ANOSOV REPRESENTATIONS AND DOMINATED SPLITTINGS

JAIRO BOCHI, RAFAEL POTRIE, AND ANDRÉS SAMBARINO

Abstract. We provide a link between Anosov representations introduced by Labourie and dominated splitting of linear cocycles. This allows us to obtain equivalent characterizations for Anosov representations and to recover recent results due to Guéritaud–Guichard–Kassel–Wienhard [GGKW] and Kapovich–Leeb–Porti [KLP_2] by different methods. We also give characterizations in terms of multicones and cone-types inspired in the work of Avila–Bochi–Yoccoz [ABY] and Bochi–Gourmelon [BG]. Finally we provide a new proof of the higher rank Morse Lemma of Kapovich–Leeb–Porti [KLP_2].

Contents

1. Introduction

The aim of this paper is to expose and exploit connections between the following two areas:

1. linear representations of discrete groups;
2. differentiable dynamical systems.

More specifically, we show that *Anosov representations* are closely related to *dominated splittings*. This relation allows us to reobtain some results about Anosov representations, and to give new characterizations of them.

Date: May 5, 2016.

2010 *Mathematics Subject Classification.* 22E40 (primary); 20F67, 37B15, 37D30, 53C35, 53D25 (secondary).

J. B. was partially supported by projects Fondecyt 1140202, Anillo ACT1103, and REDES 140138. R. P. was partially supported by CSIC grupo 618. A. S. was partially supported by IFUM and CSIC group 618.

Anosov representations where introduced by Labourie [Lab] in his study of the Hitchin component ([Hit]). They provide a stable class of discrete faithful representations of word-hyperbolic groups into semi-simple Lie groups, that unifies examples of varying nature. Since then, Anosov representations have become a main object of study, being subject of various deep results (see for example [GW], the surveys [BCS, Wie] and references therein). Recently, new characterizations of Anosov representations have been found by [GGKW, KLP_1, KLP_2, KLP_3], these characterizations considerably simplify the definition. It is now fairly accepted in the community that Anosov representations are a good generalization of convex co-compact groups to higher rank.

In differentiable dynamical systems, the notion of *hyperbolicity*, as introduced by Anosov and Smale [Sma] plays a central role. Early on, it was noted that weaker forms of hyperbolicity (partial, nonuniform, etc.) should also be studied: see [BDV] for a detailed account. One of these is the the notion of *dominated splittings*, which can be thought as a projective version of hyperbolicity: see Section 2.

As mentioned above, in this paper we benefit from the viewpoint of differentiable dynamics in the study of linear representations. For example, it turns out that a linear representation is Anosov if and only if its associated linear flow has a dominated splitting: see Subsection 4.3 for the precise statements.

Let us summarize the contents of this paper.

In Section 2 we describe the basic facts about dominated splittings that will be used in the rest of the paper. We present the characterization of dominated splittings given by [BG, Theorem A]. We rely on this theorem in different contexts throughout the paper.

In Section 3 we introduce *dominated representations* ρ of a given finitely generated group Γ into $\mathrm{GL}(d,\mathbb{R})$. The definition is simple: we require the gap between some consecutive singular values to be exponentially large with respect to the word length of the group element, that is,

$$\frac{\sigma_{p+1}(\rho(\gamma))}{\sigma_p(\rho(\gamma))} < Ce^{-\lambda|\gamma|} \quad \text{for all } \gamma \in \Gamma,$$

for some constants $p \in \{1, \dots, d-1\}$, $C > 0$, and $\lambda > 0$. We show that the existence of such a domination representation implies that the group Γ is word-hyperbolic: see Theorem 3.2. Word-hyperbolicity allows us to consider Anosov representations, and in Section 4 we show that they are exactly the same as dominated representations. Many of the results in Sections 3 and 4 are not new, appearing in the recent works of [GGKW] and [KLP_1, KLP_2, KLP_3] with different terminology. Our proofs are different and make use of the formalism of linear cocycles and most importantly of non-trivial properties of dominated splittings. At the end of Section 4 we pose a few questions where the connection with differentiable dynamics is manifest.

In Section 5 we give yet another equivalent condition for a representation to be Anosov, which uses the sofic subshift generated by the cone-types instead of the geodesic flow of the group: see Theorem 5.7. This condition is very much inspired in [ABY, BG] and provides nice ways to understand the variation of the limit maps, as well as a quite direct method to check if a representation is dominated. This criterion is used in Section 6 to reobtain a basic result from [BCLS] on the analyticity of the limit maps for an Anosov representation: see Theorem 6.1.

Section 7 shows how [BG, Theorem A] implies a Morse Lemma-type statement for the symmetric space of $\mathrm{PSL}(d,\mathbb{R})$. That result is contained in the recent work of [KLP$_2$], but we provide here a different approach. This section only relies on Section 2.

In Section 8 we replace $\mathrm{GL}(d,\mathbb{R})$ with a real-algebraic non-compact semi-simple Lie group. Representation theory of such groups allows one to reduce most of the general statement to the corresponding ones in $\mathrm{GL}(d,\mathbb{R})$. This is a fairly standard procedure and provides straightforward generalizations of the results of Sections 3 through 6. Nevertheless, more work is needed to obtain a Morse Lemma for symmetric spaces of non-compact type, and this occupies most of Section 8.

The results of this paper were announced at [EHY].

Acknowledgements. The authors would like to thank Matías Carrasco, Fanny Kassel, and Jean-François Quint for many interesting discussions. The first-named author acknowledges the kind hospitality of Universidad de la República.

2. Dominated splittings

In the 1970's, Mañé introduced the notion of *dominated splittings*, which played an important role in his solution of Smale's Stability Conjecture: see [Sam] and references therein. Independently, dominated splittings had been studied in the theory of Ordinary Differential Equations by the Russian school at least since the 1960s, where it is called *exponential separation*: see [Pa] and references therein. Dominated splittings continue to be an important subject in Dynamical Systems [CP, Sam] and Control Theory [CK].

2.1. Definition and basic properties of dominated splittings.

Let X be a compact metric space. Let $\mathbb{T}$ be either $\mathbb{Z}$ or $\mathbb{R}$. Consider a continuous action of $\mathbb{T}$ on X, that is, a family of homeomorphisms $\{\phi^t \colon X \to X\}_{t\in\mathbb{T}}$ such that $\phi^{t+s} = \phi^t \circ \phi^s$. We call $\{\phi^t\}$ a *continuous flow.*

Let E be a real vector bundle with projection map $\pi \colon E \to X$ and fibers $E_x := \pi^{-1}(x)$ of constant dimension d. We endow E with a Riemannian metric (that is, a continuous choice of an inner product on each fiber). Suppose $\{\psi^t \colon E \to E\}_{t\in\mathbb{T}}$ is an action of $\mathbb{T}$ on E by automorphisms of the vector bundle; also suppose that it that covers $\{\phi^t\}$, that is, $\pi \circ \psi^t = \phi^t \circ \pi$. So the restriction of ψ^t to each fiber E_x is a linear automorphism ψ^t_x onto $E_{\phi^t(x)}$. We say that $\{\psi^t\}$ is a *linear flow* which *fibers over* the continuous flow $\{\phi^t\}$.

The simplest situation is when $\mathbb{T} = \mathbb{Z}$ and the vector bundle is trivial, i.e., $E = X \times \mathbb{R}^d$ and $\pi(x, v) = x$; in that case the linear flow $\{\psi^t\}$ is called a *linear cocycle*, and in order to describe it is sufficient to specify the maps $\phi = \phi^1 \colon X \to X$ and $A\colon X \to \mathrm{GL}(d,\mathbb{R})$ such that $\psi^1(x, v) = (\phi(x), A(x)v)$. With some abuse of terminology, we sometimes call the pair (ϕ, A) a linear cocycle.

Suppose that the vector bundle E splits as a direct sum $E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$ of (continuous) subbundles of constant dimensions.[1] This splitting is called *invariant* under the linear flow $\{\psi^t\}$ if for all $x \in X$ and $t \in \mathbb{T}$,

$$\psi^t(E^{\mathrm{cu}}_x) = E^{\mathrm{cu}}_{\phi^t(x)}, \quad \psi^t(E^{\mathrm{cs}}_x) = E^{\mathrm{cs}}_{\phi^t(x)}.$$

[1] cu and cs stand for *center-stable* and *center-unstable*, respectively. This terminology is usual in differentiable dynamics.

Such a splitting is called *dominated* (with E^{cu} *dominating* E^{cs}) if there are constants $C > 0$, $\lambda > 0$ such that for all $x \in X$, $t > 0$, and unit vectors $v \in E^{\mathrm{cs}}_x$, $w \in E^{\mathrm{cu}}_x$ we have:

$$\frac{\|\psi^t(v)\|}{\|\psi^t(w)\|} < Ce^{-\lambda t} . \tag{2.1}$$

Note that this condition is independent of the choice of the Riemannian norm. It is actually equivalent to the following condition (see [CK, p. 156]): for all $x \in X$ and all unit vectors $v \in E^{\mathrm{cs}}_x$, $w \in E^{\mathrm{cu}}_x$ we have:

$$\lim_{t\to+\infty} \frac{\|\psi^t(v)\|}{\|\psi^t(w)\|} = 0. \tag{2.2}$$

The bundles of a dominated splitting are unique given their dimensions; more generally:

Proposition 2.1. *Suppose a linear flow $\{\psi^t\}$ has dominated splittings $E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$ and $F^{\mathrm{cu}} \oplus F^{\mathrm{cs}}$, with E^{cu} (resp. F^{cu}) dominating E^{cs} (resp. F^{cs}). If $\dim E^{\mathrm{cu}} \leqslant \dim F^{\mathrm{cs}}$ then $E^{\mathrm{cu}} \subset F^{\mathrm{cs}}$ and $E^{\mathrm{cs}} \supset E^{\mathrm{cu}}$.*

See e.g. [CP] for a proof of this and other properties of dominated splittings.

One can also define domination for invariant splittings into more than two bundles. This leads to the concept of *finest dominated splitting*, whose uniqueness is basically a consequence of Proposition 2.1; see [BDP, CP].

The existence of a dominated splitting can be characterized in terms of cone fields: see e.g. [CP]; we will use these ideas later in Section 5.

2.2. **Domination in terms of singular values.** Another indirect way to detect the existence of a dominated splitting can be formulated in terms of the "non-conformality" of the linear maps. Results of this kind were obtained in [Yo, Len] for dimension 2, and later in [BG] in more generality.[2] Let us explain this characterization.

If A is a linear map between two inner product vector spaces of dimension d, then its *singular values*

$$\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots \geqslant \sigma_d(A)$$

are the eigenvalues of the positive semidefinite operator $\sqrt{A^*A}$, ordered and repeated according to multiplicity. They equal the semiaxes of the ellipsoid obtained as the A-image of the unit ball; this is easily seen using the singular value decomposition [HJ, § 7.3].

If $p \in \{1, \dots, d-1\}$ and $\sigma_p(A) > \sigma_{p+1}(A)$, then we say that A *has a gap of index* p. In that case, we denote by $U_p(A)$ the p-dimensional subspace containing the p biggest axes of the ellipsoid $\{Av : \|v\| = 1\}$. Equivalently, $U_p(A)$ is the eigenspace of $\sqrt{AA^*}$ corresponding to the p smaller eigenvalues. We also define $S_{d-p}(A) := U_{d-p}(A^{-1})$. Note that $S_{d-p}(A)^\perp = A^{-1}(U_p(A))$ and $U_p(A)^\perp = A(S_{d-p}(A))$. See Fig. 1.

The following theorem asserts that the existence of a dominated splitting can be detected in terms of singular values, and also describes the invariant subbundles in these terms:

[2]For a recent generalization to Banach spaces, see [BM].

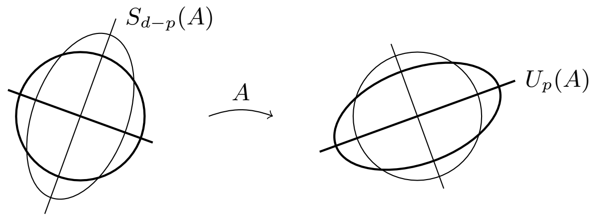


FIGURE 1. Spaces associated to a linear map A.

Theorem 2.2 (Bochi–Gourmelon [BG]). *Let $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$. Let $\{\psi^t\}_{t\in\mathbb{T}}$ be a linear flow on a vector bundle E, fibering over a continuous flow $\{\phi^t\}_{t\in\mathbb{T}}$ on a compact metric space X.*

Then the linear flow $\{\psi^t\}$ has a dominated splitting $E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$ where the dominating bundle E^{cu} has dimension p if and only if there exist $c > 0$, $\lambda > 0$ such that for every $x \in X$ and $t \geqslant 0$ we have

$$\frac{\sigma_{p+1}(\psi^t_x)}{\sigma_p(\psi^t_x)} < ce^{-\lambda t}.$$

Moreover, the bundles are given by:

$$E^{\mathrm{cu}}_x = \lim_{t\to+\infty} U_p\big(\psi^t_{\phi^{-t}(x)}\big),$$
$$E^{\mathrm{cs}}_x = \lim_{t\to+\infty} S_{d-p}\big(\psi^t_x\big),$$

and these limits are uniform.

(To make sense of the limits above it is necessary to metrize the Grassmann bundle associated to E; the particular way of doing so is irrelevant for the statement.)

