TOPOLOGICAL ENTROPY, SETS OF PERIODS AND TRANSITIVITY FOR GRAPH MAPS

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Abstract. Transitivity, the existence of periodic points and positive topological entropy can be used to characterize complexity in dynamical systems. It is known that for graphs that are not trees, for every $\varepsilon > 0$, there exist (complicate) totally transitive maps (then with cofinite set of periods) such that the topological entropy is smaller than ε (simplicity). First we will show by means of three examples that for any graph that is not a tree the relatively simple maps (with small entropy) which are totally transitive (and hence robustly complicate) can be constructed so that the set of periods is also relatively simple. To numerically measure the complexity of the set of periods we introduce a notion of a *boundary of cofiniteness*. Larger boundary of cofiniteness means simpler set of periods. With the help of the notion of boundary of cofiniteness we can state precisely what do we mean by extending the entropy simplicity result to the set of periods: *there exist relatively simple maps such that the boundary of cofiniteness is arbitrarily large (simplicity) which are totally transitive (and hence robustly complicate).* Moreover, we will show that, the above statement holds for arbitrary continuous degree one circle maps.

1. Introduction

Transitivity, the existence of infinitely many periods and positive topological entropy often characterize the complexity in dynamical systems. This paper aims at showing that totally transitive maps on graphs, despite of being complicate in the above sense can have somewhat simple sets of periods (simplicity with respect to topological entropy was already known). To be more precise and to state the main results of the paper we need to introduce some basic notation.

Let X be a topological space and let $f\colon X \longrightarrow X$ be a map. A point $x \in X$ is called a *periodic point of f of period n* if $f^n(x) = x$ and n is the minimum positive integer with this property. The set of all positive integers n such that f has a periodic point of period n is denoted by $\mathsf{Per}(f)$. A set of periods is called *cofinite* if its complement (on $\mathbb{N}$) is finite or, equivalently, it contains all positive integers larger than a given period.

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Let X be a compact space and let $f\colon X \longrightarrow X$ be a continuous map. The *topological entropy* of f is defined as in [1] and denoted by $h(f)$.

Definition 1.1. Given a topological space X, a continuous map $f\colon X \longrightarrow X$ is called *transitive* if for every pair of open subsets of X, U and V, there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$.

It is well known that when X has no isolated point, transitivity is equivalent to the existence of a dense orbit (see for instance [16]).

A map f is called *totally transitive* if all iterates of f are transitive. ◘

We are interested in totally transitive maps on graphs. A *(topological) graph* is a connected compact Hausdorff space for which there exists a finite non-empty subset whose complement is the disjoint union of a finite number subsets, each of them homeomorphic to an open interval of the real line. A *tree* is a graph without circuits or, equivalently, a uniquely arcwise connected graph. A continuous map from a graph to itself will be called a *graph map*.

A transitive graph map has positive topological entropy and dense set of periodic points [12, 13, 4, 5], with the only exception of an irrational rotation on the circle. Thus, in view of [9] every transitive map on a graph is chaotic in the sense of Devaney (except, again, for an irrational rotation on the circle). Moreover, from [3, Main Theorem] we have

Theorem 1.2. *Let G be a graph and let f be a continuous transitive map from G to itself which has periodic points. Then the following statements are equivalent:*

(a) f is totally transitive.
(b) $\mathsf{Per}(f)$ is cofinite in $\mathbb{N}$.

Thus, *totally transitive maps on graphs are complicate since they have positive topological entropy, are chaotic in the sense of Devaney and have cofinite set of periods.*

However, from [2] we know that for every graph that is not a tree and for every $\varepsilon > 0$, there exists a totally transitive map such that its topological entropy is positive but smaller than ε. Thus, *transitive maps on graphs may be relatively simple because they may have arbitrarily small positive topological entropy.*

Remark 1.3. This result is valid only for graphs that are not trees since from [6] we know that for any tree T and any transitive map $f\colon T \longrightarrow T$,

$$h(f) \geq \frac{\log 2}{\mathsf{E}(T)},$$

where $\mathsf{E}(T)$ denotes the number of endpoints of T. ◘

The aim of this paper is to study whether the simplicity phenomenon described above, that happens for the topological entropy, can be extended to the set of periods. More precisely, *is it true that when a totally transitive graph map has small positive topological entropy it also has simple set of periods (and in particular small "cofinite part" of the set of periods)?*

To measure the size of the set of periods and, in particular, of its "cofinite part" we introduce the notions of *strict boundary of cofiniteness* and *boundary of cofiniteness*.

In what follows, given $L \in \mathbb{N}$, the set $\{k \in \mathbb{N} : k \geq L\}$ will be denoted by $\mathsf{Succ}\,(L)$.

Definition 1.4. Let f be a graph map whose set of periods is cofinite. The *strict boundary of cofiniteness of* f is defined as the smallest positive integer n such that $\mathsf{Per}(f) \supset \mathsf{Succ}\,(n)$. The strict boundary of cofiniteness of f will be denoted by $\mathsf{StrBdCof}\,(f)$. ◘

Observe that, despite of the fact that the strict boundary of cofiniteness of f is a good measure of the size of the cofinite part of the set of periods of f, it does not measure at all the simplicity of the whole set of periods. Indeed, it could happen that the set of periods of graph map f is precisely $\mathbb{N}\setminus\{\mathsf{StrBdCof}\,(f)-1\}$; which cannot be called simple at all. Thus, a simplicity measure of a cofinite set of periods, apart from a measure of the size of the cofinite part of the set of periods, should incorporate a measure of the simplicity of the periods smaller than the strict boundary of cofiniteness. This is achieved by imposing low density of the periods lower than the boundary of cofiniteness. To this end we introduce the notion of *boundary of cofiniteness*.

Definition 1.5. Given a graph map f whose set of periods is cofinite consider the set

$$\mathsf{sBC}(f) := \Big\{L \in \mathsf{Per}(f) : L > 2,\ L-1 \notin \mathsf{Per}(f) \text{ and } \mathsf{Card}\big(\{1, \ldots, L-2\} \cap \mathsf{Per}(f)\big) \leq 2\log_2(L-2)\Big\}.$$

Let $L \in \mathsf{Per}(f)$, $L \geq 2$, be such that $L-1 \notin \mathsf{Per}(f)$. The periods of f which are smaller than L are called the *$L-$low periods of f*, and the density of the $L-$low periods of f is defined by

$$\mathsf{DensLowPer}_f(L) := \frac{\mathsf{Card}\big(\{1, \ldots, L-2\} \cap \mathsf{Per}(f)\big)}{L-2}.$$

Then,

$$\mathsf{sBC}(f) = \left\{L \in \mathsf{Per}(f) : L > 2,\ L-1 \notin \mathsf{Per}(f) \text{ and } \mathsf{DensLowPer}_f(L) \leq \frac{2\log_2(L-2)}{L-2}\right\}.$$

Observe that every $L \in \mathsf{Per}(f)$, $L \geq 2$, such that $L-1 \notin \mathsf{Per}(f)$ must satisfy $L \leq \mathsf{StrBdCof}\,(f)$. Hence, $\mathsf{sBC}(f) \subset \{1, 2, \ldots, \mathsf{StrBdCof}\,(f)\}$ and, thus, $\mathsf{sBC}(f)$ is finite.

When $\mathsf{sBC}(f) \neq \emptyset$ we define the *boundary of cofiniteness of* f as the number $\mathsf{BdCof}\,(f) := \max \mathsf{sBC}(f)$ (which is well defined thanks to the finiteness of $\mathsf{sBC}(f)$). ◘

The idea behind the above definition is explained in the following remark.

Remark 1.6 (on the definition of boundary of cofiniteness).

- Since the set $\mathsf{sBC}(f)$ could be empty, $\mathsf{BdCof}\,(f)$ may not be defined.

- $\mathsf{BdCof}(f) \leq \mathsf{StrBdCof}(f)$ whenever $\mathsf{BdCof}(f)$ is defined. Moreover, as we will see in the examples below, $\mathsf{StrBdCof}(f)$ may not belong to $\mathsf{sBC}(f)$ and, hence, $\mathsf{BdCof}(f)$ may be strictly smaller than $\mathsf{StrBdCof}(f)$.
- The fact that $\mathsf{BdCof}(f) \leq \mathsf{StrBdCof}(f)$ implies that the cofinite part of $\mathsf{Per}(f)$ is small (relative to $\mathsf{BdCof}(f)$) whenever $\mathsf{BdCof}(f)$ is large.
- On the other hand,

$$\mathsf{DensLowPer}_f(\mathsf{BdCof}(f)) \leq \frac{2\log_2(\mathsf{BdCof}(f) - 2)}{\mathsf{BdCof}(f) - 2},$$

  is also small (relative to $\mathsf{BdCof}(f)$) whenever $\mathsf{BdCof}(f)$ is large. Moreover,

$$\lim_{L\to\infty} \frac{2\log_2(L-2)}{L-2} = 0.$$

- Observe that every element of $\mathsf{sBC}(f)$ verifies the above two statements. The reason for defining $\mathsf{BdCof}(f)$ as the maximum of $\mathsf{sBC}(f)$ is that, since the map $\frac{2\log_2(x)}{x}$ is decreasing for $x \geq 2$, this number gives the smallest possible bound of the cofinite part of the set of periods and of the density of the set of low periods of f *simultaneously.*

Summarizing, *large boundary of cofiniteness implies simple set of periods.* □

Now we can state precisely what do we mean by extending the entropy simplicity phenomenon described above to the set of periods: *is it true that there exist totally transitive (and hence dynamically complicate) graph maps with arbitrarily large boundary of cofiniteness?*

We start by illustrating the above statement with three examples for arbitrary graphs which are not trees.

The rotation interval of a circle map of degree one is a fundamental tool to determine its set of periods. We will define this object in Section 2, where will describe in detail its relation with the set of periods of the map under consideration. In what follows we will denote the set of all liftings of all continuous circle maps of degree one by $\mathcal{L}_1$, and the rotation interval of $F \in \mathcal{L}_1$ by $\mathsf{Rot}(F)$.

Example 1.7 (the dream example)**.** *For every positive integer $n \geq 3$ there exists f_n, a totally transitive continuous circle map of degree one having a lifting $F_n \in \mathcal{L}_1$ such that* $\mathsf{Rot}(F_n) = \left[\frac{1}{2n-1}, \frac{2}{2n-1}\right]$, $\mathsf{Per}(f_n) = \mathsf{Succ}(n)$ *and* $\lim_{n\to\infty} h(f_n) = 0$. *Hence,* $\mathsf{BdCof}(f_n) = \mathsf{StrBdCof}(f_n) = n$ *and, consequently,* $\lim_{n\to\infty} \mathsf{BdCof}(f_n) = \infty$.

Furthermore, given any graph G with a circuit, the sequence of maps $\{f_n\}_{n\geq 5}$ can be extended to a sequence of continuous totally transitive self maps of G, $\{g_n\}_{n\geq 5}$, such that $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$ *and* $\lim_{n\to\infty} h(g_n) = 0$.

Remark 1.8. In this example there are no $\mathsf{BdCof}(g_n)-$low periods. Hence, $\mathsf{StrBdCof}(g_n)$ is enough to control the complexity of the set of periods. □

Example 1.9 (with persistent fixed low periods)**.** *For every positive integer $n \in \{4k+1, 4k-1 : k \in \mathbb{N}\}$ there exists f_n, a totally transitive continuous circle map of degree one having a lifting $F_n \in \mathcal{L}_1$ such that* $\mathsf{Rot}(F_n) =$

$\left[\frac{1}{2}, \frac{n+2}{2n}\right]$, $\lim_{n\to\infty} h(f_n) = 0$,

$$\mathsf{Per}(f_n) = \{2\} \cup \{q \textit{ odd} : 2k+1 \le q \le n-2\} \cup \mathsf{Succ}\,(n)$$

and $\mathsf{BdCof}\,(f_n)$ *exists and verifies* $2k+1 \le \mathsf{BdCof}\,(f_n) \le n$ *(and, hence,* $\lim_{n\to\infty} \mathsf{BdCof}\,(f_n) = \infty$*).*

Furthermore, given any graph G *with a circuit, the sequence of maps* $\{f_n\}_{n\ge 7, n \text{ odd}}$ *can be extended to a sequence of continuous totally transitive self maps of* G*,* $\{g_n\}_{n\ge 7, n \text{ odd}}$*, such that* $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$ *and* $\lim_{n\to\infty} h(g_n) = 0$.

Remark 1.10. The above example is different from the previous one since for every n there exist $\mathsf{BdCof}\,(f_n)-$low periods and, moreover, every map f_n has a constant $\mathsf{BdCof}\,(f_n)-$low period 2.

On the other hand, $\mathsf{StrBdCof}\,(f_n) = n \ne \mathsf{BdCof}\,(f_n)$. In this example this is due to the fact that the set of periods which are smaller than $\mathsf{StrBdCof}\,(f_n)$ is very large relative to the value of $\mathsf{StrBdCof}\,(f_n)$. More concretely,

$$\mathsf{DensLowPer}_{f_n}\,(\mathsf{StrBdCof}\,(f_n)) = \begin{cases} \frac{k+1}{4k-1} & \text{if } n = 4k+1, \\ \frac{k}{4k-3} & \text{if } n = 4k-1. \end{cases}$$

So, for large n,

$$\mathsf{DensLowPer}_{f_n}\,(\mathsf{StrBdCof}\,(f_n)) \approx \tfrac{1}{4} > \frac{2\log_2(n-2)}{n-2} = \frac{2\log_2(\mathsf{StrBdCof}\,(f_n) - 2)}{\mathsf{StrBdCof}\,(f_n) - 2}$$

and, hence, the strict boundary of cofiniteness does not belong to $\mathsf{sBC}(f_n)$. Furthermore, the differences $\mathsf{StrBdCof}\,(f_n) - \mathsf{BdCof}\,(f_n)$ are unbounded because

$$\mathsf{DensLowPer}_{f_n}\,(\mathsf{BdCof}\,(f_n)) \le \frac{2\log_2(\mathsf{BdCof}\,(f_n) - 2)}{\mathsf{BdCof}\,(f_n) - 2}$$

converges to zero as n goes to infinity (so, if the differences $\mathsf{StrBdCof}\,(f_n) - \mathsf{BdCof}\,(f_n)$ are bounded, then $\mathsf{DensLowPer}_{f_n}\,(\mathsf{StrBdCof}\,(f_n))$ also converges to zero as n goes to infinity; which contradicts the previous estimate). This is a concrete new motivation of our definition of boundary of cofiniteness. ◘

Example 1.11 (with non-constant low periods). *For every* $n \in \mathbb{N}$, $n \ge 3$ *there exists* f_n*, a totally transitive continuous circle map of degree one having a lifting* $F_n \in \mathcal{L}_1$ *such that*

$$\mathsf{Rot}(F_n) = \left[\tfrac{2n-1}{2n^2}, \tfrac{2n+1}{2n^2}\right] = \left[\tfrac{1}{n} - \tfrac{1}{2n^2}, \tfrac{1}{n} + \tfrac{1}{2n^2}\right],$$

$\lim_{n\to\infty} h(f_n) = 0$ *and*

$$\mathsf{Per}(f_n) = \{n\} \cup \left\{tn + k : t \in \{2, 3, \dots, \nu - 1\} \textit{ and } -\tfrac{t}{2} < k \le \tfrac{t}{2},\ k \in \mathbb{Z}\right\} \cup \mathsf{Succ}\left(n\nu + 1 - \tfrac{\nu}{2}\right)$$

with

$$\nu = \begin{cases} n & \textit{if } n \textit{ is even, and} \\ n-1 & \textit{if } n \textit{ is odd.} \end{cases}$$

Moreover, $\mathsf{StrBdCof}\,(f_n) = n\nu + 1 - \frac{\nu}{2}$ *and* $\mathsf{BdCof}\,(f_n)$ *exists and verifies* $n \le \mathsf{BdCof}\,(f_n) \le n\nu - 1 - \frac{\nu}{2}$ *(and hence,* $\lim_{n\to\infty} \mathsf{BdCof}\,(f_n) = \infty$*).*

Furthermore, given any graph G with a circuit, the sequence of maps $\{f_n\}_{n=4}^{\infty}$ can be extended to a sequence of continuous totally transitive self maps of G, $\{g_n\}_{n=4}^{\infty}$, such that $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$ *and* $\lim_{n\to\infty} h(g_n) = 0$.

Remark 1.12. This example is different from the previous two examples since for every n there exist $\mathsf{BdCof}(f_n)-$low periods but there is no constant $\mathsf{BdCof}(f_n)-$low period.

Moreover, as in the previous example, $\mathsf{StrBdCof}(f_n) \neq \mathsf{BdCof}(f_n)$,

$$\mathsf{DensLowPer}_{f_n}(\mathsf{StrBdCof}(f_n)) \approx \tfrac{1}{2}$$

and the differences $\mathsf{StrBdCof}(f_n) - \mathsf{BdCof}(f_n)$ are unbounded. ◘

Remark 1.13. In all three examples we still have $\lim_{n\to\infty} h(g_n) = 0$ despite of the fact that $h(g_n)$ is slightly larger than $h(f_n)$ for every n. ◘

Finally, we state the main theorem of the paper that shows that the above examples are not exceptional among circle maps of degree one. On the contrary, a sequence of totally transitive circle maps that unfolds an entropy simplification process also must unfold a set of periods simplification process:

Theorem A. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of totally transitive circle maps of degree one with periodic points such that* $\lim_{n\to\infty} h(f_n) = 0$. *For every n let $F_n \in \mathcal{L}_1$ be a lifting of f_n. Then,*

- $\lim_{n\to\infty} \mathsf{len}(\mathsf{Rot}(F_n)) = 0$,
- *there exists $N \in \mathbb{N}$ such that* $\mathsf{BdCof}(f_n)$ *exists for every $n \geq N$, and*
- $\lim_{n\to\infty} \mathsf{BdCof}(f_n) = \infty$.

This paper is organized as follows. In Section 2 we introduce the definitions and preliminary results for the rest of the paper; the construction of the Examples 1.7, 1.9 and 1.11 is done in the very long Section 3. Finally, Theorem A is proved in Section 4.

2. Definitions and preliminary results

In this section we essentially follow the notation and strategy of [8].

2.1. **Basic definitions.** In the rest of the paper we denote the unit circle $\{z \in \mathbb{C} : |x| = 1\}$ by $\mathbb{S}^1$.

A *topological graph* (or simply a *graph*) is a connected compact Hausdorff space X for which there exists a finite (or empty) set $V(X)$, called the *set of vertices of* X , such that either $X = \mathbb{S}^1$ and $V(X) = \emptyset$ or $X \setminus V(X)$ is the disjoint union of finitely many open subsets of X each of them homeomorphic to an open interval of the real line, called *edges of* X, with the property that the boundary of every edge consists of at most two points which are in $V(X)$. A point $z \in V(X)$ is an *endpoint of* X if there exists an open (in X) neighbourhood U of z such that $X \cap U$ is homeomorphic to the interval $[0, 1)$ being z the preimage of 0. A *circuit* of a graph X is any subset of X homeomorphic to a circle. A *tree* is a graph without circuits.

Let X be a topological space and let $f\colon X \longrightarrow X$ be a continuous map. When iterating the map f we will use the following notation: f^0 will denote the identity map (in X), and $f^n := f \circ f^{n-1}$ for every $n \in \mathbb{N}$, $n \geq 1$. For a point $x \in X$, we define the *f-orbit of x* (or simply the *orbit of x*), denoted by $\mathsf{Orb}_f(x)$, as the set $\{f^n(x) : n \geq 0\}$. A point $x \in X$ is called a *periodic point of f* if $f^n(x) = x$. In such case $\mathsf{Orb}_f(x)$ is called a *periodic orbit of f* and $\mathsf{Card}\left(\mathsf{Orb}_f(x)\right)$ is called the *period of x* (or *f-period of x* if we need to specify the map). Observe that if x is a periodic point of f of period n, then $f^k(x) \neq x$ for every $1 \leq k < n$ and if P is a periodic orbit of f, then $P = \mathsf{Orb}_f(x)$ for every $x \in P$.

The set of all positive integers n such that f has a periodic point of period n is denoted by $\mathsf{Per}(f)$.

2.2. **Rotation theory and sets of periods for circle maps of degree one.** We start by introducing the key notion to work with circle maps: the liftings. Let $\mathsf{e}\colon \mathbb{R} \longrightarrow \mathbb{S}^1$ be the natural projection which is defined by $\mathsf{e}(x) := \exp(2\pi i x)$. Given a continuous map $f\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, we say that a continuous map $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a *lifting of f* if $\mathsf{e}(F(x)) = f(\mathsf{e}(x))$ for every $x \in \mathbb{R}$. For such F, there exists $d \in \mathbb{Z}$ such that $F(x+1) = F(x) + d$ for all $x \in \mathbb{R}$, and this integer is called both the *degree of f* and the *degree of F*. If G and F are two liftings of f then $G = F + k$ for some integer k and so F and G have the same degree. We denote by $\mathcal{L}_d$ the set of all liftings of circle maps of degree d.

Next we introduce the important notion of *rotation interval* for maps from $\mathcal{L}_1$. Let $F \in \mathcal{L}_1$ and let $x \in \mathbb{R}$. The number

$$\rho_F(x) := \limsup_{n\to\infty} \frac{F^n(x) - x}{n}$$

will be called the *rotation number of x*. Moreover, the set

$$\mathsf{Rot}(F) := \{\rho_F(x) : x \in \mathbb{R}\} = \{\rho_F(x) : x \in [0,1]\}$$

will be called the *rotation interval of F*. It is well known that it is a closed interval of the real line [15].

If $F \in \mathcal{L}_1$ is a non-decreasing map, then

$$\rho_F(x) = \lim_{n\to\infty} \frac{F^n(x) - x}{n}$$

for every $x \in \mathbb{R}$ and, moreover, it is independent on x (see for instance [8]). Then this number ($\rho_F(x)$ for any $x \in \mathbb{R}$), will be called the *rotation number of F*. For every $F \in \mathcal{L}_1$ we define the *lower map* $F_l\colon \mathbb{R} \longrightarrow \mathbb{R}$ by (see Figure 1 for a graphical example)

$$F_l(x) = \inf\{F(y) : y \geq x\}$$

and the *upper map* $F_u\colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$F_u(x) = \sup\{F(y) : y \leq x\}.$$

It is easy to see (see e.g. [8]) that F_l, F_u are non-decreasing maps from $\mathcal{L}_1$.

The next theorem gives an effective way to compute the rotation interval from the rotation numbers of the upper and lower maps.

Theorem 2.1 ([8, Theorem 3.7.20])**.** *For every $F \in \mathcal{L}_1$ it follows that* $\mathsf{Rot}(F) = [\rho(F_l), \rho(F_u)]$.

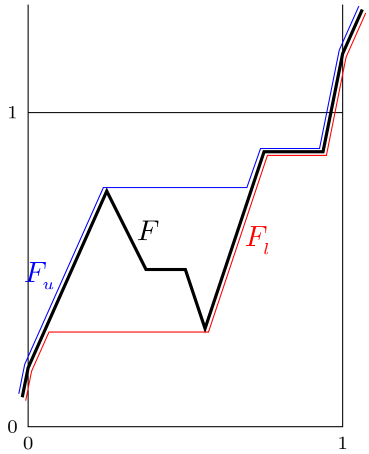


FIGURE 1. An example of a map $F \in \mathcal{L}_1$ with its lower map F_l in red and its upper map F_u in blue.

It is well known that the rotation interval of a lifting $F \in \mathcal{L}_1$ can be used to obtain information about the set of periods of the corresponding circle map. To clarify this point we will introduce the notion of *lifted orbit*.

Let f be a continuous circle map of degree d and let $F \in \mathcal{L}_d$ be a lifting of f. A set $P \subset \mathbb{R}$ will be called a *lifted orbit of F* if there exists $z \in \mathbb{S}^1$ such that $P = \mathsf{e}^{-1}\big(\mathsf{Orb}_f(z)\big)$ and $f(\mathsf{e}(x)) = \mathsf{e}(F(x))$ for every $x \in P$. Whenever z is a periodic point of f of period n, P will be called a *lifted periodic orbit of F of period n*. We will denote by $\mathsf{Per}(F)$ the set of periods of all lifted periodic orbits of F. Observe that then, $\mathsf{Per}(F) = \mathsf{Per}(f)$.

Remark 2.2. Let $F \in \mathcal{L}_1$ and let P be a lifted periodic orbit of F of period period n. Set

$$P = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$$

with $x_i < x_j$ if and only if $i < j$. The fact that $P = \mathsf{e}^{-1}\big(\mathsf{Orb}_f(z)\big)$, in this case, gives

$$\mathsf{Card}\,(P \cap [r, r+1]) = n$$

for every $r \in \mathbb{R}$ and, hence,

$$x_{kn+i} = x_i + k$$

for every $i, k \in \mathbb{Z}$.

Moreover, there exists $m \in \mathbb{Z}$ such that $F^n(x_i) = x_i + m = x_{mn+i}$ for every $x_i \in P$. Consequently,

$$\rho_F(x_i) = \tfrac{m}{n}$$

for every $x_i \in P$. ◘

From the above remark it follows that if P is a lifted periodic orbit of $F \in \mathcal{L}_1$, then all the points of P have the same rotation number. This number will be called the *rotation number of P* (or *F-rotation number of P* if we need to specify the lifting).

A lifted periodic orbit P of $F \in \mathcal{L}_1$ such that $F\big|_P$ is increasing will be called *twist*.

Remark 2.3. Let

$$P = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$$

be a twist lifted periodic orbit of $F \in \mathcal{L}_1$ of period n and rotation number m/n labelled so that $x_i < x_j$ if and only if $i < j$. By [8, Lemma 3.7.4 and Corollary 3.7.6] we have that m and n are coprime and

$$F(x_i) = x_{i+m},$$

for all $i \in \mathbb{Z}$. ◘

The next theorem due to Misiurewicz (see [17, 8]) already makes the connection between $\mathsf{Rot}(F)$ and $\mathsf{Per}(F)$. To state it, we still need to recall the *Sharkovskiĭ Ordering*.

The *Sharkovskiĭ Ordering* ${}_{\text{Sh}}\!\geq$ (the symbols ${}_{\text{Sh}}\!>$, $<_{\text{Sh}}$ and $\leq_{\text{Sh}}$ will also be used in the natural way) is a linear ordering of $\mathbb{N}_{\text{Sh}} := \mathbb{N} \cup \{2^\infty\}$ (we have to include the symbol $\{2^\infty\}$ in order to ensure the existence of supremum of every subset with respect to the ordering ${}_{\text{Sh}}\!\geq$) defined as follows:

$$\begin{array}{l}
3 \;{}_{\text{Sh}}\!> 5 \;{}_{\text{Sh}}\!> 7 \;{}_{\text{Sh}}\!> 9 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> \\
2\cdot 3 \;{}_{\text{Sh}}\!> 2\cdot 5 \;{}_{\text{Sh}}\!> 2\cdot 7 \;{}_{\text{Sh}}\!> 2\cdot 9 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> \\
4\cdot 3 \;{}_{\text{Sh}}\!> 4\cdot 5 \;{}_{\text{Sh}}\!> 4\cdot 7 \;{}_{\text{Sh}}\!> 4\cdot 9 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> \\
\qquad\qquad\vdots \\
2^n\cdot 3 \;{}_{\text{Sh}}\!> 2^n\cdot 5 \;{}_{\text{Sh}}\!> 2^n\cdot 7 \;{}_{\text{Sh}}\!> 2^n\cdot 9 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> \\
\qquad\qquad\vdots \\
2^\infty \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> 2^n \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> 16 \;{}_{\text{Sh}}\!> 8 \;{}_{\text{Sh}}\!> 4 \;{}_{\text{Sh}}\!> 2 \;{}_{\text{Sh}}\!> 1.
\end{array}$$

We introduce the following notation. Given $c, d \in \mathbb{R}$, $c \leq d$ we set

$$M(c, d) := \{n \in \mathbb{N} : c < k/n < d \text{ for some integer } k\}.$$

Let $F \in \mathcal{L}_1$ let c be an endpoint of $\mathsf{Rot}(F)$. We define the set

$$Q_F(c) := \begin{cases} \emptyset & \text{if } c \notin \mathbb{Q} \\ \{sk : k \in \mathbb{N} \text{ and } k \leq_{\text{Sh}} s_c\} & \text{if } c = r/s \text{ with } r, s \text{ coprime} \end{cases}$$

and $s_c \in \mathbb{N}_{\text{Sh}}$ is defined by the Sharkovskiĭ Theorem on the real line. Indeed, since $c = r/s$ and r and s are coprime, the map $F^s - r$ is a continuous map on the real line with periodic points. Hence, by the Sharkovskiĭ Theorem there exists an $s_c \in \mathbb{N}_{\text{Sh}}$ such that the set of periods (not lifted periods) of $F^s - r$ is precisely $\{s \in \mathbb{N} : s \leq_{\text{Sh}} s_c\}$.

Theorem 2.4. *Let f be a continuous circle map of degree one having a lifting $F \in \mathcal{L}_1$. Assume that $\mathsf{Rot}(F) = [c, d]$. Then*

$$\mathsf{Per}(f) = Q_F(c) \cup M(c, d) \cup Q_F(d).$$

2.3. **Markov graphs, Markov maps and sets of periods.** Take a finite set $V = \{v_1, v_2, \dots, v_n\}$. The pair $\mathcal{G} = (V, U)$ where $U \subset V \times V$ is called a *combinatorial oriented graph*. The elements of V are called the *vertices of* $\mathcal{G}$ and each element $(v_i, v_j) \in U$ is called an *arrow from* v_i *to* v_j. An arrow (v_i, v_j) will also be denoted by $v_i \longrightarrow v_j$, which allows us to give a graphical representation of an oriented graph. A *path of length* k is a sequence of $k+1$

vertices $v_0, v_1, \dots, v_k$ with the property that there is an arrow from every vertex to the next one. A path is denoted as $v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_k$. A *loop of length* k is a path of length k where the first and last vertex coincide: $v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{k-1} \longrightarrow v_0$.

Let X be a topological graph. Every subset of X homeomorphic to the interval $[0,1]$ will in turn be called an *interval of* X. The preimages of 0 and 1 by the homeomorphism will be called *the endpoints of* I and the set of (both) endpoints of I will be denoted by ∂I. Note that if $I \cap V(X) = \emptyset$, then $I \setminus \partial I = \mathsf{Int}(I)$.

Let X be a topological graph and let $f\colon X \longrightarrow X$ be a continuous map. A set $Q \subset X$ will be called *f-invariant* if $f(Q) \subset Q$. A *Markov invariant set* is defined to be a finite f-invariant set $Q \supset V(X)$ such that the closure of each connected component of $X \setminus Q$, called a *Q-basic interval*, is an interval of X. Observe that two different Q-basic intervals have disjoint interiors (here *interior* means the topological interior in X which is the whole Q-basic interval minus those of its endpoints which are not endpoints of the graph because $Q \supset V(X)$).

The set of all Q-basic intervals will be denoted by $\mathcal{B}(Q)$.

Definition 2.5 (Of monotonicity over an interval). Let X be a topological graph, let $f\colon X \longrightarrow X$ be a continuous map and let I be an interval of X. The map f will be said to be *monotone at* I if the set $\left(f\big|_I\right)^{-1}(y) = \{x \in I : f(x) = y\}$ is connected for every $y \in f(I)$. Clearly, since I is an interval, $\left(f\big|_I\right)^{-1}(y)$ is either a point or an interval for every $y \in f(I)$. Moreover, a simple exercise shows that the subgraph $f(I)$, in turn, must be either a point or an interval. Finally, let $g\colon X \longrightarrow X$ be another continuous map, and let J be another interval of X such that $f(I) \cap \mathsf{Int}(J) \neq \emptyset$ and g is monotone at J. A first year calculus exercise shows that $\left(f\big|_I\right)^{-1}(J) = \{x \in I : f(x) \in J\}$ is an interval and $g\big|_J \circ f\big|_I$ is monotone at $\left(f\big|_I\right)^{-1}(J)$. ◘

Remark 2.6. The above definition of monotonicity over an interval is equivalent to the usual one: $f(I)$ is a point or an interval and, in the second case, the map $\zeta \circ f\big|_I \circ \xi\colon [0,1] \longrightarrow [0,1]$ is monotone as interval map, where $\xi\colon [0,1] \longrightarrow I$ and $\zeta\colon f(I) \longrightarrow [0,1]$ are homeomorphisms (which exist because I and $f(I)$ are intervals). We prefer the more intrinsic definition given above because it is independent on the choice of the auxiliary homeomorphisms and on whether they are increasing or decreasing. This will be specially helpful when studying compositions of monotone maps over intervals like in the rest of this subsection and Subsection 2.6. ◘

Let X be a topological graph, let $f\colon X \longrightarrow X$ be a continuous map and let $Q \subset X$ be a Markov invariant set. We say that f is *Q-monotone* if f is monotone on each Q-basic interval. In such a case, Q is called a *Markov partition of X with respect to f* and f is called a *Markov map with respect to Q*.

Next we introduce the very important notion of f-covering that allows us to get a combinatorial oriented graph from a Markov partition of a Markov map.

Let X be a topological graph, let $f\colon X \longrightarrow X$ be a continuous map and let Q be a Markov partition of X with respect to f. Given $I, J \in \mathcal{B}(Q)$, we say that I *f-covers* J if $f(I) \supset J$. The *Markov graph of f with respect to Q* (or *f-graph*) is a combinatorial oriented graph whose vertices are all the Q-basic intervals and there is an arrow $I \longrightarrow J$ from the vertex (Q-basic interval) I to the vertex (Q-basic interval) J if and only if I f-covers J. The *Markov matrix of f with respect to Q* is another combinatorial object that describe the dynamical behaviour of a Markov map f and is associated in a natural way to the Markov graph of f with respect to Q. It is a $\mathsf{Card}(\mathcal{B}(Q)) \times \mathsf{Card}(\mathcal{B}(Q))$ matrix $M = (m_{I,J})_{I,J\in\mathcal{B}(Q)}$ such that

$$m_{I,J} = \begin{cases} 1 & \text{if } I \ f\text{-covers } J \\ 0 & \text{otherwise} \end{cases}.$$

The next lemma shows the relation between loops of Markov graphs and periodic points. Essentially it is [8, Corollary 1.2.8] extended to graph maps.

Given a tree T (which is uniquely arcwise connected) and a set $A \subset T$, we denote by $\langle A \rangle_T$ the *convex hull of A in T*, that is, the smallest closed connected set of T that contains A. Also, given $q \in \mathbb{N}$, the congruence classes modulo q will be $\{0, 1, \dots, q-1\}$.

Lemma 2.7. *Let X be a topological graph, let $f\colon X \longrightarrow X$ be a Markov map with respect to a Markov partition Q of X and let $\alpha = I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow I_0$ be a loop in the Markov graph of f with respect to Q. Then, there exists a fixed point $x \in I_0$ of f^n such that $f^i(x) \in I_i$ for $i = 1, 2, \dots, n-1$.*

The next result compiles [8, Lemma 1.2.12 and Theorem 2.6.4] extended to graph maps. To state it we need to introduce some more definitions.

Given two paths $\alpha = v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_k$ and $\beta = w_0 \longrightarrow w_1 \longrightarrow w_2 \longrightarrow \cdots \longrightarrow w_m$ in a combinatorial graph such that the last vertex of the first path is the first vertex of the second one (i.e., $v_k = w_0$), the *concatenation* of α and β is denoted by $\alpha\beta$ and is the path

$$v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{k-1} \longrightarrow w_0 \longrightarrow w_1 \longrightarrow w_2 \longrightarrow \cdots \longrightarrow w_m.$$

Clearly, the length of the concatenated path is the sum of the lengths of the original paths. A loop is an *n-repetition* of a (shorter) loop α if $n \geq 2$ and it is a concatenation of α with itself n times. Such a loop will be called *repetitive*. A loop which is not repetitive will also be called *simple*.

The *shift of a loop* $\alpha = v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{k-1} \longrightarrow v_0$ is defined to be the loop

$$S(\alpha) := v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{k-1} \longrightarrow v_0 \longrightarrow v_1.$$

Iteratively, we set $S^0(\alpha) := \alpha$ and, for every $m \in \mathbb{N}$, we define the *m-shift of α*, denoted by $S^m(\alpha)$, as the loop

$$v_{m \ (\mathrm{mod}\ k)} \longrightarrow v_{m+1 \ (\mathrm{mod}\ k)} \longrightarrow \cdots \longrightarrow v_{m+k-1 \ (\mathrm{mod}\ k)} \longrightarrow v_{m \ (\mathrm{mod}\ k)}.$$

Let X be a topological graph, let $f\colon X \longrightarrow X$ be a Markov map with respect to a Markov partition, let $\alpha = I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{m-1} \longrightarrow I_0$ be a loop in the Markov graph of f and let P be a periodic orbit of period m of f. We say that α and P are *associated* to each other if there exists an $x \in P$ such that $f^k(x) \in I_k$ for $k = 0, 1, \dots, m-1$. Observe that if α is associated to a periodic orbit, then so is $S^k(\alpha)$ for every $k \in \mathbb{N}$ (by replacing x by $f^k(x)$ in the above definition).

Next we will define the notion of *negative* and *positive* loop in a Markov graph of f with respect to a Markov partition. Given a loop $\alpha = I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{m-1} \longrightarrow I_0$ we inductively define a sequence of intervals $\{U_k\}_{k=0}^m$ as follows. We start by setting $U_m = I_0$. Then, for every positive integer $k = m-1, m-2, \dots, 1, 0$, we take U_k to be the unique interval contained in I_k such that $f(\mathsf{Int}(U_k)) = \mathsf{Int}(U_{k+1})$ (such intervals exist due to the monotonicity of f at I_k and because I_k f-covers $I_{k+1 \,(\mathrm{mod}\ m)} \supset U_{k+1}$). The continuity of f implies that f sends ∂U_k bijectively to ∂U_{k+1} for every $k = 0, 1, \dots, m-1$. Denote the endpoints of U_0 as u_-^0 and u_+^0 (this is equivalent to define a linear ordering in I_0; namely the orientation that gives $u_-^0 < u_+^0$). Now we label the endpoints of I_0 as r_-^0 and r_+^0 in such a way that $\langle r_-^0, u_-^0 \rangle_{I_0} \cap U_0 = \{u_-^0\}$ and $\langle r_+^0, u_+^0 \rangle_{I_0} \cap U_0 = \{u_+^0\}$. Putting all together, f^m is monotone at U_0, $f^m(\mathsf{Int}(U_0)) = \mathsf{Int}(I_0)$ and f^m sends $\{u_-^0, u_+^0\}$ bijectively to $\{r_-^0, r_+^0\}$. There are two cases:

- $f^m(u_-^0) = r_-^0$ and $f^m(u_+^0) = r_+^0$, and we call the loop α *positive,* and
- $f^m(u_-^0) = r_+^0$ and $f^m(u_+^0) = r_-^0$ in which case we call the loop α *negative.*

Note that in the positive case f^m sends U_0 to I_0 preserving the linear ordering compatible with $u_-^0 < u_+^0$, while in the negative case f^m sends U_0 to I_0 by reversing this linear ordering.

