1,1)} \text{-} \widetilde{K}|$ with one node and then take $\widetilde{D} := \widetilde{\Gamma}_\mu + \widetilde{\Gamma}_{\mu+(2,0,0,0)} + \widetilde{C} \in |\widetilde{\Lambda}_{3,\mu}|$. Let us recall that $|\widetilde{\Gamma}_{\mu+(1,1,1,1)} \text{-} \widetilde{K}|$ is the pencil associated to a *non exceptional case* $(n', \mu') := (n{+}2, \mu{+}(1,1,1,1))$ such that $2n' = \mu'^{(2)}{+}3$ ([9] Th.4.2). Its generic fiber is an elliptic curve and has a unique base point (lying on $\widetilde{s}_\circ$). Blowing it up and contracting the inverse image of $\widetilde{\Gamma}_{\mu+(1,1,1,1)}$ we get an elliptic surface, with a (unique) reduced fiber of Euler characteristic 6, namely, the image of $\widetilde{\Gamma}_{\mu+(1,1,1,1)} \text{-} \widetilde{K}$. Any other singular fiber is irreducible and has, either a cusp and Euler characteristic 2 or a node and Euler characteristic 1. According to the Lefschetz formula their Euler characteristics must add up to 6. Hence, there are at least three (and up to six) singular irreducible curves, depending on the moduli of X as an elliptic curve. Furthermore, the fact that $\mu'_1 = \mu'_2 = \mu'_3 \neq 0$ implies there are indeed six nodal irreducible curves (to be developped in a forthcoming paper). Hence the result.

At last, for any other odd $d = 2h + 3 > 3$ we take the above reduced nodal curve $\widetilde{D} \in |\widetilde{\Lambda}_{3,\mu}|$ plus h distinct generic fibers of $|\widetilde{L}_{2,\mu}|$ transverse to $\widetilde{D}$. ■

Theorem 23
Given $d \geq 2$ and $\mu \in \mathbb{N}^4$ such that $\mu_\circ{+}1 \equiv \mu_j (\text{mod}\, 2)$, the Severi variety $V_\circ|\widetilde{\Lambda}_{d,\mu}|$ is non-empty, has dimension $d\text{-}1$ and its generic element is transverse to $\widetilde{s}_\circ$.

Proof. The reduced nodal curve $\widetilde{D} \in |\widetilde{\Lambda}_{d,\mu}|$ constructed in the preceding lemma is transverse to $\widetilde{s}_\circ$ and has $p_a(\widetilde{D})$ nodes which can be deleted without disconnecting it. According to A.Tannenbaum's results ([5] 2.9, 2.13 & 2.14) there exists a flat family $\Upsilon \subset \widetilde{S} \times T$ of dimension $dimT = d\text{-}1$, with special fiber $\Upsilon_\circ = \widetilde{D}$ and generic fiber a rational irreducible nodal curve $\Upsilon_t \in V_\circ|\widetilde{\Lambda}_{d,\mu}|$ transverse to $\widetilde{s}_\circ$. In particular $V_\circ|\widetilde{\Lambda}_{d,\mu}| \neq \emptyset$ and $\dim(V_\circ|\widetilde{\Lambda}_{d,\mu}|) \geq d\text{-}1$. On the other hand one can mimic [6] 4.10 to deduce that its tangent space at any $\Upsilon_t \in V_\circ|\widetilde{\Lambda}_{d,\mu}|$ has dimension $d\text{-}1$ implying $\dim(V_\circ|\widetilde{\Lambda}_{d,\mu}|) = d\text{-}1$. ■

Corollary 24
Given $d \geq 2$ and $\mu \in \mathbb{N}^4$ such that $\mu_\circ + 1 \equiv \mu_j (\text{mod}\, 2)$ there exists a $(d\text{-}1)$-dimensional family of hyperelliptic d-tangential covers of arithmetic genus $\text{-}1 + \frac{d}{2}(\mu^{(1)} + 3)$ and degree:

1. $n := \frac{1}{2}d(\mu^{(2)} + 3) + \mu.(d, d\text{-}2, 1, 1)$ *if d is odd;*
2. $n := \frac{1}{2}d(\mu^{(2)} + 3) + \mu.(d, d\text{-}2, 2, 0)$ *if d is even.*

Proof. We can immediately check that the generic element $\widetilde{\Gamma}$ of $V_\circ|\widetilde{\Lambda}_{d,\mu}|$ satisfies the *Rational characterization of hyperelliptic tangential covers.* ■

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