In the paper [BG], the power of the Multiplicative Ergodic Theorem of Oseledets is used to basically reduce the proof of Theorem 2.2 to some angle estimates. Since we will explicitly need such estimates in other parts of this paper, we will also present a sketch of the proof of Theorem 2.2 in § A.4.

Remark 2.3. Consider the case of *complex* vector spaces, bundles, etc. Dominated splittings can be defined analogously; so can singular values and the subspaces U_p, S_{d-p}. Theorem 2.2 also extends to the complex case; indeed it can be deduced from the real case.[3]

2.3. **Domination for sequences of matrices.** Next, we describe some consequences of Theorem 2.2 for sequences of $d \times d$ matrices.[4]

Given $K > 1$, define the following compact set:

$$\mathcal{D}(K) := \big\{A \in \mathrm{GL}(d,\mathbb{R}),\ \|A\| \leqslant K,\ \|A^{-1}\| \leqslant K\big\}.$$

[3]Use that every d-dimensional complex vector space can be considered as a $2d$-dimensional real vector space, and that a hermitian inner product on the former induces a inner product on the latter in such a way that the induced norms coincide.

[4]We note that similar sequences have been considered in [GGKW], namely sequences that have what they call *coarsely linear increments* (CLI) in certain Cartan projections.

If I is a (possibly infinite) interval in $\mathbb{Z}$, we endow $\mathcal{D}(K)^I$ with the product topology, which is compact and is induced e.g. by the following metric:

$$d((A_n),(B_n)) := \sum_{n\in I} 2^{-|n|}\left(\|A_n - B_n\| + \|A_n^{-1} - B_n^{-1}\|\right) .$$

Let $p \in \{1,\dots,d-1\}$, $\mu > 0$, $c > 0$. For each interval $I \subset \mathbb{Z}$, let $\mathcal{D}(K,p,\mu,c,I)$ denote the set of sequences of matrices $(A_n) \in \mathcal{D}(K)^I$ such that for all m, $n \in I$ with $m \geqslant n$ we have

$$\frac{\sigma_{p+1}}{\sigma_p}(A_m \cdots A_{n+1}A_n) \leqslant ce^{-\mu(m-n+1)} .$$

Let us consider the case $I = \mathbb{Z}$. Let ϑ denote the shift map on the space $\mathcal{D}(K,p,\mu,c,\mathbb{Z})$, and let $A\colon \mathcal{D}(K,p,\mu,c,\mathbb{Z}) \to \mathrm{GL}(d,\mathbb{R})$ denote the projection on the zeroth coordinate. The pair (ϑ, A) determines a linear cocycle (in the sense explained in § 2.1). Note that the hypothesis of Theorem 2.2 is automatically satisfied. So we obtain:[5]

Proposition 2.4. *Fix constants $K > 1$, $p \in \{1,\dots,d-1\}$, $\mu > 0$, $c > 0$. Then, for each sequence $x = (A_n) \in \mathcal{D}(K,p,\mu,c,\mathbb{Z})$, the limits:*

$$E^{\mathrm{cu}}(x) = \lim_{n\to+\infty} U_p\big(A_{-1}A_{-2}\cdots A_{-n}\big) ,$$
$$E^{\mathrm{cs}}(x) = \lim_{n\to+\infty} S_{d-p}\big(A_{n-1}A_{n-2}\cdots A_0\big) ,$$

exist and are uniform over $\mathcal{D}(K,p,\mu,c,\mathbb{Z})$. Moreover, $\Xi^{\mathrm{cu}} \oplus \Xi^{\mathrm{cs}}$ is a dominated splitting for the linear cocycle (ϑ, A) defined above.

By a compactness argument, the theorem above yields information for finite sequences of matrices:

Lemma 2.5. *Given $K > 1$, $\mu > 0$, and $c > 0$, there exist $\ell_1 \in \mathbb{N}$ and $\delta > 0$ with the following properties. Suppose that $I \subset \mathbb{Z}$ is an interval*

and $(A_i)_{i\in I}$ is an element of $\mathcal{D}(K,p,\mu,c,I)$. If $n < k < m$ all belong to I and $\min\{k-n, m-k\} > \ell_1$ then:

$$\measuredangle\big(U_p(A_{k-1}\cdots A_{n+1}A_n),\, S_{d-p}(A_{m-1}\cdots A_{k+1}A_k)\big) > \delta .$$

Proof. The proof is by contradiction. Assume that there exist numbers $K > 1$, $\mu > 0$, $c > 0$, and sequences $\ell_j \to \infty$, $\delta_j \to 0$ such that for each j there exist an interval $I_j \subset \mathbb{Z}$, an element $\big(A_i^{(j)}\big)_{i\in I_j}$ of $\mathcal{D}(K,p,\mu,c,I_j)$, and integers $n_j < k_j < m_j$ in I_j such that $\min\{k_j - n_j, m_j - k_j\} > \ell_j$ and

$$\measuredangle\big(U_p(A_{k_j-1}\cdots A_{n_j}),\, S_{d-p}(A_{m_j-1}\cdots A_{k_j})\big) \leqslant \delta_j .$$

Shifting indices, we can assume that $k_j = 0$ for every j. By a diagonal argument, passing to subsequences we can assume that for each $i \in \mathbb{Z}$, the matrices $A_i^{(j)}$ (which are defined for sufficiently large j) converge to some matrix A_i as $j \to \infty$. The resulting sequence $x = (A_i)_{i\in\mathbb{Z}}$ belongs to $\mathcal{D}(K,p,\mu,c,\mathbb{Z})$. If $E^{\mathrm{cu}}(x)$ and $E^{\mathrm{cs}}(x)$ are the limit spaces as in Proposition 2.4, then their angle is zero, which contradicts domination. □

[5]For a similar statement with different notation and not relying on Theorem 2.2, see [GGKW, Theorem 5.3].

3. Domination implies word-hyperbolicity

In this section, we define dominated linear representations of a finitely generated group, and prove that groups that admit such representations are word-hyperbolic.

3.1. Dominated representations and the word-hyperbolicity theorem.

Let Γ be a finitely generated group. Let us fix a symmetric generating set S of Γ. We denote by $|\gamma|$ the *word-length* of $\gamma \in \Gamma$, i.e. the minimum number of elements of S required to obtain γ as a product of elements of S. The *word-metric* is defined as:

$$d(\gamma, \eta) := |\eta^{-1}\gamma| \,. \tag{3.1}$$

Then the left action of Γ into itself is isometric.

Recall that the group Γ is called *word-hyperbolic* if it is a Gromov-hyperbolic metric space when endowed with the word-metric (3.1); this does not depend on the choice of the generating set S; see [Gr, CDP, GH, BH].

A representation $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is *p-dominated* if there exists constants C, $\lambda > 0$ such that:

$$\frac{\sigma_{p+1}(\rho(\gamma))}{\sigma_p(\rho(\gamma))} \leqslant Ce^{-\lambda|\gamma|} \quad \text{for all } \gamma \in \Gamma. \tag{3.2}$$

It is easy to see that the definition does not depend on the choice of S though the constants C and λ may change.

Remark 3.1. Since $\sigma_i(A) = \sigma_{d+1-i}(A^{-1})^{-1}$ for each i, the property $|\gamma^{-1}| = |\gamma|$ implies that if a representation ρ is p-dominated, then it is also $(d-p)$-dominated.

The purpose of this section is to show:

Theorem 3.2. *If a group Γ admits a p-dominated representation onto $\mathrm{GL}(d, \mathbb{R})$ then Γ is word-hyperbolic.*

Theorem 3.2 follows from a more general result recently obtained by Kapovich, Leeb, and Porti [KLP_2, Theorem 1.4]. Their result concern not only dominated representations, but quasi-isometric embeddings of metric spaces satisfying a condition related to domination. Here we use the results from Section 2 to give a direct and more elementary proof.

Let us mention that a related but different notion of domination was recently studied by other authors [DT, GKW].

Remark 3.3 (Representations to $\mathrm{SL}(d, \mathbb{R})$). Given a representation with target group $\mathrm{GL}(d, \mathbb{R})$, we can always assume that it has its image contained in matrices with determinant ± 1 by composing with the homomorphism $A \mapsto |\det A|^{-1/d} A$, which does not affect p-domination.

Remark 3.4 (Representations to $\mathrm{PGL}(d, \mathbb{R})$). We can define p-dominated representations on $\mathrm{PGL}(d, \mathbb{R})$ in exactly the same way, since the quotients σ_{p+1}/σ_p are well-defined in the latter group. Obviously, composing any p-dominated representation on $\mathrm{GL}(d, \mathbb{R})$ with the quotient map $\pi\colon \mathrm{GL}(d, \mathbb{R}) \to \mathrm{PGL}(d, \mathbb{R})$ we obtain a p-dominated representation. Conversely, given any p-dominated representation $\rho\colon \Gamma \to \mathrm{PGL}(d, \mathbb{R})$, we can find a group $\hat{\Gamma}$, a 2-to-1 homomorphism $f\colon \hat{\Gamma} \to \Gamma$, and a p-dominated representation $\hat{\rho}\colon \hat{\Gamma} \to \mathrm{GL}(d, \mathbb{R})$ (with determinants ± 1) such that $\pi \circ \hat{\rho} = f \circ \rho$. Theorem 3.2 yields that $\hat{\Gamma}$ is word-hyperbolic, and it follows (see [GH, p. 63]) that Γ is word-hyperbolic as well.

Remark 3.5 (General semi-simple Lie groups). Using exterior powers, any p-dominated representation on $\mathrm{GL}(d,\mathbb{R})$ induces a 1-dominated representation in $\mathrm{GL}(k,\mathbb{R})$ for $k = \binom{d}{p}$. In Subsection 8.5 we shall see that every representation $\rho\colon \Gamma \to G$, where G is an arbitrary (real-algebraic non-compact) semi-simple Lie group, can be reduced to the case of $\mathrm{PGL}(d,\mathbb{R})$ for some d via a similar construction.

Remark 3.6. Given Theorem 3.2, one may wonder whether every hyperbolic group admits a dominated representation. This is far from true, since there exist hyperbolic groups that are nonlinear, i.e., admit no linear representation: see [Kap, Section 8]. On the other hand, one can ask if every linear hyperbolic group admits a dominated representation.

3.2. **Criterion for hyperbolicity.** In order to show word-hyperbolicity the following sufficient condition will be used:

Theorem 3.7 (Bowditch [Bow]). *Let Γ be a group which acts in a perfect compact metric space M such that the diagonal action of Γ on the (nonempty) space*

$$M^{(3)} := \left\{(x_1,x_2,x_3) \in M^3 : \ x_i \neq x_j \ \textit{if} \ \ i \neq j\right\}$$

is properly discontinuous and cocompact. Then Γ is word-hyperbolic.

We recall that an action of Γ in a topological space X is:

- *properly discontinuous* if given any compact subset $K \subset X$ there exists n such that if $|\gamma| > n$ then $\gamma K \cap K = \varnothing$;
- *cocompact* if there exists a compact subset $K \subset X$ such that $\Gamma x \cap K \neq \varnothing$ for every $x \in X$.

Remark 3.8. Theorem 3.7 also gives that the set M is equivariantly homeomorphic to $\partial\Gamma$. Here $\partial\Gamma$ denotes the *visual boundary* of the group Γ, defined as the set of equivalence classes of quasi-geodesic rays (i.e. quasi-geodesic maps from $\mathbb{N}$ to Γ) by the equivalence of being at finite Hausdorff distance from each other (see for example [GH, Chapitre 7] or [CDP, Chapitre 2]). The topology in $\partial\Gamma$ is given by uniform convergence of the quasi-geodesic rays (with same constants) starting in the same point. In Section 6 we will also comment on the metric structure on the boundary.

A group is called *elementary* if it is finite or virtually cyclic; elementary groups are trivially word-hyperbolic.

The converse of Theorem 3.7 applies to non-elementary word-hyperbolic groups. In the proof of Theorem 3.2 we must separate the case where the group is elementary, since elementary groups may admit dominated representations while Theorem 3.7 does not apply to them.

3.3. **Some preliminary lemmas for p-dominated representations.** We denote by $\mathscr{G}_p(\mathbb{R}^d)$ the set of all p-dimensional subspaces of $\mathbb{R}^d$, i.e. the *Grassmannian*. As we explain in detail in the Appendix A, the following formula defines a metric on the Grassmannian:

$$d(P,Q) := \cos \measuredangle(P^\perp, Q)\,.$$

(Here $\perp$ denotes orthogonal complement, and $\measuredangle$ denotes the minimal angle between pairs of vectors in the respective spaces.) Appendix A also contains a number of quantitative linear-algebraic estimates that we use in this section.

In particular, in Appendix A there are precise statements and proofs of the following results we will use:

- Lemmas A.3 and A.4 estimate the distance of $U_p(A)$ with $U_p(AB)$ and $BU_p(A)$ with $U_p(BA)$ with respect to the norms of $B^{\pm 1}$ and the gap on the singular values of A. Lemma A.10 is a reinterpretation of these facts in terms of dominated sequences of matrices.
- Lemma A.5 shows that if there is a large gap between the singular values of A then AP will be close to $U_p(A)$ for every P which makes a given angle with $S_{d-p}(A)$.
- Lemma A.7 estimates the dilatation of the action of A in $\mathscr{G}_p(\mathbb{R}^d)$ for subspaces far from the subspace which is mapped to $U_p(A)$.
- Corollary A.12 is a consequence of classical properties of dominated splittings in the context of sequences of matrices in $\mathcal{D}(K, p, \mu, c, \mathbb{N})$.