Proposition 2.8 ([8, Lemma 1.2.12 and Theorem 2.6.4])**.** *Let X be a topological graph and let $f\colon X \longrightarrow X$ be a Markov map with respect to a Markov partition Q of X. Then, the following statements hold:*

(a) Assume that P is a periodic orbit of f disjoint from Q. Then, there exists a loop α in the Markov graph of f with respect to Q which is associated to P, and any other loop in the f-graph of Q associated to P is of the form $S^k(\alpha)$ with $k \in \mathbb{N}$.

(b) Let α be a loop from the Markov graph of f with respect to Q which has a periodic orbit associated to it. Then α is either a simple loop or a 2-repetition of a simple negative loop.

2.4. **Markov graphs modulo 1 for circle maps of degree one.** As we have seen, when the topological graph is the circle $\mathbb{S}^1$ and we are dealing with maps of degree one it is better to work with liftings instead of with the original maps, specially to avoid problems with ordering (the circle does not have a linear ordering). In what follows we adapt the definitions related to Markov graphs to this approach.

Let f be a continuous circle map and let $\widetilde{Q} \subset \mathbb{S}^1$ be a finite f-invariant set with at least two elements (in fact, since $V(\mathbb{S}^1) = \emptyset$, $\widetilde{Q}$ is a Markov invariant set). Let $F \in \mathcal{L}_1$ be a lifting of f. Then the set $Q = e^{-1}\big(\widetilde{Q}\big)$ is F-invariant, is a partition of $\mathbb{R}$ and each interval produced by this partition will be called a *Q-basic interval.* Again the set of all Q-basic intervals will be denoted by $\mathcal{B}(Q)$. If the restriction of F to each Q-basic interval is monotone (as a map from the real line to itself), we say that F is Q-monotone, Q is a *Markov partition with respect to F*, and F is a *Markov map.* Given $I, J \in \mathcal{B}(Q)$, we say I is equivalent to J and denote it by $I \sim J$ if $I = J + k$ for some $k \in \mathbb{Z}$. The equivalence class of I, $\{I + k : k \in \mathbb{Z}\}$, is denoted by $[\![I]\!]$.

Now we define the *Markov graph modulo 1 of F with respect to Q*. It is a combinatorial oriented graph whose vertices are all the equivalence classes of Q-basic intervals and there is an arrow $[\![I]\!] \longrightarrow [\![J]\!]$ from $[\![I]\!]$ to $[\![J]\!]$ if and only if there is a representative $J + k$ of $[\![J]\!]$ such that $F(I) \supset J + k$. Recall that, since $F \in \mathcal{L}_1$, $F(I + \ell) = F(I) + \ell$ for every $\ell \in \mathbb{Z}$. Therefore, the Markov graph modulo 1 of F with respect to Q is well defined. Moreover, two different liftings of f have the same Markov graphs modulo 1 with respect to Q.

Remark 2.9 (On the projection of a Markov graph modulo 1 to the circle). Since the kernel of e is $\mathbb{Z}$ (that is, $\mathsf{e}(x + k) = \mathsf{e}(x)$ for every $x \in \mathbb{R}$ and $k \in \mathbb{Z}$), it follows that $\mathsf{e}(I) = \mathsf{e}(K)$ for every $K \in [\![I]\!]$. So, $\mathsf{e}\big([\![I]\!]\big) := \mathsf{e}(I)$ is well defined. Moreover, since $\widetilde{Q}$ has at least two elements, it follows that every Q-basic interval has length strictly smaller than 1. Hence, if $I \in \mathcal{B}(Q)$, $\mathsf{e}\big([\![I]\!]\big) \in \mathcal{B}(\mathsf{e}(Q))$ (that is, $\mathsf{e}\big([\![I]\!]\big)$ is a $\mathsf{e}(Q)$-basic interval in $\mathbb{S}^1$ — recall that $\mathsf{e}(Q) = \widetilde{Q}$ is a Markov invariant set of $\mathbb{S}^1$ with respect to f). Additionally, since $f(\mathsf{e}(I)) = \mathsf{e}(F(I))$, $\mathsf{e}\big([\![I]\!]\big)$ f-covers $\mathsf{e}\big([\![J]\!]\big)$ if and only if there is an arrow $[\![I]\!] \longrightarrow [\![J]\!]$ in the Markov graph modulo 1 of F.

However, the Q-monotonicity of F implies the $\mathsf{e}(Q)$-monotonicity of f *provided that the length of the F-image of every Q-basic interval is smaller than 1* (otherwise there exists $I \in \mathcal{B}(Q)$ such that $f(\mathsf{e}(I)) = \mathsf{e}(F(I)) = \mathbb{S}^1$ is not an interval). In other words, $\widetilde{Q}$ is a Markov partition of $\mathbb{S}^1$ with respect to f (and f is a Markov map with respect to $\widetilde{Q}$) whenever Q is a Markov partition with respect to F and the length of the F-image of every Q-basic interval is smaller than 1. ◻

This remark motivates the following definition: Let $F \in \mathcal{L}_1$ be a lifting of f and let Q be a Markov partition with respect to F. A Q-basic interval will be called *F-short* if the length of the interval $F(I)$ is strictly smaller than 1. Then, Q will be called a *short Markov partition with respect to F* whenever every Q-basic interval is F-short.

With this definition Remark 2.9 immediately gives the following result that relates Markov partitions in the circle with short Markov partitions with respect to liftings from $\mathcal{L}_1$.

Proposition 2.10. *Let f be a continuous circle map and let $F \in \mathcal{L}_1$ be a lifting of f. Let $Q \subset \mathbb{R}$ be a Markov partition with respect to F. Then, the following statements hold:*

(a) $\mathsf{e}(Q)$ *is a Markov partition with respect to f if and only if Q is short.*
(b) When Q is short, the Markov graph of f with respect to $\mathsf{e}(Q)$ *and the Markov graph modulo 1 of F with respect to Q coincide, provided that we identify $[\![I]\!]$ with* $\mathsf{e}\big([\![I]\!]\big)$ *for every $I \in \mathcal{B}(Q)$.*

2.5. **Markov graphs and entropy.** The Markov graph of a map is very useful to obtain information about the dynamics of graph maps. The next result, due to Block, Guckenheimer, Misiurewicz and Young [11] (see also [8, Theorem 4.4.5]), relates the topological entropy of a Markov map with the spectral radius of its associated Markov matrix. We recall that the *spectral* *radius* of a matrix M is the maximum of the moduli of all the eigenvalues of M and it will be denoted here by $\sigma(M)$.

Proposition 2.11. *Let X be a topological graph, let $\phi\colon X \longrightarrow X$ be a Markov map with respect to a Markov partition Q of X and let M be a Markov matrix of ϕ with respect to Q. Then,*

$$h(\phi) = \log \max\{\sigma(M), 1\}.$$

To compute the spectral radius of a Markov matrix we will use the *rome method* proposed in [11] (see also [8]). To this purpose we will introduce some notation.

Let $M = (m_{ij})$ be a $k \times k$ Markov matrix. Given a sequence $p = (p_j)_{j=0}^{\ell(p)}$ of elements of $\{1, 2, \dots, k\}$ we define the *width of* p, denoted by $w(p)$, as the number $\prod_{j=1}^{\ell(p)} m_{p_{j-1}p_j}$. If $w(p) \neq 0$ then p is called a *path of length* $\ell(p)$. A *loop* is a path such that $p_{\ell(p)} = p_0$ i.e. that begins and ends at the same index. The words "path" and "loop" in this setting are inherited from the analogous notions in the Markov graph.

A subset R of $\{1, 2, \dots, k\}$ is called a *rome* if there is no loop outside R, i.e. there is no path $(p_j)_{j=0}^{\ell}$ such that $p_\ell = p_0$ and $\{p_j : 0 \leq j \leq \ell\}$ is disjoint from R. For a rome R, a path $(p_j)_{j=0}^{\ell}$ is called *simple* if $p_i \in R$ for $i = 0, \ell$ and $p_i \notin R$ for $i = 1, 2, \dots, \ell - 1$. Of course, we can define a rome using the vertices in the Markov graph associated with the matrix instead of the matrix itself.

If $R = \{r_1, r_2, \dots, r_m\}$ is a rome of a matrix M then we define an $m \times m$ matrix $M_R(x)$ whose entries are real functions by setting $M_R(x) := (a_{ij}(x))$, where $a_{ij}(x) := \sum_p w(p) \cdot x^{-\ell(p)}$, where the sum is taken over all simple paths starting at r_i and ending at r_j.

Theorem 2.12 ([11, Theorem 1.7])**.** *Let $\mathbf{I}_m$ be the identity matrix of size $m \times m$. If R is a rome of cardinality m of a $k \times k$ matrix M then the characteristic polynomial of M is equal to*

$$(-1)^{k-m} x^k \det(M_R(x) - \mathbf{I}_m).$$

2.6. **Transitivity and Markov matrices.** The aim of this subsection is to establish and prove the following result:

Theorem 2.13. *Let X be a topological graph and let $f\colon X \longrightarrow X$ be a Q-expansive Markov map with respect to a Markov partition Q of X. Then f is transitive if and only if the Markov matrix of f with respect to Q is irreducible but not a permutation matrix.*

This result is well known when X is a closed interval of the real line and the map is piecewise affine (see [10, Theorem 3.1]) but we aim at extending it to the general setting of graphs. Its proof in this more general case goes along the lines of the one from [10] for the interval but we will sketch it here for completeness.

In any case we need to recall the definition of irreducibility and establish what we understand by a piecewise expansive graph map.

A $k \times k$ matrix M is called *reducible* if there exists a permutation matrix P such that

$$P^T M P = \left(\begin{array}{c|c} M_{11} & \mathbf{0} \\ \hline M_{21} & M_{22} \end{array}\right)$$

where M_{11}, M_{21} and M_{22} are block matrices, and $\mathbf{0}$ is a block matrix whose entries are all 0. A matrix $P = (p_{ij})_{i,j=1}^k$ is a permutation matrix whenever $p_{ij} \in \{0, 1\}$ for all $0 \le i, j \le k$ and in each row and in each column there is exactly one non-zero element. Observe that if P is a permutation matrix, then $P^{-1} = P^T$.

The matrix M is called *irreducible* if it is not reducible or, equivalently (see [14]), if for every $0 \le i, j \le k$ there is a natural number $n = n(i, j)$ such that the i, j-entry of M^n is strictly positive. In the case of a Markov matrix of a Markov partition of X, if we set $M^n = (m_{ij}^{(n)})_{i,j=1}^k$, then $m_{ij}^{(n)}$ is the number of paths of length n in the Markov graph starting at the vertex v_i and ending at the vertex v_j. In this context, M is irreducible if and only if there exists a path from the vertex v_i to the vertex v_j for every $0 \le i, j \le k$. In particular $f(I)$ is a (non-degenerate) interval for every basic interval I.

To define the notion of Q-expansive graph map we need to define a distance on basic intervals of graphs.

Let X be a topological graph and let $Q \subset X$ be a finite set such that $Q \supset V(X)$ and the closure of every connected component of $X \setminus Q$, called a *Q-basic interval*, is an interval of X (formally the notion of Q-basic interval is only defined for Markov partitions of graph maps but we use it here by abusing of notation for simplicity). Every Q-basic interval I can be endowed in many ways with a distance d_I verifying that the *length of* I, defined as $\max_{x,y \in I} d_I(x, y)$, is 1. For instance, we can fix a homeomorphism $\mu_I \colon I \longrightarrow [0, 1]$ and set $d_I(x, y) := |\mu_I(x) - \mu_I(y)|$ for every $x, y \in I$. Given a connected set $W \subset I$ we define the *length of* W by

$$\|W\|_I := \max \{d_I(x, y) : x, y \in W\}.$$

From above it follows that $\|I\|_I = 1$ and $\|W\|_I \le 1$.

Now we are ready to define the notion of:

Definition 2.14 (piecewise expansiveness). Let X be a topological graph and let $f \colon X \longrightarrow X$ be a Markov map with respect to a Markov invariant set Q.

We say that f is *expansive on* I if $f(I)$ is not a point and

- **when $f(I) \in \mathcal{B}(Q)$:** f verifies
$$d_{f(I)}(f(x), f(y)) = \lambda_I d_I(x, y) = d_I(x, y)$$
with $\lambda_I = 1$ for every $x, y \in I$;
- **when $f(I)$ contains more than one Q-basic interval:** there exists $\lambda_I > 1$ such that
$$d_J(f(x), f(y)) \ge \lambda_I d_I(x, y)$$
for every $x, y \in I$ such that $\langle f(x), f(y) \rangle_{f(I)} \subset J \in \mathcal{B}(Q)$.

Observe that when f is expansive on I then $f\big|_I$ is one-to-one.

We say that f is *Q-expansive* if it is expansive on every Q-basic interval. ◾

Proof of Theorem 2.13. First we will perform the simple exercise of proving that if f is transitive then the Markov matrix of f with respect to Q is irreducible but not a permutation matrix.

First we assume that the Markov matrix of f with respect to Q is a permutation matrix. This is equivalent to say that we can label the set of all Q-basic intervals as $I_0, I_1, \dots, I_{m-1}$ so that $f(I_i) = I_{i+1 \pmod m}$, and $f\big|_{I_i}$ is monotone for every $i = 0, 1, \dots, m-1$. In these conditions f cannot have a dense orbit and thus it cannot be transitive.

On the other hand, by transitivity, the image of every Q-basic interval is different from a point (otherwise, again, we get a contradiction with the existence of a dense f-orbit). Consequently, since f is Markov with respect to a Markov partition Q (in particular Q is f-invariant), it follows that for every $I \in \mathcal{B}(Q)$, $f(I)$ is an interval which is a union of Q-basic intervals and $f\big|_I$ is monotone. It follows inductively that $f^k(I)$ is a union of Q-basic intervals for every $k \geq 1$.

Now we choose two arbitrary intervals $I, J \in \mathcal{B}(Q)$. Since f is transitive there exists a positive integer n such that

$$f^n(I) \cap \mathsf{Int}(J) \supset f^n(\mathsf{Int}(I)) \cap \mathsf{Int}(J) \neq \emptyset.$$

Since $f^n(I)$ is a union of Q-basic intervals and two different Q-basic intervals have disjoint interiors, $f^n(I) \supset J$. This means that there exists a Q-basic interval $J_{n-1} \subset f^{n-1}(I)$ such that J_{n-1} f-covers J. Analogously, there exists a Q-basic interval $J_{n-2} \subset f^{n-2}(I)$ such that J_{n-2} f-covers J_{n-1}. Iterating this argument we obtain a path $I \longrightarrow J_1 \longrightarrow J_2 \longrightarrow \cdots \longrightarrow J_{n-1} \longrightarrow J$ from I to J in the Markov graph of f with respect to Q. Consequently, the Markov matrix of f with respect to Q is irreducible. This ends the proof of the "only if part" of the theorem.

Now we prove the "if part" following the ideas of [10]. So, we assume that the Markov matrix of f with respect to Q is irreducible but not a permutation matrix. The fact that the Markov matrix of f with respect to Q is not a permutation matrix tells us that

$$\{I : I \in \mathcal{B}(Q) \text{ and } I \text{ } f\text{-covers at least two basic intervals}\} \neq \emptyset.$$

Hence,

$$\lambda_f := \min\{\lambda_I : I \in \mathcal{B}(Q) \text{ and } I \text{ } f\text{-covers at least two basic intervals}\} > 1.$$

We claim that $U \cap \left(\cup_{k\geq 0} f^{-k}(Q)\right) \neq \emptyset$ for every connected non-empty open set $U \subset X$. To prove the claim assume by way of contradiction that $f^k(U) \cap Q = \emptyset$ for every $k \geq 0$. Then, there exists a sequence of Q-basic intervals $\{J_k\}_{k=0}^{\infty}$ such that $U \subset \mathsf{Int}(J_0)$ and $f^k(U) \subset \mathsf{Int}(J_k) \subset f(J_{k-1})$ for every $k \geq 1$. Moreover, from Definition 2.14 we have

$$\begin{aligned} \left\|f^{k+1}(U)\right\|_{J_{k+1}} &\geq \lambda_{J_k} \left\|f^k(U)\right\|_{J_k} \geq \lambda_{J_k}\lambda_{J_{k-1}} \left\|f^{k-1}(U)\right\|_{J_{k-1}} \geq \cdots \\ &\geq \|U\|_{J_0} \prod_{i=0}^{k} \lambda_{J_i} \end{aligned} \tag{2.1}$$

for every $k \geq 0$.

Assume that $\lambda_{J_i} \geq \lambda_f > 1$ for infinitely many indices i. Then, since $\lambda_I \geq 1$ for every $I \in \mathcal{B}(Q)$, the sequence

$$\left\{ \prod_{i=0}^{k} \lambda_{J_i} \right\}_{k=0}^{\infty}$$

is non-decreasing, and hence

$$\lim_{k\to\infty} \left\| f^{k+1}(U) \right\|_{J_{k+1}} \geq \|U\|_{J_0} \lim_{k\to\infty} \prod_{i=0}^{k} \lambda_{J_i} = \infty.$$

This is a contradiction because, for every $k \geq 0$, J_{k+1} is a Q-basic interval and $f^{k+1}(U) \subset J_{k+1} \subset f(J_k)$; which implies $\left\| f^{k+1}(U) \right\|_{J_{k+1}} \leq \left\| J_{k+1} \right\|_{J_{k+1}} = 1$.

From the part of the claim already proven, there exists $m \in \mathbb{N}$ such that $\lambda_{J_k} = 1$ (that is, $f(J_k) \in \mathcal{B}(Q)$) for every $k \geq m$. Thus, $f(J_k) = J_{k+1}$ for every $k \geq m$ because $J_{k+1} \subset f(J_k)$. Since the number of Q-basic intervals is finite, there exist $\ell \geq m$ and $t \geq 1$ such that $J_\ell = J_{\ell+t}$.

We already know that there exists a basic interval $I \in \mathcal{B}(Q)$ that f-covers at least two basic intervals. So, $I \notin \{J_\ell, J_{\ell+1}, \dots, J_{\ell+t-1}\}$, and in the Markov graph of f with respect to Q there does not exist any path starting in a Q-basic interval from $\{J_\ell, J_{\ell+1}, \dots, J_{\ell+t-1}\}$ and ending at I. This contradicts the irreducibility of the Markov matrix of f with respect to Q and ends the proof of the claim.

Since for every non-empty open set V there exists $I \in \mathcal{B}(Q)$ such that $V \cap \mathsf{Int}(I) \neq \emptyset$, to prove that f is transitive it is enough to show that for every non-empty open set $U \subset X$ and every $I \in \mathcal{B}(Q)$ there exists a positive integer n such that $f^n(U) \supset \mathsf{Int}(I)$.

Let $J \in \mathcal{B}(Q)$ be such that $U \cap \mathsf{Int}(J) \neq \emptyset$. By the above claim with U replaced by a connected component of $U \cap \mathsf{Int}(J)$, it follows that there exists $x \in (U \cap \mathsf{Int}(J)) \cap \left(\cup_{k\geq 0} f^{-k}(Q)\right)$. So, again by the claim for a connected component of $(U \cap \mathsf{Int}(J)) \setminus \{x\}$ instead of U, we obtain that

$$\mathsf{Card}\left((U \cap \mathsf{Int}(J)) \cap \left(\cup_{k\geq 0} f^{-k}(Q) \right) \right) \geq 2.$$

Therefore, there exist $x, y \in U \cap \mathsf{Int}(J)$ with $x \neq y$ and $k_x, k_y \in \mathbb{N}$ such that $\langle x, y \rangle_J \subset U \cap \mathsf{Int}(J)$, $1 \leq k_x \leq k_y$, $f^{k_x}(x), f^{k_y}(y) \in Q$, and, concerning the preimages of Q, $(U \cap \mathsf{Int}(J)) \cap f^{-k}(Q) = \emptyset$ for $k = 0, 1, \dots, k_x - 1$ and $\left((U \cap \mathsf{Int}(J)) \setminus \{x\}\right) \cap f^{-k}(Q) = \emptyset$ for $k = k_x, k_x + 1, \dots, k_y - 1$.

Consequently, as in the proof of the above claim and using the fact that f is Q-monotone, it follows inductively that there exist Q-basic intervals $J_0 = J, J_1, \dots, J_{k_y - 1}$ such that

$$\langle x, y \rangle_{J_0} \subset U \cap \mathsf{Int}(J_0),$$
$$f^k\left(\langle x, y \rangle_{J_0}\right) = \langle f^k(x), f^k(y) \rangle_{J_k} \subset \mathsf{Int}(J_k) \text{ for } k = 1, 2, \dots, k_x - 1 \text{ and}$$
$$f^k\left(\langle x, y \rangle_{J_0} \setminus \{x\}\right) = \langle f^k(x), f^k(y) \rangle_{J_k} \setminus \{f^k(x)\} \subset \mathsf{Int}(J_k) \quad \text{for } k = k_x, k_x + 1, \dots, k_y - 1$$

(recall that $f(Q) \subset Q$ and, hence, $f^k(x) \in Q$ for every $k \geq k_x$). Moreover,

$$f^{k_y}\left(\langle x, y\rangle_{J_0}\right) = \langle f^{k_y}(x), f^{k_y}(y)\rangle_{f\left(J_{k_y-1}\right)} \subset f\left(J_{k_y-1}\right)$$

with $f^{k_y}(x), f^{k_y}(y) \in Q$. On the other hand, from above it follows that $f^k(x), f^k(y) \in J_k$ for $k = 0, 1, \dots, k_y - 1$, and from Definition 2.14, $f\big|_{J_k}$ is one-to-one. Hence, $f^k(x) \neq f^k(y)$ for $k = 0, 1, \dots, k_y$, and consequently there exists $J_{k_y} \in \mathcal{B}(Q)$ such that

$$J_{k_y} \subset \langle f^{k_y}(x), f^{k_y}(y)\rangle_{f\left(J_{k_y-1}\right)} \quad \text{and} \quad f^{k_y}\left(\langle x, y\rangle_{J_0}\right) \supset J_{k_y}.$$

Since the Markov matrix of f with respect to Q is irreducible there exists a path of length $r \geq 0$ from J_{k_y} to I in the Markov graph of f with respect to Q. Then, from the definition of path and f-covering it follows that $f^r\left(J_{k_y}\right) \supset I$. Consequently,

$$f^{r+k_y}(U) \supset f^{r+k_y}\left(\langle x, y\rangle_{J_0}\right) \supset f^r\left(J_{k_y}\right) \supset I.$$

This ends the proof of the proposition. □

3. Examples

This section is devoted to construct the Examples 1.7, 1.9 and 1.11. For clarity, each example will be dealt in a subsection. Moreover, we will start with to additional subsections: the first one being introductory to explain the philosophy of the constructions that we make, while the second one is devoted to introduce a couple of specific technical auxiliary results.

3.1. **Philosophy and introduction: on how to explicitly specify the families of circle maps in the examples.** We have to construct examples of totally transitive graph maps with arbitrarily small topological entropy and an arbitrarily large boundary of cofiniteness.

To do this we start with circle maps that verify these properties and, with the help of the result from the next subsection, we extend these circle examples to any graph that is not a tree while keeping the stated properties.

A natural idea to define these circle maps is: *start by fixing a family of non-degenerate intervals, and for each of them take the circle map of degree one with the prescribed interval as rotation interval and having minimum entropy among all maps with these properties.* Indeed, *the appropriate choice of the rotation interval* and the fact that we take minimum entropy maps should guarantee that that the boundary of cofiniteness goes to infinity (because the rotation interval for these maps completely determines the set of periods), that the entropies converge to zero and, finally, the total transitivity should be inherited from the non-degeneracy of the rotation interval.

Les us see with more detail how this can be achieved. In particular this will explain the "mysterious" assumptions in the examples.

In this discussion and survey on minimum entropy maps depending on the rotation interval we will follow the approach from [7, 8].

For $c, d \in \mathbb{R}$, $c < d$ and $z > 1$ we define

$$R_{c,d}(z) := \sum_{q \in M(c,d)} z^{-q}.$$

Then, one can show that $R_{c,d}(z) = \frac{1}{2}$ has a unique solution $\beta_{c,d}$ and $\beta_{c,d} > 1$.

The following result is what makes possible the strategy proposed above.

Theorem 3.1. *Let f be a circle map of degree 1 with rotation interval $[c, d]$ with $c < d$. Then $h(f) \geq \log \beta_{c,d}$. Moreover, for every $c, d \in \mathbb{R}$, $c < d$ there exists a circle map of degree 1, $f_{c,d}$, having rotation interval $[c, d]$ and entropy $h(f_{c,d}) = \log \beta_{c,d}$.*

From the proof of this theorem it follows that the circle map $f_{c,d}$ has as a lifting (see Figure 2 for an example with $c = \frac{1}{2}$ and $d = \frac{7}{10}$)

$$G_{c,d}(x) := \begin{cases} \beta_{c,d} x + b & \text{if } 0 \leq x \leq u, \\ \beta_{c,d}(1 - x) + b + 1 & \text{if } u \leq x \leq 1, \\ G_{c,d}(x - \lfloor x \rfloor) + \lfloor x \rfloor & \text{if } x \notin [0, 1], \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the integer part function,

$$b := (\beta_{c,d} - 1)^2 \sum_{n=1}^{\infty} \lfloor nc \rfloor \beta_{c,d}^{-n-1},$$

and the continuity of $G_{c,d}$ gives $u := \frac{\beta_{c,d}+1}{2\beta_{c,d}}$.

Remark 3.2. For $c \in \mathbb{R}$, $c > 0$, and $z > 1$ we define

$$T_c(z) := \sum_{n=0}^{\infty} z^{-\lfloor \frac{n}{c} \rfloor},$$

and, for definiteness, we set $T_0(z) \equiv 0$. Then, for $c, d \in \mathbb{R}$, $c < d$, $c \in [0, 1)$, and $z > 1$ we define

$$Q_{c,d}(z) := z + 1 + 2\left(\frac{z}{z-1} - T_{1-c}(z) - T_d(z)\right)$$

(observe that if $[c, d]$ is a rotation interval then, by replacing the lifting F used to compute the rotation interval by $F - \lfloor c \rfloor$, we get the new rotation interval $\left[c - \lfloor c \rfloor, d - \lfloor c \rfloor\right]$ with $c - \lfloor c \rfloor \in [0, 1)$; this is the reason that the assumption $c \in [0, 1)$ above is not restrictive).

Concerning the map $Q_{c,d}$ one can show that

$$Q_{c,d}(z) = (z - 1)\left(1 - 2R_{c,d}(z)\right).$$

Hence, $\beta_{c,d}$ is the largest root of the equation $Q_{c,d}(z) = 0$. This observation gives a much easier way of calculating the numbers $\beta_{c,d}$. ◻

Remark 3.3 (On when $\lim \boldsymbol{\beta_{c,d}} = 1$). The numbers $\beta_{c,d}$ have the following important properties:

- If $c \leq a < b \leq d$ and $\{a, b\} \neq \{c, d\}$, then $\beta_{c,d} > \beta_{a,b}$. This implies that if we have a decreasing sequence of intervals $\{[c_n, d_n]\}_{n=0}^{\infty}$ whose lengths do not converge to 0, then the sequence $\{\beta_{c_n, d_n}\}_{n=0}^{\infty}$ is bounded away from 1.

- Assume that $c < \frac{p}{q} < d$ with p and q coprime. Then, $\beta_{c,d} > 3^{\frac{1}{q}}$. This implies that given a sequence of intervals $\{[c_n, d_n]\}_{n=0}^{\infty}$ for which there exist M such that $\min M(c_n, d_n) \leq M$ for every n (for instance $\left[\frac{n-1}{2n}, \frac{n+1}{2n}\right] = \left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\right]$), then $\beta_{c_n,d_n} > 3^{\frac{1}{M}}$ and, hence, the sequence $\{\beta_{c_n,d_n}\}_{n=0}^{\infty}$ is bounded away from 1.

Summarizing, if we want to achieve $\lim_{n\to\infty} \beta_{c_n,d_n} = 1$ for a given sequence of intervals $\{[c_n, d_n]\}_{n=0}^{\infty}$, then we need (as necessary but not sufficient conditions) that the lengths $d_n - c_n$ of the intervals converge to zero, and $\{\min M(c_n, d_n)\}_{n=0}^{\infty}$ is unbounded. ◻

Taking all the above comments and results into account, Theorem 3.1 gives the following procedure to build our examples:

- ► Choose, a sequence of closed intervals of the real line $\{[c_n, d_n]\}_{n=0}^{\infty}$ with rational endpoints such that:
  - $\lim_{n\to\infty} \beta_{c_n,d_n} = 1$ (see Remark 3.3), and
  - the boundary of cofiniteness of $M(c_n, d_n)$ (defined as the boundary of cofiniteness of a map f but replacing $\mathsf{Per}(f)$ by $M(c_n, d_n)$) exists and goes to infinity with n.
- ► Compute the numbers β_{c_n,d_n}, b and u defined above to get the map f_{c_n,d_n} determined. Observe that we automatically have

$$\lim_{n\to\infty} h\big(f_{c_n,d_n}\big) = \lim_{n\to\infty} \log \beta_{c_n,d_n} = \log \lim_{n\to\infty} \beta_{c_n,d_n} = 0.$$

- ► Check that $\mathsf{Per}(f_{c_n,d_n}) = M(c_n, d_n)$ and compute this set to get

$$\lim_{n\to\infty} \mathsf{BdCof}\left(f_{c_n,d_n}\right) = \lim_{n\to\infty} \mathsf{BdCof}\big(M(c_n, d_n)\big) = \infty.$$

- ► Show that the map f_{c_n,d_n} is totally transitive for every n, if that is the case.

This method, while giving an effective procedure to construct the sequences of maps that we are looking for, has two serious drawbacks: First, in this setting it is very difficult to show that the map f_{c_n,d_n} is totally transitive and to compute $\mathsf{Per}\big(f_{c_n,d_n}\big)$ and β_{c_n,d_n} for every n (essentially, we only can do it numerically). Second, we cannot extend these models to graphs in such a way that we can also extend the study of the transitivity, $\mathsf{Per}\big(f_{c_n,d_n}\big)$ and $h\big(f_{c_n,d_n}\big)$ from the circle (indeed, for graphs there is no analogous to the theorem stating that the topological entropy of a piecewise linear circle map such that the absolute value of its slopes is constant, is precisely the logarithm of this number).

To solve all these problems it is much better to find a Markov partition for every map f_{c_n,d_n} and use it to prove that it is totally transitive and to compute $\mathsf{Per}\big(f_{c_n,d_n}\big)$ and β_{c_n,d_n}. Moreover, this also gives a good tool to extend the circle models to graphs while keeping the transitivity, the sets of periods, and the fact that $\left\{h\big(f_{c_n,d_n}\big)\right\}_{n=0}^{\infty}$ converges to zero (despite of the fact these entropies increase a little bit).

To find these Markov partitions we note that the maps $G_{c_n,d_n}\big|_{[0,1]}$ are bimodal, and monotone on the intervals $[0, u]$ and $[u, 1]$. So, every Markov

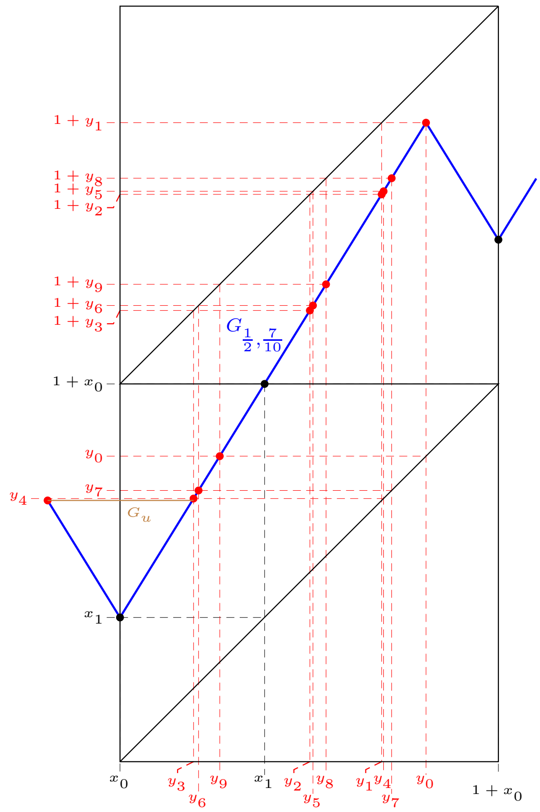


FIGURE 2. The graph of the map $G_{\frac{1}{2},\frac{7}{10}}$ used in Example 1.9 as a model for the map F_5. To better show the dynamics of the orbits, in this picture they are labelled so that $G_{\frac{1}{2},\frac{7}{10}}(x_0) = x_1$, $G_{\frac{1}{2},\frac{7}{10}}(x_1) = x_0 + 1$, $G_{\frac{1}{2},\frac{7}{10}}(y_i) = 1 + y_{i+1}$ for $i \in \{0, 1, 2, 4, 5, 7, 8\}$, $G_{\frac{1}{2},\frac{7}{10}}(y_i) = y_{i+1}$ for $i \in \{3, 6\}$ and $G_{\frac{1}{2},\frac{7}{10}}(y_9) = y_0$ (observe that, for the points y_i, this labelling is different than the linearly ordered one introduced below).

partition must include $\{0, u\}$. Moreover, the existence of such a partition depends on the finiteness of the orbits of $\mathsf{e}(0)$ and $\mathsf{e}(u)$ (on the circle). It follows that if $c_n = \frac{p_n}{q_n}$ with p_n and q_n relatively prime (respectively, $d_n = \frac{r_n}{s_n}$ with r_n and s_n relatively prime), then $\mathsf{e}(0)$ (respectively $\mathsf{e}(u)$) is a periodic point of f_{c_n,d_n} of period q_n (respectively s_n) and $Q_n = \mathsf{e}^{-1}\Big(\mathsf{Orb}_{f_{c_n,d_n}}(\mathsf{e}(0))\Big)$ (respectively $S_n = \mathsf{e}^{-1}\Big(\mathsf{Orb}_{f_{c_n,d_n}}(\mathsf{e}(u))\Big)$) is a twist lifted periodic orbit with rotation number $\frac{p_n}{q_n}$ (respectively $\frac{r_n}{s_n}$). Hence, $Q_n \cup S_n \supset \{0, u\}$ is a Markov partition with respect G_{c_n,d_n} (see Figure 2). Notice also that

$$\mathsf{Card}\,(Q_n \cap [0,1]) = \mathsf{Card}\,(Q_n \cap [0,u]) = q_n,$$
$$\mathsf{Card}\,(S_n \cap [0,1]) = \mathsf{Card}\,(S_n \cap [0,u]) = s_n$$

and $G_{c_n,d_n}\big|_{Q_n}$ (respectively $G_{c_n,d_n}\big|_{S_n}$) is determined by Remark 2.3 and the numbers p_n and q_n (respectively r_n and s_n). So, to determine the Markov partition $Q_n \cup S_n$ and $G_{c_n,d_n}\big|_{Q_n \cup S_n}$ it is enough to determine the relative positions of the points from $Q_n \cap [0,u]$ and $S_n \cap [0,u]$. It turns out that if the endpoints of the rotation interval are appropriately chosen, then there is a formula for the relative positions of the points from $Q_n \cap [0,u]$ and $S_n \cap [0,u]$ that depends solely on n. For instance (to fix ideas), in Example 1.9 we consider the family of liftings with rotation interval $\left[\frac{1}{2}, \frac{n+2}{2n}\right]$ and minimum entropy relative to the rotation interval. Then, $\mathsf{Card}\,(Q_n \cap [0,u]) = 2$ and Q_n is a twist lifted periodic orbit with rotation number $\frac{1}{2}$, $\mathsf{Card}\,(S_n \cap [0,u]) = 2n$ and S_n is a twist lifted periodic orbit with rotation number $\frac{n+2}{2n}$. Without loss of generality we can write $Q_n \cap [0,u] = \{x_0, x_1\}$ with $x_0 = 0 < x_1$ and $S_n \cap [0,u] = \{y_0, y_1, \dots, y_{2n-1}\}$ with $y_0 < y_1 < \dots < y_{2n-1} = u$. Then, by explicitly computing some particular examples of this family (as it is done in Figure 2 for $n = 5$), one can see that that the $2n+2$ points of $(Q_n \cup S_n) \cap [0,u]$ are organized in the following way:

$$x_0 = 0 < y_0 < y_1 < \dots < y_{n-3} < x_1 < y_{n-2} < y_{n-1} < \dots < y_{2n-1} = u$$

(notice that the number $n-3$ in the above formula is, indeed, the number $s_n - r_n - 1 = 2n - (n+2) - 1$).

Summarizing, the numbers p_n, q_n, r_n and s_n and the relative positions of the points of the lifted periodic orbits with rotation numbers $\frac{p_n}{q_n}$ and $\frac{r_n}{s_n}$ in $[0,u]$ specify in a totally explicit way the Markov partitions of the functions f_{c_n,d_n} (and hence, by linearity, the maps f_{c_n,d_n} themselves) in a way that easily give the totally transitive of these maps, $\mathsf{Per}(f_{c_n,d_n})$ and β_{c_n,d_n}. Moreover, we can easily extend the circle models to graphs while keeping the transitivity, the sets of periods and the fact that $h(f_{c_n,d_n})$ still converges to zero.

3.2. **Two technical auxiliary results.** The following lemma is analogous to [2, Lemma 3.6] with the additional assumption that the number of elements of the partition must be even. It will be our main tool to translate the examples from $\mathbb{S}^1$ to any graph that is not a tree (see Figure 3). Due to the additional assumption about the parity of the number of elements of the partition we will include the proof for completeness.

Lemma 3.4. *Let X be a topological graph which is not an interval and let $a, b \in V(X)$ be two different endpoints of X. Then, there exist a partition of the interval $[0,1]$, $0 = s_0 < s_1 < \dots < s_m = 1$, with $m = m(X,a,b) \geq 5$ odd, and two continuous surjective maps $\varphi_{a,b}\colon [0,1] \longrightarrow X$ and $\psi_{a,b}\colon X \longrightarrow [0,1]$ such that the following statements hold:*

(a) $\varphi_{a,b}^{-1}(W) = \{s_i : i \in \{0,1,\dots,m\}\}$, *where*

$$W := \varphi_{a,b}\big(\{s_i : i \in \{0,1,\dots,m\}\}\big) \supset V(X),$$

and $\varphi_{a,b}(0) = a$ *and* $\varphi_{a,b}(1) = b$.

(b) For every $i = 0,1,\dots,m-1$, $\varphi_{a,b}\big|_{[s_i,s_{i+1}]}$ *is injective and* $\varphi_{a,b}\big([s_i,s_{i+1}]\big)$ *is an interval which is the closure of a connected component of the punctured graph* $X \setminus W$.

(c) If $\varphi_{a,b}(s_i) = \varphi_{a,b}(s_j)$ *then* $i \equiv j \pmod 2$.

(d) $\psi_{a,b}(\varphi_{a,b}(s_i)) = 0$ *if i is even and* $\psi_{a,b}(\varphi_{a,b}(s_i)) = 1$ *if i is odd (in particular,* $\psi_{a,b}(a) = 0$ *and* $\psi_{a,b}(b) = 1$*).*

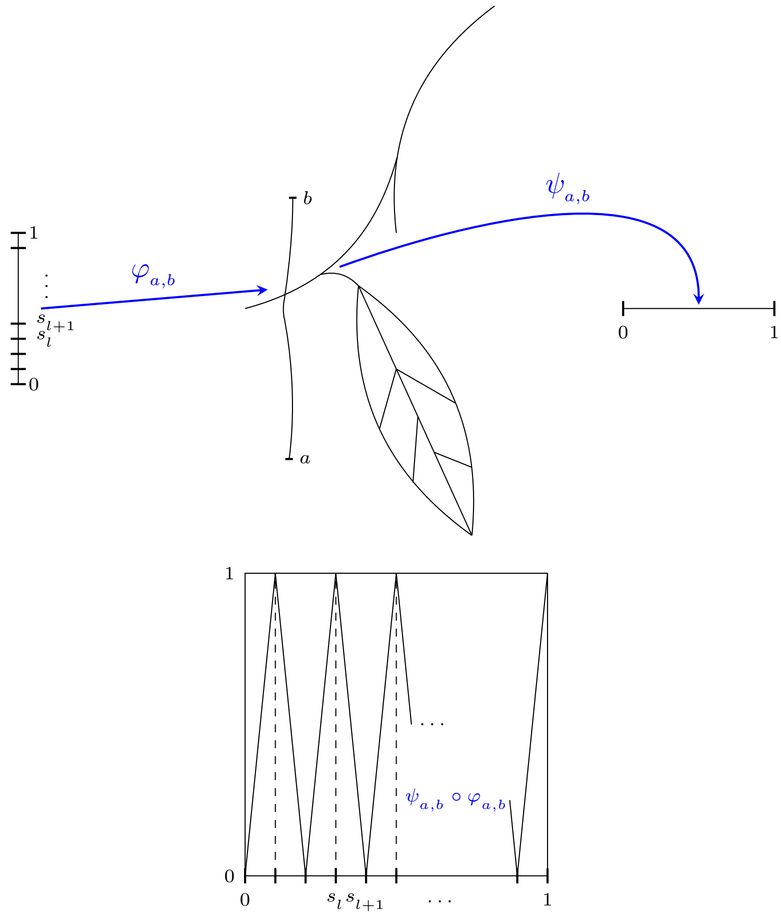


FIGURE 3. A sketch of a topological graph X and the maps from Lemma 3.4 (top picture). The map $\psi_{a,b} \circ \varphi_{a,b}$ is shown in the bottom picture.

(e) *The map* $\psi_{a,b}\big|_{\varphi_{a,b}([s_i,s_{i+1}])}$ *is injective and* $\psi_{a,b}\big(\varphi_{a,b}([s_i, s_{i+1}])\big) = [0,1]$ *for every* $i = 0, 1, \dots, m-1$. *In particular, the map* $\big(\psi_{a,b} \circ \varphi_{a,b}\big)\big|_{[s_i,s_{i+1}]}$ *is strictly monotone.*

In what follows the closure of a set $A \subset X$ will be denoted by $\mathsf{Clos}(A)$.

Proof. The existence of a surjective map $\psi_{a,b}$ which satisfies (d–e) follows easily from the existence of the partition $0 = s_0 < s_1 < \cdots < s_m = 1$ and a map $\varphi_{a,b}$ which satisfy statements (a–c) of the lemma. In particular, (d) can be guaranteed by using (c), and (e) by (b) and (d). So, we only have to show that there exist a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ with $m \geq 5$ odd, and a continuous surjective map $\varphi_{a,b}$ such that (a–c) hold.

Since X is not an interval and a and b are endpoints of X, there exist $v \in V(X) \setminus \{a, b\}$ and an interval $J \subset X$ with endpoints v and b such that $J \cap V(X) = \{v, b\}$.

Let C be an edge of X (i.e. a connected component of $X \setminus V(X)$). Clearly, $\mathsf{Clos}(C)$ is either an interval or a circuit which contains a unique vertex of

X. For every C such that $\mathsf{Clos}(C)$ is a circuit we choose a point $v_C \in C$ (that will play the role of an artificial vertex). Then we set

$$\widetilde{V}(X) := V(X) \cup \{v_C : C \text{ is an edge of } X \text{ such that } \mathsf{Clos}(C) \text{ is a circuit}\}.$$

Observe that the closure of every connected component of $X \setminus \widetilde{V}(X)$ is an interval and $J \cap \widetilde{V}(X) = J \cap V(X) = \{v, b\}$. Moreover, since $\{a, b, v\} \subset V(X) \subset \widetilde{V}(X)$ and a and b are endpoints of X, $X \setminus \widetilde{V}(X)$ has at least three connected components and 4 vertices. So, $\mathsf{Card}\bigl(\widetilde{V}(X)\bigr) \geq 4$.

Since a topological graph is path connected there exists $\varphi_{a,b} \colon [0,1] \longrightarrow X$, a path from a to b, which is continuous and onto (i.e., visits each point from X going several times trough the same edge, if necessary) and a partition of the interval $[0, 1]$, $0 = s^*_0 < s^*_1 < \cdots < s^*_n = 1$, such that the following statements hold:

(i) $\left\{s^*_j : j \in \{0, 1, \ldots, n\}\right\} := \varphi^{-1}_{a,b}\bigl(\widetilde{V}(X)\bigr)$ with $\varphi_{a,b}(s^*_0) = \varphi_{a,b}(0) = a$, $\varphi_{a,b}\bigl(s^*_{n-1}\bigr) = v$ and $\varphi_{a,b}(s^*_n) = \varphi_{a,b}(1) = b$,

(ii) for every $j = 0, 1, \ldots, n-1$, $\varphi_{a,b}\bigr|_{\left[s^*_j, s^*_{j+1}\right]}$ is injective and, hence, $\varphi_{a,b}\bigl(\bigl[s^*_j, s^*_{j+1}\bigr]\bigr)$ is an interval which is the closure of a connected component of $X \setminus \widetilde{V}(X)$,

(iii) $\varphi^{-1}_{a,b}(b) = s^*_n$ and $\varphi^{-1}_{a,b}\bigl(J \setminus \{v\}\bigr) = \bigl(s^*_{n-1}, s^*_n\bigr]$.

Let $\mathcal{E}$ be the set of all connected components of $X \setminus \widetilde{V}(X)$ which are different from $J \setminus \{b, v\}$. Then, for every $C \in \mathcal{E}$ we choose an arbitrary but fixed point $\alpha_C \in C$. By (i) and (iii),

$$\varphi^{-1}_{a,b}\bigl(\{\alpha_C : C \in \mathcal{E}\}\bigr) \subset \bigl(s^*_0, s^*_{n-1}\bigr) \setminus \left\{s^*_j : j \in \{1, \ldots, n-2\}\right\}.$$

We claim that for every $j = 0, 1, \ldots, n-2$,

$$\mathsf{Card}\Bigl(\varphi^{-1}_{a,b}\bigl(\{\alpha_C : C \in \mathcal{E}\}\bigr) \cap \bigl(s^*_j, s^*_{j+1}\bigr)\Bigr) = 1.$$

To prove the claim note that, by (i–iii),

$$\mathcal{E} = \left\{\varphi_{a,b}\bigl(\bigl(s^*_j, s^*_{j+1}\bigr)\bigr) : j \in \{0, 1, \ldots, n-2\}\right\}. \tag{3.1}$$

Hence, for every $j = 0, 1, \ldots, n-2$, $\alpha_{\varphi_{a,b}\left(\left(s^*_j, s^*_{j+1}\right)\right)} \in \varphi_{a,b}\bigl(\bigl(s^*_j, s^*_{j+1}\bigr)\bigr)$, implies

$$\emptyset \neq \varphi^{-1}_{a,b}\left(\alpha_{\varphi_{a,b}\left(\left(s^*_j, s^*_{j+1}\right)\right)}\right) \cap \bigl(s^*_j, s^*_{j+1}\bigr) \subset \varphi^{-1}_{a,b}\bigl(\{\alpha_C : C \in \mathcal{E}\}\bigr) \cap \bigl(s^*_j, s^*_{j+1}\bigr).$$

Assume that $s^1_i, s^2_i \in \varphi^{-1}_{a,b}\bigl(\{\alpha_C : C \in \mathcal{E}\}\bigr) \cap \bigl(s^*_j, s^*_{j+1}\bigr)$. By (3.1) and the definition of the points α_C,

$$\varphi_{a,b}\bigl(s^1_i\bigr), \varphi_{a,b}\bigl(s^2_i\bigr) \in \varphi_{a,b}\bigl(\bigl(s^*_j, s^*_{j+1}\bigr)\bigr) \cap \{\alpha_C : C \in \mathcal{E}\} = \left\{\alpha_{\varphi_{a,b}\left(\left(s^*_j, s^*_{j+1}\right)\right)}\right\}.$$

Consequently, $s^1_i = s^2_i$ by (ii). This proves the claim.

Now we set $m = m(X, a, b) := 2n - 1$ and by the above claim we define the partition

$$s_0 = s^*_0 = 0 < s_1 < s_2 \cdots < s_{m-2} < s_{m-1} = s_{2(n-1)} = s^*_{n-1} < s_m = s^*_n = 1$$

of the interval $[0,1]$ by:

$$s_{2j} := s^*_j, \text{ and}$$
$$s_{2j+1} \text{ is the unique point of } \varphi^{-1}_{a,b}\big(\{\alpha_C : C \in \mathcal{E}\}\big) \cap \big(s^*_j, s^*_{j+1}\big)$$

for every $j = 0, 1, 2, \dots, n-2$. With these definitions and (i), the set $\{s_0, s_1, \dots, s_m\}$ is the union of two disjoint sets:

$$\begin{aligned}\{s_0, s_2, \dots, s_{m-1}, s_m\} &= \Big\{s^*_j : j \in \{0,1,\dots,n\}\Big\} = \varphi^{-1}_{a,b}\big(\widetilde{V}(X)\big) \text{ and} \\ \{s_1, s_3, \dots, s_{m-2}\} &= \varphi^{-1}_{a,b}\big(\{\alpha_C : C \in \mathcal{E}\}\big).\end{aligned} \tag{3.2}$$

By definition $m = m(X, a, b, M)$ is odd, $\varphi_{a,b} \colon [0,1] \longrightarrow X$ is continuous and surjective and, by (i), $\varphi_{a,b}(0) = a$ and $\varphi_{a,b}(1) = b$. Moreover, since the map $\varphi_{a,b}$ is onto,

$$n+1 = \mathsf{Card}\left(\Big\{s^*_j : j \in \{0,1,\dots,n\}\Big\}\right) = \mathsf{Card}\left(\varphi^{-1}_{a,b}\big(\widetilde{V}(X)\big)\right) \geq \mathsf{Card}\big(\widetilde{V}(X)\big) \geq 4,$$

and hence, $m = 2n - 1 \geq 5$.

On the other hand, by (3.2),

$$\begin{aligned} W &= \varphi_{a,b}\big(\{s_i : i \in \{0,1,\dots,m\}\}\big) \\ &= \varphi_{a,b}\left(\varphi^{-1}_{a,b}\big(\widetilde{V}(X)\big)\right) \cup \varphi_{a,b}\left(\varphi^{-1}_{a,b}\big(\{\alpha_C : C \in \mathcal{E}\}\big)\right) \\ &= \widetilde{V}(X) \cup \{\alpha_C : C \in \mathcal{E}\} \supset V(X), \end{aligned}$$

and

$$\varphi^{-1}_{a,b}(W) = \varphi^{-1}_{a,b}\big(\widetilde{V}(X)\big) \cup \varphi^{-1}_{a,b}\big(\{\alpha_C : C \in \mathcal{E}\}\big) = \{s_i : i \in \{0,1,\dots,m\}\}.$$

Thus, (a) holds.

Statement (b) follows from (ii), Statement (a) and the fact that every interval $[s_i, s_{i+1}]$ is contained in an interval $\big[s^*_j, s^*_{j+1}\big]$.

To end the proof of the lemma it remains to prove (c). Assume that $\varphi_{a,b}(s_i) = \varphi_{a,b}(s_j)$ (or, equivalently, that there exists $\alpha \in W$ such that $s_i, s_j \in \varphi^{-1}_{a,b}(\alpha) \subset \varphi^{-1}_{a,b}(W)$). Since

$$\varphi^{-1}_{a,b}(W) = \varphi^{-1}_{a,b}\big(\widetilde{V}(X)\big) \cup \varphi^{-1}_{a,b}\big(\{\alpha_C : C \in \mathcal{E}\}\big) \text{ and } \widetilde{V}(X) \cap \{\alpha_C : C \in \mathcal{E}\} = \emptyset,$$

by (3.2), it follows that either

$$\begin{aligned} s_i, s_j &\in \varphi^{-1}_{a,b}\big(\widetilde{V}(X)\big) = \{s_0, s_2, \dots, s_{m-1}, s_m\} \text{ or} \\ s_i, s_j &\in \varphi^{-1}_{a,b}\big(\{\alpha_C : C \in \mathcal{E}\}\big) = \{s_1, s_3, \dots, s_{m-2}\}. \end{aligned}$$

On the other hand, by (iii), $s_m = s^*_n = \varphi^{-1}_{a,b}(b) \notin \{s_i, s_j\}$. Consequently, either $i, j \in \{0, 2, 4, \dots, m-1\}$ or $i, j \in \{1, 3, 5, \dots, m-2\}$, and (c) holds. □

The next lemma will be useful in dealing with piecewise expansiveness and in making possible to use Theorem 2.13 to obtain transitive graph maps.

Lemma 3.5. *Let X be a topological graph and let $f\colon X \longrightarrow X$ be a Markov map with respect to a Markov invariant set Q such that $f(I)$ is a (non-degenerate) interval for every $I \in \mathcal{B}(Q)$. Then there exists a Q-expansive (Markov) map $g\colon X \longrightarrow X$ such that $g\big|_Q = f\big|_Q$ and $g(I) = f(I)$ for every $I \in \mathcal{B}(Q)$. In particular, the Markov graphs of f and g with respect to Q coincide.*

Proof. The requirement that $g\big|_Q = f\big|_Q$ implies that it is enough to define $g\big|_I$ for every $I \in \mathcal{B}(Q)$ so that $g\big|_I$ is expansive, $g\big|_{\partial I} = f\big|_{\partial I}$ and $g\big|_I(I) = f(I)$.

Let $I \in \mathcal{B}(Q)$. The monotonicity of f on I implies that $f(I)$ is an interval that it is the union of $n \geq 1$ Q-basic intervals. Thus there exists a partition $t_0, t_1, \dots, t_n$ of I with

$$\langle t_i, t_{i+1} \rangle_I \cap \{t_0, t_1, \dots, t_n\} = \{t_i, t_{i+1}\} \quad \text{for} \quad i = 0, 1, \dots, n-1$$

(in particular, $\{t_0, t_n\} = \partial I$) and such that

$$f\big(\langle t_0, t_1 \rangle_I\big) = \langle f(t_0), f(t_1) \rangle_{f(I)}, f\big(\langle t_1, t_2 \rangle_I\big) = \langle f(t_1), f(t_2) \rangle_{f(I)}, \dots,$$
$$f\big(\langle t_{n-1}, t_n \rangle_I\big) = \langle f(t_{n-1}), f(t_n) \rangle_{f(I)}$$

are pairwise different basic intervals. Clearly, for every $i \in \{0, 1, \dots, n-1\}$,

$$\left\{ \frac{d_I(x, t_i)}{d_I(t_i, t_{i+1})} : x \in \langle t_i, t_{i+1} \rangle_I \right\} = \left[0, \frac{d_I(t_{i+1}, t_i)}{d_I(t_i, t_{i+1})} \right] = [0, 1], \text{ and}$$
$$\left\{ d_{f\left(\langle t_i, t_{i+1} \rangle_I\right)}(z, f(t_i)) : z \in f\big(\langle t_i, t_{i+1} \rangle_I\big) \right\} = [0, 1]$$

because $f\big(\langle t_i, t_{i+1} \rangle_I\big) \in \mathcal{B}(Q)$, and hence

$$d_{f\left(\langle t_i, t_{i+1} \rangle_I\right)}(f(t_{i+1}), f(t_i)) = \left\| f\big(\langle t_i, t_{i+1} \rangle_I\big) \right\|_{f\left(\langle t_i, t_{i+1} \rangle_I\right)} = 1.$$

Then, for $i \in \{0, 1, \dots, n-1\}$ and $x \in \langle t_i, t_{i+1} \rangle_I$ we define $g\big|_{\langle t_i, t_{i+1} \rangle_I}(x)$ to be the unique point from $\langle f(t_i), f(t_{i+1}) \rangle_{f(I)}$ that verifies

$$d_{f\left(\langle t_i, t_{i+1} \rangle_I\right)}(g\big|_{\langle t_i, t_{i+1} \rangle_I}(x), f(t_i)) = \frac{1}{d_I(t_i, t_{i+1})} d_I(x, t_i).$$

Observe that this formula defines

$$g\big|_{\langle t_i, t_{i+1} \rangle_I}(t_i) = f(t_i) \quad \text{and} \quad g\big|_{\langle t_i, t_{i+1} \rangle_I}(t_{i+1}) = f(t_{i+1}).$$

Hence $g\big|_I$ is well defined and continuous and $g\big|_{\{t_0, t_1, \dots, t_n\}} = f\big|_{\{t_0, t_1, \dots, t_n\}}$. In particular, $g\big|_{\partial I} = f\big|_{\partial I}$. Moreover, for every $i \in \{0, 1, \dots, n-1\}$, $g\big|_{\langle t_i, t_{i+1} \rangle_I}$ is one-to-one (and hence monotone), and

$$g\big|_I\big(\langle t_i, t_{i+1} \rangle_I\big) = \langle f(t_i), f(t_{i+1}) \rangle_{f(I)} = f\big(\langle t_i, t_{i+1} \rangle_I\big).$$

Consequently $g\big|_I(I) = f(I)$.

To end the proof of the lemma only it remains to show that $g\big|_I$ is expansive. By using appropriately the triangle inequality and the monotonicity of $g\big|_{\langle t_i, t_{i+1} \rangle_I}$ is not difficult to see that

$$d_{f\left(\langle t_i, t_{i+1} \rangle_I\right)}\big(g\big|_I(x), g\big|_I(y)\big) = \frac{1}{d_I(t_i, t_{i+1})} d_I(x, y)$$

for every $x, y \in \langle t_i, t_{i+1} \rangle_I$ with $i \in \{0, 1, \ldots, n-1\}$. Thus, in the special case when $n = 1$ (that is, when $I = \langle t_0, t_1 \rangle_I \in \mathcal{B}(Q)$ and $f(I) \in \mathcal{B}(Q)$), we have

$$d_{f(I)}\big(g\big|_I(x), g\big|_I(y)\big) = d_I(x, y)$$

for every $x, y \in I$, because $d_I(t_i, t_{i+1}) = \|I\|_I = 1$. Hence $g\big|_I$ is expansive on I.

When $n > 1$ then $g\big|_I$ also is expansive on I by setting

$$\lambda_I = \min\left\{\tfrac{1}{d_I(t_i, t_{i+1})} : i \in \{0, 1, \ldots, n-1\}\right\} > 1.$$

□

3.3. **Example with persistent fixed low periods.** This subsection is devoted to construct and prove

Example 1.9. *For every positive integer $n \in \{4k+1, 4k-1 : k \in \mathbb{N}\}$ there exists f_n, a totally transitive continuous circle map of degree one having a lifting $F_n \in \mathcal{L}_1$ such that $\mathsf{Rot}(F_n) = \left[\frac{1}{2}, \frac{n+2}{2n}\right]$, $\lim_{n\to\infty} h(f_n) = 0$,*

$$\mathsf{Per}(f_n) = \{2\} \cup \{p \text{ odd} : 2k+1 \le p \le n-2\} \cup \mathsf{Succ}\,(n)$$

and $\mathsf{BdCof}\,(f_n)$ exists and verifies $2k+1 \le \mathsf{BdCof}\,(f_n) \le n$ (and, hence, $\lim_{n\to\infty} \mathsf{BdCof}\,(f_n) = \infty$).

Furthermore, given any graph G with a circuit, the sequence of maps $\{f_n\}_{n\ge 7, n \text{ odd}}$ can be extended to a sequence of continuous totally transitive self maps of G, $\{g_n\}_{n\ge 7, n \text{ odd}}$, such that $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$ and $\lim_{n\to\infty} h(g_n) = 0$.

Example 1.9 will be split into Theorem 3.6 which shows the existence of the circle maps f_n by constructing them along the lines of Subsection 3.1, and Theorem 3.7 that extends these maps to a generic graph that is not a tree. The proof of Theorem 3.6, in turn, will use a proposition that computes the Markov graph modulo 1 of the liftings F_n.

The auxiliary Figure 4 illustrates the construction of the orbits P_n, Q_n and the map F_n from the next theorem in a particular case.

Theorem 3.6. *Let $n \in \{4k+1, 4k-1 : k \in \mathbb{N}\}$ and let*

$$Q_n = \{\ldots, x_{-1}, x_0, x_1, x_2, \ldots\} \subset \mathbb{R}, \quad \text{and}$$
$$P_n = \{\ldots y_{-1}, y_0, y_1, y_2, \ldots, y_{2n-1}, y_{2n}, y_{2n+1}, \ldots\} \subset \mathbb{R}$$

be infinite sets such that the points of P_n and Q_n are intertwined so that

$$0 = x_0 < y_0 < y_1 < \cdots y_{n-3} < x_1 < y_{n-2} < \cdots < y_{2n-1} < x_2 = 1,$$

and $x_{i+2\ell} = x_i + \ell$ and $y_{i+2n\ell} = y_i + \ell$ for every $i, \ell \in \mathbb{Z}$.

We define a lifting $F_n \in \mathcal{L}_1$ such that, for every $i \in \mathbb{Z}$, $F_n(x_i) = x_{i+1}$ and $F_n(y_i) = y_{i+n+2}$, and F_n is affine between consecutive points of $P_n \cup Q_n$. Then, Q_n (respectively P_n) is a twist lifted periodic orbit of F_n of period 2 (respectively $2n$) with rotation number $\frac{1}{2}$ (respectively $\frac{n+2}{2n}$). Moreover, the map F_n has $\mathsf{Rot}(F_n) = \left[\frac{1}{2}, \frac{n+2}{2n}\right]$ as rotation interval.

Let $f_n : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the continuous map which has F_n as a lifting. Then, f_n is totally transitive, $\lim_{n\to\infty} h(f_n) = 0$,

$$\mathsf{Per}(f_n) = \mathsf{Per}(F_n) = \{2\} \cup \{q \text{ odd} : 2k+1 \le q \le n-2\} \cup \mathsf{Succ}\,(n)$$

and $\mathsf{BdCof}\,(f_n)$ exists and verifies $2k+1 \le \mathsf{BdCof}\,(f_n) \le n$.

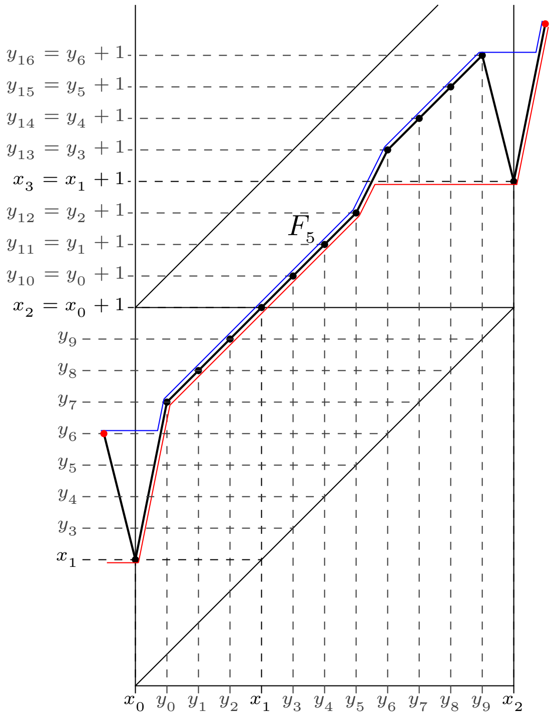


FIGURE 4. A possible choice of the points of $(P_5 \cup Q_5) \cap [0,1]$ from Theorem 3.6 and Proposition 3.8, and the graph of the corresponding map F_5. The lower map $(F_5)_l$ is drawn in red and the upper map $(F_5)_u$ in blue.

Theorem 3.7. *Let G be a graph with a circuit. Then, the sequence of maps $\{f_n\}_{n\geq 7,\ n \text{ odd}}$ from Theorem 3.6 can be extended to a sequence of continuous totally transitive self maps of G, $\{g_n\}_{n\geq 7,\ n \text{ odd}}$, such that $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$ and $\lim_{n\to\infty} h(g_n) = 0$.*

Before proving Theorem 3.6, in the next proposition, we study the Markov graph modulo 1 of the liftings F_n (see the auxiliary Figure 4). Given $m \in \mathbb{Z}$ and $q \in \mathbb{N}$, to simplify the notation, we will denote $m \pmod q$ by $\{m\}_q$.

Proposition 3.8 ($\mathcal{B}(P_n \cup Q_n)$ and the F_n-Markov graph modulo 1)**.** *In the assumptions of Theorem 3.6 we set*

$$J_0 := [x_0, y_0],\ J_1 := [y_{n-3}, x_1],\ J_2 := [x_1, y_{n-2}], \quad \text{and} \quad J_3 := [y_{2n-1}, x_2],$$

and

$$I_i := \Big[y_{\{n+1+i(n+2)\}_{2n}}, y_{\{n+2+i(n+2)\}_{2n}}\Big] \quad \text{for} \quad i \in \{0,1,\ldots,2n-3\}.$$

Then the following statements hold:

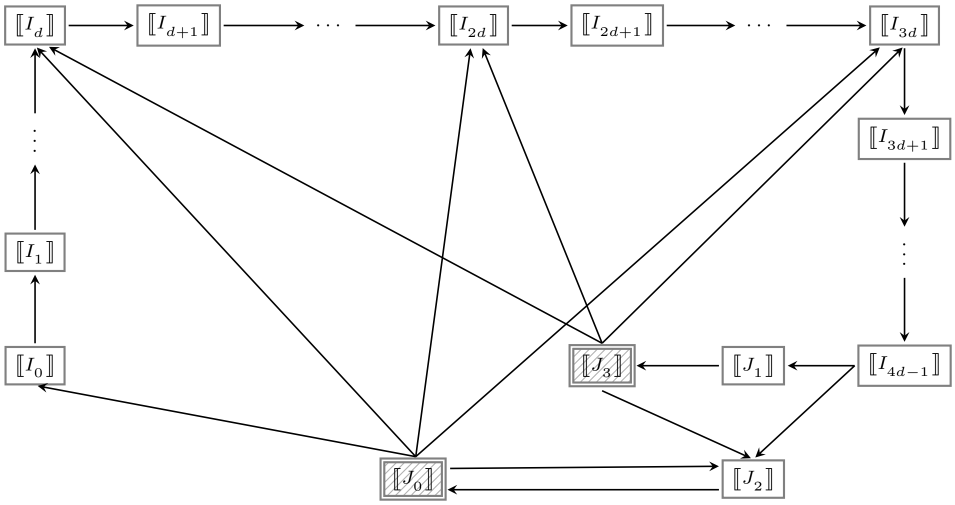
(a) We have,

$$\mathcal{B}(P_n \cup Q_n) = \{I_i + \ell : i \in \{0,1,\ldots,2n-3\},\ \ell \in \mathbb{Z}\} \cup \{J_i + \ell : i \in \{0,1,2,3\},\ \ell \in \mathbb{Z}\}.$$

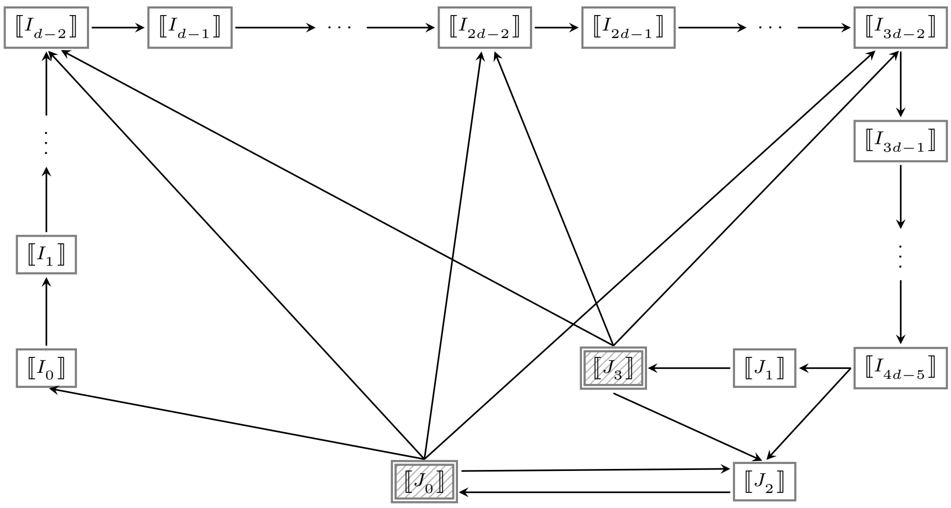
and, in particular,

$$\Big(\bigcup_{i=0}^{2n-3} I_i\Big) \cup \Big(\bigcup_{j=0}^{3} J_j\Big) = [0,1].$$

(b) When $n = 4k-1$ for some $k \in \mathbb{N}$, the Markov graph modulo 1 of F_n is:



where $d := 2k-1$. Otherwise, when $n = 4k+1$ for some $k \in \mathbb{N}$, the Markov graph modulo 1 of F_n is:



where $d := 2k+1$.

(c) $h(f_n) = \log \rho_n$, where $\rho_n > 1$ is the largest root of the polynomial

$$T_n(x) = x^{2n}(x^2-1) - 2x^{\frac{3n+1}{2}} - 2x^{n+1} - 2x^{\frac{n+3}{2}} - x^2 - 1.$$

Proof. It is obvious from the assumptions that $J_0, J_1, J_2, J_3 \in \mathcal{B}\big(P_n \cup Q_n\big)$ and $\mathsf{Card}\big((P_n \cup Q_n) \cap [0,1]\big) = 2n+3$. Hence, there are $2n-2$ $P_n \cup Q_n$-basic intervals contained in the interval $[0,1]$ different from J_0, J_1, J_2 and J_3. On the other hand, $n+2$ and $2n$ are coprime and, hence, there exist $2n-2$ pairwise different intervals I_i. Thus, to prove (a) it is enough to show that all the intervals I_i are $P_n \cup Q_n$-basic. This amounts showing that

$$\left\{I_i = [y_{\{n+1+i(n+2)\}_{2n}}, y_{\{n+2+i(n+2)\}_{2n}}] : i \in \{0, 1, \dots, 2n-3\}\right\} \subset$$
$$\big\{[y_0, y_1], [y_1, y_2], \dots, [y_{n-4}, y_{n-3}], [y_{n-2}, y_{n-1}], [y_{n-1}, y_n], \dots, [y_{2n-2}, y_{2n-1}]\big\}.$$

More concretely, we have to see that

$$\big\{\{n+1+i(n+2)\}_{2n} : i \in \{0, 1, \dots, 2n-3\}\big\} \cap \{n-3, 2n-1\} = \emptyset$$

because $\{n+2+i(n+2)\}_{2n} = \{n+1+i(n+2)\}_{2n} + 1$ provided that $\{n+1+i(n+2)\}_{2n} \neq 2n-1$.

Assume by way of contradiction that there exist $i \in \{0, 1, \dots, 2n-3\}$, $\ell \in \mathbb{Z}$ and $a \in \{n-3, 2n-1\}$ such that

$$n+1+i(n+2) = a + \ell 2n \quad \Longleftrightarrow \quad i(n+2) = (2\ell-1)n + (a-1).$$

Then, since $n \in \{4k+1, 4k-1 : k \in \mathbb{N}\}$ is odd, it follows that i has the same parity as a. So, there exists $t \in \{0, 1, \dots, n-2\}$ such that $i = 2t$ when $a = n-3$, and $i = 2t+1$ when $a = 2n-1$. In any case,

$$i(n+2) = (2\ell-1)n + (a-1) \quad \Longleftrightarrow \quad 4t = 2n(\ell-t-1) + (2n-4).$$

Since $0 \le 4t \le 4n-8$ it follows that $4t = 2n-4$ (that is, $\ell-t-1$ must be 0) which implies $\frac{n}{2} - 1 = t \in \mathbb{Z}$; a contradiction. So, the intervals I_i are $P_n \cup Q_n$-basic and, hence, (a) holds.

Now we will compute the Markov graph modulo 1 of F_n to prove (b) Recall that, by definition,

$$y_{i+2n\ell} = y_i + \ell = y_{\{i+2n\ell\}_{2n}} + \ell \quad \text{and} \quad F_n(y_i) = y_{i+n+2}$$

for every $i, \ell \in \mathbb{Z}$. For convenience we set $\widetilde{I}_i = \left[y_{n+1+i(n+2)}, y_{n+2+i(n+2)}\right]$ for every $i \in \{0, 1, \dots, 2n-2\}$. Hence, by the part of the lemma already proven, for $i \in \{0, 1, \dots, 2n-3\}$ we have $[\![I_i]\!] = \left[\!\!\left[\widetilde{I}_i\right]\!\!\right]$, $\widetilde{I}_i \in \mathcal{B}(P_n \cup Q_n)$, and

$$F_n\left(\widetilde{I}_i\right) = \left[F_n(y_{n+1+i(n+2)}), F_n(y_{n+2+i(n+2)})\right] = \widetilde{I}_{i+1} \tag{3.3}$$

because $F_n\big|_{P_n}$ is increasing and F_n is affine on $P_n \cup Q_n$-basic intervals. Moreover,

$$F_n\left(\widetilde{I}_{2n-3}\right) = \widetilde{I}_{2n-2} = \left[y_{n-3+2n(n+1)}, y_{n-2+2n(n+1)}\right] = \\ (J_1 + (n+1)) \cup (J_2 + (n+1)).$$

Consequently, the Markov graph modulo 1 of F_n has the following subgraph:

$$[\![I_0]\!] \longrightarrow [\![I_1]\!] \longrightarrow \cdots \longrightarrow [\![I_{2n-3}]\!] \begin{array}{l} \nearrow [\![J_1]\!] \\ \searrow [\![J_2]\!] \end{array}. \tag{3.4}$$

To completely determine the Markov graph modulo 1 of F_n we still need to compute the images of the intervals J_0, J_1, J_2 and J_3. We have $x_{i+2\ell} = x_i + \ell$ and $F_n(x_i) = x_{i+1}$ for every $i, \ell \in \mathbb{Z}$ and, hence,

(i) $F_n(J_0) = [x_1, y_{n+2}] = J_2 \cup [y_{n-2}, y_{n+2}]$;
(ii) $F_n(J_1) = [y_{2n-1}, x_2] = J_3$;
(iii) $F_n(J_2) = [x_2, y_{2n}] = J_0 + 1$; and
(iv) $F_n(J_3) = [x_3, y_{3n+1}] = [x_1, y_{n+1}] + 1 = (J_2 + 1) \cup ([y_{n-2}, y_{n+1}] + 1)$.

We will end the proof in the case $n = 4k - 1$ and $d = 2k - 1 = \frac{n-1}{2}$. The proof in the other case follows analogously.

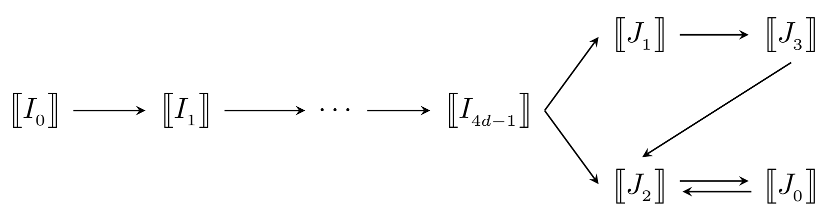
In this case we have $2n - 3 = 4d - 1$. Moreover, for $\ell \geq 1$ we have

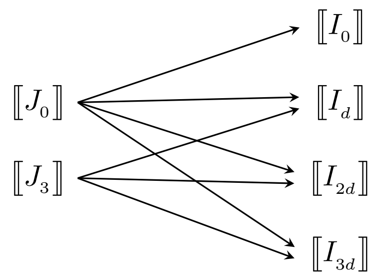
$$\ell d(n + 2) = dn\ell + 2d\ell = 2kn\ell - n\ell + (4k - 1)\ell - \ell = k\ell 2n - \ell.$$

Hence,

$$\{n + 1 + \ell d(n + 2)\}_{2n} = \{n + 1 - \ell + k\ell 2n\}_{2n} = n + 1 - \ell, \text{ and}$$
$$\{n + 2 + \ell d(n + 2)\}_{2n} = \{n + 2 - \ell + k\ell 2n\}_{2n} = n + 2 - \ell.$$

So, $I_{\ell d} = [y_{n+1-\ell}, y_{n+2-\ell}]$ for $\ell = 0, 1, 2, 3$ (observe that the interval I_{4d} is not defined because $4d > 2n - 3$ and, on the other hand, $[y_{n+1-\ell}, y_{n+2-\ell}]$ with $\ell = 4$ is not $P_n \cup Q_n$-basic). Consequently, from (3.4) and (i–iv) above we get that the Markov graph modulo 1 of F_n is the union of the following two subgraphs:





This ends the proof of (b).