Throughout the rest of this section, let $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ a p-dominated representation with constants $C \geqslant 1$, $\lambda > 1$ (c.f. relation (3.2)). Let:

$$K \coloneqq \max_{g \in S} \|\rho(g)\| \,, \tag{3.3}$$

where S is the finite symmetric generating set of Γ fixed before. Also fix $\ell_0 \in \mathbb{N}$ such that:

$$Ce^{-\lambda \ell_0} < 1 \,, \tag{3.4}$$

and, in particular (recalling the notation introduced in § 2.2), the spaces $U_p(\rho(\gamma)) \in \mathscr{G}_p(\mathbb{R}^d)$ and $S_{d-p}(\rho(\gamma)) \in \mathscr{G}_{d-p}(\mathbb{R}^d)$ are well-defined whenever $|\gamma| \geqslant \ell_0$.

Suppose that γ, η are large elements in Γ such that the spaces $U_p(\rho(\gamma))$ and $U_p(\rho(\eta))$ are not too close; then the next two lemmas respectively assert that $d(\gamma, \eta)$ is comparable to $|\gamma| + |\eta|$, and that $U_p(\rho(\gamma))$ and $S_{d-p}(\rho(\eta^{-1}))$ are transverse.

Lemma 3.9. *There exist constants $\nu \in (0,1)$, $c_0 > 0$, $c_1 > 0$ with the following properties. If γ, $\eta \in \Gamma$ are such that $|\gamma|$, $|\eta| \geqslant \ell_0$ (where ℓ_0 is as in (3.4)) then:*

$$d(\gamma, \eta) \geqslant \nu(|\gamma| + |\eta|) - c_0 - c_1 \left|\log d\big(U_p(\rho(\gamma)), U_p(\rho(\eta))\big)\right| \,.$$

Proof. Consider two elements γ, $\eta \in \Gamma$ with word-length at least ℓ_0. Assume $|\gamma| \leqslant |\eta|$, the other case being analogous. Applying Lemma A.3 to $A = \rho(\eta)$ and $B = \rho(\eta^{-1}\gamma)$, and using (3.2) and (3.3), we obtain:

$$d\big(U_p(\rho(\eta)), U_p(\rho(\gamma))\big) \leqslant K^{2|\eta^{-1}\gamma|}\, Ce^{-\lambda|\eta|} \,, \tag{3.5}$$

or equivalently,

$$d(\gamma, \eta) = |\eta^{-1}\gamma| \geqslant \frac{\lambda|\eta| - \log C + \left|\log d\big(U_p(\rho(\eta)), U_p(\rho(\gamma))\big)\right|}{2 \log K} \,.$$

Using that $|\eta| \geqslant (|\gamma| + |\eta|)/2$, we obtain the lemma. □

Lemma 3.10. *For every $\varepsilon > 0$ there exist $\ell_1 \geqslant \ell_0$ and $\delta > 0$ with the following properties: If γ, $\eta \in \Gamma$ are such that:*

(i) *$|\gamma|$, $|\eta| > \ell_1$, and*
(ii) *the distance $d\big(U_p(\rho(\gamma)), U_p(\rho(\eta))\big) > \varepsilon$,*

then it follows that $\measuredangle\big(U_p(\rho(\gamma)), S_{d-p}(\rho(\eta^{-1}))\big) > \delta$.

Proof. Consider two elements γ, $\eta \in \Gamma$ with word-length at least ℓ_0. Let $\varepsilon > 0$, and suppose the hypothesis (ii) is satisfied. Write $\gamma = g_1 \cdots g_n$ with each g_i in the symmetric generator set of Γ in such a way that n is minimal, that is, $n = |\gamma|$.

Similarly, write $\eta = h_1 \cdots h_m$ with each h_i in the symmetric generator set of Γ in such a way that $|\eta| = m$.

Let $\gamma_i := g_1 \cdots g_i$ and $\eta_i := h_1 \cdots h_i$. Note that for $j > i$ we have $d(\gamma_i, \gamma_j) = |\gamma_i^{-1}\gamma_j| = |g_{i+1} \cdots g_j| = j - i$ so that the sequence $\{\gamma_i\}$ is a geodesic from id to γ. The same holds for $\{\eta_i\}$ (but note that $\{\eta_i^{-1}\}$ is not necessarily a geodesic).

By the domination condition (3.2) and Lemma A.10, we can find $\ell_* = \ell_*(\varepsilon) > \ell_0$ such that if $n \geqslant i > \ell_*$ then

$$d(U_p(\rho(\gamma_i)), U_p(\rho(\gamma))) < \varepsilon/3\,,$$

and analogously, if $m \geqslant j > \ell_*$ then

$$d(U_p(\rho(\eta_j)), U_p(\rho(\eta))) < \varepsilon/3.$$

In particular, if both conditions hold then hypothesis (ii) implies that

$$d(U_p(\rho(\gamma_i)), U_p(\rho(\eta_j)) > \varepsilon/3. \tag{3.6}$$

Let $\nu \in (0,1)$, c_0, c_1 be given by Lemma 3.9. Let $c := \max\{2\ell_*, c_0 + c_1 \log(3/\varepsilon)\}$. We claim that

$$d(\gamma_i, \eta_j) = |\eta_j^{-1}\gamma_i| \geqslant \nu(i+j) - c \quad \text{for all } i \in \{1, \dots, n\},\ j \in \{1, \dots, m\}. \tag{3.7}$$

To prove this, consider first the case when $j \leqslant \ell_*$. Then

$$d(\gamma_i, \eta_j) \geqslant d(\gamma_i, \mathrm{id}) - d(\eta_j, \mathrm{id}) = i - j \geqslant i + j - 2\ell_* \geqslant \nu(i+j) - c\,,$$

as claimed. The case $i \leqslant \ell_*$ is dealt with analogously. The remaining case where i and j are both bigger than ℓ_* follows from Lemma 3.9 and property (3.6). This proves (3.7).

As a consequence of (3.2) and (3.7), we obtain:

$$\frac{\sigma_{p+1}}{\sigma_p}(\eta_j^{-1}\gamma_i) \leqslant \hat{C} e^{-\mu(i+j)} \quad \text{for all } i \in \{1, \dots, n\},\ j \in \{1, \dots, m\}, \tag{3.8}$$

where $\hat{C} := Ce^{\lambda c}$ and $\mu := \lambda\nu$. Now consider the following sequence of matrices:

$$\big(A_{-n}, \dots, A_{-1}, A_0, A_1, \dots, A_{m-1}\big) := \big(\rho(g_n), \dots, \rho(g_1), \rho(h_1^{-1}), \rho(h_2^{-1}), \dots, \rho(h_m^{-1})\big)\,.$$

So $A_{-1}A_{-2} \cdots A_{-i} = \gamma_i$ and $A_{j-1}A_{j-2} \cdots A_0 = \eta_j^{-1}$. It follows from (3.2) and (3.8), together with the facts that $\hat{C} > C$ and $\mu < \lambda$, that the sequence of matrices defined above belongs to the set $\mathcal{D}(K, p, \mu, \hat{C}, I)$, where $I := \{-n, \dots, m-1\}$ and K is defined by equation (3.3). Now let ℓ_1 and δ be given by Lemma 2.5. Then

$$\measuredangle\big(U_p(A_{-1} \cdots A_{-n}), S_{d-p}(A_{m-1} \cdots A_0)\big) > \delta\,,$$

provided that $|\gamma| = n$ and $|\eta| = m$ are both bigger than ℓ_1. This concludes the proof. □

3.4. **Candidate for boundary of Γ.** In this subsection we define a candidate for the role of M in Theorem 3.7 using the domination of the representation and show some of the topological properties of M required for applying Theorem 3.7. The set we shall consider is the following:

$$M := \bigcap_{n \geqslant \ell_0} \overline{\{U_p(\rho(\gamma)) : \ |\gamma| \geqslant n\}} \subset \mathscr{G}_p(\mathbb{R}^d)\,, \tag{3.9}$$

where ℓ_0 is as in (3.4). This set has been considered before and named *limit set* by Benoist (see [Be, Section 6]) in the Zariski dense context and extended in [GGKW, Definition 5.1] to a more general setting.

The first properties to be established about M are the following ones:

Proposition 3.11. *The set M is compact, non-empty, and $\rho(\Gamma)$-invariant.*

Proof. The fact that M is compact and non-empty follows at once since it is a decreasing intersection of non-empty closed subsets of the compact space $\mathscr{G}_p(\mathbb{R}^d)$.

Let us show that M is $\rho(\Gamma)$-invariant. Fix $\eta \in \Gamma$ and $P \in M$. Choose a sequence (γ_n) in Γ such that $|\gamma_n| \to \infty$ and $U_p(\rho(\gamma_n)) \to P$. Note that the spaces $U_p(\rho(\eta\gamma_n))$ are defined for large enough n (namely, whenever $|\gamma_n| \geqslant \ell_0 - |\eta|$); moreover, by Lemma A.4 and the domination condition (3.2), we have:

$$d\big(\rho(\eta)U_p(\rho(\gamma_n)), U_p(\rho(\eta\gamma_n))\big) \leqslant \|\rho(\eta)\| \, \|\rho(\eta)^{-1}\| \, C \, e^{-\lambda|\gamma_n|} \to 0 \quad \text{as } n \to \infty.$$

This shows that $U_p(\rho(\eta\gamma_n)) \to \rho(\eta)P$ as $n \to \infty$, and in particular $\rho(\eta)P \in M$, as we wished to prove. □

If one manages to show that $M^{(3)}$ is non-empty then one of the assumptions of Theorem 3.7 is satisfied:

Lemma 3.12. *If the set M has at least 3 points then it is perfect.*

We first show the following lemma:

Lemma 3.13. *Given $\varepsilon > 0$, $\varepsilon' > 0$, there exists $\ell > \ell_0$ with the following properties: If $\eta \in \Gamma$ is such that $|\eta| > \ell$ (where ℓ_0 is as in (3.4)) and $P \in M$ is such that*

$$d\big(P, \, U_p(\rho(\eta^{-1}))\big) > \varepsilon \,,$$

then

$$d\big(\rho(\eta)P, \, U_p(\rho(\eta))\big) < \varepsilon' \,.$$

Proof. Let $\ell_1 \geqslant \ell_0$ and $\delta > 0$ be given by Lemma 3.10, depending on ε. Let $\ell > \ell_1$ be such that $Ce^{-\lambda\ell} < (\varepsilon' \sin\delta)/2$. where C and λ are the domination constants, as in (3.2). Now fix $\eta \in \Gamma$ and $P \in M$ such that $|\eta| > \ell$ and $d\big(P, U_p(\rho(\eta^{-1}))\big) > \varepsilon$. Choose a sequence (γ_n) in Γ such that $|\gamma_n| \to \infty$ and $U_p(\rho(\gamma_n)) \to P$. Without loss of generality, we can assume that for each n we have $|\gamma_n| > \ell_1$ and

$$d\big(U_p(\rho(\gamma_n)), U_p(\rho(\eta^{-1}))\big) > \varepsilon.$$

It follows from Lemma 3.10 that

$$\measuredangle\big(U_p(\rho(\gamma_n)), S_{d-p}(\rho(\eta))\big) > \delta.$$

Using Lemma A.5 and the domination condition (3.2), we obtain:

$$d\Big(\rho(\eta)\big(U_p(\rho(\gamma_n))\big) \, , U_p(\rho(\eta))\Big) < \tfrac{\sigma_{p+1}}{\sigma_p}\big(\rho(\eta)\big) \, \frac{1}{\sin\delta} < \frac{\varepsilon'}{2} \,.$$

Letting $n \to \infty$ yields $d\big(\rho(\eta)P \, , U_p(\rho(\eta))\big) \leqslant \varepsilon'/2$. □

Proof of Lemma 3.12. Let P_1, P_2, P_3 be three distinct points in M, and let $\varepsilon' > 0$. We will show that the $2\varepsilon'$-neighborhood of P_1 contains another element of M.

Let $\varepsilon := \frac{1}{2}\min_{i \neq j} d(P_i, P_j)$. Let $\ell > \ell_0$ be given by Lemma 3.13, depending on ε and ε'. Choose $\eta \in \Gamma$ such that $|\eta| > \ell$ and $d\big(U_p(\rho(\eta)), P_1\big) < \varepsilon'$. Consider the space $U_p(\rho(\eta^{-1}))$; it can be ε-close to at most one of the spaces P_1, P_2, P_3. In other words, there are different indices i_1, $i_2 \in \{1, 2, 3\}$ such that for each $j \in \{1, 2\}$ we have:

$$d\big(P_{i_j} \, , U_p(\rho(\eta^{-1}))\big) > \varepsilon$$

In particular, by Lemma 3.13,

$$d\big(\rho(\eta)P_{i_j}, P_1\big) \leqslant d\big(\rho(\eta)P_{i_j}, U_p(\rho(\eta))\big) + \varepsilon' < 2\varepsilon' \,.$$

By Proposition 3.11, the spaces $\rho(\eta)P_{i_1}$ and $\rho(\eta)P_{i_2}$ belong to M. Since at most one of these can be equal to P_1, we conclude that the $2\varepsilon'$-neighborhood of P_1 contains another element of M, as we wished to prove. □

So we would like to assure that M has at least 3 points (provided Γ is non-elementary). For this, we need first to digress a little and show that virtually abelian groups cannot have a dominated representations unless they are virtually cyclic.