To prove (c) we will use Propositions 2.10 and 2.11, and Theorem 2.12.

First notice that (3.3) and (i–iv) above imply that $P_n \cup Q_n$ is a short Markov partition with respect to F_n. Then, Propositions 2.10 and 2.11, imply that f_n is a Markov map and

$$h(f_n) = \log \max\{\sigma(M_n), 1\}$$

where M_n is the Markov matrix of f_n with respect to $\mathsf{e}\big(P_n \cup Q_n\big)$. Moreover, we can identify the set $\mathcal{B}\big(\mathsf{e}\big(P_n \cup Q_n\big)\big)$ with the set of all equivalence classes of $P_n \cup Q_n$-basic intervals (i.e. the set of all vertices of the Markov graph modulo 1 of F_n). Then, the matrix M_n coincides with the transition matrix of the Markov graph modulo 1 of F_n which, by definition and Proposition 2.10, is a $\mathsf{Card}\big(\mathcal{B}\big(\mathsf{e}\big(P_n \cup Q_n\big)\big)\big) \times \mathsf{Card}\big(\mathcal{B}\big(\mathsf{e}\big(P_n \cup Q_n\big)\big)\big)$ matrix $M_n = (m_{[\![I]\!],[\![J]\!]})_{[\![I]\!],[\![J]\!] \in \mathcal{B}(\mathsf{e}(P_n \cup Q_n))}$ such that

$$m_{[\![I]\!],[\![J]\!]} = \begin{cases} 1 & \text{if } [\![I]\!]\ f\text{-covers } [\![J]\!] \\ 0 & \text{otherwise} \end{cases}.$$

To compute $\sigma(M_n)$ we will use Theorem 2.12 with

$$\mathsf{Rom}_n = \{\mathsf{r}_1 = [\![J_0]\!], \mathsf{r}_2 = [\![J_3]\!]\}$$

as a rome (being their elements marked in the statement with a box with double border and sloping lines background pattern). Moreover, recall that $M_{\mathsf{Rom}_n}(x) = (a_{ij}(x))$ is the matrix such that $a_{ij}(x) = \sum_p x^{-\ell(p)}$, where the sum is taken over all simple paths starting at r_i and ending at r_j (since M_n is a matrix of zeroes and ones the width of every path is 1). Then, the matrix $M_{\mathsf{Rom}_n}(x)$ is:

$$\begin{cases} \begin{pmatrix} x^{-2} + x^{-4d-2} + \alpha(x) & x^{-4d-2} + \alpha(x) \\ x^{-2} + \alpha(x) & \alpha(x) \end{pmatrix} & \text{when } n = 4k-1 \text{ and } d = \frac{n-1}{2} \text{ for some } k \in \mathbb{N}, \text{ and} \\ \begin{pmatrix} x^{-2} + x^{-4d+2} + \alpha(x) & x^{-4d+2} + \alpha(x) \\ x^{-2} + \alpha(x) & \alpha(x) \end{pmatrix} & \text{when } n = 4k+1 \text{ and } d = \frac{n+1}{2} \text{ for some } k \in \mathbb{N}, \end{cases}$$

where

$$\alpha(x) = \begin{cases} x^{-3d-2} + x^{-2d-2} + x^{-d-2} & \text{when } n = 4k-1 \text{ and } d = \frac{n-1}{2} \text{ for some } k \in \mathbb{N}, \text{ and} \\ x^{-3d} + x^{-2d} + x^{-d} & \text{when } n = 4k+1 \text{ and } d = \frac{n+1}{2} \text{ for some } k \in \mathbb{N}. \end{cases}$$

By Theorem 2.12, the characteristic polynomial $T_n(x)$ of M_n is

$$T_n(x) = (-1)^{2n} x^{2n+2} \det(M_{\mathsf{Rom}_n}(x) - \mathbf{I}_2) = x^{2n}(x^2-1) - 2x^{\frac{3n+1}{2}} - 2x^{n+1} - 2x^{\frac{n+3}{2}} - x^2 - 1,$$

where $\mathbf{I}_2$ is the unit matrix 2×2.

By direct inspection of the Markov graph modulo 1 of F_n (see (b)), given any two vertices $[\![K]\!]$ and $[\![L]\!]$ in the graph, there exists a path from $[\![K]\!]$ to $[\![L]\!]$. This means that the transition matrix M_n of the Markov graph modulo 1 of F_n is non-negative and irreducible. Then, by the Perron-Frobenius Theorem, $\sigma(M_n) > 1$ is the largest eigenvalue of M_n. Hence, $\sigma(M_n)$ is the largest root (larger than one) of T_n. □

Proof of Theorem 3.6. Since $y_{i+2n\ell} = y_i + \ell$ and $F_n(y_i) = y_{i+n+2}$ for every $i, \ell \in \mathbb{Z}$, it follows that

$$F_n^{2n}(y_i) = y_{i+2n(n+2)} = y_i + n + 2$$

for every $i \in \mathbb{Z}$. Moreover, let $j \in \{1, 2, \dots, 2n-1\}$ and assume that

$$F_n^j(y_i) - y_i = y_{i+j(n+2)} - y_i = \ell \in \mathbb{Z}.$$

Then, $y_{i+j(n+2)} = y_i + \ell = y_{i+2n\ell}$ and, thus, $j(n+2) = 2n\ell$; a contradiction because $n+2$ and $2n$ are relatively prime. Hence, P_n is a lifted periodic orbit of F_n of period $2n$ and rotation number $\frac{n+2}{2n}$. Moreover, $F_n\big|_{P_n}$ is increasing and, thus, P_n is a twist lifted periodic orbit of F_n of period $2n$ and rotation number $\frac{n+2}{2n}$. In a similar manner, Q_n is a twist lifted periodic orbit of F_n of period 2 and rotation number $\frac{1}{2}$.

Now we will show tat $\mathsf{Rot}(F_n) = \left[\frac{1}{2}, \frac{n+2}{2n}\right]$ by using Theorem 2.1. To this end we need to compute the rotation number of the lower map $(F_n)_l$ and of the upper map $(F_n)_u$ (see Figures 1 and 4). Observe that

$$F_n(y_{2n-5}) = y_{3n-3} = y_{n-3} + 1 < x_1 + 1 < y_{n-2} + 1 = y_{3n-2} = F_n(y_{2n-4}).$$

Hence, there exists a unique $u_l^n \in (y_{2n-5}, y_{2n-4})$ such that $F_n\bigl(u_l^n\bigr) = x_1 + 1$. So,

$$(F_n)_l(x) = \inf\{F_n(y) : y \geq x\} = \begin{cases} \inf\bigl[F_n(x), +\infty\bigr) = F_n(x) & \text{for } x \in \bigl[0, u_l^n\bigr], \\ \inf\bigl[x_1 + 1, +\infty\bigr) = x_1 + 1 & \text{for } x \in \bigl[u_l^n, 1\bigr], \\ (F_n)_l(x - \lfloor x \rfloor) + \lfloor x \rfloor & \text{if } x \notin [0,1]. \end{cases}$$

On the other hand, $F_n(x_0) = x_1 < y_{n+1} < y_{n+2} = F_n(y_0)$, which implies that there exists a unique $u_u^n \in (x_0, y_0)$ such that $F_n\bigl(u_u^n\bigr) = y_{n+1} = F_n(y_{2n-1}) - 1$. So,

$$(F_n)_u(x) = \sup\{F_n(y) : y \leq x\} = \begin{cases} \sup\bigl[-\infty, y_{n+1}\bigr) = y_{n+1} & \text{for } x \in \bigl[0, u_u^n\bigr], \\ \sup\bigl[-\infty, F_n(x)\bigr) = F_n(x) & \text{for } x \in \bigl[u_u^n, y_{2n-1}\bigr], \\ \sup\bigl[-\infty, F_n(y_{2n-1})\bigr) = y_{n+1} + 1 & \text{for } x \in \bigl[y_{2n-1}, 1\bigr], \\ (F_n)_u(x - \lfloor x \rfloor) + \lfloor x \rfloor & \text{if } x \notin [0,1]. \end{cases}$$

To compute $\rho\bigl((F_n)_l\bigr)$ observe that

$$P_n \cap [0,1] \subset \bigl[u_u^n, y_{2n-1}\bigr] \quad \text{and, hence,} \quad (F_n)_u\bigr|_{P_n} = F_n\bigr|_{P_n}.$$

So, $\rho\bigl((F_n)_u\bigr) = \rho_{F_n}(P_n) = \frac{n+2}{2n}$. In a similar way, $Q_n \cap [0,1] \subset \bigl[0, u_l^n\bigr]$, $(F_n)_l\bigr|_{Q_n} = F_n\bigr|_{Q_n}$ and $\rho\bigl((F_n)_l\bigr) = \rho_{F_n}(Q_n) = \frac{1}{2}$. Hence, $\mathsf{Rot}(F_n) = \bigl[\frac{1}{2}, \frac{n+2}{2n}\bigr]$ by Theorem 2.1.

Next we will compute $\mathsf{Per}(f_n)$. By Theorem 2.4 we have

$$\mathsf{Per}(F_n) = Q_{F_n}\bigl(\tfrac{1}{2}\bigr) \cup M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right) \cup Q_{F_n}\bigl(\tfrac{n+2}{2n}\bigr).$$

We will compute separately the sets $Q_{F_n}\bigl(\frac{1}{2}\bigr)$ and $M\left(\frac{1}{2}, \frac{n+2}{2n}\right) \cup Q_{F_n}\bigl(\frac{n+2}{2n}\bigr)$, starting with $Q_{F_n}\bigl(\frac{1}{2}\bigr)$.

To compute $Q_{F_n}\bigl(\frac{1}{2}\bigr)$ we will use Proposition 2.8 with X and f replaced by $\mathbb{S}^1$ and f_n, respectively. Hence, we will use the Markov graph of f_n. However, from the proof of Proposition 3.8 we already know that $P_n \cup Q_n$ is a short Markov partition with respect to F_n and, by Proposition 2.10, f_n is a Markov map with respect to the Markov partition $\mathsf{e}\bigl(P_n \cup Q_n\bigr)$. Moreover, the Markov graph of f_n with respect to $\mathsf{e}\bigl(P_n \cup Q_n\bigr)$ and the Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$ coincide, provided that we identify $[\![I]\!]$ with $\mathsf{e}\bigl([\![I]\!]\bigr)$ for every $I \in \mathcal{B}\bigl(P_n \cup Q_n\bigr)$. Thus, to perform our arguments we will use Proposition 3.8 and the Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$ instead of the Markov graph of f_n with respect to $\mathsf{e}\bigl(P_n \cup Q_n\bigr)$.

By Proposition 3.8, in the Markov graph modulo 1 of F_n there is no simple loop of length 4 and the only repetitive loop of length 4 is the 2-repetition of $[\![J_0]\!] \longrightarrow [\![J_2]\!] \longrightarrow [\![J_0]\!]$. Since

$$F_n(x_1) = x_2 = x_0 + 1 < y_0 + 1 = y_{2n} = F_n(y_{n-2}) \text{ and}$$
$$F_n(x_0) = x_1 < y_{n+2} = F_n(y_0),$$

$[\![J_0]\!] \longrightarrow [\![J_2]\!] \longrightarrow [\![J_0]\!]$ is a positive loop. Then, $4 \notin \mathsf{Per}(F_n) \supset Q_{F_n}\!\left(\frac{1}{2}\right)$ by Proposition 2.8(b). Therefore, with the notation from the definition of $Q_F(c)$ in Subsection 2.2, we have $s = 2$ and $s_{1/2} = 1$. Consequently, $Q_{F_n}\!\left(\frac{1}{2}\right) = \{2\}$.

Now we compute $M\left(\frac{1}{2}, \frac{n+2}{2n}\right) \cup Q_{F_n}\!\left(\frac{n+2}{2n}\right)$. Since $\frac{n+2}{2n} - \frac{1}{2} = \frac{1}{n}$, it follows that for every all $q \in \mathbb{N}$, $q > n$, there exists $p \in \mathbb{Z}$ such that $\frac{1}{2} < \frac{p}{q} < \frac{n+2}{2n}$. On the other hand, since n is odd, $(n+1)/2 \in \mathbb{Z}$ and $\frac{1}{2} < \frac{(n+1)/2}{n} < \frac{n+2}{2n}$. Summarizing,

$$M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right) \supset \mathsf{Succ}\,(n) \supset \{2n\ell : \ell \in \mathbb{N}\} \supset Q_{F_n}\!\left(\tfrac{n+2}{2n}\right).$$

Thus,

$$M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right) \cup Q_{F_n}\!\left(\tfrac{n+2}{2n}\right) = M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right) = \mathsf{Succ}\,(n) \cup \left\{q \in M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right) \,:\, q < n\right\}.$$

Now we need to compute $\left\{q \in M\left(\frac{1}{2}, \frac{n+2}{2n}\right) \,:\, q < n\right\}$. To this end, assume that $\frac{1}{2} < \frac{p}{q} < \frac{n+2}{2n}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, $q \le n-1$. We claim that q is odd. Otherwise, $q = 2\ell \le n-1$ with $\ell \in \mathbb{N}$. Then, the expression $\frac{1}{2} < \frac{p}{2\ell} < \frac{n+2}{2n}$ is equivalent to

$$\ell < p < \ell + \frac{2\ell}{n} \le \ell + \frac{n-1}{n} < \ell + 1;$$

a contradiction. This proves the claim.

Now assume that $q = 2\ell + 1 \le n-2$ with $\ell \in \mathbb{N}$ (recall that n is odd). We have,

$$\frac{1}{2} < \frac{p}{2\ell+1} < \frac{n+2}{2n}$$

which is equivalent to

$$\ell + \frac{1}{2} = \frac{2\ell+1}{2} < p < (\ell+2)\frac{n(2\ell+1) + 2(2\ell+1)}{2n(\ell+2)} \le (\ell+2)\frac{n(2\ell+1) + 2(n-2)}{2n\ell+4n} = (\ell+2)\frac{2n\ell+3n-4}{2n\ell+4n} < \ell+2.$$

Consequently, $p = \ell + 1$ and, hence,

$$\frac{1}{2} < \frac{\ell+1}{2\ell+1} < \frac{n+2}{2n}.$$

The second inequality is equivalent to

$$2n\ell + 2n = 2n(\ell+1) < (n+2)(2\ell+1) = 2n\ell + n + 2(2\ell+1).$$

Thus, since $n = 4k \pm 1$ with $k \in \mathbb{N}$, this is equivalent to

$$2\ell + 1 \ge \frac{n+1}{2} = \frac{4k+r}{2} = 2k + \frac{r}{2} \quad \text{with} \quad r \in \{0, 2\}.$$

Hence, since $q = 2\ell + 1$ is odd,

$$2\ell + 1 \ge 2k + 1.$$

Summarizing, we have seen that

$$\left\{q \in M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right) \,:\, q < n\right\} = \{2\ell+1 : \ell \in \mathbb{N} \text{ and } 2k+1 \le 2\ell+1 \le n-2\}$$

and, consequently,

$$\begin{aligned}\mathsf{Per}(f_n) = \mathsf{Per}(F_n) &= Q_{F_n}\big(\tfrac{1}{2}\big) \cup M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right) \cup Q_{F_n}\big(\tfrac{n+2}{2n}\big)\\ &= \{2\} \cup \mathsf{Succ}\,(n) \cup \left\{q \in M\left(\tfrac{1}{2}, \tfrac{n+2}{2n}\right)\,:\, q < n\right\}\\ &= \{2\} \cup \mathsf{Succ}\,(n) \cup \{q \text{ odd} : 2k+1 \le q \le n-2\}\,.\end{aligned}$$

Moreover, $\mathsf{StrBdCof}\,(f_n) = n$ and $2k+1 \in \mathsf{sBC}(f_n)$. So, $\mathsf{BdCof}\,(f_n)$ exists and verifies $2k+1 \le \mathsf{BdCof}\,(f_n) \le n$.

Next we will show that f_n is totally transitive by using Theorem 2.13. We already know that f_n is a Markov map with respect to the Markov partition $\mathsf{e}\big(P_n \cup Q_n\big)$, and the transition matrix of the Markov graph of f_n with respect to $\mathsf{e}\big(P_n \cup Q_n\big)$ coincides with the transition matrix of the Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$. From the proof of Proposition 3.8(c), it follows that this transition matrix is non-negative and irreducible. By direct inspection (see Proposition 3.8(b)) it follows that the vertex $[\![J_0]\!]$ of the Markov graph modulo 1 of F_n is the beginning of 4 arrows. That is, the transition matrix of the Markov graph of f_n with respect to $\mathsf{e}\big(P_n \cup Q_n\big)$ has a row with 4 non-zero elements and, thus, it cannot be a permutation matrix.

To use Theorem 2.13 we also need to know that f_n is a $\mathsf{e}\big(P_n \cup Q_n\big)$-expansive Markov map in the sense of Definition 2.14. Since we already know that f_n is a Markov map we have to show that f_n is expansive on every $\mathsf{e}\big(P_n \cup Q_n\big)$-basic interval. To do this we need two ingredients, a distance d_I on every $\mathsf{e}\big(P_n \cup Q_n\big)$-basic interval I and an appropriate way of writing the fact that the maps F_n are affine on every basic interval.

Let $[a, b] \in \mathcal{B}\big(P_n \cup Q_n\big)$. The fact that $F_n\big|_{[a,b]}$ is affine can be written as

$$\frac{|F_n(x) - F_n(y)|}{|F_n(a) - F_n(b)|} = \frac{|y - x|}{b - a} \tag{3.5}$$

for every $x, y \in [a, b]$.

A distance d_I on every basic interval $I \in \mathcal{B}\big(\mathsf{e}\big(P_n \cup Q_n\big)\big)$ can be defined as follows. Write $I = \mathsf{e}\big([x_I, y_I]\big)$ where $[x_I, y_I]$ is a $P_n \cup Q_n$-basic interval. Then, for every $x, y \in [x_I, y_I]$, we define

$$d_I(\mathsf{e}(x), \mathsf{e}(y)) := \frac{|x - y|}{|x_I - y_I|}.$$

Observe that $I = \mathsf{e}\big([x_I, y_I]\big)$ implies

$$f_n(I) = \mathsf{e}\big(F_n\left([x_I, y_I]\right)\big) = \mathsf{e}\big(\langle F_n(x_I), F_n(y_I)\rangle_{\mathbb{R}}\big).$$

Consider first the case $f_n(I) \in \mathcal{B}\big(\mathsf{e}\big(P_n \cup Q_n\big)\big)$ (which is equivalent to $\langle F_n(x_I), F_n(y_I)\rangle_{\mathbb{R}} \in \mathcal{B}(P_n \cup Q_n)$). Hence, since F_n is affine on $[x_I, y_I]$,

$$\begin{aligned}d_{f_n(I)}\big(f_n(\mathsf{e}(x)), f_n(\mathsf{e}(y))\big) &= d_{f_n(I)}\big(\mathsf{e}(F_n(x)), \mathsf{e}(F_n(y))\big) =\\ &\frac{|F_n(x) - F_n(y)|}{|F_n(x_I) - F_n(y_I)|} = \frac{|x - y|}{y_I - x_I} = d_I(\mathsf{e}(x), \mathsf{e}(y)).\end{aligned}$$

Now assume that $f_n(I) = \mathsf{e}\big(\langle F_n(x_I), F_n(y_I)\rangle_{\mathbb{R}}\big)$ contains more than one $\mathsf{e}\big(P_n \cup Q_n\big)$-basic interval and let $x, y \in [x_I, y_I]$ be such that

$$\langle f_n(\mathsf{e}(x)), f_n(\mathsf{e}(y))\rangle_{f_n(I)} \subset J = \mathsf{e}\big([x_J, y_J]\big) \quad \text{with} \quad [x_J, y_J] \in \mathcal{B}(P_n \cup Q_n).$$

In this case we set

$$\lambda_I := \min \left\{ \tfrac{|F_n(x_I) - F_n(y_I)|}{y_J - x_J} : [x_J, y_J] \in \mathcal{B}(P_n \cup Q_n) \text{ and } [x_J, y_J] \subset \langle F_n(x_I), F_n(y_I) \rangle_{\mathbb{R}} \right\},$$

and we have

$$\begin{aligned} d_J\big(f_n(\mathsf{e}(x)), f_n(\mathsf{e}(y))\big) &= d_J\big(\mathsf{e}(F_n(x)), \mathsf{e}(F_n(y))\big) = \\ \frac{|F_n(x) - F_n(y)|}{y_J - x_J} &= \frac{|F_n(x_I) - F_n(y_I)|}{y_J - x_J} \frac{|F_n(x) - F_n(y)|}{|F_n(x_I) - F_n(y_I)|} \geq \\ &\lambda_I \frac{|y - x|}{y_I - x_I} = \lambda_I d_I(\mathsf{e}(x), \mathsf{e}(y)). \end{aligned}$$

So, we have proved that f_n is $\mathsf{e}\big(P_n \cup Q_n\big)$-expansive, and thus, f_n is transitive by Theorem 2.13. Moreover, since $\mathsf{Per}(f_n) \supset \mathsf{Succ}\,(n)$, $\mathsf{Per}(f_n)$ is cofinite and f_n is totally transitive by Theorem 1.2.

To prove that $\lim_{n\to\infty} h(f_n) = 0$ we will use Proposition 3.8(c) which states that $h(f_n) = \log \rho_n$, where $\rho_n > 1$ is the largest root of the polynomial

$$T_n(x) = x^{2n}(x^2 - 1) - 2x^{\frac{3n+1}{2}} - 2x^{n+1} - 2x^{\frac{n+3}{2}} - x^2 - 1.$$

Set

$$\begin{aligned} q_n(x) &:= x^{2n}(x^2 - 1), \\ t_n(x) &:= 2x^{\frac{3n+1}{2}} + 2x^{n+1} + 2x^{\frac{n+3}{2}} + x^2 + 1, \text{ and} \\ \xi_n(x) &:= \frac{q(x)}{t(x)} = \frac{x^{\frac{n-1}{2}}(x^2 - 1)}{2 + 2x^{\frac{-n+1}{2}} + 2x^{-n+1} + x^{\frac{-3n+3}{2}} + x^{-\frac{3n+1}{2}}} \text{ for } x > 0. \end{aligned}$$

With this notation, the expression $0 = T_n(\rho_n) = q_n(\rho_n) - t_n(\rho_n)$ is equivalent to $\xi_n(\rho_n) = 1$. Observe that, for $x \geq 1$,

(3.6)
$$\xi_n(x) = \frac{x^{\frac{n-1}{2}}(x^2 - 1)}{2 + 2x^{\frac{-n+1}{2}} + 2x^{-n+1} + x^{\frac{-3n+3}{2}} + x^{-\frac{3n+1}{2}}} \geq \frac{x^{\frac{n-1}{2}}(x^2 - 1)}{8},$$

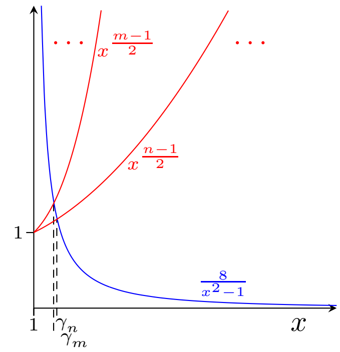
$$\text{and} \qquad \frac{x^{\frac{n-1}{2}}(x^2 - 1)}{8} = 1 \iff x^{\frac{n-1}{2}} = \frac{8}{x^2 - 1}.$$

Now we remark that

(i) The map $x \mapsto \frac{8}{x^2-1}$ is strictly decreasing on $(1, +\infty)$, $\lim_{x\to 1^+} \frac{8}{x^2-1} = +\infty$ and $\lim_{x\to\infty} \frac{8}{x^2-1} = 0$.

(ii) For every n odd and every $x \geq 1$, the map $x \mapsto x^{\frac{n-1}{2}}$ is strictly increasing and $x^{\frac{n-1}{2}}\big|_{x=1} = 1$.

(iii) For every $n, m \in \mathbb{N}$, n, m odd, $n < m$ and $x > 1$, $x^{\frac{n-1}{2}} < x^{\frac{m-1}{2}}$.



Then, for each n odd, there exists a unique real number $\gamma_n > 1$ such that $\gamma_n^{\frac{n-1}{2}} = \frac{8}{\gamma_n^2-1}$, $x^{\frac{n-1}{2}} > \frac{8}{x^2-1}$ for every $x > \gamma_n$, the sequence $\{\gamma_n\}_n$ is strictly decreasing and $\lim_{n\to\infty} \gamma_n = 1$. Hence, by (3.6)

$$\xi(x) \geq \frac{x^{\frac{n-1}{2}}(x^2-1)}{8} > 1$$

for every $x > \gamma_n$. Consequently, $\rho_n \leq \gamma_n$ for every n odd and, thus,

$$\lim_{n\to\infty} \log \rho_n \leq \lim_{n\to\infty} \log \gamma_n = 0.$$

□

Next we prove Theorem 3.7 by "exporting" the maps F_n from Theorem 3.6 to any arbitrary graph.

Proof of Theorem 3.7. If $G = \mathbb{S}^1$ then there is nothing to prove since Theorem 3.6 already gives the desired sequence of maps. So, we assume that $G \neq \mathbb{S}^1$.

Let $n \geq 7$ odd, and let F_n, P_n, Q_n and f_n be as in Theorem 3.6 and Proposition 3.8. Recall also that I_i, $0 \leq i \leq 2n-3$ and J_j, $0 \leq j \leq 3$ are $P_n \cup Q_n$-basic intervals which generate all the equivalence classes of $P_n \cup Q_n$-basic intervals.

Next we fix the general notation to be used in this proof: Let C be a circuit of G, let $I \subset C$ be an interval such that $I \cap V(G) = \emptyset$ and let $\eta \colon \mathbb{S}^1 \longrightarrow C$ be a homeomorphism such that

$$I \supset \left(\bigcup_{\substack{i=0\\ i\neq 2}}^{2n-3} \widetilde{I}_i \right) \cup \eta(\mathsf{e}(P_n \cup Q_n)) \quad \text{and} \quad C \setminus \mathsf{Int}(I) = \widetilde{I}_2,$$

where $\widetilde{I}_i := \eta(\mathsf{e}(I_i))$ for $i \in \{0, 1, \dots, 2n-3\}$. Clearly, $X := G \setminus \mathsf{Int}(I) \supset \widetilde{I}_2$ is a subgraph of G (see Figure 5). Observe that $\tilde{z}_0^2, \tilde{z}_1^2 \in I$ are endpoints (and thus vertices) of X but they cannot be vertices of G because $I \cap V(G) = \emptyset$. Moreover, $V(X) = V(G) \cup \{\tilde{z}_0^2, \tilde{z}_1^2\}$.

Recall that, for every $i \in \{0, 1, \dots, 2n-3\}$, the endpoints of I_i are $y_{\{n+1+i(n+2)\}_{2n}} < y_{\{n+2+i(n+2)\}_{2n}}$. Then, for $j = 0, 1$ we set

$$\tilde{z}_j^i := \eta\big(\mathsf{e}\big(y_{\{n+j+1+i(n+2)\}_{2n}}\big)\big)$$

so that, $\partial \widetilde{I}_i = \{\tilde{z}_0^i, \tilde{z}_1^i\}$.

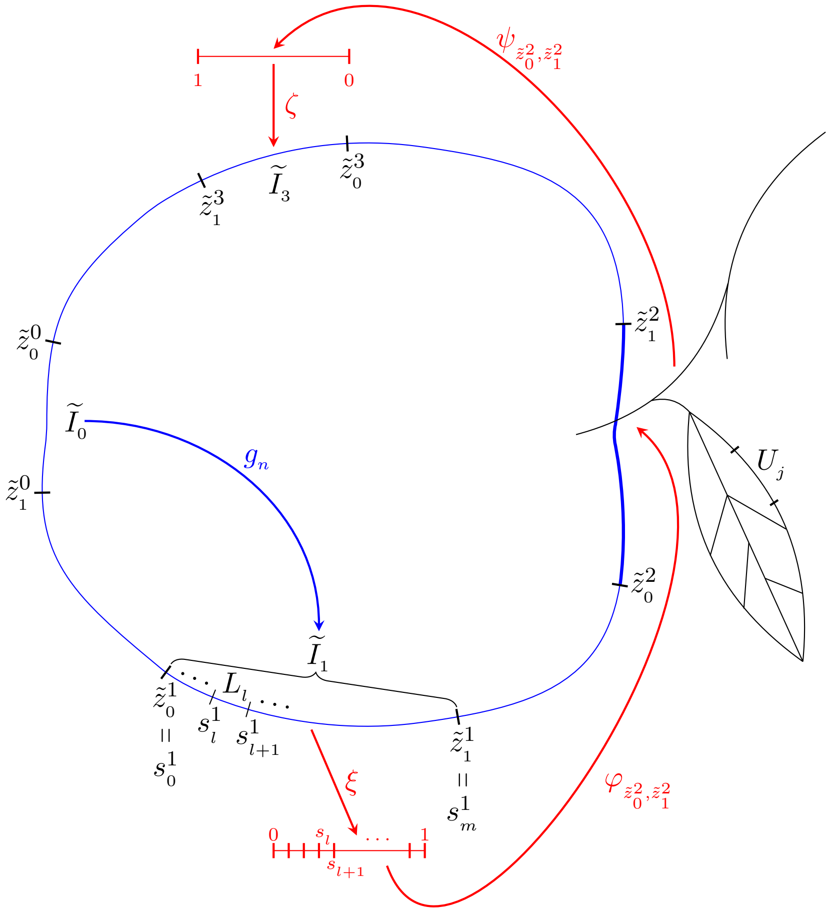


FIGURE 5. A topological graph G and the definition of g_n. The circuit C (with apple shape) is drawn in blue. Then, the interval I is the (thin) path in C from $\tilde{z}_1^2$ to $\tilde{z}_0^2$ counter-clockwise ($\partial I = \{\tilde{z}_0^2, \tilde{z}_1^2\}$) and the interval $C \setminus \mathsf{Int}(I) = \widetilde{I}_2$ is the **thick** path in C from $\tilde{z}_1^2$ to $\tilde{z}_0^2$ clockwise.

Under the assumptions of Theorem 3.6 with $n \geq 7$ odd we have

$$0 = x_0 < y_3 < y_4 < y_7 < y_8 < y_{n+5} < y_{n+6} \leq y_{2n-1} < 1.$$

Hence,

$$I_1 = [y_3, y_4], \quad I_2 = [y_{n+5}, y_{n+6}] \quad \text{and} \quad I_3 = [y_7, y_8]$$

are pairwise disjoint and, consequently, so are $\widetilde{I}_1$, $\widetilde{I}_2$ and $\widetilde{I}_3$.

To define the maps $g_n \colon G \longrightarrow G$ for every n odd, $n \geq 7$, we will use Lemma 3.4 for the subgraph X with a replaced by $\tilde{z}_0^2$ and b replaced by $\tilde{z}_1^2$ (see Figure 3).

We define the map $g_n \colon G \longrightarrow G$ (see Figure 5) by:

$$g_n(x) := \begin{cases} \varphi_{\tilde{z}_0^2,\tilde{z}_1^2}(\xi(x)) & \text{if } x \in \widetilde{I}_1; \\ \zeta\big(\psi_{\tilde{z}_0^2,\tilde{z}_1^2}(x)\big) & \text{if } x \in X; \\ (\eta \circ f_n \circ \eta^{-1})(x) & \text{if } x \in I \setminus \mathsf{Int}\big(\widetilde{I}_1\big). \end{cases}$$

Recall that $\widetilde{I}_1$, $\widetilde{I}_2$ and $\widetilde{I}_3$ are pairwise disjoint and, for every $\ell \in \mathbb{Z}$ and $j \in \{0,1\}$, $\mathsf{e}\big(y_{\{n+j+1+\ell(n+2)\}_{2n}}\big) = \mathsf{e}\big(y_{n+j+1+\ell(n+2)}\big)$ because $y_{i+2n\ell} = y_i + \ell$ for every $i, \ell \in \mathbb{Z}$. Hence, by Lemma 3.4,

$$\begin{aligned}
\varphi_{\tilde{z}^2_0,\tilde{z}^2_1}\big(\xi(\tilde{z}^1_j)\big) &= \varphi_{\tilde{z}^2_0,\tilde{z}^2_1}(j) = \tilde{z}^2_j = \eta\big(\mathsf{e}\big(y_{n+j+1+2(n+2)}\big)\big)\\
&= \eta\big(\mathsf{e}\big(F_n\big(y_{n+j+1+(n+2)}\big)\big)\big) = \eta\big(f_n\big(\mathsf{e}\big(y_{n+j+1+(n+2)}\big)\big)\big)\\
&= \eta\big(f_n\big(\eta^{-1}\big(\eta\big(\mathsf{e}\big(y_{n+j+1+(n+2)}\big)\big)\big)\big)\big) = \eta\circ f_n\circ\eta^{-1}\big(\tilde{z}^1_j\big), \text{ and}\\
\zeta\big(\psi_{\tilde{z}^2_0,\tilde{z}^2_1}\big(\tilde{z}^2_j\big)\big) &= \zeta(j) = \tilde{z}^3_j = \eta\big(\mathsf{e}\big(y_{n+j+1+3(n+2)}\big)\big)\\
&= \eta\big(\mathsf{e}\big(F_n\big(y_{n+j+1+2(n+2)}\big)\big)\big) = \eta\big(f_n\big(\mathsf{e}\big(y_{n+j+1+2(n+2)}\big)\big)\big)\\
&= \eta\big(f_n\big(\eta^{-1}\big(\eta\big(\mathsf{e}\big(y_{n+j+1+2(n+2)}\big)\big)\big)\big)\big) = \eta\circ f_n\circ\eta^{-1}\big(\tilde{z}^2_j\big)
\end{aligned}$$

for $j \in \{0,1\}$. So, g_n is continuous because the maps $\varphi_{\tilde{z}^2_0,\tilde{z}^2_1}\circ\xi\big|_{\widetilde{I}_1}$, $\zeta\circ\psi_{\tilde{z}^2_0,\tilde{z}^2_1}\big|_X$, and $\eta\circ f_n\circ\eta^{-1}\big|_{I\setminus\mathsf{Int}(\widetilde{I}_1)}$ are continuous.

To be able to compute and use a Markov partition for the map g_n we introduce the following notation. Set

$$\begin{aligned}
s^1_i &= \xi^{-1}(s_i) \text{ for } i = 0,1,\dots,m, \text{ and}\\
R_n &= \eta\big(\mathsf{e}\big(Q_n \cup P_n\big)\big) \cup \big\{s^1_i : i \in \{0,1,\dots,m\}\big\} \cup\\
&\qquad \Big\{\varphi_{\tilde{z}^2_0,\tilde{z}^2_1}(s_i) : i \in \{0,1,\dots,m\}\Big\}
\end{aligned} \tag{3.7}$$

and observe that

$$\begin{aligned}
&\eta\big(\mathsf{e}\big(Q_n \cup P_n\big)\big) =\\
&\qquad \eta(\mathsf{e}(Q_n)) \cup \Big\{\tilde{z}^i_j : i \in \{0,1,\dots,2n-3\} \text{ and } j \in \{0,1\}\Big\} \subset\\
&\qquad\qquad I \setminus \mathsf{Int}\big(\widetilde{I}_1\big),\\
&\qquad \big\{s^1_i : i \in \{0,1,\dots,m\}\big\} \subset \widetilde{I}_1 \text{ and}\\
&\qquad\qquad \Big\{\varphi_{\tilde{z}^2_0,\tilde{z}^2_1}(s_i) : i \in \{0,1,\dots,m\}\Big\} \subset X.
\end{aligned}$$

Moreover, by Lemma 3.4(a),

$$R_n \supset \Big\{\varphi_{\tilde{z}^2_0,\tilde{z}^2_1}(s_i) : i \in \{0,1,\dots,m\}\Big\} \supset V(X) \supset V(G).$$

Hence, R_n will be a Markov invariant set provided that it is g_n-invariant and the closure of each connected component of $G \setminus R_n$ is an interval in G.

Fix a point $x \in R_n$. Then, $g_n(x)$ is

$$(3.8) \quad \begin{cases} \eta\big(f_n\big(\eta^{-1}(\eta(\mathsf{e}(t)))\big)\big) = \eta\big(f_n(\mathsf{e}(t))\big) = \\ \qquad\qquad \eta\big(\mathsf{e}\big(F_n(t)\big)\big) \in \eta\big(\mathsf{e}\big(Q_n \cup P_n\big)\big) \\ \qquad \text{if } x = \eta(\mathsf{e}(t)) \in I \setminus \mathsf{Int}\big(\widetilde{I}_1\big) \text{ with } t \in Q_n \cup P_n \\ \qquad \text{(here we use Theorem 3.6)}, \\ \varphi_{\tilde{z}_0^2,\tilde{z}_1^2}\big(\xi\big(s_i^1\big)\big) = \varphi_{\tilde{z}_0^2,\tilde{z}_1^2}(s_i) \in \Big\{\varphi_{\tilde{z}_0^2,\tilde{z}_1^2}(s_i) : i \in \{0,1,\ldots,m\}\Big\} \\ \qquad \text{if } x = s_i^1 \in \widetilde{I}_1 \text{ with } i \in \{0,1,\ldots,m\}, \\ \zeta\Big(\psi_{\tilde{z}_0^2,\tilde{z}_1^2}\Big(\varphi_{\tilde{z}_0^2,\tilde{z}_1^2}(s_i)\Big)\Big) \in \zeta\big(\{0,1\}\big) = \big\{\tilde{z}_0^3, \tilde{z}_1^3\big\} \in \eta\big(\mathsf{e}\big(Q_n \cup P_n\big)\big) \\ \qquad \text{if } x = \varphi_{\tilde{z}_0^2,\tilde{z}_1^2}(s_i) \in X \text{ with } i \in \{0,1,\ldots,m\} \\ \qquad \text{(here we use Lemma 3.4(d))}. \end{cases}$$

In either case, $g_n(x) \in R_n$ and, consequently, R_n is g_n-invariant.

Let K be a connected component of $G \setminus R_n$. Since I is an interval with endpoints $\big\{\tilde{z}_0^2, \tilde{z}_1^2\big\} \subset R_n$, either $\mathsf{Clos}(K) \subset I$ or $\mathsf{Clos}(K) \subset X$. In the first case, $\mathsf{Clos}(K)$ is clearly an interval. Now assume that $K \subset \mathsf{Clos}(K) \subset X$. Clearly, K is a connected component of $X \setminus R_n = X \setminus \Big\{\varphi_{\tilde{z}_0^2,\tilde{z}_1^2}(s_i) : i \in \{0,1,\ldots,m\}\Big\}$. Since the map $\varphi_{\tilde{z}_0^2,\tilde{z}_1^2} : [0,1] \longrightarrow X$ is surjective and

$$K \cap \Big\{\varphi_{\tilde{z}_0^2,\tilde{z}_1^2}(s_i) : i \in \{0,1,\ldots,m\}\Big\} = \emptyset,$$

by Lemma 3.4(a), $\varphi_{\tilde{z}_0^2,\tilde{z}_1^2}\big((s_i, s_{i+1})\big) = K$ for some $i \in \{0,1,\ldots,m-1\}$. Hence, by Lemma 3.4(b), $\mathsf{Clos}(K) = \varphi_{\tilde{z}_0^2,\tilde{z}_1^2}\big([s_i, s_{i+1}]\big)$ is an interval. This shows that R_n is a Markov invariant set for g_n.

The R_n-basic intervals are:

$\widetilde{I}_0, \widetilde{I}_3, \widetilde{I}_4, \ldots, \widetilde{I}_{2n-3} \subset I \setminus \mathsf{Int}\big(\widetilde{I}_1\big)$

$\widetilde{J}_j := \eta\big(\mathsf{e}\big(J_j\big)\big) \subset I \setminus \mathsf{Int}\big(\widetilde{I}_1\big)$ for $j = 0,1,2,3$,

$L_i := \langle s_i^1, s_{i+1}^1 \rangle_{\tilde{I}_1} = \xi^{-1}([s_i, s_{i+1}]) \subset \widetilde{I}_1$ for $i = 0,1,\ldots,m-1$, and

$\varphi_{\tilde{z}_0^2,\tilde{z}_1^2}\big([s_i, s_{i+1}]\big) \subset X$ for $i = 0,1,\ldots,m-1$.

Moreover, unlike other R_n-basic intervals, the elements of

$$\Big\{\varphi_{\tilde{z}_0^2,\tilde{z}_1^2}\big([s_i, s_{i+1}]\big) : i \in \{0,1,\ldots,m-1\}\Big\}$$

may coincide. The (pairwise different) elements of this set will be denoted by $U_0, U_1, \ldots, U_r$ with $r \leq m-1$, so that

$$\{U_0, U_1, \ldots, U_r\} = \Big\{\varphi_{\tilde{z}_0^2,\tilde{z}_1^2}\big([s_i, s_{i+1}]\big) : i \in \{0,1,\ldots,m-1\}\Big\} \subset X.$$

Next we will show that g_n is a Markov map with respect to R_n and we will compute the Markov graph of g_n with respect to R_n. More precisely, we will show that g_n is monotone at every basic interval and derive the Markov graph of g_n with respect to R_n from the Markov graph of f_n with respect to $\mathsf{e}\big(P_n \cup Q_n\big)$ which, by the proof of Theorem 3.6, coincides with the Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$ (see Proposition 3.8(b)) provided that we identify $[\![I]\!]$ with $\mathsf{e}([\![I]\!]) = \mathsf{e}(I)$ for every $I \in \mathcal{B}(P_n \cup Q_n)$.

We start by observing that if $K \in \{\widetilde{I}_0, \widetilde{I}_3, \widetilde{I}_4, \dots, \widetilde{I}_{2n-3}, \widetilde{J}_0, \widetilde{J}_1, \widetilde{J}_2, \widetilde{J}_3\}$, (that is, $K \in \mathcal{B}(R_n)$ with $K \subset I \setminus \mathsf{Int}(\widetilde{I}_1)$), then $\eta^{-1}(K) \in \mathcal{B}(\mathsf{e}(Q_n \cup P_n))$ and $g_n(K) = (\eta \circ f_n)(\eta^{-1}(K))$, which is equivalent to $\eta^{-1}(g_n(K)) = f_n(\eta^{-1}(K))$. Consequently, g_n is monotone on K because f_n is a Markov map with respect to $\mathsf{e}(P_n \cup Q_n)$ (see the proof of Theorem 3.6) and, for every interval $L \in \{\widetilde{I}_0, \widetilde{I}_3, \widetilde{I}_4, \dots, \widetilde{I}_{2n-3}, \widetilde{J}_0, \widetilde{J}_1, \widetilde{J}_2, \widetilde{J}_3\} \cup \{\widetilde{I}_1\}$, it follows that K g_n-covers L if and only if $\eta^{-1}(K)$ f_n-covers $\eta^{-1}(L)$. With the help of Proposition 3.8(b) this gives the Markov graph of g_n on the intervals $J \in \mathcal{B}(R_n)$ with $J \subset I \setminus \mathsf{Int}(\widetilde{I}_1)$ and shows that $\widetilde{I}_0$ g_n-covers L_i for $i = 0, 1, \dots, m-1$ (to illustrate this fact see Figure 6 which shows the Markov graph of g_n in the case $n = 4k-1$ and $d = 2k-1$ with $k \in \mathbb{N}$).

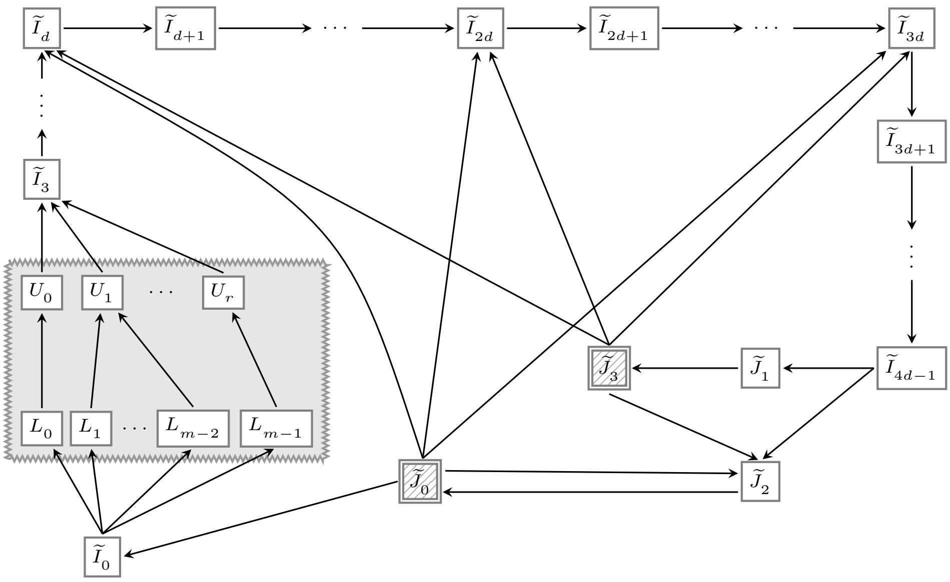


FIGURE 6. The Markov graph of g_n in the case when $n = 4k-1$ and $d = 2k-1$ with $k \in \mathbb{N}$ (being the other case when $n = 4k+1$ and $d = 2k+1$). The part of the Markov graph of g_n with respect to R_n which differs from the Markov graph of f_n with respect to $\mathsf{e}(P_n \cup Q_n)$ is shown inside a grey box with a zigzag border (see Proposition 3.8(b)). The arrows between the intervals L_i and U_j are just illustrative.

Now we consider the intervals L_i. Clearly, by Lemma 3.4(b),

$$g_n(L_i) = \varphi_{\tilde{z}_0^2, \tilde{z}_1^2}(\xi(L_i)) = \varphi_{\tilde{z}_0^2, \tilde{z}_1^2}([s_i, s_{i+1}]) \in \{U_0, U_1, \dots, U_r\}$$

and g_n is monotone on L_i. This shows that in the Markov graph of g_n every interval L_i g_n-covers a unique interval U_j but different intervals L_i can g_n-cover the same interval U_j (see again Figure 6).

Finally, we consider the intervals U_j. Clearly, by Lemma 3.4(e),

$$g_n(U_j) = \zeta\big(\psi_{\tilde z_0^2,\tilde z_1^2}(U_j)\big) = \zeta\big(\psi_{\tilde z_0^2,\tilde z_1^2}\big(\varphi_{\tilde z_0^2,\tilde z_1^2}([s_{i_j}, s_{i_j+1}])\big)\big) = \zeta([0,1]) = \widetilde{I}_3$$

and g_n is monotone on U_j. This shows that in the Markov graph of g_n every interval U_j g_n-covers a unique interval $\widetilde{I}_3$ (see once more Figure 6).

We just have seen that g_n is a Markov map with respect to R_n such that $f(K)$ is a (non-degenerate) interval for every $K \in \mathcal{B}(R_n)$. Then, by Lemma 3.5, the map g_n can be modified without altering $g_n\big|_{R_n}$ and $g_n(K)$ for every $K \in \mathcal{B}(R_n)$ in such a way that g_n becomes R_n-expansive. So, we can use Theorem 2.13 to prove that g_n is transitive. The Markov graph of g_n (see Figure 6 and Proposition 3.8(b)) tells us that the Markov matrix of g_n with respect to R_n is not a permutation matrix because there is at least one basic interval which g_n-covers more than one basic interval (for instance the interval $\widetilde{J}_0$ that g_n-covers 4 different intervals). Moreover, by direct inspection of the Markov graph of g_n, given any two vertices $\widetilde{I}$ and $\widetilde{J}$ in the graph, there exists a path from $\widetilde{I}$ to $\widetilde{J}$. This means that the transition matrix of the Markov graph of g_n is non-negative and irreducible. Thus, g_n is transitive by Theorem 2.13.

Next we will show that $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$ (which will also be helpful in showing that g_n is totally transitive).

In what follows, given $q \in \mathbb{N}$ and $A \subset \mathbb{N}$ we will denote the set $\{q\ell : \ell \in A\}$ by $q \cdot A$.

Observe that, by Theorem 3.6, $q \in \mathsf{Per}(f_n)$, $q \neq 2$ implies $q \geq 2k+1 > \frac{n}{2}$. Thus, for every $\ell \in \mathbb{N} \setminus \{1\}$, $\ell q > n$ and, hence, $\ell q \in \mathsf{Succ}\,(n) \subset \mathsf{Per}(f_n)$. Consequently, again by Theorem 3.6,

$$\mathsf{Per}(f_n) = \{2\} \cup \bigcup_{\substack{q\in\mathsf{Per}(f_n)\\ q\neq 2}} q\cdot\{1\} \subset \{2\} \cup \bigcup_{\substack{q\in\mathsf{Per}(f_n)\\ q\neq 2}} q\cdot\mathbb{N} =$$
$$\mathsf{Per}(f_n) \cup \bigcup_{\substack{q\in\mathsf{Per}(f_n)\\ q\neq 2}} q\cdot\big(\mathbb{N}\setminus\{1\}\big) \subset \mathsf{Per}(f_n) \cup \mathsf{Succ}\,(n) = \mathsf{Per}(f_n).$$

So, to prove that $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$ it is enough to show that

$$\mathsf{Per}(g_n) = \{2\} \cup \bigcup_{\substack{q\in\mathsf{Per}(f_n)\\ q\neq 2}} q\cdot\mathbb{N}.$$

First we will show that $2 \in \mathsf{Per}(g_n)$. Recall that, by Theorem 3.6, Q_n is a lifted periodic orbit of F_n of period 2, which implies that $\mathsf{e}(Q_n)$ is a periodic orbit of f_n of period 2. Moreover, $\eta\big(\mathsf{e}(Q_n)\big) \subset I \setminus \mathsf{Int}\big(\widetilde{I}_1\big)$ and, hence, $g_n(y) = \big(\eta\circ f_n\circ\eta^{-1}\big)(y)$ for every $y \in \eta\big(\mathsf{e}(Q_n)\big)$ (here and in the rest of the proof that $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$ we use the fact that $g_n\big|_{R_n}$ has not been modified). Thus, $\eta\big(\mathsf{e}(Q_n)\big)$ is a periodic orbit of g_n of period 2.

Let Y be a periodic orbit of g_n of period $p \neq 2$. We have to see that $p = \ell q$ with $\ell \in \mathbb{N}$ and $q \in \mathsf{Per}(f_n)$, $q \neq 2$. In the same way as before, $\eta\big(\mathsf{e}(P_n)\big)$ is a periodic orbit of g_n of period $2n$, and $\mathsf{e}\big(P_n\big)$ is a periodic orbit of f_n of period $2n$. So, if $Y = \eta(\mathsf{e}(P_n))$, $p \in \mathsf{Per}(f_n)$, $p \neq 2$ and we are done in this case.

In the rest of the proof we assume that $Y \neq \eta(\mathsf{e}(P_n))$. Then, $Y \cap R_n = \emptyset$. Indeed, otherwise,

$$\emptyset \neq g_n^2(Y \cap R_n) \subset g_n^2(Y) \cap g_n^2(R_n) \subset \\ g_n^2(Y) \cap \eta(\mathsf{e}(Q_n \cup P_n)) = Y \cap \Big(\eta(\mathsf{e}(Q_n)) \cup \eta(\mathsf{e}(P_n))\Big)$$

by (3.7) and (3.8). Thus, $Y \cap \eta(\mathsf{e}(Q_n)) \neq \emptyset$ because $Y \neq \eta(\mathsf{e}(P_n))$ implies $Y \cap\ \eta(\mathsf{e}(P_n)) = \emptyset$. Then, since both Y and $\eta(\mathsf{e}(Q_n))$ are periodic orbits of g_n, it follows that $Y = \eta(\mathsf{e}(Q_n))$; a contradiction because we are assuming that $p \neq 2$. Therefore we have shown that Y is disjoint from R_n. Consequently, by Proposition 2.8, there is a loop $\lambda = K_0 \longrightarrow K_1 \longrightarrow \cdots \longrightarrow K_{p-1} \longrightarrow K_0$ in the Markov graph of g_n of length p associated to Y.

Now we define a projection $\pi\colon \mathcal{B}(R_n) \longrightarrow \mathcal{B}(\mathsf{e}(P_n \cup Q_n))$ from the set of basic intervals of g_n to the set of basic intervals of f_n in the following way. For every $K \in \mathcal{B}(R_n)$ we set:

$$\pi(K) := \begin{cases} \eta^{-1}(K) & \text{if } K \subset I \setminus \widetilde{I}_1; \\ \mathsf{e}(I_1) & \text{if } K \subset \widetilde{I}_1; \\ \mathsf{e}(I_2) & \text{if } K \subset X. \end{cases}$$

It is clear by construction (see Figure 6 and Proposition 3.8(b)) that if there is an arrow $J \longrightarrow L$ in the Markov graph of g_n, then there is an arrow $\pi(J) \longrightarrow \pi(L)$ in the Markov graph of f_n. Moreover, since λ is a loop of length p in the Markov graph of g_n, the projection of λ

$$\pi(\lambda) := \pi(K_0) \longrightarrow \pi(K_1) \longrightarrow \cdots \longrightarrow \pi(K_{p-1}) \longrightarrow \pi(K_0)$$

is a loop in the Markov graph of f_n of the same length. By Lemma 2.7, there exists $x \in \pi(K_0)$ such that $f_n^i(x) \in \pi(K_i)$, $0 \leq i \leq p-1$ and $f^p(x) = x$. Then, the f_n-period of x is q, a divisor of p, so that $p = \ell q$ with $\ell \in \mathbb{N}$ and $q \in \mathsf{Per}(f_n)$. To end the proof that $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$ only it remains to show that $q \neq 2$.

By way of contradiction we assume that $q = 2$ (so that the f_n-orbit of x is $\{x, f_n(x)\}$). Clearly, in this case, $\ell \geq 2$ (and, hence, $p > q$) because $p \neq 2$. On the other hand, as it it has been already justified after enumerating the R_n-basic intervals, the Markov graph of f_n with respect to $\mathsf{e}(P_n \cup Q_n)$ coincides with the Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$ provided that we identify $[\![I]\!]$ with $\mathsf{e}([\![I]\!]) = \mathsf{e}(I)$ for every $I \in \mathcal{B}(P_n \cup Q_n)$. Consequently, Proposition 3.8(b) gives the Markov graph of f_n with respect to $\mathsf{e}(P_n \cup Q_n)$.

First we consider the case when $\{x, f_n(x)\} \cap \mathsf{e}(P_n \cup Q_n) = \emptyset$. By Proposition 2.8(a), there exists a loop associated to $\{x, f_n(x)\}$ in the Markov graph of f_n. By Proposition 3.8(b), $\{x, f_n(x)\}$ is associated to the loop $\mathsf{e}(J_0) \longrightarrow \mathsf{e}(J_2) \longrightarrow \mathsf{e}(J_0)$, which is the unique loop of length 2 in the Markov graph of f_n. Moreover, since $\{x, f_n(x)\} \cap \mathsf{e}(P_n \cup Q_n) = \emptyset$, and $f_n^i(x) \in \pi(K_i)$ for $0 \leq i \leq p-1$ and $f^p(x) = x$, it follows that $\pi(\lambda)$ is an ℓ-repetition of $\mathsf{e}(J_0) \longrightarrow \mathsf{e}(J_2) \longrightarrow \mathsf{e}(J_0)$. Hence, in view of the definition of π, we have that $\pi^{-1}(\pi(\lambda)) = \lambda$ and, thus, λ is an ℓ-repetition of the loop

$$\widetilde{J}_0 = \pi^{-1}(\mathsf{e}(J_0)) \longrightarrow \widetilde{J}_2 = \pi^{-1}(\mathsf{e}(J_2)) \longrightarrow \widetilde{J}_0 = \pi^{-1}(\mathsf{e}(J_0)).$$

By Proposition 2.8(b) applied to λ it follows that $\ell = 2$ and $\widetilde{J}_0 \longrightarrow \widetilde{J}_2 \longrightarrow \widetilde{J}_0$ must be negative. However, with the notation of Proposition 3.8 it follows that $F_n\big|_{J_0}$ and $F_n\big|_{J_2}$ are strictly increasing and

$$F_n(J_0) = [x_1, y_{n+2}] \supsetneq [x_1, y_{n-2}] = J_2 \quad \text{and} \quad F_n(J_2) = [x_2, y_{2n}] = J_0 + 1.$$

Thus, the loop $\widetilde{J}_0 \longrightarrow \widetilde{J}_2 \longrightarrow \widetilde{J}_0$ is positive; a contradiction.

Now we consider the case when $\{x, f_n(x)\} \cap \mathsf{e}\big(P_n \cup Q_n\big) \neq \emptyset$. Clearly, since $\{x, f_n(x)\}$, $\mathsf{e}\big(Q_n\big)$ and $\mathsf{e}\big(P_n\big)$ are periodic orbits of f_n, we have either $\{x, f_n(x)\} = \mathsf{e}\big(Q_n\big)$ or $\{x, f_n(x)\} = \mathsf{e}\big(P_n\big)$. Furthermore, since the f_n period of $\mathsf{e}\big(P_n\big)$ is $2n$, it follows that $\{x, f_n(x)\} = \mathsf{e}\big(Q_n\big)$. On the other hand, observe that the only intervals of $\mathcal{B}\big(\mathsf{e}\big(P_n \cup Q_n\big)\big)$ containing a point from $\mathsf{e}\big(Q_n\big)$ are $\mathsf{e}(J_0)$, $\mathsf{e}(J_1)$, $\mathsf{e}(J_2)$ and $\mathsf{e}(J_3)$ (see Proposition 3.8). Thus, since $f_n^i(x) \in \mathsf{e}\big(Q_n\big) \cap \pi(K_i)$ for $0 \leq i \leq p-1$, it follows that

$$\pi(K_i) \in \{\mathsf{e}(J_0), \mathsf{e}(J_1), \mathsf{e}(J_2), \mathsf{e}(J_3)\} \quad \text{for} \quad 0 \leq i \leq p-1.$$

Moreover, as it can be checked in Proposition 3.8(b), $\mathsf{e}(J_1)$ is not f_n-covered by any of these four intervals. So, $\mathsf{e}(J_1)$ cannot appear in $\pi(\lambda)$. In a similar way, since $\mathsf{e}(J_3)$ is not f_n covered by $\mathsf{e}(J_0)$ and $\mathsf{e}(J_2)$ and is f_n-covered only by $\mathsf{e}(J_1)$ which, as we already know, does not take part in $\pi(\lambda)$, $\mathsf{e}(J_1)$ does not appear in $\pi(\lambda)$. Consequently, $\pi(K_i) \in \{\mathsf{e}(J_0), \mathsf{e}(J_2)\}$ for $0 \leq i \leq p-1$ and, as in the previous case, $\pi^{-1}(\pi(\lambda)) = \lambda$. So,

$$K_i \in \big\{\widetilde{J}_0 = \pi^{-1}(\mathsf{e}(J_0)), \widetilde{J}_2 = \pi^{-1}(\mathsf{e}(J_2))\big\} \quad \text{for} \quad 0 \leq i \leq p-1.$$

We recall that the only loop in the Markov graph of g_n consisting only on intervals $\widetilde{J}_0$ and $\widetilde{J}_2$ is $\widetilde{J}_0 \longrightarrow \widetilde{J}_2 \longrightarrow \widetilde{J}_0$ (see Figure 6) and that this loop is positive. Thus, either $\lambda = \widetilde{J}_0 \longrightarrow \widetilde{J}_2 \longrightarrow \widetilde{J}_0$ or λ is an ℓ-repetition of $\widetilde{J}_0 \longrightarrow \widetilde{J}_2 \longrightarrow \widetilde{J}_0$. In view of Proposition 2.8(b), this last option is not possible because, in that case, λ would be a repetition of a positive loop and hence $\lambda = \widetilde{J}_0 \longrightarrow \widetilde{J}_2 \longrightarrow \widetilde{J}_0$ and $p = 2$; a contradiction. This ends the proof that $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$.

On the other hand, since $\mathsf{Per}(g_n) = \mathsf{Per}(f_n) \supset \mathsf{Succ}\,(n)$ and g_n is transitive, it follows that $\mathsf{Per}(g_n)$ is cofinite in $\mathbb{N}$ and g_n is totally transitive by Theorem 1.2.

To end the proof of the theorem we will estimate $h(g_n)$ in a similar way as in Proposition 3.8. So, we choose $\mathsf{Rom}_n = \{\widetilde{J}_0, \widetilde{J}_3\}$ as a rome in both cases: $n = 4k+1$ and $n = 4k-1$. Then, the matrix $M_{\mathsf{Rom}_n}(x)$ is:

$$\begin{cases} \begin{pmatrix} x^{-2} + mx^{-4d-2} + \alpha(x) & mx^{-4d-2} + \alpha(x) \\ x^{-2} + \alpha(x) & \alpha(x) \end{pmatrix} & \text{when } n = 4k-1 \text{ and } d = \frac{n-1}{2} \text{ for some } k \in \mathbb{N}, \text{ and} \\[2ex] \begin{pmatrix} x^{-2} + mx^{-4d+2} + \alpha(x) & mx^{-4d+2} + \alpha(x) \\ x^{-2} + \alpha(x) & \alpha(x) \end{pmatrix} & \text{when } n = 4k+1 \text{ and } d = \frac{n+1}{2} \text{ for some } k \in \mathbb{N}, \end{cases}$$

where

$$\alpha(x) = \begin{cases} x^{-3d-2} + x^{-2d-2} + x^{-d-2} & \text{when } n = 4k-1 \text{ and } d = \frac{n-1}{2} \text{ for some } k \in \mathbb{N}, \text{ and} \\ x^{-3d} + x^{-2d} + x^{-d} & \text{when } n = 4k+1 \text{ and } d = \frac{n+1}{2} \text{ for some } k \in \mathbb{N}. \end{cases}$$

Finally we are ready to compute the characteristic polynomial $P_n(x)$ of the Markov matrix of g_n with respect to R_n by using Theorem 2.12. As in Proposition 3.8 it turns out that it is the same in both cases: $n = 4k-1$ and $d = \frac{n-1}{2}$ or $n = 4k+1$ and $d = \frac{n+1}{2}$. We get

$$P_n(x) = x^{2n}(x^2-1) - 2x^{\frac{3n+1}{2}} - 2x^{n+1} - 2x^{\frac{n+3}{2}} - mx^2 - m$$

and $h(g_n) = \log \rho_n$ where ρ_n is the largest root (larger than one) of P_n.

The polynomial P_n is very similar to the polynomial T_n in Proposition 3.8. Thus, reasoning as at the end of the proof of Theorem 3.6, we conclude that, for each n odd, there exists a real number $\gamma_n \geq \rho_n$ such that the sequence $\{\gamma_n\}_n$ is strictly decreasing and $\lim_{n\to\infty} \gamma_n = 1$. Consequently, $\lim_{n\to\infty} h(g_n) = 0$ by Proposition 2.11. □

3.4. **Example with low non-constant periods.** This subsection is devoted to construct and prove

Example 1.11. *For every $n \in \mathbb{N}$, $n \geq 3$ there exists f_n, a totally transitive continuous circle map of degree one having a lifting $F_n \in \mathcal{L}_1$ such that*

$$\mathsf{Rot}(F_n) = \left[\tfrac{2n-1}{2n^2}, \tfrac{2n+1}{2n^2}\right] = \left[\tfrac{1}{n} - \tfrac{1}{2n^2}, \tfrac{1}{n} + \tfrac{1}{2n^2}\right],$$

$\lim_{n\to\infty} h(f_n) = 0$ and

$$\mathsf{Per}(f_n) = \{n\} \cup \left\{tn + k : t \in \{2,3,\dots,\nu-1\} \text{ and } -\tfrac{t}{2} < k \leq \tfrac{t}{2},\ k \in \mathbb{Z}\right\} \cup \mathsf{Succ}\left(n\nu + 1 - \tfrac{\nu}{2}\right)$$

with

$$\nu = \begin{cases} n & \text{if } n \text{ is even, and} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, $\mathsf{StrBdCof}(f_n) = n\nu + 1 - \frac{\nu}{2}$ and $\mathsf{BdCof}(f_n)$ exists and verifies $n \leq \mathsf{BdCof}(f_n) \leq n\nu - 1 - \frac{\nu}{2}$ (and hence, $\lim_{n\to\infty} \mathsf{BdCof}(f_n) = \infty$).

Furthermore, given any graph G with a circuit, the sequence of maps $\{f_n\}_{n=4}^{\infty}$ can be extended to a sequence of continuous totally transitive maps $g_n \colon G \longrightarrow G$ such that $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$ and $\lim_{n\to\infty} h(g_n) = 0$.

As in the previous subsection, Example 1.11 will be split into Theorem 3.9 which shows the existence of the circle maps f_n by constructing them along the lines of Subsection 3.1, and Theorem 3.10 that extends these maps to a generic graph that is not a tree. The proof of Theorem 3.9, in turn, will use a proposition that computes the Markov graph modulo 1 of the liftings F_n.

Theorem 3.9. *Let $n \in \mathbb{N}$, $n \geq 3$, $p = 2n-1$, $r = 2n+1$ and $q = 2n^2$, and let*

$$Q_n = \{\dots x_{-1}, x_0, x_1, x_2, \dots, x_{q-1}, x_q, x_{q+1}, \dots\} \subset \mathbb{R}, \quad \text{and}$$

$$P_n = \{\dots y_{-1}, y_0, y_1, y_2, \dots, y_{q-1}, y_q, y_{q+1}, \dots\} \subset \mathbb{R}$$

be infinite sets such that the points of P_n and Q_n are intertwined so that

$$\begin{array}{l}
x_0 = 0 < x_1 < \cdots < x_{n-1} < \\
\begin{array}{ccccccccccccc}
x_n & < \cdots < & x_{p+n-1} & & & < & & & y_0 & < \cdots < & y_n & < \\
x_{p+n} & < \cdots < & x_{2p+n-1} & < & y_{n+1} & < \cdots < & y_{2n} & < & y_r & < \cdots < & y_{r+n} & < \\
x_{2p+n} & < \cdots < & x_{3p+n-1} & < & y_{r+n+1} & < \cdots < & y_{r+2n} & < & y_{2r} & < \cdots < & y_{2r+n} & < \\
\vdots & < \cdots < & \vdots & < & \vdots & < \cdots < & \vdots & < & \vdots & < \cdots < & \vdots & < \\
x_{(n-1)p+n} & < \cdots < & x_{np+n-1} & < & y_{(n-2)r+n+1} & < \cdots < & y_{(n-2)r+2n} & < & y_{(n-1)r} & < \cdots < & y_{(n-1)r+n} & <
\end{array} \\
\hfill x_q = 1,
\end{array}$$

and $x_{i+q\ell} = x_i + \ell$ and $y_{i+q\ell} = y_i + \ell$ for every $i, \ell \in \mathbb{Z}$.

We define a lifting $F_n \in \mathcal{L}_1$ such that, for every $i \in \mathbb{Z}$, $F_n(x_i) = x_{i+p}$ and $F_n(y_i) = y_{i+r}$, and F_n is affine between consecutive points of $P_n \cup Q_n$. Then, Q_n and P_n are twist lifted periodic orbits of F_n both of period q such that Q_n has rotation number $\frac{p}{q}$ and P_n has rotation number $\frac{r}{q}$. Moreover, the map F_n has $\mathsf{Rot}(F_n) = \left[\frac{p}{q}, \frac{r}{q}\right]$ as rotation interval.

Let $f_n \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the continuous map which has F_n as a lifting. Then, f_n is totally transitive, $\lim_{n\to\infty} h(f_n) = 0$,

$$\begin{aligned}\mathsf{Per}(f_n) = \{n\} \cup \\ & \left\{tn + k : t \in \{2, 3, \dots, \nu - 1\} \text{ and } -\tfrac{t}{2} < k \le \tfrac{t}{2},\ k \in \mathbb{Z}\right\} \cup \\ & \qquad\qquad \mathsf{Succ}\left(n\nu + 1 - \tfrac{\nu}{2}\right).\end{aligned}$$

Moreover, $\mathsf{StrBdCof}(f_n) = n\nu + 1 - \frac{\nu}{2}$ and $\mathsf{BdCof}(f_n)$ exists and verifies $n \le \mathsf{BdCof}(f_n) \le n\nu - 1 - \frac{\nu}{2}$ (and hence, $\lim_{n\to\infty} \mathsf{BdCof}(f_n) = \infty$).

Theorem 3.10. *Let G be a graph with a circuit. Then, the sequence of maps $\{f_n\}_{n=4}^{\infty}$ from Theorem 3.9 can be extended to a sequence of continuous totally transitive self maps of G, $\{g_n\}_{n=4}^{\infty}$, such that $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$ and $\lim_{n\to\infty} h(g_n) = 0$.*

Before proving Theorem 3.9 we will study the Markov graph modulo 1 of the liftings F_n.

Proposition 3.11 ($\mathcal{B}(P_n \cup Q_n)$ and the F_n-Markov graph modulo 1)**.** *In the assumptions of Theorem 3.9 we have*

(a) The Markov graph modulo 1 of F_n is the one shown in Figure 7, where the double arrows arriving to the the boxes in grey mean that there is an arrow arriving to each class of basic intervals modulo 1 in the box.

(b) $h(f_n) = \log \rho_n$, where $\rho_n > 1$ is the largest root of the polynomial

$$T_n(x) = \kappa_2(x)\big(x^{2q} + 1\big) + \kappa_1(x)x^{q+n} - 2\big(x^{4n} - 2x^{2n-1} + 1\big),$$

where

$$\kappa_2(x) = x^{4n} - 2x^{3n} - x^{2n+1} - 2x^{2n} - 3x^{2n-1} - 2x^n + 1,$$

and

$$\kappa_1(x) = 4x^{2n} + 2x^{n+1} + 4x^n + 2x^{n-1} + 4.$$

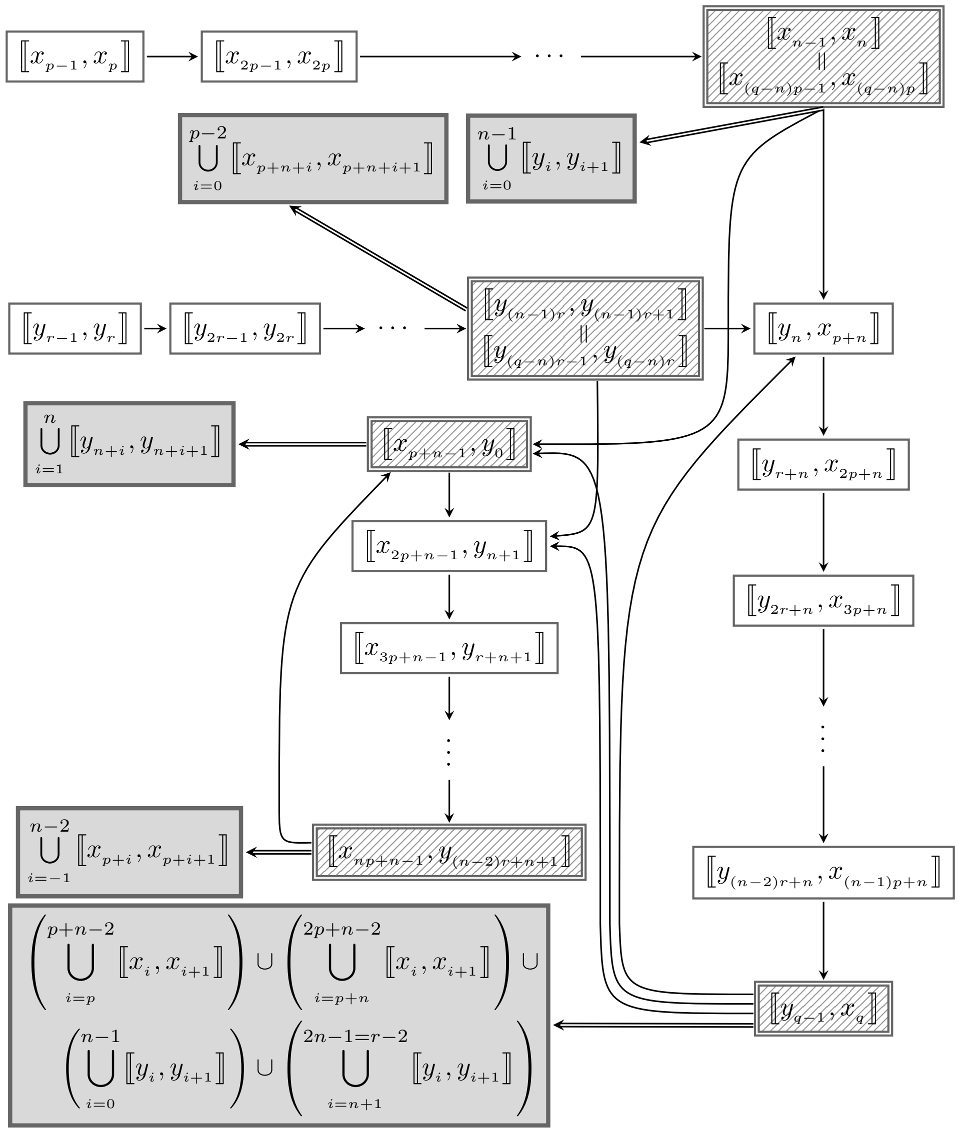


FIGURE 7. The Markov graph modulo 1 of the map F_n from Theorem 3.9. The double arrows arriving to the the boxes in grey mean that there is an arrow arriving to each interval in the box.

Proof. We start by proving (a) but before it is helpful to introduce a new auxiliary definition. Let $\alpha = [\![I_0]\!] \longrightarrow [\![I_1]\!] \longrightarrow [\![I_2]\!] \longrightarrow \cdots \longrightarrow [\![I_k]\!]$ be a path in the Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$. We say that α is a *road* if $[\![I_i]\!]$ only F_n-covers $[\![I_{i+1}]\!]$ for $i = 0, 1, 2, \ldots, k-1$ (that is $[\![I_i]\!] \longrightarrow [\![I_{i+1}]\!]$ is the unique arrow beginning at $[\![I_i]\!]$ in the whole Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$) and $[\![I_k]\!]$ F_n-covers more than one equivalence class (modulo 1) of $P_n \cup Q_n$-basic intervals. In the trivial case when $k = 0$ a road consists on a single class (modulo 1) which F_n-covers more than one equivalence class.

The Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$ can be decomposed in the following roads:

(3.9.a) $[\![x_{p-1}, x_p]\!] \longrightarrow [\![x_{2p-1}, x_{2p}]\!] \longrightarrow \cdots \longrightarrow [\![x_{(q-n)p-1}, x_{(q-n)p}]\!] = [\![x_{n-1}, x_n]\!]$
is a road of length $q-n-1$ which is formed by all the $q-n$ equivalence classes (modulo 1) of $P_n \cup Q_n$-basic intervals having both endpoints in Q_n. More concretely,

$$\begin{aligned}\{[\![x_{\ell p-1}, x_{\ell p}]\!] : \ell \in \{1,2,\dots,q-n\}\} = \\ \{[\![x_\ell, x_{\ell+1}]\!] : \ell \in \{0,1,\dots,q-1\}\setminus \\ \{ip+n-1 : i \in \{1,2,\dots,n\}\}\}.\end{aligned} \tag{3.9}$$

(3.9.b) The class $[\![x_{n-1}, x_n]\!]$ F_n-covers $[\![x_{n-1+p}, y_0]\!]$, $[\![y_n, x_{n+p}]\!]$ and all the classes $[\![y_i, y_{i+1}]\!]$ for $i = 0,1,\dots,n-1$.