Lemma 3.14. *Let $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ be a p-dominated representation. Let Γ' be a finite index subgroup of Γ, and let $m \geqslant 2$. Then there exists no surjective homomorphism $\varphi\colon \Gamma' \to \mathbb{Z}^m$ with finite kernel.*

Proof. Assume for a contradiction that a Γ contains a subgroup Γ' with the property of admitting a homomorphism φ onto $\mathbb{Z}^m$ with finite kernel, where $m \geqslant 2$. Let Z be the standard symmetric generating set for $\mathbb{Z}^m$, with cardinality $2m$; then $\varphi^{-1}(Z)$ is a finite symmetric generating set for Γ'. Note that the restriction of ρ to Γ' is p-dominated, since this is a finite index subgroup of Γ. For simplicity of notation, we assume that $\Gamma = \Gamma'$, with generating set $S = \varphi^{-1}(Z)$. We can also assume that $p \leqslant d/2$, by Remark 3.1. Fix the constants C, λ, K, and ℓ_0 satisfying (3.2), (3.3), and (3.4).

Fix $g \in \Gamma$ such that $\varphi(g)$ is an element of Z, say $(1, 0, \dots, 0)$. Since the representation is p-dominated and $(g^n)_{n\in\mathbb{Z}}$ is a geodesic in Γ, it follows from Lemma 2.5 that there exist $\ell_1 > \ell_0$ and $\delta > 0$ such that

$$\measuredangle\big(U_p(\rho(g^n)), S_{d-p}(\rho(g^n))\big) > \delta \quad \text{for all } n > \ell_1.$$

Since $d - p \geqslant p$ we have $S_{d-p}(A) = U_{d-p}(A^{-1}) \supset U_p(A^{-1})$; recalling (A.3) we obtain:

$$d\big(U_p(\rho(g^n)), U_p(\rho(g^{-n}))\big) \geqslant \sin\delta \quad \text{for all } n > \ell_1. \tag{3.10}$$

Fix $n > \ell_1$. Since $m \geqslant 2$, we can find $\gamma_0, \gamma_1, \dots, \gamma_{4n} \in \Gamma$ with the following properties:

$$\gamma_0 = g^n, \quad \gamma_{4n} = g^{-n}, \quad |\gamma_{i+1}^{-1}\gamma_i| = 1, \quad |\gamma_i| \geqslant n.$$

Indeed, we can take a preimage under φ of an appropriate path in $\mathbb{Z}^m$, sketched in Fig. 2 for the case $m = 2$.

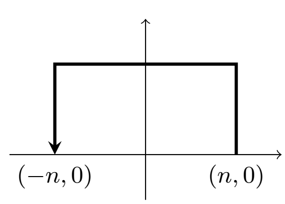


FIGURE 2. A path in $\mathbb{Z}^2$.

Now we estimate:

$$\begin{aligned} d\big(U_p(\rho(g^n)), U_p(\rho(g^{-n}))\big) &\leqslant \sum_{i=0}^{4n-1} d\big(U_p(\rho(\gamma_i)), U_p(\gamma_{i+1})\big) \\ &\leqslant \sum_{i=0}^{4n-1} K^{2|\gamma_{i+1}^{-1}\gamma_i|}\, C\, e^{-\lambda|\gamma_i|} \qquad \text{(by estimate (3.5))} \\ &\leqslant 4nK^2Ce^{-\lambda n}\,. \end{aligned}$$

Taking n large enough, we contradict (3.10). This proves the lemma. ☐

Now we are ready to obtain the topological property of M required by Theorem 3.7. (Later we will have to check the hypotheses about the action on $M^{(3)}$.) Recall that Γ is elementary if it is finite or virtually cyclic (i.e. virtually $\mathbb{Z}$).

Proposition 3.15. *If Γ is non-elementary then the set M is perfect.*

Proof. By Lemma 3.12, it is enough to show that M is infinite. Recall that M is non-empty. We assume by contradiction that M is finite, say $M = \{P_1, \dots, P_k\}$. We assume for convenience that $|\det \rho(\gamma)| = 1$ for every $\gamma \in \Gamma$ (see Remark 3.3).

By Proposition 3.11, the set M is $\rho(\Gamma)$-invariant. Consider the subgroup $\Gamma' \subset \Gamma$ of those elements such that $\rho(\gamma)P_i = P_i$ for all i. Then Γ' is a finite-index subgroup of Γ. Consider the homomorphism $\varphi\colon \Gamma' \to \mathbb{R}^k$ defined by

$$\gamma \in \Gamma' \mapsto (\log|\det \rho(\gamma)|_{P_i}|)_{i=1,\dots,k} \quad .$$

This morphism is well defined since invariance of the subspaces P_i implies that the determinant is multiplicative.

The image of φ is a subgroup of $\mathbb{R}^k$ and therefore is abelian, without torsion, and finitely generated (since so is Γ'); therefore this image is isomorphic to $\mathbb{Z}^m$ for some m. It follows that the kernel of φ has to be infinite, since otherwise Γ would be elementary or would admit a morphism with finite kernel onto $\mathbb{Z}^m$ for some $m \geqslant 2$ contradicting Lemma 3.14.

Take a sequence $\gamma_n \in \operatorname{Ker}\varphi$ with $|\gamma_n| \to \infty$. Passing to a subsequence, we assume that $U_p(\rho(\gamma_n))$ converges to some element of M, say P_1. By considering a further subsequence, we can also assume that there is a sequence $g_n \in S$ (where S is the fixed finite symmetric generating set of Γ) such that

$$\gamma_n = g_1 g_2 \cdots g_{\ell_n} \quad \text{where} \quad \ell_n = |\gamma_n| \quad .$$

Now, the domination condition and the fact that the determinant of every element is ± 1 implies that the determinant along a subspace which is close to the most expanded one has to be exponentially large; this is expressed by Lemma A.9 which gives precisely that

$$|\det(\rho(\gamma_n^{-1})|_{P_1})| \leqslant ce^{-\varepsilon \ell_n} \quad \text{for some constants } c > 0,\ \varepsilon > 0.$$

However, for every n the left-hand side is 1, since $\gamma_n \in \operatorname{Ker}\varphi$. This contradiction concludes the proof. ☐

3.5. **Proper discontinuity.** Given a triple $T = (P_1, P_2, P_3) \in M^{(3)}$, let us denote

$$|T| := \min_{i \neq j} d(P_i, P_j)\,. \tag{3.11}$$

Note that for any $\delta > 0$, the set $\big\{T \in M^{(3)} : |T| \geqslant \delta\big\}$ is a compact subset of $M^{(3)}$; conversely, every compact subset of $M^{(3)}$ is contained in a subset of that form.

Proposition 3.16. *For every $\delta > 0$ there exists $\ell \in \mathbb{N}$ such that if $T \in M^{(3)}$ satisfies $|T| > \delta$ and $\eta \in \Gamma$ satisfies $|\eta| > \ell$, then we have $|\rho(\eta)T| < \delta$.*

Proof. Given $\delta > 0$, let ℓ be given by Lemma 3.13 with $\varepsilon = \varepsilon' = \delta/2$. Now consider $(P_1, P_2, P_3) \in M^{(3)}$ such that $|T| > \delta$ and $\eta \in \Gamma$ such that $|\eta| > \ell$. Note that $d(U_p(\eta^{-1}), P_i) > \delta/2$ for at least two of the spaces P_1, P_2, P_3 – say, P_1 and P_2. Lemma 3.13 yields $d\big(\rho(\eta)P_i, U_p(\rho(\eta))\big) < \delta/2$ for each $i = 1, 2$. In particular, $d\big(\rho(\eta)P_1, \rho(\eta)P_2\big) < \delta$ and so $|\rho(\eta)T| < \delta$, as we wanted to show. □

3.6. **Cocompactness.** The purpose of this subsection is to prove the following proposition which will complete the proof of Theorem 3.2. Recall notation (3.11).

Proposition 3.17. *There exists $\varepsilon > 0$ such that for every $T \in M^{(3)}$ there exists $\gamma \in \Gamma$ such that $|\rho(\gamma)T| \geqslant \varepsilon$.*

We first show the following lemma:

Lemma 3.18 (Expansivity)**.** *There exist constants $\delta > 0$ and $\ell \in \mathbb{N}$ with the following properties. For every $P \in M$ there exists $\gamma \in \Gamma$ with $|\gamma| \leqslant \ell$ such that if P', P'' belong to the δ-neighborhood of P in $\mathscr{G}_p(\mathbb{R}^d)$ then*

$$d(\rho(\gamma)P', \rho(\gamma)P'') \geqslant 2d(P', P'') .$$

Proof. By compactness of M, it is sufficient to prove that for every $P \in M$ there exists $\gamma \in \Gamma$ such that if P', P'' belong to a sufficiently small neighborhood of P in $\mathscr{G}_p(\mathbb{R}^d)$ then $d(\rho(\gamma)P', \rho(\gamma)P'') \geqslant 2d(P', P'')$.

Given $P \in M$, by definition there exists a sequence (γ_i) such that $n_i := |\gamma_i| \to \infty$ and $U_p(\rho(\gamma_i)) \to P$. Using a diagonal argument, we can assume that $\gamma_i = g_1 g_2 \cdots g_{n_i}$. Consider the sequence of matrices $(A_0, A_1, \dots)$ given by $A_n = \rho(g_{n-1}^{-1})$ By the domination condition (3.2), the sequence belongs to some $\mathcal{D}(\mu, c, K, d{-}p, \mathbb{N})$. Note that $U_p(\rho(g_1 g_2 \cdots g_n)) = S_p(A_{n-1} \dots A_0)$ also converges to P, so for simplicity of notation we can assume that $n_i = i$.

Applying Corollary A.12 to the sequence of matrices, we find $\tilde{P} \in \mathscr{G}_{d-p}(\mathbb{R}^d)$ such that, for all $n > 0$,

$$\measuredangle\big(\rho(\gamma_n^{-1})\tilde{P}, \rho(\gamma_n^{-1})P\big) \geqslant \alpha \,,$$

$$\frac{\|\rho(\gamma_n^{-1})|_P\|}{\mathbf{m}(\rho(\gamma_n^{-1})|_{\tilde{P}})} < \tilde{c} e^{-\tilde{\mu} n} \,,$$

where α, $\tilde{c}$, $\tilde{\mu}$ are positive constants that do not depend on P. Let $b > 0$ be given by Lemma A.7, depending on α. Fix k such that $b\tilde{c}^{-1} e^{\tilde{\mu} k} > 2$, and let $\gamma = \gamma_k^{-1}$. Applying Lemma A.7 to $A = \rho(\gamma)$, we conclude that for all P', P'' in a sufficiently small neighborhood of P in $\mathscr{G}_p(\mathbb{R}^d)$ we have

$$d(\rho(\gamma)P', \rho(\gamma)P'') \geqslant 2d(P', P'') \,,$$

as we wanted to show. □

Proof of Proposition 3.17. Let δ and ℓ be given by Lemma 3.18. Let

$$\varepsilon := \inf \left\{ d\big(\rho(\gamma)P, \rho(\gamma)P'\big) : \ \gamma \in \Gamma, \ |\gamma| \leqslant \ell, \ P, P' \in \mathscr{G}_p(\mathbb{R}^d), \ d(P, P') \geqslant \frac{\delta}{2} \right\} .$$

So $0 < \varepsilon \leqslant \delta/2$. We claim that

$$\forall T \in M^{(3)} \ \exists \gamma \in \Gamma \text{ such that } |\rho(\gamma)T| \geqslant \min\{2|T|, \varepsilon\} \,. \tag{3.12}$$

Indeed, given $T = (P_1, P_2, P_3) \in M^{(3)}$, we can suppose that $|T| < \varepsilon$, otherwise we simply take $\gamma = \mathrm{id}$. Permuting indices if necessary we can assume that $d(P_1, P_2) = |T|$. We apply Lemma 3.18 and find $\gamma \in \Gamma$ such that the action of $\rho(\gamma)$ on $N_\delta(P_1)$ (the δ-neighborhood of P_1) expands distances by a factor of at least 2. Since $\varepsilon \leqslant \delta/2$, for each pair $\{i \neq j\} \subset \{1, 2, 3\}$ we have

$$\{P_i, P_j\} \subset N_\delta(P_1) \quad \text{or} \quad d(P_i, P_j) \geqslant \frac{\delta}{2}\,.$$

So $d(\rho(\gamma)P_i, \rho(\gamma)P_j) \geqslant \min\{2|T|, \varepsilon\}$, thus proving the claim (3.12). Now the proposition follows by an obvious recursive argument. □

3.7. **Conclusion.** Now we join the pieces and obtain the main result of this section:

Proof of Theorem 3.2. Consider a p-dominated representation $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$. If Γ is an elementary group then it is word-hyperbolic and there is nothing to prove. So assume that Γ is non-elementary. Using the representation ρ, we define an action of Γ on $\mathscr{G}_p(\mathbb{R}^d)$. Consider the set $M \subset \mathscr{G}_p(\mathbb{R}^d)$ defined by (3.9), which is perfect (by Proposition 3.15) and invariant under the action of Γ (by Proposition 3.11). The diagonal action of Γ on $M^{(3)}$ is properly discontinuous (by Proposition 3.16) and cocompact (by Proposition 3.17). Therefore Theorem 3.7 assures that Γ is word-hyperbolic. □

4. Anosov representations and dominated representations

The main goal of this section is to show that being p-dominated (c.f. condition (3.2)) and satisfying the Anosov condition as defined by Labourie [Lab] (and extended by Guichard-Wienhard [GW] to arbitrary hyperbolic groups) are equivalent. That equivalence (among others) is contained in the results of [KLP_1, KLP_2, KLP_3]. Our approach also yields a slightly different characterization directly related to dominated splittings (see Proposition 4.6).