(3.10.a) $[\![y_{r-1}, y_r]\!] \longrightarrow [\![y_{2r-1}, y_{2r}]\!] \longrightarrow \cdots \longrightarrow [\![y_{(q-n)r-1}, y_{(q-n)r}]\!] = [\![y_{(n-1)r}, y_{(n-1)r+1}]\!]$
is a road of length $q-n-1$ which is formed by all the $q-n$ equivalence classes (modulo 1) of $P_n \cup Q_n$-basic intervals having both endpoints in P_n. More concretely,

$$\begin{aligned}\{[\![y_{\ell r-1}, y_{\ell r}]\!] : \ell \in \{1,2,\dots,q-n\}\} = \\ \{[\![y_\ell, y_{\ell+1}]\!] : \ell \in \{0,1,\dots,q-1\}\setminus \\ \{ir+n : i \in \{0,1,\dots,n-1\}\}\}.\end{aligned} \tag{3.10}$$

(3.10.b) The class $[\![y_{(n-1)r}, y_{(n-1)r+1}]\!]$ F_n-covers $[\![y_n, x_{p+n}]\!]$, $[\![x_{2p+n-1}, y_{n+1}]\!]$ and all the classes $[\![x_{p+n+i}, x_{p+n+i+1}]\!]$ for $i = 0,1,\dots,p-2$.

(3.11.a) $[\![y_n, x_{p+n}]\!] \longrightarrow [\![y_{r+n}, x_{2p+n}]\!] \longrightarrow \cdots \longrightarrow [\![y_{(n-1)r+n}, x_{np+n}]\!] = [\![y_{q-1}, x_q]\!]$
is a road of length $n-1$ which is formed by all the n equivalence classes (modulo 1) of $P_n \cup Q_n$-basic intervals verifying that each class has a representative basic interval whose first endpoint belongs to P_n and the second one to Q_n.

(3.11.b) The class $[\![y_{q-1}, x_q]\!]$ (negatively) F_n-covers $[\![x_{p+n-1}, y_0]\!]$, $[\![y_n, x_{p+n}]\!]$, $[\![x_{2p+n-1}, y_{n+1}]\!]$, and all the classes

$$\begin{aligned}&[\![x_i, x_{i+1}]\!] \quad \text{for} \quad i = p, p+1, \dots, p+n-2,\\ &\qquad\qquad p+n, p+n+1, \dots, 2p+n-2, \text{ and}\\ &[\![y_i, y_{i+1}]\!] \quad \text{for} \quad i = 0,1,\dots,n-1,\\ &\qquad\qquad n+1, n+1, \dots, 2n-1 = r-2.\end{aligned}$$

(3.12.a) $[\![x_{p+n-1}, y_0]\!]$ is a trivial road.

(3.12.b) The class $[\![x_{p+n-1}, y_0]\!]$ F_n-covers $[\![x_{2p+n-1}, y_{n+1}]\!]$ and all the classes $[\![y_{n+i}, y_{n+i+1}]\!]$ for $i = 1,2,\dots,n$.

(3.13.a) $[\![x_{2p+n-1}, y_{n+1}]\!] \longrightarrow [\![x_{3p+n-1}, y_{r+n+1}]\!] \longrightarrow \cdots \longrightarrow [\![x_{np+n-1}, y_{(n-2)r+n+1}]\!]$
is a road of length $n-2$ which is formed by all the $n-1$ equivalence classes (modulo 1) of $P_n \cup Q_n$-basic intervals verifying that each class

has a representative basic interval whose first endpoint belongs to Q_n and the second one to P_n except for $[\![x_{p+n-1}, y_0]\!]$, which gives the previous trivial road.

(3.13.b) The class $[\![x_{np+n-1}, y_{(n-2)r+n+1}]\!]$ F_n-covers $[\![x_{p+n-1}, y_0]\!]$ and all the classes $[\![x_{p+i}, x_{p+i+1}]\!]$ for $i = -1, 0, 1, \dots, n-2$.

We will prove Statements (3.12.a,b). Statements (3.13.a) through (3.16.b) follow analogously. We will start by proving that

$$[\![x_{(q-n)p-1}, x_{(q-n)p}]\!] = [\![x_{n-1}, x_n]\!]$$

and (3.9), which shows that the path from (3.12.a) is formed by all the $q-n$ equivalence classes (modulo 1) of $P_n \cup Q_n$-basic intervals having both endpoints in Q_n.

To do this we will use the following facts (which are easy to check) about the numbers $n,\ p,\ r$ and $q : np = q-n,\ nr = q+n,\ rp \equiv -1 \pmod{q}$ and $(p,q) = (r,q) = 1$.

Observe that $(p,q) = 1$ implies $\{\ell p\}_q \in \{1, 2, \dots, q-1\}$ for every $\ell \in \{0, 1, \dots, q-1\}$ and, hence, $\{\ell p - 1\}_q \in \{0, 1, \dots, q-2\}$. Summarizing,

$$\{\ell p\}_q = \{\ell p - 1\}_q + 1 \quad \text{whenever} \quad \ell \in \{1, 2, \dots, q-1\}. \tag{3.14}$$

Since $np = q-n$, we have $(q-n)p - 1 = q(p-1) + (n-1)$. Hence, $\{(q-n)p-1\}_q = n-1$, and $[\![x_{(q-n)p-1}, x_{(q-n)p}]\!] = [\![x_{n-1}, x_n]\!]$.

Now we will prove (3.9). Fix $\ell \in \{1, 2, \dots, q-n\}$. From (3.14) it follows that $\left[x_{\ell p-1}, x_{\ell p}\right]$ and $\left[x_{\{\ell p-1\}_q}, x_{\{\ell p-1\}_q+1}\right] = \left[x_{\ell p-1}, x_{\ell p}\right] - \left\lfloor \frac{\ell p-1}{q} \right\rfloor$[1] are basic intervals provided that their endpoints ($x_{\ell p-1}$ and $x_{\ell p}$ in the first case and $x_{\{\ell p-1\}_q}$ and $x_{\{\ell p-1\}_q+1}$ in the second one) are consecutive in $P_n \cup Q_n$. In view of the crucial assumption on the relative positions of the points of $P_n \cup Q_n$ from Theorem 3.9, this happens whenever $\{\ell p - 1\}_q \neq ip + n - 1$ with $i \in \{1, 2, \dots, n\}$. By way of contradiction assume that $\{\ell p-1\}_q = ip+n-1$ for some $i \in \{1, 2, \dots, n\}$. By using again (3.14) this is equivalent to

$$\{\ell p\}_q = ip + n \quad \Longleftrightarrow \quad (\ell - i)p + kq = n$$

for some $k \in \mathbb{Z}$. The last equality holds if and only if

$$(\ell - i, k) \in \{(q - n + tq, 1 - p - tp) : t \in \mathbb{Z}\}$$

(recall that $np = q-n$). Since $1 \le \ell \le q-n$ and $1 \le i \le n$, it follows that

$$1 - n \le \ell - i = q - n + tq \le q - n - 1;$$

a contradiction because $q-n+tq < 1-n$ for $t < 0$ and $q-n+tq > q-n-1$ for $t \ge 0$. So, we already know that the path from (3.12.a) is formed by all the $q-n$ equivalence classes (modulo 1) of $P_n \cup Q_n$-basic intervals having both endpoints in Q_n. To see that this path is a road and to prove (3.12.b) we need to compute the images of the corresponding $P_n \cup Q_n$-basic intervals.

Since $F_n(x_i) = x_{i+p}$ and $x_{i+q\ell} = x_i + \ell$ for every $i, \ell \in \mathbb{Z}$; and F_n is monotone on every interval from $\mathcal{B}(P_n \cup Q_n)$ (bearing in mind the assumption on the relative ordering of the points of $P_n \cup Q_n$ in Theorem 3.9) we see that

---

[1]This equality follows from $\ell p - 1 = q \cdot \left\lfloor \frac{\ell p-1}{q} \right\rfloor + \{\ell p - 1\}_q$ and $x_{i+q\ell} = x_i + \ell$ for every $i, \ell \in \mathbb{Z}$.

$F_n\left([x_{\ell p-1}, x_{\ell p}]\right) = \left[x_{(\ell+1)p-1}, x_{(\ell+1)p}\right]$ for $\ell = 1, 2, \dots, q-n-1$. On the other hand,

$$F_n\left([x_{n-1}, x_n]\right) = [x_{n-1+p}, x_{n+p}] = \\ [x_{n-1+p}, y_0] \cup \left(\bigcup_{i=0}^{n-1} [y_i, y_{i+1}]\right) \cup [y_n, x_{n+p}].$$

Thus, Statements (3.12.a,b) hold.

To end the proof of Statement (a) (Figure 7) observe that there are exactly $2q$ $P_n \cup Q_n$-basic intervals in the interval $[0,1]$ and, hence, there exist $2q$ equivalence classes (modulo 1) of $P_n \cup Q_n$-basic intervals. So, the above list of roads given in Statements (3.12–16.a) displays all vertices in the Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$ (classified according to roads). The arrows between vertices in this Markov graph are those given by the previous roads and the arrows beginning at the last class of every road given in Statements (3.12–16.b) All these vertices and arrows between them are, precisely, the ones packaged in Figure 7.

To prove (b), as in the previous subsection, we will use Propositions 2.10 and 2.11, and Theorem 2.12. Notice that Statements (3.12–16.a,b) above imply that $P_n \cup Q_n$ is a short Markov partition with respect to F_n. Then, as before, f_n is a Markov map, the Markov matrix M_n of f_n with respect to $\mathsf{e}(P_n \cup Q_n)$ is non-negative and irreducible and, by Propositions 2.10 and 2.11, $h(f_n) = \log \sigma(M_n)$ where, by the Perron-Frobenius Theorem, $\sigma(M_n)$ is the largest eigenvalue of M_n and, hence, the largest root (larger than one) of the characteristic polynomial of M_n. So, to end the proof of the proposition we need to compute the characteristic polynomial of M_n.

As before, we identify the set $\mathcal{B}(\mathsf{e}(P_n \cup Q_n))$ with the set of all equivalence classes of $P_n \cup Q_n$-basic intervals (i.e. the set of all vertices of the Markov graph modulo 1 of F_n). Then, the matrix M_n coincides with the transition matrix of the Markov graph modulo 1 of F_n given in Figure 7.

To compute T_n we will use Theorem 2.12 with

$$\mathsf{Rom}_n = \Big\{\mathsf{r}_1 = [\![x_{n-1}, x_n]\!], \mathsf{r}_2 = [\![x_{p+n-1}, y_0]\!], \\ \mathsf{r}_3 = [\![x_{np+n-1}, y_{(n-2)r+n+1}]\!], \\ \mathsf{r}_4 = [\![y_{(n-1)r}, y_{(n-1)r+1}]\!], \mathsf{r}_5 = [\![y_{q-1}, x_q]\!]\Big\}$$

as a rome (being their elements marked in Figure 7 with a box with double border and sloping lines background pattern). Then, we recall that $M_{\mathsf{Rom}_n}(x) = (a_{ij}(x))$ where $a_{ij}(x) = \sum_p x^{-\ell(p)}$, and the sum is taken over all simple paths starting at r_i and ending at r_j (since M_n is a matrix of zeroes and ones the width of every path is 1). From (a) and Figure 7 we have

$$M_{\mathsf{Rom}_n}(x) = \begin{pmatrix} 0 & x^{-1} & 0 & a_{14}(x) & x^{-n} \\ 0 & 0 & x^{-(n-1)} & a_{24}(x) & 0 \\ a_{31}(x) & x^{-1} & 0 & 0 & 0 \\ a_{41}(x) & 0 & x^{-(n-1)} & 0 & x^{-n} \\ a_{51}(x) & x^{-1} & x^{-(n-1)} & a_{54}(x) & x^{-n} \end{pmatrix}$$

where (recall that $p = 2n-1$, $n+np = q$, $r = 2n+1$, $nr = q+n$ and $pr = 2q-1$):

$$\begin{aligned}
a_{14}(x) &= \sum_{i=0}^{n-1} x^{-\{n+ip\}_q} = \sum_{i=0}^{n-1} x^{-(n+ip)} = x^{-n}(1+\alpha(x));\\
a_{24}(x) &= \sum_{i=1}^{n} x^{-\{n+(n+i)p\}_q} = \sum_{i=1}^{n} x^{-ip} = x^{-np} + \alpha(x);\\
a_{31}(x) &= \sum_{i=-1}^{n-2} x^{-\{n+2+(p+i)r\}_q} = x^{-(n+1+q-r)} + \sum_{i=0}^{n-2} x^{-(n+1+ir)}\\
&= x^{-(n+1+q-r)} + x^{-(n+1)}\beta(x) = x^{-(n+1)}\big(x^{-(q-r)} + \beta(x)\big);\\
a_{41}(x) &= \sum_{i=0}^{p-2} x^{-\{n+2+(p+n+i)r\}_q} = \sum_{i=0}^{p-2} x^{-\{(i+1)r\}_q}\\
&= \sum_{i=0}^{n-2} x^{-(i+1)r} + \sum_{i=n-1}^{p-2} x^{-((i+1)r-q)}\\
&= x^{-r}\beta(x) + \sum_{i=0}^{n-2} x^{-((n+i)r-q)} = x^{-r}\beta(x) + \sum_{i=0}^{n-2} x^{-(n+ir)}\\
&= \big(x^{-r} + x^{-n}\big)\beta(x);\\
a_{51}(x) &= \big(a_{31}(x) - x^{-(n+1+q-r)}\big) + a_{41}(x) = \big(x^{-(n+1)} + x^{-r} + x^{-n}\big)\beta(x);\\
a_{54}(x) &= a_{14}(x) + \big(a_{24}(x) - x^{-np}\big) = x^{-n}(1+\alpha(x)) + \alpha(x)\\
&= x^{-n} + \big(1 + x^{-n}\big)\alpha(x);
\end{aligned}$$

with

$$\begin{aligned}
\alpha(x) &= \sum_{i=1}^{n-1} x^{-ip} = \frac{x^{-np} - x^{-p}}{x^{-p} - 1} = \frac{x^{-(n-1)p} - 1}{1 - x^{p}}, \text{ and}\\
\beta(x) &= \sum_{i=0}^{n-2} x^{-ir} = \frac{x^{-(n-1)r} - 1}{x^{-r} - 1} = \frac{x^{-(n-2)r} - x^{r}}{1 - x^{r}}.
\end{aligned}$$

Next we explain the above computations for the matrix $M_{\mathsf{Rom}_n}(x)$. All entries in this matrix can be easily deduced from Figure 7 except for the entries $a_{14}(x)$, $a_{24}(x)$, $a_{31}(x)$, $a_{41}(x)$, $a_{51}(x)$ and $a_{54}(x)$ (these "complicate" terms of $M_{\mathsf{Rom}_n}(x)$ are determined by the partition of the Markov graph modulo 1 of F_n in roads and the fact that we have chosen the last vertex of each road to be a member of the rome). They correspond to simple paths of the form (see (3.9) and (3.10)) either

$$\begin{aligned}
\mathsf{r}_j &\longrightarrow [\![x_i, x_{i+1}]\!] = [\![x_{\ell p-1}, x_{\ell p}]\!] \longrightarrow [\![x_{(\ell+1)p-1}, x_{(\ell+1)p}]\!] \longrightarrow \cdots\\
&\qquad\qquad \longrightarrow [\![x_{(q-n)p-1}, x_{(q-n)p}]\!] = \mathsf{r}_1, \text{ or}\\
\mathsf{r}_j &\longrightarrow [\![y_i, y_{i+1}]\!] = [\![y_{\ell r-1}, y_{\ell r}]\!] \longrightarrow [\![y_{(\ell+1)r-1}, y_{(\ell+1)r}]\!] \longrightarrow \cdots\\
&\qquad\qquad \longrightarrow [\![y_{(q-n)r-1}, y_{(q-n)r}]\!] = \mathsf{r}_4,
\end{aligned}$$

for some $j \in \{1, 2, 3, 4, 5\}$. We claim that the length of the first one of the above paths is $\{n + 2 + ir\}_q$ and the length of the second one is $\{n + ip\}_q$.

Then, entry $a_{14}(x)$ can be obtained as follows: Statement (3.12.b) tells us that $\mathsf{r}_1 = [\![x_{n-1}, x_n]\!]$ F_n-covers all the classes $[\![y_i, y_{i+1}]\!]$ for $i = 0, 1, \dots, n-1$, and (3.10) sows that $[\![y_i, y_{i+1}]\!] = [\![y_{\ell r-1}, y_{\ell r}]\!]$ for some $\ell \in \{1, 2, \dots, q-n\}$. Hence, every of such paths is a simple path from r_1 to r_4 and, by the claim, it contributes $x^{-\{n+ip\}_q}$ to the entry $a_{14}(x)$. Thus,

$$a_{14}(x) = \sum_{i=0}^{n-1} x^{-\{n+ip\}_q} = \sum_{i=0}^{n-1} x^{-(n+ip)}$$

because $n + ip \le n + (n-1)p \le q - p$ and, hence, $\{n + ip\}_q = n + ip$. The other "complicate" entries: $a_{24}(x)$, $a_{31}(x)$, $a_{41}(x)$, $a_{51}(x)$ and $a_{54}(x)$ can be justified analogously.

Now we prove the first statement of the claim; the second one follows analogously. The path

$$[\![x_i, x_{i+1}]\!] = [\![x_{\ell p-1}, x_{\ell p}]\!] \longrightarrow [\![x_{(\ell+1)p-1}, x_{(\ell+1)p}]\!] \longrightarrow \cdots \longrightarrow [\![x_{(q-n)p-1}, x_{(q-n)p}]\!] = [\![x_{n-1}, x_n]\!]$$

can also be written as

$$[\![x_i, x_{i+1}]\!] \longrightarrow [\![x_{i+p}, x_{i+1+p}]\!] \longrightarrow \cdots \longrightarrow [\![x_{i+dp}, x_{i+1+dp}]\!] = [\![x_{n-1}, x_n]\!],$$

which clearly has length d for some $d \in \{0, 1, \dots, q-1\}$. We have to show that $d = \{n + 1 + ir\}_q$. Since $np = q - n$ and $rp \equiv -1 \pmod q$,

$$i + (n+1+ir)p = i + q + (p-n) + irp = q + i(1+rp) + (n-1) \equiv n - 1 \pmod q.$$

So, the path

$$\mathsf{r}_j \longrightarrow [\![x_i, x_{i+1}]\!] = [\![x_{\ell p-1}, x_{\ell p}]\!] \longrightarrow [\![x_{(\ell+1)p-1}, x_{(\ell+1)p}]\!] \longrightarrow \cdots \longrightarrow [\![x_{(q-n)p-1}, x_{(q-n)p}]\!] = [\![x_{n-1}, x_n]\!]$$

has length $\{n + 1 + ir\}_q + 1 = \{n + 2 + ir\}_q$ because, according to (3.12.a), the length of the path is $\{n + 1 + ir\}_q \le q - n - 1 < q - 1$. This ends the proof of the claim.

By Theorem 2.12, the characteristic polynomial (ignoring the sign) of M_n is

$$\pm x^{2q} \det(M_{\mathsf{Rom}_n} - \mathbf{I}_5) = \frac{\kappa_2(x)x^{2q} + \kappa_1(x)x^{q+n} + \kappa_2(x) - 2\left(x^{4n} - 2x^{2n-1} + 1\right)}{\left(x^{2n-1} - 1\right)\left(x^{2n+1} - 1\right)} = \frac{T_n(x)}{\left(x^{2n-1} - 1\right)\left(x^{2n+1} - 1\right)}.$$

Clearly, the largest root (larger than one) of the characteristic polynomial of M_n coincides with the largest root (larger than one) of the numerator of $\pm x^{2q} \det(M_{\mathsf{Rom}_n} - \mathbf{I}_5)$ which is $T_n(x)$. □

Proof of Theorem 3.9. In a similar way to the proof of Theorem 3.6 we see that Q_n and P_n are twist lifted periodic orbits of F_n both of period q such that Q_n has rotation number $\frac{p}{q}$ and P_n has rotation number $\frac{r}{q}$.

The proof that $\mathsf{Rot}(F_n) = \left[\frac{p}{q}, \frac{r}{q}\right]$ also follows as in Theorem 3.6 with the following differences. There exists a unique $u_l^n \in (x_{np+n-1}, y_{(n-2)r+n+1})$ with $F_n\bigl(u_l^n\bigr) = x_p + 1 = F_n(x_0) + 1$ ($F_n\Bigl(\bigl[x_{np+n-1}, y_{(n-2)r+n+1}\bigr]\Bigr) = 1 + \bigl[x_{p-1}, y_0\bigr]$) such that

$$(F_n)_l(x) = \inf\{F_n(y) : y \geq x\} = \begin{cases} F_n(x) & \text{for } x \in \bigl[0, u_l^n\bigr], \\ x_p + 1 & \text{for } x \in \bigl[u_l^n, 1\bigr], \\ (F_n)_l(x - \lfloor x \rfloor) + \lfloor x \rfloor & \text{if } x \notin [0,1]; \end{cases}$$

and a unique $u_u^n \in (x_{p+n-1}, y_0)$ with $F_n\bigl(u_u^n\bigr) = y_{2n} = y_{r-1} = F_n(y_{q-1}) - 1$ ($F_n\bigl([x_{p+n-1}, y_0]\bigr) = [x_{2p+n-1}, y_r]$) such that

$$(F_n)_u(x) = \sup\{F_n(y) : y \leq x\} = \begin{cases} y_{2n} & \text{for } x \in \bigl[0, u_u^n\bigr], \\ F_n(x) & \text{for } x \in \bigl[u_u^n, y_{q-1}\bigr], \\ y_{2n} + 1 & \text{for } x \in \bigl[y_{q-1}, 1\bigr], \\ (F_n)_u(x - \lfloor x \rfloor) + \lfloor x \rfloor & \text{if } x \notin [0,1]. \end{cases}$$

In this situation we have $P_n \cap [0,1] \subset [u_u^n, y_{q-1}]$, $(F_n)_u\big|_{P_n} = F_n\big|_{P_n}$ and, hence, $\rho\bigl((F_n)_u\bigr) = \rho_{F_n}(P_n) = \frac{r}{q}$. In a similar way, $Q_n \cap [0,1] \subset [0, u_l^n]$, $(F_n)_l\big|_{Q_n} = F_n\big|_{Q_n}$ and, hence, $\rho\bigl((F_n)_l\bigr) = \rho_{F_n}(Q_n) = \frac{p}{q}$. Consequently, $\mathsf{Rot}(F_n) = \left[\frac{p}{q}, \frac{r}{q}\right]$ by Theorem 2.1.

To compute the set $\mathsf{Per}(f_n)$ we will start by computing $M\Bigl(\frac{p}{q}, \frac{r}{q}\Bigr)$. We claim that

$$M\Bigl(\tfrac{p}{q}, \tfrac{r}{q}\Bigr) = \{n\} \cup \bigl\{tn + k : t \in \{2, 3, \ldots, \nu - 1\} \text{ and } -\tfrac{t}{2} < k \leq \tfrac{t}{2},\ k \in \mathbb{Z}\bigr\} \cup \mathsf{Succ}\bigl(n\nu + 1 - \tfrac{\nu}{2}\bigr)$$

with

$$\nu = \begin{cases} n & \text{if } n \text{ is even, and} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

In what follows, to simplify the notation, we will denote

$$\mathcal{K}_n := \{1 - \tfrac{\nu}{2}, 2 - \tfrac{\nu}{2}, \ldots, 0, 1, \ldots, n - \tfrac{\nu}{2}\}.$$

Taking into account that $n\nu - \frac{\nu}{2} = n(\nu - 1) + \Bigl(n - \frac{\nu}{2}\Bigr)$ and

$$\mathbb{N} = \{1, 2, \ldots, 2n - \tfrac{\nu}{2}\} \cup \mathsf{Succ}\bigl(n\nu + 1 - \tfrac{\nu}{2}\bigr) \cup \{tn + k : t \in \{2, 3, \ldots, \nu - 1\},\ k \in \mathcal{K}_n\},$$

the claim follows directly from

(i) $M\left(\frac{p}{q}, \frac{r}{q}\right) \cap \{1, 2, \ldots, 2n - \frac{\nu}{2}\} = \{n\}$,

(ii) $M\left(\frac{p}{q}, \frac{r}{q}\right) \supset \mathsf{Succ}\left(n\nu + 1 - \frac{\nu}{2}\right)$, and

(iii) $M\left(\frac{p}{q}, \frac{r}{q}\right) \cap \{tn + k : t \in \{2, 3, \ldots, \nu - 1\} \text{ and } k \in \mathcal{K}_n\} =$
$$\left\{tn + k : t \in \{2, 3, \ldots, \nu - 1\} \text{ and } -\tfrac{t}{2} < k \leq \tfrac{t}{2},\ k \in \mathbb{Z}\right\}.$$

Moreover, to prove these three statements note that the elements of $M\left(\frac{p}{q}, \frac{r}{q}\right)$ are those $m \in \mathbb{N}$ for which there exists $\ell \in \mathbb{N}$ such that

$$\frac{2n-1}{2n^2} < \frac{\ell}{m} < \frac{2n+1}{2n^2}. \tag{3.15}$$

Simple computations show that

$$0 < \tfrac{1}{n+k} \leq \frac{2n-1}{2n^2} < \frac{1}{n} < \frac{2n+1}{2n^2} \leq \tfrac{2}{n+k} < \tfrac{1}{k}$$

for every $k \in \{1, 2, \ldots, n-1\}$. Thus (i) holds.

To prove (ii) we write

$$\mathsf{Succ}\left(n\nu + 1 - \tfrac{\nu}{2}\right) = \{tn + k : t \in \mathbb{N},\ t \geq \nu \text{ and } k \in \mathcal{K}_n\}$$

(recall that $n\nu - \frac{\nu}{2} = n(\nu-1) + (n - \nu/2)$). Moreover, (3.15) with $m = tn+k$ is equivalent to

$$(2n-1)k - tn < 2(\ell - t)n^2 < (2n+1)k + tn. \tag{3.16}$$

Assume first that either n is even or $k \leq n - \frac{\nu}{2} - 1$. In this case (3.16) with $\ell = t$ holds because $t \geq \nu$, $1 - \frac{\nu}{2} \leq k \leq n - \frac{\nu}{2}$ and

$$\begin{aligned}
&(2n-1)k - tn \leq \\
&(2n-1)\left\{\begin{array}{ll} \left(n - \frac{\nu}{2}\right) & \text{when } n \text{ is even} \\ \left(n - \frac{\nu}{2} - 1\right) & \text{when } n \text{ is odd and } k \leq n - \frac{\nu}{2} - 1 \end{array}\right\} - \nu n = \\
&\qquad (2n-1)\tfrac{\nu}{2} - \nu n = -\tfrac{\nu}{2} < 0 < 2n + 1 - \tfrac{\nu}{2} = \\
&\qquad\qquad (2n+1)\left(1 - \tfrac{\nu}{2}\right) + \nu n \leq (2n+1)k + tn.
\end{aligned}$$

Thus, $tn + k \in M\left(\frac{p}{q}, \frac{r}{q}\right)$ in this case. Now we assume that n is odd and $k = n - \frac{\nu}{2}$. Then, (3.16) with $\ell = t + 1$ holds:

$$\begin{aligned}
&(2n-1)k - tn \leq (2n-1)\left(n - \tfrac{\nu}{2}\right) - \nu n = \tfrac{3n-1}{2} < 2n^2 < \\
&\qquad 2n^2 + \left(n - \tfrac{\nu}{2}\right) = (2n+1)\left(n - \tfrac{\nu}{2}\right) + \nu n \leq (2n+1)k + tn,
\end{aligned}$$

and (ii) follows.

Next we prove (iii). First notice that when $n = 3$, then $\nu = 2$ and, hence,

$$\begin{aligned}
&\{tn + k : t \in \{2, 3, \ldots, \nu - 1\} \text{ and } k \in \mathcal{K}_n\} = \\
&\qquad \left\{tn + k : t \in \{2, 3, \ldots, \nu - 1\} \text{ and } -\tfrac{t}{2} < k \leq \tfrac{t}{2},\ k \in \mathbb{Z}\right\} = \emptyset.
\end{aligned}$$

So, (iii) holds trivially in this case.

In the rest of the proof of (iii) we assume that $n \geq 4$ and we will again use (3.16). Observe that

$$-2n^2 < -2n^2 + 2n\Big(n+\tfrac{3}{2}-\nu\Big) + \Big(\tfrac{\nu}{2}-1\Big) = (2n-1)\Big(1-\tfrac{\nu}{2}\Big) - (\nu-1)n \leq$$
$$(2n-1)k - tn < 2(\ell-t)n^2 < (2n+1)k + tn \leq$$
$$(2n+1)\Big(n-\tfrac{\nu}{2}\Big) + (\nu-1)n = 2n^2 - \tfrac{\nu}{2} < 2n^2$$

because $t \in \{2,3,\dots,\nu-1\}$ and $1-\frac{\nu}{2} \leq k \leq n-\frac{\nu}{2}$. This implies $\ell - t = 0$ and, then, (3.16) becomes

$$(2n-1)k - tn < 0 < (2n+1)k + tn,$$

which is equivalent to

$$-\frac{tn}{2n+1} < k < \frac{tn}{2n-1} \quad \text{and} \quad k \in \mathcal{K}_n.$$

Observe that, for every $t \in \mathbb{N}$,

$$-\frac{t}{2} < -\frac{tn}{2n+1} < 0 < \frac{t}{2} < \frac{tn}{2n-1}.$$

To prove (iii) we will show that the following three statements hold:

(iii.1) $\max\Big(\mathbb{Z} \cap \Big(-\frac{tn}{2n+1}, \frac{t}{2}\Big]\Big) = \left\{\begin{array}{ll} \frac{t}{2} & \text{if } t \text{ is even} \\ \frac{t-1}{2} & \text{if } t \text{ is odd}\end{array}\right\} \in \mathcal{K}_n.$

(iii.2) $\frac{tn}{2n-1} < 1 + \max\Big(\mathbb{Z} \cap \Big(-\frac{tn}{2n+1}, \frac{t}{2}\Big]\Big).$

(iii.3) $\min\Big(\mathbb{Z} \cap \Big(-\frac{tn}{2n+1}, \frac{t}{2}\Big]\Big) = \left\{\begin{array}{ll} 1-\frac{t}{2} & \text{if } t \text{ is even} \\ \frac{1-t}{2} & \text{if } t \text{ is odd}\end{array}\right\} \in \mathcal{K}_n.$

First we will show that (iii) follows from the above statements and then we will prove them. From (iii.3) we immediately get that

$$\min\Big(\mathbb{Z} \cap \Big(-\tfrac{tn}{2n+1}, \tfrac{t}{2}\Big]\Big) - 1 = \left\{\begin{array}{ll} -\frac{t}{2} & \text{if } t \text{ is even} \\ -\frac{t+1}{2} & \text{if } t \text{ is odd}\end{array}\right\} \leq -\tfrac{t}{2}.$$

Consequently, by (iii.1–3),

$$\mathbb{Z} \cap \left(-\frac{tn}{2n+1}, \frac{t}{2}\right] = \mathcal{K}_n \cap \left(-\frac{tn}{2n+1}, \frac{t}{2}\right] \quad \text{and}$$
$$\mathbb{Z} \cap \left(-\frac{t}{2}, -\frac{tn}{2n+1}\right] = \mathbb{Z} \cap \left(\frac{t}{2}, \frac{tn}{2n-1}\right) = \emptyset.$$

So, since $\mathcal{K}_n \subset \mathbb{Z}$,

$$\mathbb{Z} \cap \left(-\frac{t}{2}, \frac{t}{2}\right] = \mathbb{Z} \cap \left(-\frac{tn}{2n+1}, \frac{tn}{2n-1}\right) \supset \mathcal{K}_n \cap \left(-\frac{tn}{2n+1}, \frac{tn}{2n-1}\right) \supset$$
$$\mathcal{K}_n \cap \left(-\frac{tn}{2n+1}, \frac{t}{2}\right] = \mathbb{Z} \cap \left(-\frac{tn}{2n+1}, \frac{t}{2}\right] = \mathbb{Z} \cap \left(-\frac{t}{2}, \frac{t}{2}\right],$$

which gives $\mathbb{Z} \cap \Big(-\frac{t}{2}, \frac{t}{2}\Big] = \mathcal{K}_n \cap \Big(-\frac{tn}{2n+1}, \frac{tn}{2n-1}\Big)$. Thus, (iii) holds and hence the claim, provided that (iii.1–3) are verified.

We start by checking that (iii.1) holds. The fact that

$$\max\Big(\mathbb{Z} \cap \Big(-\tfrac{tn}{2n+1}, \tfrac{t}{2}\Big]\Big) = \left\{\begin{array}{ll} \frac{t}{2} & \text{if } t \text{ is even,} \\ \frac{t-1}{2} & \text{if } t \text{ is odd,}\end{array}\right.$$

is obvious. So, we have to see that $\max\left(\mathbb{Z} \cap \left(-\frac{tn}{2n+1}, \frac{t}{2}\right]\right) \in \mathcal{K}_n$. Note that $2n > 2n-1 \geq 2\nu - 1$ and thus,

$$\tfrac{t}{2} \leq \tfrac{\nu-1}{2} < n - \tfrac{\nu}{2} = \max \mathcal{K}_n$$

because $t \leq \nu - 1$. So, since $0 \in \mathcal{K}_n$ and $0 < \max\left(\mathbb{Z} \cap \left(-\frac{tn}{2n+1}, \frac{t}{2}\right]\right)$, the statement (iii.1) holds.

To show (iii.2), again since $t \leq \nu - 1$ we have

$$2tn < 2tn + (2n - \nu) \leq 2n(t+1) - (t+1) = (2n-1)(t+1)$$

which is equivalent to

$$\frac{tn}{2n-1} < \frac{t+1}{2}.$$

So, by (iii.1),

$$\frac{tn}{2n-1} < \frac{t+1}{2} \leq 1 + \max\left(\mathbb{Z} \cap \left(-\tfrac{tn}{2n+1}, \tfrac{t}{2}\right]\right).$$

Now we prove (iii.3). By assumption we have $t \leq \nu - 1 \leq n - 1$. Hence,

$$(2n+1)(t+1) > (2n+1)t > 2nt > 2nt - 2n + n >$$
$$2n(t-1) + (t-1) = (2n+1)(t-1) > (2n+1)(t-2).$$

This gives

$$-\frac{t}{2} < -\frac{tn}{2n+1} < 1 - \frac{t}{2} = \frac{2-t}{2} \text{ when } t \text{ is even, and}$$
$$-\frac{t+1}{2} = \frac{1-t}{2} - 1 < -\frac{tn}{2n+1} < \frac{1-t}{2} \text{ when } t \text{ is odd,}$$

which proves that

$$\min\left(\mathbb{Z} \cap \left(-\tfrac{tn}{2n+1}, \tfrac{t}{2}\right]\right) = \begin{cases} 1 - \frac{t}{2} & \text{if } t \text{ is even,} \\ \frac{1-t}{2} & \text{if } t \text{ is odd.} \end{cases}$$

Furthermore, we need to show that $\min\left(\mathbb{Z} \cap \left(-\frac{tn}{2n+1}, \frac{t}{2}\right]\right) \in \mathcal{K}_n$. If $t = \nu - 1$, since ν is always even we have

$$\min\left(\mathbb{Z} \cap \left(-\frac{tn}{2n+1}, \frac{t}{2}\right]\right) = \frac{1-t}{2} = \frac{1-(\nu-1)}{2} = 1 - \frac{\nu}{2} \in \mathcal{K}_n.$$

When $t \leq \nu - 2$,

$$\min \mathcal{K}_n = \frac{2-\nu}{2} \leq -\frac{t}{2} < -\frac{tn}{2n+1}.$$

Consequently, (iii.3) holds as before because

$$\min\left(\mathbb{Z} \cap \left(-\tfrac{tn}{2n+1}, \tfrac{t}{2}\right]\right) \leq 0 \in \mathcal{K}_n.$$

This ends the proof of (iii) and the claim with it.

Finally are ready to compute the set $\mathsf{Per}(f_n)$ by using the above claim. By Theorem 2.4,

$$\mathsf{Per}(f_n) = Q_{F_n}\left(\tfrac{p}{q}\right) \cup M\left(\tfrac{p}{q}, \tfrac{r}{q}\right) \cup Q_{F_n}\left(\tfrac{r}{q}\right)$$

and, from the above claim,

$$Q_{F_n}\left(\tfrac{p}{q}\right) \cup Q_{F_n}\left(\tfrac{r}{q}\right) \subset q\mathbb{N} \subset \mathsf{Succ}\left(2n^2\right) \subset \mathsf{Succ}\left(n\nu + 1 - \tfrac{\nu}{2}\right) \subset M\left(\tfrac{p}{q}, \tfrac{r}{q}\right)$$

because, independently of the parity of n, $n\nu + 1 - \frac{\nu}{2} < n^2 + 1 < 2n^2$. Consequently, $\mathsf{Per}(f_n) = M\left(\frac{p}{q}, \frac{r}{q}\right)$, which, together with the claim, proves the statement about the set $\mathsf{Per}(f_n)$.

Notice that $2\frac{\nu}{2} \le n$ implies $\frac{\nu}{2} - n \le -\frac{\nu}{2}$. Hence, since ν is always even,

$$\max\left\{tn + k : t \in \{2, 3, \dots, \nu - 1\} \text{ and } -\tfrac{t}{2} < k \le \tfrac{t}{2},\ k \in \mathbb{Z}\right\} = \\ (\nu - 1)n + \tfrac{\nu-2}{2} = \nu n + \left(\tfrac{\nu}{2} - n\right) - 1 \le \nu n - \tfrac{\nu}{2} - 1.$$

Then, $\nu n - \frac{\nu}{2} \notin \mathsf{Per}(f_n)$ and thus, $\mathsf{StrBdCof}(f_n) = n\nu + 1 - \frac{\nu}{2}$. On the other hand, $n \in \mathsf{sBC}(f_n)$ and therefore, $\mathsf{BdCof}(f_n)$ exists and verifies

$$n \le \mathsf{BdCof}(f_n) \le n\nu - 1 - \tfrac{\nu}{2}.$$

Next we show that f_n is totally transitive. As in the previous example we have that $P_n \cup Q_n$ is a short Markov partition with respect to F_n. Then, f_n is an expansive Markov map with respect to the partition $\mathsf{e}(P_n \cup Q_n)$, and the transition matrix of the Markov graph of f_n with respect to the partition $\mathsf{e}(P_n \cup Q_n)$ coincides with the transition matrix of the Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$. Moreover, this transition matrix is non-negative and irreducible, and from the proof of Proposition 3.11(a) (see also Figure 7) it follows that there exist five vertices in the Markov graph modulo 1 of F_n (indeed all ends of roads) which are the beginning of more than one arrow. That is, the transition matrix of the Markov graph of f_n with respect to $\mathsf{e}(P_n \cup Q_n)$ is not a permutation matrix. Then, f_n is transitive by Theorem 2.13 and, since $\mathsf{Per}(f_n) \supset \mathsf{Succ}\left(n\nu + 1 - \frac{\nu}{2}\right)$, $\mathsf{Per}(f_n)$ is cofinite and f_n is totally transitive by Theorem 1.2.