In the final subsection we discuss relations with characterizations of [GGKW], and pose some questions.

We first introduce the notion of Anosov representations into $\mathrm{GL}(d, \mathbb{R})$ which requires introducing the geodesic flow of a hyperbolic group.

4.1. **The geodesic flow.** In order to define the Anosov property for a representation of a hyperbolic group, we need to recall the *Gromov geodesic flow* of Γ.

Given a word-hyperbolic group Γ we can define its visual boundary $\partial\Gamma$ (c.f. Remark 3.8). Denote $\partial^{(2)}\Gamma := \{(x, y) \in \partial\Gamma \times \partial\Gamma :\ x \neq y\}$. We define a flow on the space $\widetilde{U\Gamma} := \partial^{(2)}\Gamma \times \mathbb{R}$, called the *lifted geodesic flow* by the formula $\tilde{\phi}^t(x, y, s) := (x, y, s + t)$.

A function $c : \partial^{(2)}\Gamma \times \Gamma \to \mathbb{R}$ such that

$$c(\gamma_0\gamma_1, x, y) = c(\gamma_0, \gamma_1(x, y)) + c(\gamma_1, x, y) \quad \text{for any } \gamma_0, \gamma_1 \in \Gamma \text{ and } (x, y) \in \partial^{(2)}\Gamma$$

is called a *cocycle*. Recall that every $\gamma \in \Gamma \backslash \{\mathrm{id}\}$ acts on $\partial\Gamma$ leaving only two fixed points, an attractor γ^+ and a repeller γ^-. Let us say that a cocycle c is *positive* if $c(\gamma, \gamma^-, \gamma^+) > 0$ for every $\gamma \in \Gamma \backslash \{\mathrm{id}\}$.

Given such a cocycle, we can define an action of Γ on $\widetilde{U\Gamma}$ by $\gamma \cdot (x, y, s) = (\gamma \cdot x, \gamma \cdot y, s - c(x, y, \gamma))$, which obviously commutes with the lifted geodesic flow. Gromov [Gr] (see also [Mat, Ch, Mi]) proved that there exists a positive cocycle such that the latter action is properly discontinuous and cocompact. This allows to define the *geodesic flow* ϕ^t of Γ on $U\Gamma := \widetilde{U\Gamma}/\Gamma$, the *unit tangent bundle* of Γ.

There is a metric on $\widetilde{U\Gamma}$, well-defined up to Hölder equivalence, so that Γ acts by isometries, the lifted geodesic flow acts by bi-Lipschitz homeomorphisms, and its flow lines are quasi-geodesics.

Remark 4.1. If $\Gamma = \pi_1(M)$ is the fundamental group of a negatively curved closed manifold M then the geodesic flow on the unit tangent bundle UM is hyperbolic and equivalent to the abstract geodesic flow defined above. In that case, the unit tangent bundle of the universal cover $\tilde{M}$ is homeomorphic to $\widetilde{U\Gamma}$ by means of the Hopf parametrization. For details, see [Led].

Lemma 4.2. *For any compact set $K \subset \widetilde{U\Gamma}$, there exist $a > 0$ and $\kappa > 1$ such that if $t \in \mathbb{R}$ and $\gamma \in \Gamma$ satisfy*

$$\tilde{\phi}^t(K) \cap \gamma(K) \neq \varnothing$$

then

$$\kappa^{-1}|t| - a \leqslant |\gamma| \leqslant \kappa|t| + a\,.$$

Proof. Take a ball $B(u_0, r)$ containing K. The Svarc-Milnor lemma (see for example [BH, Proposition I.8.19]) implies that the map $\gamma \in \Gamma \mapsto \gamma u_0 \in \widetilde{U\Gamma}$ is a quasi-isometry, so there exist $\kappa > 1$, $b > 0$ such that for all $\gamma_1, \gamma_2 \in \Gamma$ we have

$$\kappa^{-1} d(\gamma_1, \gamma_2) - b \leqslant d(\gamma_1 u_0, \gamma_2 u_0) \leqslant \kappa d(\gamma_1, \gamma_2) + b$$

In particular,

$$\kappa^{-1}|\gamma| - b \leqslant d(\gamma u_0, u_0) \leqslant \kappa|\gamma| + b\,.$$

Now assume that t and γ satisfy $\tilde{\phi}^t(K) \cap \gamma(K) \neq \varnothing$, that is, there exist $u_1, u_2 \in K$ such that $\tilde{\phi}^t u_1 = \gamma u_2$. So[6]

$$|t| \overset{*}{=} d(\tilde{\phi}^t u_1, u_1) = d(\gamma u_2, u_1) \begin{cases} \leqslant d(\gamma u_0, u_0) + 2r \\ \geqslant d(\gamma u_0, u_0) - 2r \end{cases}$$

The desired inequalities follow. □

4.2. **Equivariant maps and the definition of Anosov representations.** Let $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ be a representation of a word-hyperbolic group Γ. The definitions here can be adapted for representations into general semisimple Lie groups and the results are equivalent. In order to be able to present our results in a more elementary manner, we have deferred the introduction of the general context to section 8.

We say that the representation ρ is *p-convex* if there exist continuous maps $\xi : \partial\Gamma \to \mathscr{G}_p(\mathbb{R}^d)$ and $\theta : \partial\Gamma \to \mathscr{G}_{d-p}(\mathbb{R}^d)$ such that:

- (**transversality:**) for every $x \neq y \in \partial\Gamma$ we have $\xi(x) \oplus \theta(y) = \mathbb{R}^d$,
- (**equivariance:**) for every $\gamma \in \Gamma$ we have $\xi(\gamma \cdot x) = \rho(\gamma)\xi(x)$ and $\theta(\gamma \cdot x) = \rho(\gamma)\theta(x)$.

[6] To get exact equality in $(*)$ we need the construction in [Mi], for which orbits of the flow are geodesics. If instead we use [Ch] or [Mat], the orbits of the flow are quasi-geodesics, so the equality $(*)$ is only approximate, which is sufficient for our purpose.

Using the representation ρ, it is possible to construct a linear flow ψ^t over the geodesic flow ϕ^t of Γ as follows. Consider the lifted geodesic flow $\tilde{\phi}^t$ on $\widetilde{U\Gamma}$, and define a linear flow on $\tilde{E} := \widetilde{U\Gamma} \times \mathbb{R}^d$ by:

$$\tilde{\psi}^t((x, y, s), v) := (\tilde{\phi}^t(x, y, s), v),$$

Now consider the action of Γ on $\tilde{E}$ given by:

$$\gamma \cdot ((x, y, s), v) := (\gamma \cdot (x, y, s), \rho(\gamma)v)$$

where the action of Γ in $\widetilde{U\Gamma}$ is the one explained in Subsection 4.1. It follows that $\tilde{\psi}^t$ induces in $E_\rho := \tilde{E}/_\Gamma$ (which is a vector bundle over $U\Gamma$) a linear flow ψ^t which covers ϕ^t. See Fig. 3.

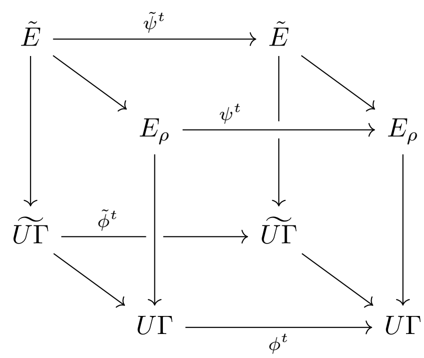


Figure 3. A commutative diagram. The ↓ arrows are vector bundle projections. The ↘ arrows are quotient maps w.r.t. the corresponding actions of Γ.

When the representation ρ is p-convex, by equivariance there exists a ψ^t-invariant splitting of the form $E_\rho = \Xi \oplus \Theta$; it is obtained taking the quotient of the bundles $\tilde{\Xi}(x, y, s) := \xi(x)$ and $\tilde{\Theta}(x, y, s) := \theta(y)$ with respect with the Γ-action.

We say that a p-convex representation ρ is *p-Anosov* if the splitting $E_\rho = \Xi \oplus \Theta$ is a dominated splitting for the linear bundle automorphism ψ^t, with Ξ dominating Θ. This is equivalent to the fact that the bundle $\mathrm{Hom}(\Theta, \Xi)$ is uniformly contracted by the flow induced by ψ^t (see [BCLS]).

Conversely, dominated splittings for the linear flow ψ^t must be of the form $\Xi \oplus \Theta$ as above: see Propositions 4.6 and 4.9 below.

Let us mention that by [GW, Theorem 1.5], if the image of the representation ρ is Zariski dense, being p-Anosov is a direct consequence of being p-convex.

Remark 4.3. As before, it is possible to use exterior powers to transform a p-Anosov representation into a 1-Anosov one. The latter are called *projective Anosov* and are discussed in Section 8 as it is shown that any Anosov representation to an arbitrary semisimple Lie group can be transformed into a projective Anosov one. See also [BCLS, Section 2.3].

4.3. **Equivalence between the definitions.** Suppose that $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is a representation of a word-hyperbolic group Γ. We will show that ρ is p-dominated if and only if it is p-Anosov. Note that the definition of p-Anosov representation requires the group to be word-hyperbolic. On the other hand, we have shown in Section 3 that if ρ is p-dominated then the group Γ is automatically word-hyperbolic. So in fact both definitions coincide regardless of the group Γ.

Let us first show the following:

Lemma 4.4. *Endow E_ρ with a Riemannian metric*[7]*. Then there exist constants $\kappa > 1$, $a > 0$, and $C > 1$ with the following properties:*

(i) *For every $z \in U\Gamma$ and $t \in \mathbb{R}$, there exists $\gamma \in \Gamma$ such that:*

$$\kappa^{-1}|t| - a \leqslant \quad |\gamma| \quad \leqslant \kappa|t| + a \quad \text{and} \tag{4.1}$$

$$C^{-1}\sigma_p(\psi_z^t) \leqslant \sigma_p(\rho(\gamma)) \leqslant C\sigma_p(\psi_z^t) \quad \text{for every } p = 1, \dots, d-1. \tag{4.2}$$

(ii) *Conversely, for every $\gamma \in \Gamma$ there exists $z \in U\Gamma$ and $t \in \mathbb{R}$ such that (4.1) and (4.2) hold.*

Proof. Using the covering map $\tilde{E} \to E_\rho$, we lift the fixed Riemannian metric on E_ρ, obtaining a Riemannian metric $\|\cdot\|_*$ on $\tilde{E}$ with respect to which the action of Γ is isometric. On the other hand, since the vector bundle $\tilde{E}$ is trivial, we can also endow it with the euclidian metric $\|\cdot\|$ on the fibers. If $K \subset \widetilde{U\Gamma}$ is a compact set then there exists $C_K > 1$ such that for every $v \in \tilde{E}$

$$C_K^{-1}\|v\| \leqslant \|v\|_* \leqslant C_K\|v\| \quad \text{for every } v \in \tilde{E}.$$

By Lemma A.1, a bounded change of inner product has a bounded effect on the singular values. It follows that for all $\tilde{z} \in K$, $t \in \mathbb{R}$, and $\gamma \in \Gamma$ such that $\tilde{\phi}^t(\tilde{z}) \in \gamma(K)$, if z is the projection of $\tilde{z}$ in $U\Gamma$, inequality (4.2) holds for $C = C_K^2$.

In order to prove part (i), let K be a compact subset of $\widetilde{U\Gamma}$ that intersects every Γ-orbit. Then the desired inequalities follow directly from Lemma 4.2 and the previous remarks.

Now let us prove part (ii). Consider the action of Γ on the compact metric space $\partial\Gamma$. Fix a positive $\delta < \frac{1}{4}\operatorname{diam}\partial\Gamma$. Let $K := \{(x, y, 0) : x, y \in \partial\Gamma,\ d(x, y) \geqslant \delta\}$; this is a compact subset of $\widetilde{U\Gamma}$. Given $\gamma \in \Gamma$, it is a simple exercise to show that there exist x, $y \in \partial\Gamma$ such that $d(x, y) \geqslant \delta$ and $d(\gamma^{-1}x, \gamma^{-1}y) \geqslant \delta$. Therefore, letting $\tilde{z} := (x, y, 0)$ and $t := c(x, y, \gamma)$, we conclude with the same arguments as above. □

Now let us prove the equivalence between p-dominated and p-Anosov representations. We first show:

Proposition 4.5. *Let $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ be a p-Anosov representation. Then ρ is p-dominated.*

Proof. Since ρ is a p-Anosov representation, Theorem 2.2 implies that there exists $C, \lambda > 0$ such that for every $z \in U\Gamma$ and $t > 0$ we have:

$$\frac{\sigma_{p+1}(\psi_z^t)}{\sigma_p(\psi_z^t)} < Ce^{-\lambda t}.$$

Using part (i) of Lemma 4.4, we can find constants $C', \lambda' > 0$ so that for every $\gamma \in \Gamma$ we have:

$$\frac{\sigma_{p+1}(\rho(\gamma))}{\sigma_p(\rho(\gamma))} < C'e^{-\lambda'|\gamma|}.$$

This means that ρ is p-dominated. □

[7] The Riemannian metric allows us to consider singular values for the linear maps ψ_x^t.

Note that the proof above only uses the fact that the linear flow ψ^t has a dominated splitting, so we obtain:

Proposition 4.6. *If the linear flow ψ^t on E_ρ has a dominated splitting with dominating bundle of dimension p then ρ is p-dominated.*

To prove that p-domination implies the p-Anosov property, we shall first show the existence of the equivariant maps ξ, θ. This is a relatively easy consequence of what is done in Section 3 (see Remark 3.8). The equivariant maps exist under an even weaker hypothesis, as shown in [GGKW, Theorem 5.2]. We provide here a proof for completeness.

Recall that a (a,b)-quasi-geodesic in Γ is a sequence $\{\gamma_n\}$ so that

$$a^{-1}|n-m| - b < d(\gamma_n,\gamma_m) < a|n-m| + b\,.$$

We denote by $\mathcal{Q}^{\mathrm{id}}_{(a,b)}$ the set of (a,b)-quasi-geodesics such that $\gamma_0 = \mathrm{id}$.

Lemma 4.7. *Let $\rho:\Gamma\to \mathrm{GL}(d,\mathbb{R})$ be a representation such that for some $a,b>0$ we have:*

$$\sup_{\gamma_n\in\mathcal{Q}^{\mathrm{id}}_{(a,b)}} \sum_{n\geqslant n_0} \frac{\sigma_{p+1}(\rho(\gamma_n))}{\sigma_p(\rho(\gamma_n))} \xrightarrow[n_0\to\infty]{} 0\ . \tag{4.3}$$

Then there exists an equivariant continuous map $\xi:\partial\Gamma\to\mathscr{G}_p(\mathbb{R}^d)$ defined by

$$\xi(x) = \lim_n U_p(\rho(\gamma_n)),$$

where $\{\gamma_n\}$ is any (a,b)-quasi-geodesic ray representing $x\in\partial\Gamma$.

Proof. For each $x\in\partial\Gamma$, choose $\{\gamma_n^x\}\in\mathcal{Q}^{\mathrm{id}}_{(a,b)}$ representing x. We define

$$\xi(x) = \lim_n U_p(\rho(\gamma_n^x))\ .$$

To see that this limit exists, let C_0 be an upper bound of $\|\rho(g)\|\|\rho(g^{-1})\|$ for $g\in S$ a finite generating set of Γ and use Lemma A.3 to see that

$$d(U_p(\rho(\gamma_n^x)),U_p(\rho(\gamma_{n-1}^x)))\leqslant C_0^{d(\gamma_n,\gamma_{n-1})}\frac{\sigma_{p+1}(\rho(\gamma_{n-1}^x))}{\sigma_p(\rho(\gamma_{n-1}^x))}$$

This implies that $U_p(\rho(\gamma_n^x))$ is a Cauchy sequence and therefore has a limit. The fact that the limit does not depend on the chosen (a,b)-quasi-geodesic follows directly from a similar estimate using Lemma A.3.

Since the estimates are uniform, this becomes a uniform limit as one changes $x\in\partial\Gamma$, providing continuity of the maps (recall the topology in $\partial\Gamma$ introduced in Remark 3.8). Equivariance follows from Lemma A.4.

□

Remark 4.8. If ρ is p-dominated then it satisfies the hypothesis of Lemma 4.7, since the terms in the sum of (4.3) are uniformly exponentially small.

Proposition 4.9. *Let $\rho:\Gamma\to\mathrm{GL}(d,\mathbb{R})$ be a p-dominated representation. Then ρ is p-Anosov.*

Proof. If a representation is p-dominated, then it is also $(d-p)$-dominated (see Remark 3.1). Lemma 4.7 then provides two equivariant continuous maps $\xi:\partial\Gamma\to\mathscr{G}_p(\mathbb{R}^d)$ and $\theta:\partial\Gamma\to\mathscr{G}_{d-p}(\mathbb{R}^d)$.

The fact that $\xi(x)\oplus\theta(y)=\mathbb{R}^d$ for $x\neq y\in\partial\Gamma$ is a direct consequence of Lemma 3.10 and the definition of the maps ξ and θ given by Lemma 4.7.

Using part (ii) of Lemma 4.4, we obtain an exponential gap in the singular values of ψ^t, and by Theorem 2.2, the splitting $\xi \oplus \theta$ is dominated. Therefore the representation ρ is p-Anosov. $\square$

4.4. **Some questions.** Given a matrix $A \in \mathrm{GL}(d, \mathbb{R})$, let

$$\chi_1(A) \geqslant \chi_2(A) \geqslant \cdots \geqslant \chi_d(A)$$

denote the absolute values of its eigenvalues, repeated according to multiplicity.

Given a finitely generated group, let

$$\ell(\eta) := \inf_\eta |\eta^{-1}\gamma\eta| = \inf_\eta d(\gamma\eta, \eta)$$

(i.e., the *translation length*). If Γ is word-hyperbolic then there exists a constant $a > 0$ such that for every $\gamma \in \Gamma$ we have:

$$\ell(\gamma) - a \leqslant \lim_{n\to\infty} \frac{|\gamma^n|}{n} \leqslant \ell(\gamma)\,; \tag{4.4}$$

see [CDP, p. 119].

Note that if $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is a p-dominated representation then there exists constants $C' > 0$, $\lambda > 0$ such that for all $\gamma \in \Gamma$ we have:

$$\frac{\chi_{p+1}(\rho(\gamma))}{\chi_p(\rho(\gamma))} < C' e^{-\lambda' \ell(\gamma)}\,. \tag{4.5}$$

Indeed, if the domination condition (3.2) holds then the group Γ is word-hyperbolic by Theorem 3.2 and, using (4.4), we obtain:

$$\frac{\chi_{p+1}(\rho(\gamma))}{\chi_p(\rho(\gamma))} = \lim_{n\to\infty} \left(\frac{\sigma_{p+1}(\rho(\gamma^n))}{\sigma_p(\rho(\gamma^n))}\right)^{1/n} \leqslant \lim_{n\to\infty} \left(Ce^{-\lambda|\gamma^n|}\right)^{1/n} \leqslant C' e^{-\lambda \ell(\gamma)}\,,$$

for $C' := e^{a\lambda}$.

Condition (4.5) is invariant under conjugacies, while condition (3.2) does not enjoy this property.

It is natural to pose the following question:

Question 4.10. Let $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ be a representation of a word-hyperbolic group Γ. Suppose that there exists constants $p \in \{1, \dots, d-1\}$, $C' > 0$, $\lambda > 0$ such that relation (4.5) holds. Does it follow that ρ is p-dominated?

Guéritaud, Guichard, Kassel, and Wienhard have shown that for p-convex representations, the question above has a positive answer, even relaxing condition (4.5): see [GGKW, Theorem 1.6].

In terms of the linear flow $\{\psi^t\}$, condition (4.5) means that for every periodic orbit $\mathcal{O}$ of $\{\phi^t\}$, say of period $\ell(\mathcal{O})$, there exists a gap between the p-th and $p+1$-th moduli of the eigenvalues of $\psi^{\ell(\mathcal{O})}$ which is exponentially large with respect to $\ell(\mathcal{O})$. Question 4.10 can be reformulated in the general context of linear flows over hyperbolic dynamics; however, that question has a negative answer: see for example [Go]. Therefore a positive answer to Question 4.10 would require a finer use of the fact that the linear flow comes from a representation.

The following important result was obtained by Bonatti, Díaz, and Pujals [BDP]: if a diffeomorphism f of a compact manifold has that property that all sufficiently small C^1-perturbations have dense orbits, then the derivative cocycle Df admits a dominated splitting. This is an example of a general principle in Differentiable Dynamics, tracing back to Palis–Smale Stability Conjecture: robust dynamical properties often imply some uniform property for the derivative. Coming back to the context of linear representations, one can try to apply the same principle. For example, if a representation ρ of a hyperbolic group is robustly faithful and discrete (or robustly quasi-isometric), does it follow that ρ is p-dominated for some p?

5. Characterizing dominated representations in terms of multicones

The main result of this section is Theorem 5.7, which gives another characterization of dominated representations. Related results have been obtained in [ABY, BG]. In dimension two, such results have been used to study how domination can break along a deformation: see [ABY, § 4].

As a consequence of Theorem 5.7, domination obeys a "local-to-global" principle, a fact that was first shown in [KLP_1] by different methods.

5.1. **Sofic linear cocycles and a general multicone theorem.** In this subsection we introduce a special class of linear cocycles called *sofic*. Then we state a necessary and sufficient condition for the existence of a dominated splitting for these cocycles, generalizing the "multicone theorems" of [ABY, BG].

Let $\mathcal{G}$ be a *graph*, or more precisely, a finite directed multigraph. This means that we are given finite sets $\mathcal{V}$ and $\mathcal{E}$ whose elements are called respectively *vertices* and *edges*, and that each edge has two (not necessarily different) associated vertices, called its *tail* and its *head*. A *bi-infinite walk* on $\mathcal{G}$ is a two-sided sequence of edges $(e_n)_{n\in\mathbb{Z}}$ such that for each $n \in \mathbb{Z}$, the head of e_n equals the tail of e_{n+1}.

The graph $\mathcal{G}$ is called *labeled* if in addition each edge has an associated *label*, taking values in some finite set $\mathcal{L}$. Let $(e_n)_{n\in\mathbb{Z}}$ be a bi-infinite walk on $\mathcal{G}$; then its *label sequence* is defined as $(\ell_n)_{n\in\mathbb{Z}}$ where each ℓ_n is the label of the edge e_n. Let Λ be the set of all label sequences; this is a closed, shift-invariant subset of $\mathcal{L}^{\mathbb{Z}}$. Let $T\colon \Lambda \to \Lambda$ denote the restriction of the shift map. Then T is called a *sofic shift*, and the labeled graph from which it originates is called a *presentation* of T. We refer the reader to [LM] for examples, properties, and alternative characterizations of sofic shifts. Let us just remark that every subshift of finite type is a sofic shift, and every sofic shift is a factor of a subshift of finite type.

Let us say that a graph is *recurrent* if it is a union of directed cycles. Given a (labeled) graph $\mathcal{G}$, let $\mathcal{G}^*$ denote the maximal recurrent (labeled) subgraph, or equivalently, the subgraph containing all the bi-infinite walks on $\mathcal{G}$. Note that the sofic shifts presented by $\mathcal{G}$ and $\mathcal{G}^*$ are exactly the same. Therefore we may always assume that $\mathcal{G}$ is recurrent, if necessary.

Fix a sofic shift and a presentation as above. Let $d \geqslant 2$ be an integer. Given a family of matrices $(A_\ell)_{\ell\in\mathcal{L}}$ in $\mathrm{GL}(d,\mathbb{R})$, consider the map $A\colon \Lambda \to \mathrm{GL}(d,\mathbb{R})$ defined by $A((\ell_n)_{n\in\mathbb{Z}}) := A_{\ell_0}$. We call the pair (T, A) a *sofic linear cocycle*. We are interested in the existence of dominated splitting for such cocycles.

A *multicone of index* p is an open subset of the projective space $\mathbb{P}(\mathbb{R}^d) = \mathscr{G}_1(\mathbb{R}^d)$ that contains the projectivization of some p-plane and does not intersect the projectivization of some $(d-p)$-plane. Such a multicone is called *tame* if it has finitely many connected components, and these components have disjoint closures.

Suppose that for each vertex v of the graph $\mathcal{G}$ it is given a multicone $M_v \subset \mathbb{P}(\mathbb{R}^d)$ of index p; then we say that $(M_v)_{v\in\mathcal{V}}$ is *a family of multicones of index* p. We say that this family is *strictly invariant* (with respect to the sofic linear cocycle) if for each edge $e \in \mathcal{E}$ we have[8]

$$A_\ell(M_{v_0}) \Subset M_{v_1} ,$$

where ℓ is the label of e, v_0 is the tail of e, and v_1 is the head of e.

Theorem 5.1. *Let T be a sofic shift with a fixed presentation. Consider a sofic linear cocycle (T, A). The following statements are equivalent:*

(a) *the cocycle (T, A) has a dominated splitting $E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$ where the dominating bundle E^{cu} has dimension p;*

(b) *there exists a strictly invariant family of multicones of index p.*

Moreover, in (b) we can always choose a family composed of tame multicones.

This theorem implies, for example, [ABY, Theorem 2.2]; to see this, note that the full shift on N symbols can be presented by the graph with a single vertex and self-loops labeled $1, \dots, N$. Theorem 5.1 also extends [ABY, Theorem 2.3] and [BG, Theorem B] (in the case of finite Σ).