Next we need to show that $\lim_{n\to\infty} h(f_n) = 0$. We will use the notation of Proposition 3.11(b) and we write

$$T_n(x) = \kappa_2(x)\left(x^{2q} + 1\right) + \kappa_1(x)x^{q+n} - 2\kappa_0(x)$$

with $\kappa_0(x) := x^{4n} - 2x^{2n-1} + 1$. Then, for $x \ge 1$ we have the following easy bounds:

$$\kappa_2(x) = x^{4n} - 2x^{3n} - x^{2n+1} - 2x^{2n} - 3x^{2n-1} - 2x^n + 1 > \\ x^{4n} - 10x^{3n} = x^{3n}(x^n - 10),$$
$$\kappa_1(x) = 4x^{2n} + 2x^{n+1} + 4x^n + 2x^{n-1} + 4 > 12x^{n-1}, \text{ and}$$
$$\kappa_0(x) = x^{4n} - 2x^{2n-1} + 1 \le x^{4n} - 1 < x^{4n}.$$

Hence, now for $x > \sqrt[n]{10} > 1$,

$$T_n(x) > x^{3n}(x^n - 10)\left(x^{2q} + 1\right) + 12x^{n-1}x^{q+n} - 2x^{4n} = \\ x^{3n}(x^n - 10)\left(x^{2q} + 1\right) + 2x^{4n}\left(6x^{2n(n-1)-1} - 1\right) > \\ x^{3n}(x^n - 10)\left(x^{2q} + 1\right) > 0.$$

Therefore, $\rho_n \le \sqrt[n]{10}$ and

$$0 \le \lim_{n\to\infty} h(f_n) = \lim_{n\to\infty} \log \rho_n \le \lim_{n\to\infty} \log \sqrt[n]{10} = 0.$$

□

Proof of Theorem 3.10. As in the proof of proof of Theorem 3.7 we may assume that $G \neq \mathbb{S}^1$ since otherwise Theorem 3.9 already gives the desired sequence of maps.

The proof in the case $G \neq \mathbb{S}^1$ goes along the lines of the proof of Theorem 3.7 and most of the details will be omitted. We will only summarize the parts of the proof which are different from the proof of Theorem 3.7, and the ones needed to fix the notation.

We fix a circuit C of G and an interval $I \subset C$ such that $I \cap V(G) = \emptyset$. Also, we choose a homeomorphism $\eta\colon \mathbb{S}^1 \longrightarrow C$ such that

$$C \setminus \mathsf{Int}(I) = \langle \eta(\mathsf{e}(y_{(n-2)r+n})), \eta(\mathsf{e}(x_{(n-1)p+n})) \rangle_C, \text{ and}$$
$$I \supset \eta\big(\mathsf{e}\big(P_n \cup Q_n\big)\big) \cup \bigcup_{\substack{[x,y] \in \mathcal{B}(P_n \cup Q_n) \\ [x,y] \notin [\![y_{(n-2)r+n}, x_{(n-1)p+n}]\!]}} \langle \eta(\mathsf{e}(x)), \eta(\mathsf{e}(y)) \rangle_C .$$

For simplicity, in the rest of the proof we will use the following notation: Given $x, y \in \mathbb{R}$ we denote by $\langle\!\langle x, y \rangle\!\rangle$ the convex hull (in C) of $\{\eta(\mathsf{e}(x)), \eta(\mathsf{e}(y))\}$ (which, of course, coincides with $\eta\big(\mathsf{e}\big(\langle x, y\rangle_{\mathbb{R}}\big)\big)$). With this notation, the $\eta\big(\mathsf{e}\big(P_n \cup Q_n\big)\big)$-basic intervals in C are

$$\big\{ \langle\!\langle x, y \rangle\!\rangle : \langle x, y \rangle_{\mathbb{R}} \in \mathcal{B}\big(P_n \cup Q_n\big) \big\}$$

(see Figure 8). Clearly, if $[\![x, y]\!] = [\![\tilde{x}, \tilde{y}]\!]$, then $\langle\!\langle x, y \rangle\!\rangle = \langle\!\langle \tilde{x}, \tilde{y} \rangle\!\rangle$.

Observe that $\langle\!\langle y_{(n-2)r+n}, x_{(n-1)p+n} \rangle\!\rangle$ plays the role of $\widetilde{I}_2$ in the proof of Theorem 3.7 (see Figure 5) and consequently, $\langle\!\langle y_{q-1}, x_q \rangle\!\rangle$ plays the role of the interval $\widetilde{I}_3$ while $\langle\!\langle y_{(n-4+j)r+n}, x_{(n-3+j)p+n} \rangle\!\rangle$ play the role of $\widetilde{I}_j$ for $j = 0, 1$. Note that all the intervals are well defined since $n \geq 4$ and they are pairwise disjoint because of the ordering of points defined in Theorem 3.9.

We set $X := G \setminus \mathsf{Int}(I) \supset \langle\!\langle y_{(n-2)r+n}, x_{(n-1)p+n} \rangle\!\rangle$, and $V(X) = V(G) \cup \{a, b\}$ with $a := \eta\big(\mathsf{e}\big(y_{(n-2)r+n}\big)\big)$ and $b := \eta\big(\mathsf{e}\big(x_{(n-1)p+n}\big)\big)$. Then, as before, we use Lemma 3.4 for the subgraph X (see Figure 3). Let $m = m(X, a, b) \geq 5$ be odd, consider the partition $0 = s_0 < s_1 < \dots < s_m = 1$, and let the maps $\varphi_{a,b}\colon [0,1] \longrightarrow X$ and $\psi_{a,b}\colon X \longrightarrow [0,1]$ be as in Lemma 3.4. Moreover, as before, we define two arbitrary but fixed homeomorphisms $\zeta\colon [0,1] \longrightarrow \langle\!\langle y_{q-1}, x_q \rangle\!\rangle$ and $\xi\colon \langle\!\langle y_{(n-3)r+n}, x_{(n-2)p+n} \rangle\!\rangle \longrightarrow [0,1]$ such that

$$\zeta(0) = \eta(\mathsf{e}(y_{q-1})),\ \zeta(1) = \eta(\mathsf{e}(x_q)),\ \xi\big(\eta\big(\mathsf{e}\big(y_{(n-3)r+n}\big)\big)\big) = 0 \text{ and}$$
$$\xi\big(\eta\big(\mathsf{e}\big(x_{(n-2)p+n}\big)\big)\big) = 1$$

(see Figure 5 for an analogous situation).

Equipped with all these definitions, for $n \geq 4$ we set

$$g_n(x) := \begin{cases} \varphi_{a,b}(\xi(x)) & \text{if } x \in \langle\!\langle y_{(n-3)r+n}, x_{(n-2)p+n} \rangle\!\rangle; \\ \zeta\big(\psi_{a,b}(x)\big) & \text{if } x \in X; \\ (\eta \circ f_n \circ \eta^{-1})(x) & \text{if } x \in I \setminus \mathsf{Int}\langle\!\langle y_{(n-3)r+n}, x_{(n-2)p+n} \rangle\!\rangle. \end{cases}$$

Then, as in the proof of Theorem 3.7 we can easily but tediously show that g_n is a Markov map with respect to the partition

$$R_n = \eta\big(\mathsf{e}\big(Q_n \cup P_n\big)\big) \cup \big\{\xi^{-1}(s_i) : i \in \{0, 1, \dots, m\}\big\} \cup$$

$$\left\{\varphi_{a,b}(s_i) : i \in \{0, 1, \dots, m\}\right\},$$

whose R_n-basic intervals are:

$$\Big\{\langle\!\langle x, y\rangle\!\rangle : \\ [x,y] \in \mathcal{B}\big(P_n \cup Q_n\big) \setminus \big([\![y_{(n-3)r+n}, x_{(n-2)p+n}]\!] \cup [\![y_{(n-2)r+n}, x_{(n-1)p+n}]\!]\big) \\ \Big\} \subset I \setminus \mathsf{Int}\langle\!\langle y_{(n-3)r+n}, x_{(n-2)p+n}\rangle\!\rangle,$$

$$\left\{L_i := \xi^{-1}([s_i, s_{i+1}]) : i \in \{0, 1, \dots, m-1\}\right\} \subset \langle\!\langle y_{(n-3)r+n}, x_{(n-2)p+n}\rangle\!\rangle, \text{ and}$$

$$\{U_0, U_1, \dots, U_t\} = \left\{\varphi_{a,b}\big([s_i, s_{i+1}]\big) : i \in \{0, 1, \dots, m-1\}\right\} \subset X.$$

Next we will derive the Markov graph of g_n with respect to R_n from the Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$, which coincides with the Markov graph of f_n with respect to $\mathsf{e}\big(P_n \cup Q_n\big)$ provided that we identify $\langle\!\langle x, y\rangle\!\rangle$ with $\mathsf{e}\big([\![x, y]\!]\big) = \mathsf{e}\big([x, y]\big)$ and this with $[\![x, y]\!]$ for every $[x, y] \in \mathcal{B}\big(P_n \cup Q_n\big)$ (see the proof of Proposition 3.11). Clearly, the Markov graph of g_n on the intervals $\langle\!\langle x, y\rangle\!\rangle$ such that $[x, y] \in \mathcal{B}\big(P_n \cup Q_n\big)$ and

$$[x, y] \notin [\![y_{(n-4)r+n}, x_{(n-3)p+n}]\!] \cup [\![y_{(n-3)r+n}, x_{(n-2)p+n}]\!] \cup [\![y_{(n-2)r+n}, x_{(n-1)p+n}]\!]$$

is isomorphic to the Markov graph modulo 1 of F_n restricted to the corresponding intervals $[\![x, y]\!]$ (see Figure 8, Proposition 3.11(a) and Figure 7). Also, by construction, the interval $\langle\!\langle y_{(n-4)r+n}, x_{(n-3)p+n}\rangle\!\rangle$ g_n-covers all the intervals $L_0, L_1, \dots, L_{m-1} \subset \langle\!\langle y_{(n-3)r+n}, x_{(n-2)p+n}\rangle\!\rangle$. Thus, in the Markov graph of g_n there is an arrow from $\langle\!\langle y_{(n-4)r+n}, x_{(n-3)p+n}\rangle\!\rangle$ to each one of the intervals $L_0, L_1, \dots, L_{m-1}$ (see Figure 8). Moreover, every interval L_i g_n-covers a unique interval U_j but different intervals L_i can g_n-cover the same interval U_j, and every interval U_j g_n-covers the same interval $\langle\!\langle y_{q-1}, x_q\rangle\!\rangle$. Hence, the Markov graph of g_n with respect to R_n is the one shown in Figure 8 where, again, the double arrows arriving to the boxes in grey mean that there is an arrow arriving to each basic interval in the box and the arrows between the intervals L_i and U_j are just illustrative. The part of the Markov graph of g_n with respect to R_n which differs from the Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$ is shown inside a grey box with a zigzag border.

As before, by Lemma 3.5, the map g_n can be modified without altering $g_n\big|_{R_n}$ and $g_n(K)$ for every $K \in \mathcal{B}(R_n)$ in such a way that g_n becomes R_n-expansive. So, we can use again Theorem 2.13 to prove that g_n is transitive. The Markov graph of g_n tells us that the Markov matrix of g_n with respect to R_n is not a permutation matrix because there are six basic intervals which g_n-cover more than one basic interval. Moreover, by direct inspection of the Markov graph of g_n, given any two vertices in the graph, there exists a path from the first to the second one. This means that the transition matrix of the Markov graph of g_n is non-negative and irreducible. Thus, g_n is transitive by Theorem 2.13.

Concerning the set of periods, it is easy to see that in this example,

$$\mathsf{Per}(f_n) = \bigcup_{w \in \mathsf{Per}(f_n)} w \cdot \mathbb{N}.$$

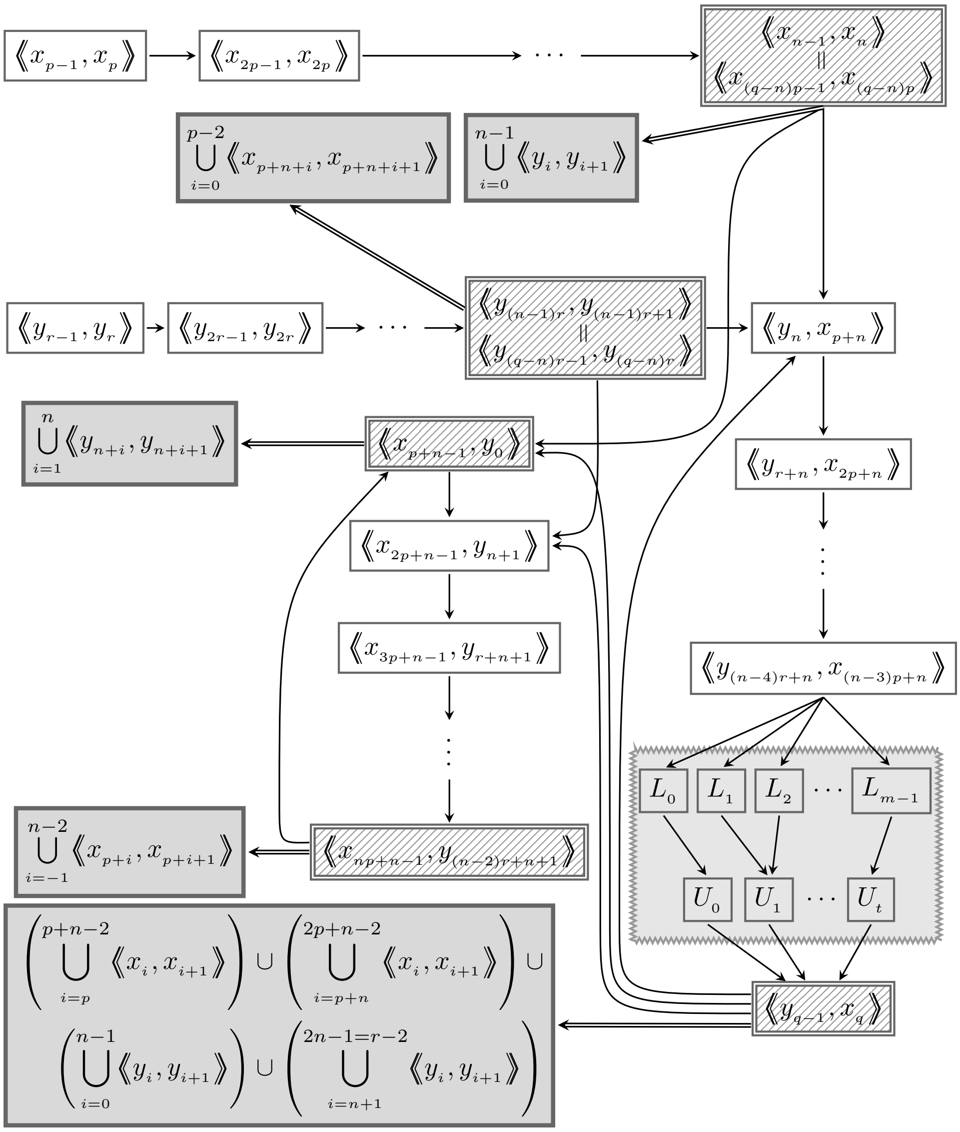


FIGURE 8. The Markov graph of g_n. The vertices which are intervals used in the Markov graph modulo 1 of F_n must be identified with their images by $\eta \circ \mathsf{e}$. Also, the arrows from the vertices L_l to the vertices U_j inside the grey box circled by a zigzag shape are symbolic because they cannot be determined precisely.

So,

$$\mathsf{Per}(g_n) = \bigcup_{w \in \mathsf{Per}(f_n)} w \cdot \mathbb{N}$$

exactly as in the proof of Theorem 3.7 with the difference that, here, the situation is much simpler because $2 \notin \mathsf{Per}(f_n)$. This proves that the set of periods does not change: $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$. Therefore, $\mathsf{Per}(g_n) \supset \mathsf{Succ}\left(n\nu + 1 + \frac{\nu}{2}\right)$

which implies that $\mathsf{Per}(g_n)$ is cofinite and, by Theorem 1.2, g_n is totally transitive.

Now we will estimate $h(g_n)$ by using Proposition 2.11 and Theorem 2.12 to show that $\lim_{n\to\infty} h(g_n) = 0$. We use the rome which corresponds to the one used in the proof of Proposition 3.11:

$$\widetilde{\mathsf{Rom}}_n = \Big\{\widetilde{\mathsf{r}}_1 = \langle\!\langle x_{n-1}, x_n \rangle\!\rangle, \widetilde{\mathsf{r}}_2 = \langle\!\langle x_{p+n-1}, y_0 \rangle\!\rangle, \widetilde{\mathsf{r}}_3 = \langle\!\langle x_{np+n-1}, y_{(n-2)r+n+1} \rangle\!\rangle, \widetilde{\mathsf{r}}_4 = \langle\!\langle y_{(n-1)r}, y_{(n-1)r+1} \rangle\!\rangle, \widetilde{\mathsf{r}}_5 = \langle\!\langle y_{q-1}, x_q \rangle\!\rangle\Big\}$$

being their elements marked in Figure 8 with a box with double border and sloping lines background pattern. Observe that the simple paths from r_i to r_j in the Markov graph of f_n, computed in the proof of Proposition 3.11, are in one-to-one correspondence with the simple simple paths from $\widetilde{\mathsf{r}}_i$ to $\widetilde{\mathsf{r}}_j$ in the Markov graph of g_n, except for the simple paths ending at r_5 and $\widetilde{\mathsf{r}}_5$. Indeed, every simple path in the Markov graph of f_n ending at r_5 is of the form

$$\mathsf{r}_\ell \longrightarrow [\![y_n, x_{p+n}]\!] \longrightarrow [\![y_{r+n}, x_{2p+n}]\!] \longrightarrow \cdots \longrightarrow [\![y_{(n-4)r+n}, x_{(n-3)p+n}]\!] \longrightarrow [\![y_{(n-3)r+n}, x_{(n-2)p+n}]\!] \longrightarrow [\![y_{(n-2)r+n}, x_{(n-1)p+n}]\!] \longrightarrow \mathsf{r}_5$$

with $\ell \in \{1,4,5\}$. However, this path corresponds to the following m paths of the same length in the Markov graph of g_n:

$$\widetilde{\mathsf{r}}_\ell \longrightarrow [\![y_n, x_{p+n}]\!] \longrightarrow [\![y_{r+n}, x_{2p+n}]\!] \longrightarrow \cdots \longrightarrow [\![y_{(n-4)r+n}, x_{(n-3)p+n}]\!] \longrightarrow L_i \longrightarrow U_j \longrightarrow \widetilde{\mathsf{r}}_5$$

with $i = 0,1,\ldots,m-1$ and $j \in \{0,1,\ldots,t\}$ because $L_0, L_1, \ldots, L_{m-1}$ are pairwise different intervals. So, every non-zero term in the fifth column of the matrix $M_{\mathsf{Rom}_n}(x)$ associated to the Markov graph of f_n (see the proof of Proposition 3.11), which is x^{-n}, must be replaced by mx^{-n} in the matrix $M_{\widetilde{\mathsf{Rom}}_n}(x)$ associated to the Markov graph of g_n, and these are the only changes when comparing $M_{\mathsf{Rom}_n}(x)$ with $M_{\widetilde{\mathsf{Rom}}_n}(x)$. Therefore,

$$M_{\widetilde{\mathsf{Rom}}_n}(x) = \begin{pmatrix} 0 & x^{-1} & 0 & a_{14}(x) & mx^{-n} \\ 0 & 0 & x^{-(n-1)} & a_{24}(x) & 0 \\ a_{31}(x) & x^{-1} & 0 & 0 & 0 \\ a_{41}(x) & 0 & x^{-(n-1)} & 0 & mx^{-n} \\ a_{51}(x) & x^{-1} & x^{-(n-1)} & a_{54}(x) & mx^{-n} \end{pmatrix}$$

where $a_{14}(x), a_{24}(x), a_{31}(x), a_{41}(x), a_{51}(x)$ and $a_{54}(x)$ are the same as in the proof of Proposition 3.11.

By Theorem 2.12, the characteristic polynomial (ignoring the sign) of the transition matrix of the Markov graph of g_n is

$$\pm x^{2(q-1)+m+t} \det(M_{\widetilde{\mathsf{Rom}}_n}(x) - \mathbf{I}_5) = x^{t+m-2}\, \frac{\widetilde{T}_n(x)}{\big(x^{2n-1}-1\big)\big(x^{2n+1}-1\big)}$$

where

$$\widetilde{T}_n(x) = \widetilde{\kappa}_2(x)(x^{2q}+1) + \widetilde{\kappa}_1(x)x^q - (m+1)\widetilde{\kappa}_0(x),$$

$$\widetilde{\kappa}_2(x) = x^{4n} - (m+1)\left[x^{3n} + x^{2n} + x^n\right] - x^{2n+1} - (m+2)x^{2n-1} + 1,$$
$$\widetilde{\kappa}_1(x) = (m-1)\left[x^{4n}+1\right] + 2(m+1)\left[x^{3n} + x^{2n} + x^n\right] + 2\left[x^{2n+1} + x^{2n-1}\right],$$

and

$$\widetilde{\kappa}_0(x) = x^{4n} - 2x^{2n-1} + 1.$$

Then, as in the proof of Theorem 3.9, we use the following easy bounds for $\widetilde{\kappa}_2(x)$, $\widetilde{\kappa}_1(x)$ and $\widetilde{\kappa}_0(x)$, which are valid for $x \geq 1$:

$$\begin{aligned} &\widetilde{\kappa}_2(x) > x^{4n} - (4m+6)x^{3n} = x^{3n}(x^n - (4m+6)),\\ &\widetilde{\kappa}_1(x) > (7m+9)x^n, \text{ and}\\ &\widetilde{\kappa}_0(x) = x^{4n} - 2x^{2n-1} + 1 \leq x^{4n} - 1 < x^{4n}. \end{aligned}$$

Hence, now for $x > \sqrt[n]{4m+6} > 1$,

$$\begin{aligned} \widetilde{T}_n(x) > x^{3n}(x^n - (4m+6))\big(x^{2q}+1\big) + (7m+9)x^n x^q - (m+1)x^{4n} =\\ x^{3n}(x^n - (4m+6))\big(x^{2q}+1\big) + (m+1)x^{4n}\Big(\tfrac{7m+9}{m+1}x^{n(2n-3)} - 1\Big) >\\ x^{3n}(x^n - (4m+6))\big(x^{2q}+1\big) > 0. \end{aligned}$$

Therefore, $\tilde{\rho}_n$, the largest root of $\widetilde{T}_n(x)$ verifies $\rho_n \leq \sqrt[n]{4m+6}$ and hence,

$$0 \leq \lim_{n\to\infty} h(g_n) = \lim_{n\to\infty} \log\tilde{\rho}_n \leq \lim_{n\to\infty} \log\sqrt[n]{4m+6} = 0$$

because m is a fixed number that depends on the topology of the graph and is independent on n. □

3.5. **The dream example.** The last example that we construct consists of maps without low periods:

Example 1.7. *For every positive integer $n \geq 3$ there exists f_n, a totally transitive continuous circle map of degree one having a lifting $F_n \in \mathcal{L}_1$ such that $\mathsf{Rot}(F_n) = \left[\frac{1}{2n-1}, \frac{2}{2n-1}\right]$, $\mathsf{Per}(f_n) = \mathsf{Succ}(n)$ and $\lim_{n\to\infty} h(f_n) = 0$. Hence, $\mathsf{BdCof}(f_n) = \mathsf{StrBdCof}(f_n) = n$ and $\lim_{n\to\infty} \mathsf{BdCof}(f_n) = \infty$.*

Furthermore, given any graph G with a circuit, the sequence of maps $\{f_n\}_{n\geq 3}$ can be extended to a sequence of continuous totally transitive maps $g_n \colon G \longrightarrow G$ such that $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$ and $\lim_{n\to\infty} h(g_n) = 0$.

As in the previous subsections, Example 1.7 will be split into a theorem that shows the existence of the circle maps f_n by constructing them along the lines of Subsection 3.1, and a theorem that extends these maps to a generic graph that is not a tree. The proof of these results will, in turn, use a proposition that computes the Markov graph modulo 1 of the liftings F_n.

Theorem 3.12. *Let $n \in \mathbb{N}$, $n \geq 3$, and let*

$$Q_n = \{\dots x_{-1}, x_0, x_1, x_2, \dots, x_{2n-2}, x_{2n-1}, x_{2n}, \dots\} \subset \mathbb{R}, \quad \textit{and}$$
$$P_n = \{\dots y_{-1}, y_0, y_1, y_2, \dots, y_{2n-2}, y_{2n-1}, y_{2n}, \dots\} \subset \mathbb{R}$$

be infinite sets such that the points of P_n and Q_n are intertwined so that
(3.17)

$$\begin{array}{ccccccc} 0 = x_0 < x_1 < \cdots < x_{n-1} < y_0 < & x_n & < & y_1 & < & y_2 & < \\ & x_{n+1} & < & y_3 & < & y_4 & < \\ & \vdots & & \vdots & & \vdots & \\ & x_{2n-2} & < & y_{2n-3} & < & y_{2n-2} & < x_{2n-1} = 1 \end{array}$$

and $x_{i+(2n-1)\ell} = x_i + \ell$ and $y_{i+(2n-1)\ell} = y_i + \ell$ for every $i, \ell \in \mathbb{Z}$.

We define a lifting $F_n \in \mathcal{L}_1$ such that, for every $i \in \mathbb{Z}$, $F_n(x_i) = x_{i+1}$ and $F_n(y_i) = y_{i+2}$, and F_n is expansive between consecutive points of $P_n \cup Q_n$. Then, Q_n and P_n are twist lifted periodic orbits of F_n both of period $2n-1$ such that Q_n has rotation number $\frac{1}{2n-1}$ and P_n has rotation number $\frac{2}{2n-1}$. Moreover, F_n has $\mathsf{Rot}(F_n) = \left[\frac{1}{2n-1}, \frac{2}{2n-1}\right]$ as rotation interval.

Let $f_n : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the continuous map which has F_n as a lifting. Then, f_n is totally transitive, $\mathsf{Per}(f_n) = \mathsf{Succ}\,(n)$ and $\lim_{n\to\infty} h(f_n) = 0$. Hence, $\mathsf{BdCof}\,(f_n) = \mathsf{StrBdCof}\,(f_n) = n$ and $\lim_{n\to\infty} \mathsf{BdCof}\,(f_n) = \infty$.

Remark 3.13. Our choice of the rotation interval in this example was influenced by the Farey sequence of order $2n-1$ (which is the ordered sequence of rationals $\frac{p}{q}$ such that $0 \le p \le q \le 2n-1$, $(p,q) = 1$). It follows that two elements $\frac{p}{q} < \frac{r}{s}$ in a Farey sequence are consecutive (called *Farey neighbours*) if and only if $qr - ps = 1$. Hence, the endpoints of the rotation interval of Example 1.7 and Theorem 3.12 belong to the Farey sequence of order $2n-1$ and the elements of this sequence between them are

$$\frac{1}{2n-1} < \frac{1}{2n-2} < \frac{1}{2n-3} < \frac{1}{2n-4} < \cdots < \frac{1}{n} < \frac{2}{2n-1}.$$

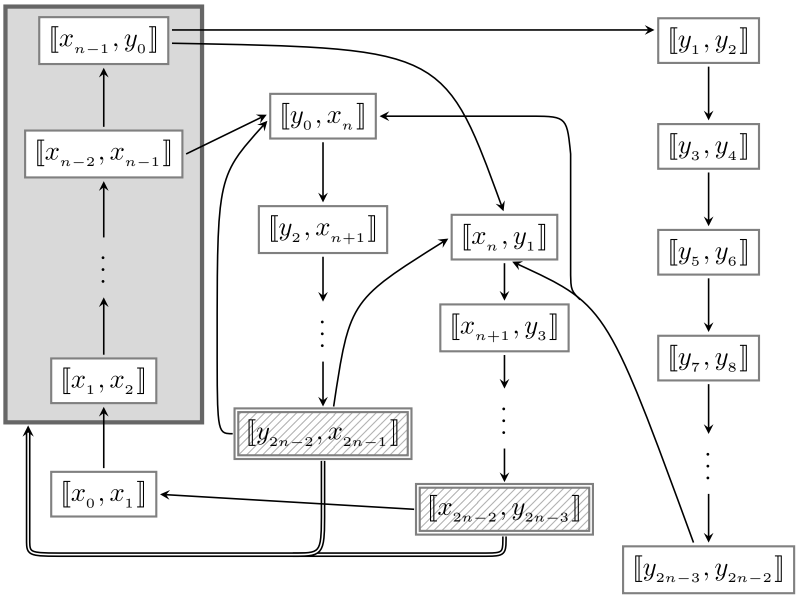
This tells us that $\mathsf{Per}(f_n) \cap \{1, 2, \ldots, 2n-2\} = \{n, n+1, \ldots, 2n-2\}$ which was the kind of set of periods we were looking for. ◘

Theorem 3.14. *Let G be a graph with a circuit. Then, the sequence of maps $\{f_n\}_{n=5}^{\infty}$ from Theorem 3.12 can be extended to a sequence of continuous totally transitive self maps of G, $\{g_n\}_{n=5}^{\infty}$, such that $\mathsf{Per}(g_n) = \mathsf{Per}(f_n)$ and $\lim_{n\to\infty} h(g_n) = 0$.*

Before proving Theorem 3.12 we will study the Markov graph modulo 1 of the liftings F_n.

Proposition 3.15 ($\mathcal{B}(P_n \cup Q_n)$ and the F_n-Markov graph modulo 1)**.** *In the assumptions of Theorem 3.12 we have:*

(a) The Markov graph modulo 1 of F_n is:



where the double arrows arriving to the the box in grey mean that there is an arrow arriving to each interval in the box.

(b) $h(f_n) = \log \rho_n$, *where* $\rho_n > 1$ *is the largest root of the polynomial*

$$T_n(x) = \left(x^{4n-2} - 1\right)(x-1) - 2x^n\left(x^{2n-1} - 1\right).$$

Proof. The proof that the Markov graph modulo 1 of F_n is the one depicted in (a) follows easily from the computation of the images of the basic intervals. To do this recall that, for every $i, \ell \in \mathbb{Z}$, $F_n(x_i) = x_{i+1}$, $F_n(y_i) = y_{i+2}$, $x_{i+(2n-1)\ell} = x_i + \ell$ and $y_{i+(2n-1)\ell} = y_i + \ell$. Moreover, F_n is strictly monotone (in fact affine) between consecutive points of $P_n \cup Q_n$. Then, by using (3.17) to determine the basic intervals we get

(i) $F_n([x_i, x_{i+1}]) = [x_{i+1}, x_{i+2}]$ for $i \in \{0, 1, \ldots, n-3\}$.

(ii) $F_n([x_{n-2}, x_{n-1}]) = [x_{n-1}, x_n] = [x_{n-1}, y_0] \cup [y_0, x_n]$.

(iii) $F_n([x_{n-1}, y_0]) = [x_n, y_2] = [x_n, y_1] \cup [y_1, y_2]$.

(iv) $F_n([y_{2i}, x_{n+i}]) = [y_{2(i+1)}, x_{n+i+1}]$ for $i \in \{0, 1, \ldots, n-2\}$.

(v) $F_n([y_{2n-2}, x_{2n-1}]) = [x_{2n}, y_{2n}] = [x_1, y_1] + 1 =$

$$\left(\left(\bigcup_{i=1}^{n-2}[x_i, x_{i+1}]\right) \cup [x_{n-1}, y_0] \cup [y_0, x_n] \cup [x_n, y_1]\right) + 1 \in$$

$$\left(\bigcup_{i=1}^{n-2}[\![x_i, x_{i+1}]\!]\right) \cup [\![x_{n-1}, y_0]\!] \cup [\![y_0, x_n]\!] \cup [\![x_n, y_1]\!].$$

Moreover, $[\![y_{2n-2}, x_{2n-1}]\!]$ is the only class of basic intervals where F_n is decreasing.

(vi) $F_n([x_{n+i}, y_{2i+1}]) = [x_{n+(i+1)}, y_{2(i+1)+1}]$ for $i \in \{0, 1, \ldots, n-3\}$.

(vii) $F_n([x_{2n-2}, y_{2n-3}]) = [x_{2n-1}, y_{2n-1}] = [x_0, y_0] + 1 =$
$$\left(\bigcup_{i=0}^{n-2}([x_i, x_{i+1}] + 1)\right) \cup \big([x_{n-1}, y_0] + 1\big) \in \left(\bigcup_{i=0}^{n-2}[\![x_i, x_{i+1}]\!]\right) \cup [\![x_{n-1}, y_0]\!].$$
(viii) $F_n([y_{2i+1}, y_{2(i+1)}]) = [y_{2(i+1)+1}, y_{2(i+2)}]$ for $i \in \{0, 1, \ldots, n-3\}$.
(ix) $F_n([y_{2n-3}, y_{2n-2}]) = [y_{2n-1}, y_{2n}] = [y_0, y_1] + 1 =$
$$\big([y_0, x_n] + 1\big) \cup \big([x_n, y_1] + 1\big) \in [\![y_0, x_n]\!] \cup [\![x_n, y_1]\!].$$

Then, (a) (the Markov graph modulo 1 of F_n) is a direct translation of the above list of images to the language of combinatorial graphs.

Now we prove (b). In this case, clearly,
$$\mathsf{Rom}_n = \{\mathsf{r}_1 = [\![x_{2n-2}, y_{2n-3}]\!], \mathsf{r}_2 = [\![y_{2n-2}, x_{2n-1}]\!]\}$$
as a rome of two elements (being their elements marked in (a) with a box with double border and sloping lines background pattern). Then, the matrix $M_{\mathsf{Rom}_n}(x)$ is:
$$\begin{pmatrix} \sum_{i=0}^{n-1} x^{-(n+i)} + \sum_{i=0}^{n-1} x^{-(2n-1+i)} & \sum_{i=1}^{n-1} x^{-(n+i)} + \sum_{i=0}^{n-1} x^{-(2n+i)} \\ x^{-(n-1)} + \sum_{i=0}^{n-2} x^{-(n+i)} + \sum_{i=0}^{n-2} x^{-(2n-1+i)} & x^{-n} + \sum_{i=1}^{n-2} x^{-(n+i)} + \sum_{i=0}^{n-2} x^{-(2n+i)} \end{pmatrix}$$
$$= \begin{pmatrix} x^{-(2n-1)} + \alpha(x) & -x^{-n} + x^{-(3n-1)} + \alpha(x) \\ x^{-(n-1)} - x^{-(3n-2)} + \alpha(x) & -x^{-(2n-1)} + \alpha(x) \end{pmatrix}$$
with
$$\alpha(x) := \sum_{i=0}^{2n-2} x^{-(n+i)}.$$
Then, by Theorem 2.12, the characteristic polynomial of the Markov matrix of F_n is
$$(-1)^{4n-4}x^{4n-2}\det\big(M_{\mathsf{Rom}_n}(x) - \mathbf{I}_2\big) = \frac{\big(x^{4n-2}-1\big)(x-1) - 2x^n\big(x^{2n-1}-1\big)}{x-1}.$$
Therefore, (b) holds. □

Proof of Theorem 3.12. In a similar way to the proof of Theorem 3.6 we see that Q_n and P_n are twist lifted periodic orbits of F_n both of period $2n-1$ such that Q_n has rotation number $\frac{1}{2n-1}$ and P_n has rotation number $\frac{2}{2n-1}$.

The proof that $\mathsf{Rot}(F_n) = \left[\frac{1}{2n-1}, \frac{2}{2n-1}\right]$ also follows as in Theorem 3.6 except that in this example the upper and lower maps are as follows: There exists a unique $u_l^n \in \big(x_{2n-2}, y_{2n-3}\big)$ such that
$$F_n\big(u_l^n\big) = x_1 + 1 = F_n\big(x_0\big) + 1 = F_n(1)$$
(see (vii) from the proof of Proposition 3.15). Then,
$$(F_n)_l(x) = \inf\{F_n(y) : y \geq x\} =$$

$$\begin{cases} F_n(x) & \text{for } x \in \left[0, u_l^n\right], \\ x_1 + 1 & \text{for } x \in \left[u_l^n, 1\right], \\ (F_n)_l\big(x - \lfloor x \rfloor\big) + \lfloor x \rfloor & \text{if } x \notin [0,1]. \end{cases}$$

Also, there exists a unique $u_u^n \in \big(x_{n-1}, y_0\big)$ such that $F_n\big(u_u^n\big) = y_1 = F_n\big(y_{2n-2}\big) - 1$ (see (iii) and (ix) from the proof of Proposition 3.15). Then,

$$(F_n)_u(x) = \sup\{F_n(y) : y \le x\} =$$
$$\begin{cases} y_1 & \text{for } x \in \left[0, u_u^n\right], \\ F_n(x) & \text{for } x \in \left[u_u^n, y_{2n-2}\right], \\ y_1 + 1 & \text{for } x \in \left[y_{2n-2}, 1\right], \\ (F_n)_u\big(x - \lfloor x \rfloor\big) + \lfloor x \rfloor & \text{if } x \notin [0,1]. \end{cases}$$

Then, as in the previous two examples, we have $P_n \cap [0,1] \subset [u_u^n, y_{2n-2}]$, $(F_n)_u\big|_{P_n} = F_n\big|_{P_n}$, $\rho\big((F_n)_u\big) = \rho_{F_n}(P_n) = \frac{2}{2n-1}$, $Q_n \cap [0,1] \subset [0, u_l^n]$, $(F_n)_l\big|_{Q_n} = F_n\big|_{Q_n}$, and $\rho\big((F_n)_l\big) = \rho_{F_n}(Q_n) = \frac{1}{2n-1}$. Consequently, from Theorem 2.1, $\mathsf{Rot}(F_n) = \left[\frac{1}{2n-1}, \frac{2}{2n-1}\right]$.