As a complement to Theorem 5.1, let us explain how to obtain the dominating bundle E^{cu} in terms of the multicones:

Proposition 5.2. *Consider a sofic linear cocycle (T, A) with a dominated splitting $E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$ where the dominating bundle E^{cu} has dimension p. Let $\{M_v\}_v$ be a strictly invariant family of multicones. Consider an element $x = (\ell_n)_{n\in\mathbb{Z}} \in \Lambda$, that is, the label sequence of a bi-infinite walk $(e_n)_{n\in\mathbb{Z}}$. Let v_n be the tail of the edge e_n. Let $(P_n)_{n\in\mathbb{Z}}$ be a sequence in $\mathscr{G}_p(\mathbb{R}^d)$ such that for each n, the projectivization of P_n is contained in the multicone M_{v_n}. Then*

$$E^{\mathrm{cu}}(x) = \lim_{n\to+\infty} A_{\ell_{-1}} A_{\ell_{-2}} \cdots A_{\ell_{-n}}(P_{-n}) .$$

Moreover, the speed of convergence can be estimated independently of x, (e_n), and (P_n), and is the same for all nearby sofic linear cocycles $(T, \tilde{A})$.

One can prove Theorem 5.1 and Proposition 5.2 by adapting the proof of [BG, Theorem B].

5.2. **Cone types for word-hyperbolic groups.** Cone types were originally introduced by Cannon [Ca] for groups of hyperbolic isometries.

Let Γ be a finitely generated group with a fixed finite symmetric generating set S. Let $|\cdot|$ and $d(\cdot,\cdot)$ denote word-length and the word-metric, respectively.

The *cone type* of an element $\gamma \in \Gamma$ is defined as:

$$C^+(\gamma) := \{\eta \in \Gamma : \ |\eta\gamma| = |\eta| + |\gamma|\} .$$

For example, $C^+(\gamma) = \Gamma$ iff $\gamma = \mathrm{id}$.

Remark 5.3. Actually the usual definition is different:

$$C^-(\gamma) := \{\eta \in \Gamma : \ |\gamma\eta| = |\eta| + |\gamma|\} .$$

But working with one definition is essentially equivalent to working with the other, because $C^-(\gamma) = [C^+(\gamma^{-1})]^{-1}$. If we need to distinguish between the two, we shall call them *positive* and *negative* cone types.

[8] $X \Subset Y$ denotes that the closure of X is contained in the interior of Y.

A fundamental fact is that word-hyperbolic groups have only finitely many cone types: see [BH, p. 455] or [CDP, p. 145]. In fact, there is a constant k (depending only on the hyperbolicity constant of the group) such that for any $\gamma \in \Gamma$, the cone type $C^+(\gamma)$ is uniquely determined by the k-prefix[9] of a shortest word representation of γ.

Given a cone type C and $a \in S \cap C$, we can define a cone type

$$aC := C^+(a\gamma),$$

where $\gamma \in \Gamma$ is such that $C^+(\gamma) = C$.

Lemma 5.4. *aC is well-defined.*

Though the lemma is contained in [CDP, p. 147, Lemme 4.3], let us provide a proof for the reader's convenience:

Proof. Suppose that $C^+(\gamma) = C^+(\gamma') = C$ and $a \in S \cap C$; we need to prove that $C^+(a\gamma) \subset C^+(a\gamma')$. Take $\eta \in C^+(a\gamma)$, so $|\eta a\gamma| = |\eta| + |a\gamma| = |\eta| + 1 + |\gamma|$. In follows that $|\eta a| = |\eta| + 1$ and so $|\eta a\gamma| = |\eta a| + |\gamma|$, that is $\eta a \in C$. In particular $|\eta a\gamma'| = |\eta a| + |\gamma'| = |\eta| + |a\gamma|$, proving that $\eta \in C^+(a\gamma')$. ☐

We associate to (Γ, S) a labeled graph $\mathcal{G}$ (in the sense of § 5.1) called the *geodesic automaton* and defined as follows:

- the vertices are the cone types of Γ;
- there is an edge $C_1 \xrightarrow{a} C_2$ from vertex C_1 to vertex C_2, labeled by a generator $a \in S$, iff $a \in C_1$ and $C_2 = aC_1$.

Remark 5.5. Replacing each vertex C by C^{-1} (a negative cone type) and each edge $C_1 \xrightarrow{a} C_2$ by $C_1^{-1} \xrightarrow{a^{-1}} C_2^{-1}$, we obtain the graph described in [BH, p. 456].

Let us explain the relation with geodesics. Consider a *geodesic segment* $(\gamma_0, \gamma_1, \dots, \gamma_\ell)$ that is, a sequence of elements of Γ such that $d(\gamma_n, \gamma_m) = |n-m|$, and assume that $\gamma_0 = \mathrm{id}$. Then there are generators $a_0, \dots a_{\ell-1}$ such that $\gamma_n = a_0 a_1 \cdots a_{n-1}$. Note that for each n, the following is an edge of the geodesic automaton graph $\mathcal{G}$:

$$C^+(\gamma_n^{-1}) \xrightarrow{a_n^{-1}} C^+(\gamma_{n+1}^{-1})$$

Thus we obtain a (finite) walk on $\mathcal{G}$ starting from the vertex $C^+(\mathrm{id})$. Conversely, for each such walk we may associate a geodesic segment starting at the identity.

Let us define also the *recurrent geodesic automaton* as the maximal recurrent subgraph $\mathcal{G}^*$ of $\mathcal{G}$; its vertices are called *recurrent cone types*. Similarly to what was explained above, for each two-sided geodesic on Γ passing through the identity element we can associate a bi-infinite walk on $\mathcal{G}^*$, and vice-versa.

Using the fact that the geodesic automation is a finite graph, it is simple to obtain the following property:[10]

Lemma 5.6. *Let Γ be an infinite word-hyperbolic group, with a fixed set of generators. Let Γ^* be the union of all two-sided geodesics passing through the identity element. Then for some finite c, the set Γ^* is c-dense on Γ, that is, for every $\gamma \in \Gamma$ there exists $\eta \in \Gamma^*$ such that $d(\eta, \gamma) \leqslant c$.*

[9] Or k-suffix in the case of negative cone types.

[10] Alternatively, one can deduce the lemma from the "bounded dead-end depth" property [Bog].

5.3. **Multicones for dominated representations.** Let Γ be a word-hyperbolic group (with a fixed finite symmetric generating set), and let $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ be a representation.

Recall from § 5.1 that a multicone of index p is an open subset of $\mathbb{P}(\mathbb{R}^d)$ that contains the projectivization of some p-plane and does not intersect the projectivization of some $(d-p)$-plane.

If for each recurrent cone type C it is given a multicone $M(C)$ of index p, then we say that $\{M(C)\}_C$ is a *family of multicones* for Γ. We say that the this family is *strictly invariant* with respect to ρ if for each edge $C_1 \xrightarrow{g} C_2$ of the geodesic automaton graph, we have:

$$\rho(g)(M(C_1)) \Subset M(C_2)\,.$$

Theorem 5.7. *A representation of a word-hyperbolic group is p-dominated if and only if it has a strictly invariant family of multicones of index p. Moreover, we can always choose a family composed of tame multicones.*

Proof. Fix a word-hyperbolic group Γ. We can assume that Γ is infinite, otherwise the theorem is vacuously true. Fix a finite symmetric generating set S. Consider the associated recurrent geodesic automaton $\mathcal{G}^*$, and the sofic shift $T\colon \Lambda \to \Lambda$ presented by this labeled graph. Then a sequence (a_n) in $S^{\mathbb{Z}}$ belongs to Λ if and only if the sequence $(\gamma_n)_{n\in\mathbb{Z}}$ defined by:

$$\gamma_n := \begin{cases} a_0^{-1}a_1^{-1}\cdots a_{n-1}^{-1} & \text{if } n>0,\\ \mathrm{id} & \text{if } n=0,\\ a_{-1}a_{-2}\cdots a_n & \text{if } n<0, \end{cases}$$

is a geodesic on Γ. The union of (the traces of) all such geodesics is a set $\Gamma^* \subset \Gamma$ which by Lemma 5.6 is c-dense in Γ for some finite c. Consider the family of matrices $(\rho(a))_{a\in S}$, and let (T,A) be the induced sofic linear cocyle. Then, for each $x=(a_n)\in X$, if (γ_n) is the geodesic defined above then $A(T^{n-1}x)\cdots A(x) = \rho(\gamma_n^{-1})$ for every positive n.

Let $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ be a representation. By Remark 3.1, ρ is p-dominated iff it is $(d-p)$-dominated. Since the set Γ^* is k-dense, ρ is $(d-p)$-dominated iff:

$$\exists C>0\ \exists\lambda>0\ \forall\gamma\in\Gamma^* \text{ we have } \frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma^{-1})) \leqslant Ce^{-\lambda|\gamma|}\,.$$

Note that this condition holds iff the sofic linear cocycle (T,A) has a dominated splitting with a dominating bundle of dimension p; this follows from Theorem 2.2 and the previous observations. Also note that a strictly invariant family of multicones for the sofic linear cocycle (T,A) is exactly the same as a strictly invariant family of multicones for the representation ρ. Therefore Theorem 5.1 allows us to conclude. □

Theorem 5.7 also has the following well known consequence (see [GW, Lab]).

Corollary 5.8. *Among representations $\Gamma \to \mathrm{GL}(d,\mathbb{R})$, the p-dominated ones form an open set.*

Proof. Given a p-dominated representation, take a strictly invariant family of multicones. Then the same family is also strictly invariant under all nearby representations. □

As a complement to Theorem 5.7, let us explain how to determine the equivariant map $\xi\colon \partial\Gamma \to \mathscr{G}_p(\mathbb{R}^d)$ (defined in Section 4) in terms of multicones.

Proposition 5.9. *Consider a p-dominated representation $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$, and let $\{M(C)\}_C$ be a strictly invariant family of multicones. Consider any geodesic ray $(\gamma_n)_{n\in\mathbb{N}}$ with $\gamma_0 = \mathrm{id}$, and let $x \in \partial\Gamma$ be the associated boundary point. Let $(P_n)_{n\geqslant 1}$ be a sequence in $\mathscr{G}_p(\mathbb{R}^d)$ such that for each n, the projectivization of P_n is contained in the multicone $M(C^+(\gamma_n))$. Then*

$$\xi(x) = \lim_{n\to\infty} \rho(\gamma_n^{-1}) P_n \, . \tag{5.1}$$

Moreover, the speed of convergence can be estimated independently of x, (γ_n), and (P_n), and is the same for all nearby representations.

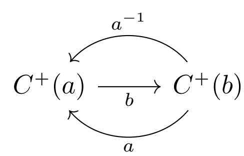
Proof. It suffices to translate Proposition 5.2 to the context of representations. □

Remark 5.10. Recall from Remark 3.4 that it also makes sense to define p-dominated representations into $\mathrm{PGL}(d,\mathbb{R})$. All that was said in this subsection applies verbatim to that case, since $\mathrm{PGL}(d,\mathbb{R})$ also acts on $\mathbb{P}(\mathbb{R}^d)$.

5.4. **An example.** Consider the free product $\Gamma := \mathbb{Z}_3 * \mathbb{Z}_2$ (which is isomorphic to $\mathrm{PGL}(2,\mathbb{Z})$) with a presentation:

$$\Gamma = \left\langle a, b \mid a^3 = b^2 = \mathrm{id} \right\rangle.$$

Then there are only 2 recurrent cone types, namely $C^+(a)$ and $C^+(b)$, and the recurrent geodesic automaton is:



Fix $\lambda > 1$. Consider the following pair of matrices in $\mathrm{SL}(2,\mathbb{R})$:

$$A := D^{-1} R_{\pi/3} D \quad \text{and} \quad B := R_{\pi/2} \, ,$$

where:

$$D := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} .$$

Since $A^3 = B^2 = -\mathrm{Id}$, we can define a representation $\rho\colon \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ by setting $\rho(a) := A$, $\rho(b) := B$. We claim that if λ is sufficiently large (namely, $\lambda > \sqrt[4]{2}$) then this representation is dominated. Indeed, it is possible to find a strictly invariant family of multicones as in Fig. 4.

6. Analytic variation of limit maps

The purpose of this section is to give another proof of a theorem from [BCLS], which establishes that the equivariant limit maps ξ, η (defined in Section 4) depend analytically on the representation. This fact is useful to show that some quantities such as entropy vary analytically with respect to the representation, which in turn is important to obtain certain rigidity results (see [BCLS, PS]).

While the original approach of [BCLS] used the formalism of [HPS], we present here a more direct proof, based on the tools discussed in the previous section. We remark that an alternative approach using the implicit function theorem and

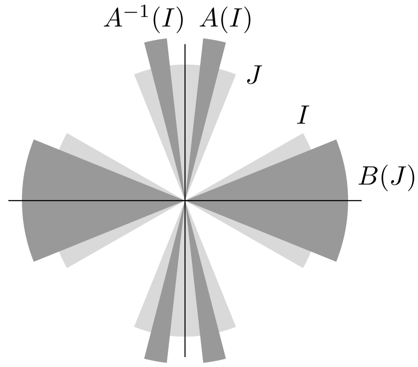


FIGURE 4. A strictly invariant multicone for the representation $\rho\colon \mathbb{Z}_3 * \mathbb{Z}_2 \to \mathrm{PSL}(d,\mathbb{R})$. We have $A(I) \Subset J$, $A^{-1}(I) \Subset J$, $B(J) \Subset I$.

avoiding complexification can be found in [Ru], though the context is different and the results are not exactly the same.