Now we prove that $\mathsf{Per}(f_n) = \mathsf{Succ}\,(n)$. Observe that

$$M\Big(\tfrac{1}{2n-1}, \tfrac{2}{2n-1}\Big) \supset \mathsf{Succ}\,(2n)$$

because $\mathsf{len}(\mathsf{Rot}(F_n)) = \frac{1}{2n-1}$, and

$$\{2n-1\} \subset Q_{F_n}\left(\tfrac{1}{2n-1}\right) \cup Q_{F_n}\left(\tfrac{2}{2n-1}\right) \subset (2n-1)\mathbb{N} \subset \mathsf{Succ}\,(2n-1)\,.$$

On the other hand, in view of Remark 3.13,

$$M\Big(\tfrac{1}{2n-1}, \tfrac{2}{2n-1}\Big) \cap \{1, 2, \dots, 2n-1\} = \{n, n+1, \dots, 2n-2\}.$$

Consequently, by Theorem 2.4,

$$\begin{aligned} \mathsf{Per}(f_n) = Q_{F_n}\left(\tfrac{1}{2n-1}\right) \cup M\Big(\tfrac{1}{2n-1}, \tfrac{2}{2n-1}\Big) \cup Q_{F_n}\left(\tfrac{2}{2n-1}\right) = \\ \Big(Q_{F_n}\left(\tfrac{1}{2n-1}\right) \cup Q_{F_n}\left(\tfrac{2}{2n-1}\right)\Big) \cup \Big(M\Big(\tfrac{1}{2n-1}, \tfrac{2}{2n-1}\Big) \cap \{1, 2, \dots, 2n-1\}\Big) \cup \\ \Big(M\Big(\tfrac{1}{2n-1}, \tfrac{2}{2n-1}\Big) \cap \mathsf{Succ}\,(2n)\Big) \supset \\ \{2n-1\} \cup \{n, n+1, \dots, 2n-2\} \cup \mathsf{Succ}\,(2n) = \mathsf{Succ}\,(n) \supset \\ Q_{F_n}\left(\tfrac{1}{2n-1}\right) \cup M\Big(\tfrac{1}{2n-1}, \tfrac{2}{2n-1}\Big) \cup Q_{F_n}\left(\tfrac{2}{2n-1}\right). \end{aligned}$$

So, clearly, $\mathsf{BdCof}\,(f_n) = \mathsf{StrBdCof}\,(f_n) = n$ and $\lim_{n\to\infty} \mathsf{BdCof}\,(f_n) = \infty$.

Next we show that f_n is totally transitive. As in the previous examples, $P_n \cup Q_n$ is a short Markov partition with respect to F_n. Then, f_n is an expansive Markov map with respect to the Markov partition $\mathsf{e}\big(P_n \cup Q_n\big)$, and the transition matrix of the Markov graph of f_n with respect to $\mathsf{e}\big(P_n \cup Q_n\big)$ coincides with the transition matrix of the Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$. Moreover, this transition matrix is non-negative and irreducible, and from Proposition 3.15(a) it follows that there exists a vertex in the Markov graph modulo 1 of F_n which is the beginning of more than one arrow (for instance $[\![x_{2n-2}, y_{2n-3}]\!]$). That is, the transition matrix of the Markov graph of f_n with respect to $\mathsf{e}\bigl(P_n \cup Q_n\bigr)$ is not a permutation matrix. Then, f_n is transitive by Theorem 2.13 and $\mathsf{Per}(f_n)$ is cofinite because $\mathsf{Per}(f_n) = \mathsf{Succ}\,(n)$. Hence, f_n is totally transitive by Theorem 1.2.

To end the proof of the theorem we need to show that $\lim_{n\to\infty} h(f_n) = 0$. With the notation of Proposition 3.15(b) we have $\rho_n > 1$ and

$$0 = T_n(\rho_n) = \bigl(\rho_n^{4n-2} - 1\bigr)(\rho_n - 1) - 2\rho_n^n\bigl(\rho_n^{2n-1} - 1\bigr),$$

which is equivalent to

$$\rho_n^{4n-2}(\rho_n - 1) = 2\rho_n^n\bigl(\rho_n^{2n-1} - 1\bigr) + (\rho_n - 1).$$

So, for $x > 1$, we consider the equation

$$x^{4n-2}(x-1) = 2x^n\bigl(x^{2n-1}-1\bigr)+(x-1) \Longleftrightarrow x^{n-1} = 2\frac{x^{2n-1}-1}{x^{2n-1}(x-1)} + \frac{1}{x^{3n-1}}.$$

Observe that

$$2\frac{x^{2n-1}-1}{x^{2n-1}(x-1)} + \frac{1}{x^{3n-1}} = \frac{2\frac{x^{2n-1}-1}{x^{2n-1}} + \frac{x-1}{x^{3n-1}}}{x-1} < \frac{3}{x-1}.$$

Now we proceed as in the proof of Theorem 3.6 (see also the figure in page 37):

(i) The map $x \mapsto \frac{3}{x-1}$ is strictly decreasing on the interval $(1, +\infty)$, $\lim_{x\to1^+} \frac{3}{x-1} = +\infty$ and $\lim_{x\to\infty} \frac{3}{x-1} = 0$

(ii) For every $n \geq 3$ and every $x \geq 1$, the map $x \mapsto x^{n-1}$ is strictly increasing and $x^{n-1}\bigr|_{x=1} = 1$.

(iii) For every $n, m \in \mathbb{N}$, $3 \leq n < m$ and $x > 1$, $x^{n-1} < x^{m-1}$.

Then, for each $n \geq 3$, there exists a unique real number $\gamma_n > 1$ such that $\gamma_n^{n-1} = \frac{3}{\gamma_n - 1}$ and $x^{n-1} > \frac{3}{x-1}$ for every $x > \gamma_n$, the sequence $\{\gamma_n\}_n$ is strictly decreasing and $\lim_{n\to\infty} \gamma_n = 1$. Hence,

$$\frac{x^{4n-2}(x-1)}{2x^n\bigl(x^{2n-1}-1\bigr)+(x-1)} = \frac{x^{n-1}}{2\frac{x^{2n-1}-1}{x^{2n-1}(x-1)} + \frac{1}{x^{3n-1}}} > \frac{x^{n-1}}{\frac{3}{x-1}} > 1$$

for every $x > \gamma_n$. Consequently, $T_n(x) > 0$ for every $x > \gamma_n$ and, hence, $\rho_n \leq \gamma_n$ for every $n \geq 3$, and $\lim_{n\to\infty} \log\rho_n \leq \lim_{n\to\infty} \log\gamma_n = 0$. □

Proof of Theorem 3.14. As in the proof of proof of Theorem 3.7 we may assume that $G \neq \mathbb{S}^1$ since otherwise Theorem 3.9 already gives the desired sequence of maps.

The proof in the case $G \neq \mathbb{S}^1$ goes along the lines of the proof of Theorems 3.7 and 3.10, and most of the details will be omitted.

We fix a circuit C of G and an interval $I \subset C$ such that $I \cap V(G) = \emptyset$. Also, we choose a homeomorphism $\eta\colon \mathbb{S}^1 \longrightarrow C$ such that

$$C\setminus\mathsf{Int}(I) = \langle\!\langle y_5, y_6\rangle\!\rangle, \text{ and}$$

$$I \supset \eta\bigl(\mathsf{e}(P_n \cup Q_n)\bigr) \cup \bigcup_{\substack{[x,y]\in\mathcal{B}(P_n\cup Q_n)\\ [x,y]\notin[\![y_5,y_6]\!]}} \langle\!\langle x, y\rangle\!\rangle.$$

where, as in Theorem 3.10, $\langle\!\langle x, y\rangle\!\rangle$ denotes the convex hull (in C) of the set $\{\eta(\mathsf{e}(x)), \eta(\mathsf{e}(y))\}$. Observe that $\langle\!\langle y_5, y_6\rangle\!\rangle$ plays the role of $\widetilde{I}_2$ in the proof of Theorem 3.7 (see Figure 5) and consequently, by Proposition 3.15(a), $\langle\!\langle y_7, y_8 \rangle\!\rangle$ plays the role of the interval $\widetilde{I}_3$ while $\langle\!\langle y_{2j+1}, y_{2j+2} \rangle\!\rangle$ play the role of $\widetilde{I}_j$ for $j = 0, 1$. Note that all the intervals are well defined since $n \geq 5$ and they are pairwise disjoint because of the ordering of points defined in Theorem 3.12.

We set $X := G \setminus \mathsf{Int}(I) \supset \langle\!\langle y_5, y_6 \rangle\!\rangle$, and $V(X) = V(G) \cup \{a, b\}$ with $a := \eta(\mathsf{e}(y_5))$ and $b := \eta(\mathsf{e}(y_6))$. Then, as before, we use Lemma 3.4 for the subgraph X (see Figure 3). Let $m = m(X, a, b) \geq 5$ be odd, consider the partition $0 = s_0 < s_1 < \cdots < s_m = 1$, and let the maps $\varphi_{a,b} \colon [0,1] \longrightarrow X$ and $\psi_{a,b} \colon X \longrightarrow [0,1]$ be as in Lemma 3.4. Also, we define two arbitrary but fixed homeomorphisms $\zeta \colon [0,1] \longrightarrow \langle\!\langle y_7, y_8 \rangle\!\rangle$ and $\xi \colon \langle\!\langle y_3, y_4 \rangle\!\rangle \longrightarrow [0,1]$ such that

$$\zeta(0) = \eta(\mathsf{e}(y_7)),\ \zeta(1) = \eta(\mathsf{e}(y_8)),\ \xi\big(\eta(\mathsf{e}(y_3))\big) = 0 \text{ and } \xi\big(\eta(\mathsf{e}(y_4))\big) = 1$$

(see Figure 5 for an analogous situation).

With all these definitions, for $n \geq 5$ we set

$$g_n(x) := \begin{cases} \varphi_{a,b}(\xi(x)) & \text{if } x \in \langle\!\langle y_3, y_4 \rangle\!\rangle; \\ \zeta\big(\psi_{a,b}(x)\big) & \text{if } x \in X; \\ (\eta \circ f_n \circ \eta^{-1})(x) & \text{if } x \in I \setminus \mathsf{Int}\langle\!\langle y_3, y_4 \rangle\!\rangle, \end{cases}$$

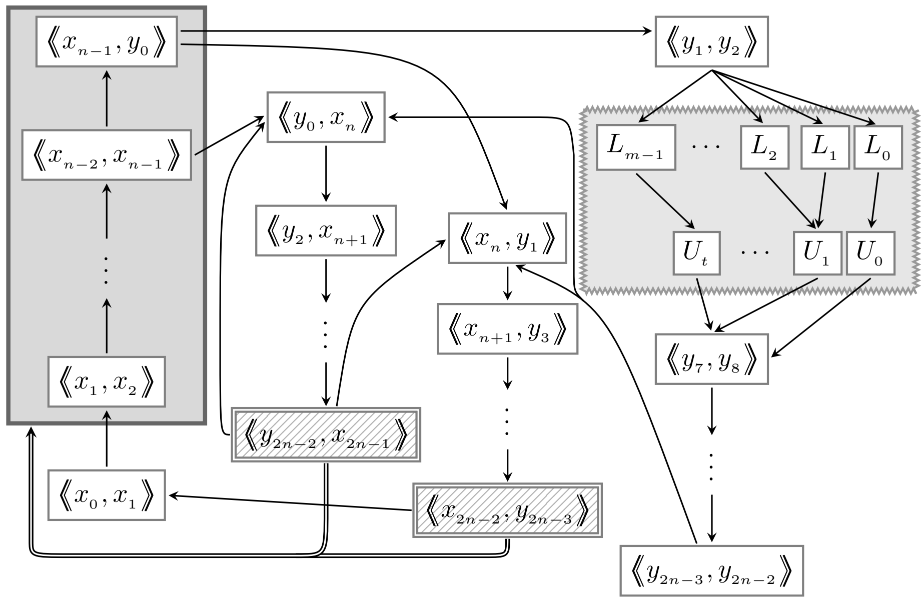
and, as in the proof of Theorem 3.7 we can easily show that g_n is a Markov map with respect to the partition

$$R_n = \eta\big(\mathsf{e}\big(Q_n \cup P_n\big)\big) \cup \big\{\xi^{-1}(s_i) : i \in \{0, 1, \dots, m\}\big\} \cup \\ \big\{\varphi_{a,b}(s_i) : i \in \{0, 1, \dots, m\}\big\},$$

whose R_n-basic intervals are:

$$\Big\{\langle\!\langle x, y \rangle\!\rangle : \\ [x, y] \in \mathcal{B}\big(P_n \cup Q_n\big) \setminus \big([\![y_3, y_4]\!] \cup [\![y_5, y_6]\!]\big) \\ \Big\} \subset I \setminus \mathsf{Int}\langle\!\langle y_3, y_4 \rangle\!\rangle,$$
$$\big\{L_i := \xi^{-1}([s_i, s_{i+1}]) : i \in \{0, 1, \dots, m-1\}\big\} \subset \langle\!\langle y_3, y_4 \rangle\!\rangle, \text{ and}$$
$$\{U_0, U_1, \dots, U_t\} = \big\{\varphi_{a,b}\big([s_i, s_{i+1}]\big) : i \in \{0, 1, \dots, m-1\}\big\} \subset X.$$

Next we will derive the Markov graph of g_n with respect to R_n (recall that it can be obtained from the Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$, which coincides with the Markov graph of f_n with respect to $\mathsf{e}\big(P_n \cup Q_n\big)$ provided that we identify $\langle\!\langle x, y \rangle\!\rangle$ with $\mathsf{e}\big([\![x, y]\!]\big) = \mathsf{e}([x, y])$ and this with $[\![x, y]\!]$ for every $[x, y] \in \mathcal{B}\big(P_n \cup Q_n\big)$ — see Proposition 3.15(a)):



(the part of the Markov graph of g_n with respect to R_n which differs from the Markov graph modulo 1 of F_n with respect to $P_n \cup Q_n$ is shown inside a grey box with a zigzag border).

As before, by Lemma 3.5 and Theorem 2.13, the map g_n can be modified without altering $g_n\big|_{R_n}$ and $g_n(K)$ for every $K \in \mathcal{B}(R_n)$ to become R_n-expansive and transitive. Moreover,

$$\mathsf{Per}(f_n) = \mathsf{Succ}\,(n) = \bigcup_{w \in \mathsf{Succ}(n)} w \cdot \mathbb{N} = \mathsf{Per}(g_n)$$

(see the proof of Theorem 3.7 but here, as in the previous example, the situation is much simpler because $2 \notin \mathsf{Per}(f_n)$). Consequently, $\mathsf{Per}(g_n)$ is cofinite and, by Theorem 1.2, g_n is totally transitive.

Now we will estimate $h(g_n)$ with the same techniques as before to show that $\lim_{n\to\infty} h(g_n) = 0$. We use the rome which corresponds to the one used in the proof of Proposition 3.15(b):

$$\widetilde{\mathsf{Rom}}_n = \{\tilde{\mathsf{r}}_1 = \langle\langle x_{2n-2}, y_{2n-3} \rangle\rangle, \tilde{\mathsf{r}}_2 = \langle\langle y_{2n-2}, x_{2n-1} \rangle\rangle\}.$$

Then, we see by direct inspection that the matrix $M_{\widetilde{\mathsf{Rom}}_n}(x)$ is:

$$\begin{pmatrix} \sum_{i=0}^{n-1} x^{-(n+i)} + m \sum_{i=0}^{n-1} x^{-(2n-1+i)} & \sum_{i=1}^{n-1} x^{-(n+i)} + m \sum_{i=0}^{n-1} x^{-(2n+i)} \\ x^{-(n-1)} + \sum_{i=0}^{n-2} x^{-(n+i)} + m \sum_{i=0}^{n-2} x^{-(2n-1+i)} & x^{-n} + \sum_{i=1}^{n-2} x^{-(n+i)} + m \sum_{i=0}^{n-2} x^{-(2n+i)} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=0}^{n-1} x^{-(n+i)} + m \sum_{i=-1}^{n-2} x^{-(2n+i)} & \sum_{i=1}^{n-1} x^{-(n+i)} + m \sum_{i=0}^{n-1} x^{-(2n+i)} \\ \sum_{i=-1}^{n-2} x^{-(n+i)} + m \sum_{i=-1}^{n-3} x^{-(2n+i)} & \sum_{i=0}^{n-2} x^{-(n+i)} + m \sum_{i=0}^{n-2} x^{-(2n+i)} \end{pmatrix}$$

By Theorem 2.12, the characteristic polynomial of the Markov matrix of g_n is

$$(-1)^{4n-5+m+t}x^{4n-3+m+t}\det\big(M_{\widetilde{\mathsf{Rom}}_n}(x)-\mathbf{I}_2\big)=\pm x^{m+t-1}\frac{\widetilde{T}_n(x)}{x-1}$$

with

$$\widetilde{T}_n(x)=\big(x^{4n-2}-m\big)(x-1)-x^{2n-1}\big(2x^n-x-1\big)-mx^n\big(x^{n-1}(x+1)-2\big).$$

To show that $\lim_{n\to\infty} h(f_n)=0$, as we did before, we consider the equation

$$x^{4n-2}(x-1)=x^{2n-1}\big(2x^n-x-1\big)+mx^n\big(x^{n-1}(x+1)-2\big)+m(x-1)$$
$$\iff \qquad x^{n-1}=\frac{2x^n-x-1}{x^n(x-1)}+m\frac{x^{n-1}(x+1)-2}{x^{2n-1}(x-1)}+\frac{m}{x^{3n-1}}$$

for $x>1$. Moreover,

$$\frac{2x^n-x-1}{x^n(x-1)}+m\frac{x^{n-1}(x+1)-2}{x^{2n-1}(x-1)}+\frac{m}{x^{3n-1}}<$$
$$\frac{\frac{2x^n-x-1}{x^n}+2m\frac{x^n-1}{x^{2n-1}}+m\frac{x-1}{x^{3n-1}}}{x-1}<\frac{2+2m+m}{x-1}.$$

So, as in the proof of Theorems 3.6 and 3.12 (see the figure and the arguments in pages 37 and 67), for every $n\geq 5$ there exists a unique real number $\gamma_n>1$ such that $\gamma_n^{n-1}=\frac{3m+2}{\gamma_n-1}$ and $x^{n-1}>\frac{3m+2}{x-1}$ for every $x>\gamma_n$, the sequence $\{\gamma_n\}_n$ is strictly decreasing and $\lim_{n\to\infty}\gamma_n=1$. Thus, for every $x>\gamma_n$,

$$\frac{x^{4n-2}(x-1)}{x^{2n-1}\big(2x^n-x-1\big)+mx^n\big(x^{n-1}(x+1)-2\big)+m(x-1)}=$$
$$\frac{x^{n-1}}{\frac{2x^n-x-1}{x^n(x-1)}+m\frac{x^{n-1}(x+1)-2}{x^{2n-1}(x-1)}+\frac{m}{x^{3n-1}}}>\frac{x^{n-1}}{\frac{3m+2}{x-1}}>1,$$

which implies that $\widetilde{T}_n(x)>0$ for every $x>\gamma_n$. Hence, if $\tilde{\rho}_n>1$ denotes the largest root of $\widetilde{T}_n(x)$, it follows that $\tilde{\rho}_n\leq\gamma_n$ and, consequently,

$$\lim_{n\to\infty}h(f_n)=\lim_{n\to\infty}\log\tilde{\rho}_n\leq\lim_{n\to\infty}\log\gamma_n=0$$

by Proposition 2.11. □

4. Proof of Theorem A

Proof of Theorem A. Fix $L\in\mathbb{N}$, $L>8$. Since $\lim_{n\to\infty}h(f_n)=0$, there exists $N\in\mathbb{N}$ such that

$$h(f_n)<\frac{3\log\sqrt{2}}{L}.$$

for every $n\geq N$. In the rest of the proof we consider a fixed but arbitrary $n\geq N$ and we denote $\mathsf{Rot}(F_n)=[c_n,d_n]$.

We claim that

$$M(c_n,d_n)\subset\mathsf{Succ}\,(L+1)=\{k\in\mathbb{N}:k\geq L+1\}\,. \tag{4.1}$$

To prove this note that for every $q \in M(c_n, d_n)$ there exists $\frac{r}{s} \in (c_n, d_n)$ with $r \in \mathbb{Z}$ and $s \in \mathbb{N}$ coprime such that $q = \ell s$ with $\ell \in \mathbb{N}$. In this situation, $h(f_n) \geq \frac{\log 3}{s}$ by [8, Corollary 4.7.7]. Hence,

$$\frac{\log 3}{q} \leq \frac{\log 3}{s} \leq h(f_n) < \frac{3 \log \sqrt{2}}{L} < \frac{\log 3}{L}.$$

Consequently, $q > L$ and the claim holds.

From the claim we get that $\operatorname{Int}(\operatorname{Rot}(F_n)) \cap \{k/L : k \in \mathbb{Z}\} = \emptyset$. This implies that $\operatorname{len}(\operatorname{Rot}(F_n)) \leq 1/L$ for every $n \geq N$. So, it follows that $\lim_{n\to\infty} \operatorname{len}(\operatorname{Rot}(F_n)) = 0$.

By Theorem 2.4 and the above claim,

$$\begin{aligned} \operatorname{Per}(F_n) &= Q_{F_n}(c_n) \cup M(c_n, d_n) \cup Q_{F_n}(d_n) \\ &\subset \operatorname{Succ}(L+1) \cup Q_{F_n}(c_n) \cup Q_{F_n}(d_n). \end{aligned} \tag{4.2}$$

In view of the above inclusion for the set $\operatorname{Per}(f_n)$ we need to study the intersections

$$\{1, 2, \ldots, L\} \cap Q_{F_n}(c_n) \quad \text{and} \quad \{1, 2, \ldots, L\} \cap Q_{F_n}(d_n).$$

We will divide this study in three claims, according to different situations for c_n and d_n.

Claim 1. *If $\alpha \notin \mathbb{Q}$ then $\{1, 2, \ldots, L\} \cap Q_{F_n}(\alpha) = \emptyset$.*

This claim follows immediately from the definition of $Q_{F_n}(\alpha) = \emptyset$.

Claim 2. *Assume that $\alpha = \frac{r}{s}$ with $r \in \mathbb{Z}$ and $s \in \mathbb{N}$ coprime, and $s \geq L$. Then, $\{1, 2, \ldots, L\} \cap Q_{F_n}(\alpha) \subset \{L\} \cap \{s\}$.*

Again by the definition of $Q_{F_n}(\alpha)$, in this case we have

$$Q_{F_n}(\alpha) = \{sk : k \in \mathbb{N} \text{ and } k \leq_{\text{Sh}} s_\alpha\} \subset s\mathbb{N} = \{sk : k \in \mathbb{N}\}.$$

Since $s \geq L$, for every $k \in \mathbb{N}$, $k \geq 2$ we have $sk \geq 2L > L$. Hence,

$$\{1, 2, \ldots, L\} \cap Q_{F_n}(\alpha) \subset \{1, 2, \ldots, L\} \cap \{sk : k \in \mathbb{N}\} = \{L\} \cap \{s\}.$$

Claim 3. *Assume that $\alpha = \frac{r}{s}$ with $r \in \mathbb{Z}$ and $s \in \mathbb{N}$ coprime, and $s < L$. Then, $\operatorname{Card}\Big(\{L-2, L-1, L\} \cap Q_{F_n}(\alpha)\Big) \leq 1$ and*

$$\{1, 2, \ldots, L-1\} \cap Q_{F_n}(\alpha) \subset \Big\{s \cdot 2^\ell : \ell \in \{0, 1, 2, \ldots, m\}\Big\}$$

where $m \geq 0$ is the integer part of $\log_2\left(\frac{L-1}{s}\right)$.

To prove this claim assume first that $Q_{F_n}(\alpha)$ contains an element of the form $s \cdot t \cdot 2^\ell$ with $t \geq 3$ odd and $\ell \in \mathbb{Z}^+$. From the definition of $Q_{F_n}(\alpha)$ it follows that then the map $F_n^s - r$ has a periodic point of period $t \cdot 2^\ell$ (as a map of the real line). Hence, by Lemmas 4.4.15 and 4.4.16 and Theorem 3.12.17 of [8] (see also [8, page 264]),

$$h(f_n) = \frac{1}{s} h(f_n^s) \geq \frac{1}{s} \frac{1}{2^\ell} \log \lambda_t$$

where λ_t is the largest root of the polynomial $x^t - 2x^{t-2} - 1$. It is well known that $\lambda_t > \sqrt{2}$ (see [8, page 232]). So,

$$\frac{3 \log \sqrt{2}}{L} > h(f_n) > \frac{\log \sqrt{2}}{s2^\ell}$$

which implies $s \cdot t \cdot 2^\ell \geq s \cdot 3 \cdot 2^\ell > L$. So, for every set $\mathsf{A} \subset \{1, 2, \dots, L\}$,

$$\begin{aligned} \mathsf{A} \cap Q_{F_n}(\alpha) &\subset \mathsf{A} \cap s\mathbb{N} \\ &= \mathsf{A} \cap \left(s\mathbb{N} \setminus \left\{ s \cdot t \cdot 2^\ell : \ell \in \mathbb{Z}^+ \text{ and } t \geq 3 \text{ odd} \right\}\right) \\ &= \mathsf{A} \cap \left\{ s \cdot 2^\ell : \ell \in \mathbb{Z}^+ \right\}. \end{aligned} \tag{4.3}$$

Since $s < L$ and m is the integer part of $\log_2 \left(\frac{L-1}{s}\right)$ it follows that $m \geq 0$ and

$$2^m \leq \tfrac{L-1}{s} < 2^{m+1}. \tag{4.4}$$

Then, from (4.3) with $\mathsf{A} = \{1, 2, \dots, L-1\}$ we obtain

$$\begin{aligned} \{1, 2, \dots, L-1\} \cap Q_{F_n}(\alpha) \subset \{1, 2, \dots, L-1\} \cap \left\{ s \cdot 2^\ell : \ell \in \mathbb{Z}^+ \right\} \\ = \left\{ s \cdot 2^\ell : \ell \in \{0, 1, 2, \dots, m\} \right\}, \end{aligned}$$

which proves the second statement of the claim.

Now we will prove the first one. We start by assuming that $s \cdot 2^m \leq L - 3$. From (4.4) we have

$$s \cdot 2^{m+2} = 2\left(s \cdot 2^{m+1}\right) \geq 2L > L.$$

Consequently, by (4.3) with $\mathsf{A} = \{L-2, L-1, L\}$,

$$\begin{aligned} &\{L-2, L-1, L\} \cap Q_{F_n}(\alpha) \subset \\ &\qquad \{L-2, L-1, L\} \cap \left\{ s \cdot 2^\ell : \ell \in \mathbb{Z}^+ \right\} \subset \\ &\qquad\qquad \{L-2, L-1, L\} \cap \{s \cdot 2^{m+1}\}. \end{aligned}$$

Now we assume that $s \cdot 2^m \in \{L-2, L-1\}$. Then,

$$s \cdot 2^{m+1} = 2\left(s \cdot 2^m\right) \geq 2(L-2) = L + (L-4) > L + 4$$

because $L > 8$. Consequently, again by (4.3) with $\mathsf{A} = \{L-2, L-1, L\}$,

$$\begin{aligned} &\{L-2, L-1, L\} \cap Q_{F_n}(\alpha) \subset \\ &\qquad \{L-2, L-1, L\} \cap \left\{ s \cdot 2^\ell : \ell \in \mathbb{Z}^+ \right\} = \\ &\quad \{L-2, L-1, L\} \cap \left\{ s \cdot 2^\ell : \ell \in \{0, 1, 2, \dots, m\} \right\} = \\ &\qquad\qquad \{L-2, L-1, L\} \cap \{s \cdot 2^m\} \end{aligned}$$

whenever $m = 0$ or $m > 0$ and $s \cdot 2^{m-1} \leq L - 3$. Now we have to show that we cannot simultaneously have $m > 0$ and $s \cdot 2^{m-1} \geq L - 2$. Otherwise, as above,

$$L - 1 \geq s \cdot 2^m = 2\left(s \cdot 2^{m-1}\right) \geq 2(L-2) > L + 4;$$

a contradiction. This ends the proof of Claim 3.

From the above three claims we obtain

$$\mathsf{Card}\left(\{L-2, L-1, L\} \cap \left(Q_{F_n}(c_n) \cup Q_{F_n}(d_n)\right)\right) \le 2 \tag{4.5}$$

and, consequently, $\{L-2, L-1, L\} \not\subset Q_{F_n}(c_n) \cup Q_{F_n}(d_n)$. Thus,

$$\{L-2, L-1, L\} \not\subset \mathsf{Per}(f_n)$$

by (4.2). So, for every $n \ge N$ we set

$$\kappa_n := \min\left(\{L-2, L-1, L\} \setminus \mathsf{Per}(f_n)\right), \text{ and}$$
$$\nu_n := \min\left(\mathsf{Per}(f_n) \cap \mathsf{Succ}\,(\kappa_n + 1)\right).$$

The inequality (4.5) is crucial for this proof. It allows us to define κ_n and, hence, ν_n and tells us that $\mathsf{StrBdCof}\,(f_n) \ge \nu_n$ (because, as we will see, $\nu_n - 1 \notin \mathsf{Per}(f_n)$). This is implicitly used in the rest of the proof of the theorem.

To end the proof of the theorem it is enough to show that $\nu_n \in \mathsf{sBC}(f_n)$ for every $n \ge N$. Indeed, by Definition 1.5, $\mathsf{BdCof}\,(f_n)$ exists and

$$L - 1 \le \kappa_n + 1 \le \nu_n \le \mathsf{BdCof}\,(f_n) \tag{4.6}$$

for every $n \ge N$. Consequently, $\lim_{n\to\infty} \mathsf{BdCof}\,(f_n) = \infty$.

Let us prove that $\nu_n \in \mathsf{sBC}(f_n)$ for every $n \ge N$. By Definition 1.5 we have to show that $\nu_n \in \mathsf{Per}(f)$, $\nu_n > 2$, $\nu_n - 1 \notin \mathsf{Per}(f)$ and

$$\mathsf{Card}\big(\{1, \dots, \nu_n - 2\} \cap \mathsf{Per}(f)\big) \le 2\log_2(\nu_n - 2). \tag{4.7}$$

Since $L > 8$, from the definition of ν_n we get

$$7 < L - 1 \le \nu_n \in \mathsf{Per}(f_n).$$

The following claim will be useful in the rest of the proof. It improves the knowledge of the set $\{1, \dots, \nu_n - 2\} \cap \mathsf{Per}(f)$.

Claim 4. $\{\kappa_n, \kappa_n + 1, \dots, \nu_n - 1\} \cap \mathsf{Per}(f_n) = \emptyset$.

When $\nu_n = \kappa_n + 1$, the claim holds because $\nu_n - 1 = \kappa_n \notin \mathsf{Per}(f_n)$ by the definition of κ_n. Now we prove the claim in the case $\nu_n > \kappa_n + 1$. We have $\{\kappa_n + 1, \dots, \nu_n - 1\} \subset \mathsf{Succ}\,(\kappa_n + 1)$ and, hence,

$$\{\kappa_n + 1, \dots, \nu_n - 1\} \cap \mathsf{Per}(f_n) = \emptyset$$

by the minimality of ν_n. Moreover, $\kappa_n \notin \mathsf{Per}(f_n)$ by definition. Hence,

$$\{\kappa_n, \kappa_n + 1, \dots, \nu_n - 1\} \cap \mathsf{Per}(f_n) = \emptyset,$$

which ends the proof of Claim 4.

Claim 4, in particular, tells us that $\nu_n - 1 \notin \mathsf{Per}(f_n)$. Hence, to end the proof of $\nu_n \in \mathsf{sBC}(f_n)$, we have to prove the inequality (4.7). By Claim 4, (4.2) and (4.1) (notice that $\kappa_n - 1 \le L - 1$ because, by definition, $\kappa_n \le L$),

$$\begin{aligned}
&\mathsf{Card}\left(\{1, \dots, \nu_n - 2\} \cap \mathsf{Per}(f_n)\right) = \\
&\qquad \mathsf{Card}\left(\{1, \dots, \kappa_n - 1\} \cap \mathsf{Per}(f_n)\right) = \\
&\qquad \mathsf{Card}\left(\{1, \dots, \kappa_n - 1\} \cap \left(Q_{F_n}(c_n) \cup Q_{F_n}(d_n)\right)\right) \le \\
&\mathsf{Card}\left(\{1, \dots, \kappa_n - 1\} \cap Q_{F_n}(c_n)\right) + \mathsf{Card}\left(\{1, \dots, \kappa_n - 1\} \cap Q_{F_n}(d_n)\right).
\end{aligned}$$

So, to prove (4.7) it is enough to show that

(4.8)
$$\operatorname{Card}\Big(\{1,\dots,\kappa_n-1\}\cap Q_{F_n}(c_n)\Big)+\operatorname{Card}\Big(\{1,\dots,\kappa_n-1\}\cap Q_{F_n}(d_n)\Big) \le 2\log_2(\nu_n-2).$$

To this end, we have to compute appropriate upper bounds of the two summands in the last expression.

Again, let $\alpha \in \{c_n, d_n\}$ denote an arbitrary endpoint of $\operatorname{Rot}(F_n)$. In the assumptions of Claims 1 and 2 we have

(4.9)
$$\operatorname{Card}\Big(\{1,\dots,\kappa_n-1\}\cap Q_{F_n}(\alpha)\Big) \le \operatorname{Card}\Big(\{1,\dots,L\}\cap Q_{F_n}(\alpha)\Big) = \emptyset.$$

Now suppose that the assumptions of Claim 3 hold. We want to prove the following estimate:

$$\begin{aligned}\operatorname{Card}\Big(\{1,\dots,\kappa_n-1\}\cap Q_{F_n}(\alpha)\Big) &\le \log_2\left(\tfrac{\nu_n-2}{s}\right)+1\\ &\le \log_2(\nu_n-2)+1.\end{aligned} \tag{4.10}$$

Assume first that $s\cdot 2^m \le \nu_n-2$. Then, by the second statement of Claim 3, we get

$$\operatorname{Card}\Big(\{1,\dots,\kappa_n-1\}\cap Q_{F_n}(\alpha)\Big) \le \operatorname{Card}\Big(\{1,\dots,L-1\}\cap Q_{F_n}(\alpha)\Big) \le m+1 \le \log_2\left(\tfrac{\nu_n-2}{s}\right)+1.$$

Now assume that $\nu_n-2 < s\cdot 2^m$. By (4.6) and (4.4),

$$L-3 \le \kappa_n-1 \le \nu_n-2 < s\cdot 2^m \le L-1.$$

Consequently, by (4.3),

$$\operatorname{Card}\Big(\{1,\dots,\kappa_n-1\}\cap Q_{F_n}(\alpha)\Big) \le \operatorname{Card}\Big(\{1,\dots,\kappa_n-1\}\cap \Big\{s\cdot 2^\ell : \ell\in\mathbb{Z}^+\Big\}\Big) \le \operatorname{Card}\Big\{s\cdot 2^\ell : \ell\in\{0,1,2,\dots,m-1\}\Big\} = m.$$

Moreover, $s\cdot 2^{m-1} \le \nu_n-2$ since, otherwise, from the above inequalities and using again the fact that $L>8$ we obtain

$$L+2 < L+(L-6) = 2(L-3) \le 2(\nu_n-2) < 2\cdot s\cdot 2^{m-1} = s\cdot 2^m \le L-1;$$

a contradiction. So, $m-1 \le \log_2\left(\frac{\nu_n-2}{s}\right)$. Putting all together we get,

$$\operatorname{Card}\Big(\{1,\dots,\kappa_n-1\}\cap Q_{F_n}(\alpha)\Big) \le m \le \log_2\left(\tfrac{\nu_n-2}{s}\right)+1.$$

This ends the proof of (4.10).

Now we are ready to prove (4.8). First assume that at most one of the endpoints of $\operatorname{Rot}(F_n)$ satisfies the assumptions of Claim 3. By (4.9) and (4.10),

$$\operatorname{Card}\Big(\{1,\dots,\kappa_n-1\}\cap Q_{F_n}(c_n)\Big)+\operatorname{Card}\Big(\{1,\dots,\kappa_n-1\}\cap Q_{F_n}(d_n)\Big) \le \log_2(\nu_n-2)+1 < 2\log_2(\nu_n-2)$$

because $7 < \nu_n$ implies $\log_2(\nu_n - 2) > 1$.

It remains to consider the case when both endpoints of $\mathsf{Rot}(F_n) = [c_n, d_n]$ satisfy the assumptions of Claim 3. That is, $c_n = \frac{r_n}{s_n}$ with $r_n \in \mathbb{Z}$ and $s_n \in \mathbb{N}$ coprime, $d_n = \frac{q_n}{t_n}$ with $q_n \in \mathbb{Z}$ and $t_n \in \mathbb{N}$ coprime, and $s_n, t_n \leq L - 1$. Observe that if $s_n, t_n \leq 3$, from above and the fact that $L > 8$ we get

$$\tfrac{1}{6} \leq d_n - c_n = \mathsf{len}\left(\mathsf{Rot}(F_n)\right) \leq \tfrac{1}{L} < \tfrac{1}{8};$$

a contradiction. Hence, either s_n or t_n is larger than 3. Assume for definiteness that $s_n \geq 4$. Then, by (4.10),

$$\begin{aligned}\mathsf{Card}\Big(\{1, \dots, \kappa_n - 1\} \cap Q_{F_n}(c_n)\Big) + \mathsf{Card}\Big(\{1, \dots, \kappa_n - 1\} \cap Q_{F_n}(d_n)\Big) \leq \\ \log_2\left(\tfrac{\nu_n - 2}{s_n}\right) + \log_2(\nu_n - 2) + 2 \leq \\ \log_2\left(\tfrac{\nu_n - 2}{4}\right) + \log_2(\nu_n - 2) + 2 = 2\log_2(\nu_n - 2).\end{aligned}$$

This ends the proof of (4.8) and, hence, that $\nu_n \in \mathsf{sBC}(f_n)$. □

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