A family of representations $\{\rho_u\}_{u\in D}$ of a word-hyperbolic group Γ into $\mathrm{GL}(d,\mathbb{R})$ is called *real analytic* if the parameter space is a real analytic manifold, and for each $\gamma \in \Gamma$, the map $u \in D \mapsto \rho_u(\gamma) \in \mathrm{GL}(d,\mathbb{R})$ is real analytic. (Of course, it suffices to check the later condition for a set of generators.)

The boundary $\partial\Gamma$ of the group Γ admits a distance function within a "canonical" Hölder class ([CDP, Chapitre 11]). For a fixed metric in this class, the geodesic flow defined previously is Lipschitz. The limit maps $\xi_\rho\colon \partial\Gamma \to \mathscr{G}_p(\mathbb{R}^d)$ and $\theta_\rho\colon \partial\Gamma \to \mathscr{G}_{d-p}(\mathbb{R}^d)$ are well known to be Hölder continuous (see Theorem A.13). Note that, since $\mathscr{G}_p(\mathbb{R}^d)$ is an analytic manifold, one can endow the space $C^\alpha(\partial\Gamma, \mathscr{G}_p(\mathbb{R}^d))$ of α-Hölder maps with a Banach manifold structure; so analyticity of a map from an analytic manifold to $C^\alpha(\partial\Gamma, \mathscr{G}_p(\mathbb{R}^d))$ makes sense.

Theorem 6.1 (Theorem 6.1 of [BCLS])**.** *Let $\{\rho_u\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})\}_{u\in D}$ be a real analytic family of representations. Suppose that $0 \in D$ and ρ_0 is p-dominated. Then there exists a neighborhood $D' \subset D$ of 0 so that for every $u \in D'$, the representation ρ_u is p-dominated, and moreover $u \mapsto \xi_{\rho_u}$ defines a real-analytic map from D' to $C^\alpha(\partial\Gamma, \mathscr{G}_p(\mathbb{R}^d))$, for some $\alpha > 0$.*

Let us provide a proof. Corollary 5.8 ensures that for every u sufficiently close to 0, the domination ρ_u is also p-dominated.

As in [BCLS] (see also [Hub, Proposition A.5.9]), it is enough to show *transverse real analyticity*. More precisely, we need to show that the map $F\colon D' \times \partial\Gamma \to \mathscr{G}_p(\mathbb{R}^d)$ given by $F(u,x) := \xi_{\rho_u}(x)$ has the following properties:

(i) it is α-Hölder continuous;
(ii) for every $x \in \partial\Gamma$, the map $F(\cdot, x)\colon D' \to \mathscr{G}_p(\mathbb{R}^d)$ is real analytic.

Hölder continuity is a standard property of dominated splittings (see for example [CP, Section 4.4]), and is independent of the analyticity of the family. For completeness, and since we could not find the specific statement in the literature, we included a sketch of the proof in the appendix: see Corollary A.14. In the case at hand, it follows that property (i) above is satisfied, for some neighborhood $D' \ni 0$ and some uniform Hölder exponent $\alpha > 0$.

So we are left to prove the analyticity property (ii). To proceed further, we consider the complexification of the representations. We can assume without loss of generality that D' is a neighborhood of 0 in some $\mathbb{R}^k$. Let $\{g_1, \dots, g_m\}$ be a finite generating set of Γ. For each i, the Taylor expansion of $u \mapsto \rho_u(g_i)$ around 0 converges a polydisk $\hat{D}_i$ in $\mathbb{C}^k$ centered at 0; we keep the same symbol for the extended map. Take a smaller polydisk $D \subset \bigcap_i \hat{D}_i$ also centered at 0 such that for $u \in D$, the complex matrix $\rho_u(g_i)$ is invertible. Since these maps are analytic, and every relation of Γ is obeyed when u is real, the same happens for all $u \in \hat{D}$. So we have constructed a complex analytic family of representations $\rho_u \colon \Gamma \to \mathrm{GL}(d, \mathbb{C})$, where u takes values in a neighborhood $\hat{D}$ of 0 in $\mathbb{C}^k$.

Recall from Remark 2.3 that dominated splittings make sense in the complex case; so do dominated representations, with the exact same definition (3.2). Actually, if ι denotes the usual homomorphism that embeds $\mathrm{GL}(d, \mathbb{C})$ into $\mathrm{GL}(2d, \mathbb{R})$, then a representation $\rho \colon \Gamma \to \mathrm{GL}(d, \mathbb{C})$ is p-dominated in the complex sense iff $\iota \circ \rho$ is $2p$-dominated in the real sense. In particular, the openness property of Corollary 5.8 also holds in the complex case. So, reducing $\hat{D}$ if necessary, we assume that each ρ_u is p-dominated.

Proposition 6.2. *For each $x \in \partial\Gamma$, the map $u \mapsto \xi_u(x)$ from $\hat{D}$ to $\mathscr{G}_p(\mathbb{C}^d)$ is complex analytic.*

Proof. It suffices to check analyticity around $u = 0$. For each $x \in \partial\Gamma$, choose a geodesic ray $(\gamma_n^x)_{n \in \mathbb{N}}$ in Γ starting at the identity element and converging to x. We consider the sequence of maps from $\hat{D}$ to $\mathscr{G}_p(\mathbb{C}^d)$ defined as follows:

$$\varphi_n^x(u) := \rho_u(\gamma_n^x)^{-1}(\xi_0(\gamma_n^x(x))) .$$

Each of these maps is complex analytic, since so is $u \mapsto \rho_u(\gamma_n^x)$.

We claim that for each $x \in \partial\Gamma$, convergence $\varphi_n^x(u) \to \xi_u(x)$ holds uniformly in a neighborhood of 0. Since $\mathscr{G}_p(\mathbb{C}^d)$ may be considered as a closed subset of $\mathscr{G}_{2p}(\mathbb{R}^{2d})$, it is sufficient to check convergence in the latter set. This convergence follows from Proposition 5.9 and the fact that for u in a small neighborhood V of 0, the representations are still dominated, and with the same multicones (see Corollary 5.8). Note that ξ_0 always belongs to the corresponding multicone because the multicones remain unchanged.

Being a uniform limit of complex-analytic maps, the map $u \mapsto \xi_u(x)$ is complex analytic on V (see for example [Hö, Corollary 2.2.4]). ☐

Restricting to the real parameters, the proposition yields the property (ii) that we were left to check. So the proof of Theorem 6.1 is complete.

7. Geometric consequences of Theorem 2.2: A Morse Lemma for $\mathrm{PSL}(d, \mathbb{R})$'s symmetric space

In this section we explain how Theorem 2.2 (and more precisely Proposition 2.4) has a deep geometric meaning for the symmetric space of $\mathrm{PSL}(d, \mathbb{R})$. This is a version of the Morse Lemma recently proved by [KLP_2]. Because of this application, one is tempted to call Theorem 2.2 a *twisted Morse Lemma*[11].

[11] Lenz [Len], who previously obtained a weaker version of Theorem 2.2 for $\mathrm{SL}(2, \mathbb{R})$, had already noted that that result was related to the classical Morse Lemma for the hyperbolic plane.

The exposition is purposely pedestrian for two reasons: 1) it is intended for the reader unfamiliar with symmetric spaces; 2) it mimics, in the case of $\mathrm{PSL}(d,\mathbb{R})$, the general structure theory of semi-simple Lie groups, in order to ease the way to Section 8.

The reader familiar with semi-simple Lie groups should jump to subsection 7.12, or even Section 8, for a proof of the Morse Lemma due to [KLP_2], for symmetric spaces of non-compact type, using dominated splittings. Specific references for subsections 7.1-7.11 are, for example, [Ebe], [GJT], [Hel], [Lang].

This section is independent of sections 3–6.

7.1. **A Cartan subalgebra.** Fix an inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^d$ and denote by o its homothety class, i.e. $\langle\cdot,\cdot\rangle$ up to positive scalars. One has then the adjoint involution $T\mapsto T^{\mathrm{t}}$ defined by $\langle Tv,w\rangle=\langle v,T^{\mathrm{t}}w\rangle$ (note that $^{\mathrm{t}}$ only depends on o). This involution splits the vector space $\mathfrak{sl}(d,\mathbb{R})$ of traceless $d\times d$ matrices as

$$\mathfrak{sl}(d,\mathbb{R})=\mathfrak{p}^o\oplus\mathfrak{k}^o$$

where

$$\mathfrak{p}^o=\{T\in\mathfrak{sl}(d,\mathbb{R}):T^{\mathrm{t}}=T\}\text{ and }\mathfrak{k}^o=\{T\in\mathfrak{sl}(d,\mathbb{R}):T^{\mathrm{t}}=-T\}.$$

The subspace $\mathfrak{k}^o$ is a Lie algebra so we can consider its associated Lie group $K^o=\exp\mathfrak{k}^o$, consisting on (the projectivisation of) determinant 1 matrices preserving the class o. The subspace $\mathfrak{p}^o$, of traceless matrices diagonalizable on a o-orthogonal basis, is not a Lie algebra.

Fix then a o-orthogonal set of d lines $\mathcal{E}$ and denote by $\mathfrak{a}\subset\mathfrak{p}^o$ those matrices diagonalizable in the chosen set $\mathcal{E}$. This is an abelian algebra, called a *Cartan subalgebra* of $\mathfrak{sl}(d,\mathbb{R})$; its associated Lie group $\exp\mathfrak{a}$ consists on (the projectivisation of) determinant 1 matrices diagonalizable on $\mathcal{E}$ with positive eigenvalues. For $a\in\mathfrak{a}$ and $u\in\mathcal{E}$, we will denote by $\lambda_u(a)$ the eigenvalue for a associated to the eigenline u. Note that λ_u is linear on a and hence we can think of $\lambda_u\in\mathfrak{a}^*$, the dual space of $\mathfrak{a}$.

7.2. **The action of $\mathfrak{a}$ on $\mathfrak{sl}(d,\mathbb{R})$.** The action of $\mathfrak{a}$ on $\mathfrak{sl}(d,\mathbb{R})$ given by $(a,T)\mapsto[a,T]=aT-Ta$ is also diagonalizable. Indeed, the set of (projective) traceless transformations $\{\varepsilon_{uv},\phi_{uv}:u\neq v\in\mathcal{E}\}$ defined by

$$\varepsilon_{uv}(v)=u\text{ and }\varepsilon_{uv}|\mathcal{E}-\{v\}=0,$$

and

$$\phi_{uv}|u=t\,\mathrm{id},\ \phi_{uv}|v=-t\,\mathrm{id}\ \text{ and }\phi_{uv}|\mathcal{E}-\{u,v\}=0,$$

contains a linearly independent set[12] of eigenlines of $[a,\cdot]$, specifically $[a,\phi_{uv}]=0$ and

$$[a,\varepsilon_{uv}]=\alpha_{uv}(a)\varepsilon_{uv}=(\lambda_u(a)-\lambda_v(a))\varepsilon_{uv}.$$

The set of functionals

$$\Sigma=\{\alpha_{uv}\in\mathfrak{a}^*:u\neq v\in\mathcal{E}\}$$

is called a *root system* (or simply the *roots*) of $\mathfrak{a}$. For $\alpha\in\Sigma\cup\{0\}$, one usually denotes by $\mathfrak{sl}(d,\mathbb{R})_\alpha$ the eigenspace associated to α,

$$\mathfrak{sl}(d,\mathbb{R})_\alpha=\{T\in\mathfrak{sl}(d,\mathbb{R}):[a,T]=\alpha(a)T,\ \forall a\in\mathfrak{a}\},$$

[12]Redundancy only appears in the set $\{\phi_{uv}:u\neq v\in\Sigma\}$.

and one has[13]

$$\mathfrak{sl}(d,\mathbb{R}) = \bigoplus_{\alpha\in\Sigma\cup\{0\}} \mathfrak{sl}(d,\mathbb{R})_\alpha = \mathfrak{a}\oplus\bigoplus_{\alpha\in\Sigma}\mathfrak{sl}(d,\mathbb{R})_\alpha. \tag{7.1}$$

7.3. **Expansion/contraction.** The closure of a connected component of

$$\mathfrak{a} - \bigcup_{\alpha\in\Sigma}\ker\alpha$$

is called *a closed Weyl chamber*. Fix a closed Weyl chamber and denote it by $\mathfrak{a}^+$; this is not a canonical choice: it is equivalent to choosing an order on the set $\mathcal{E}$. Indeed, consider the subset of *positive roots* defined by $\mathfrak{a}^+$:

$$\Sigma^+ = \{\alpha\in\Sigma : \alpha|\mathfrak{a}^+ \geqslant 0\},$$

then one can set $u > v$ if $\alpha_{uv}\in\Sigma^+$.[14]

Let $\mathfrak{n}^+$ be Lie algebra defined by

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Sigma^+}\mathfrak{sl}(d,\mathbb{R})_\alpha = \bigoplus_{u>v}\varepsilon_{uv}.$$

By definition, $\mathfrak{a}\oplus\mathfrak{n}^+$ is the subspace of $\mathfrak{sl}(d,\mathbb{R})$ which is non-expanded by $\operatorname{Ad}\exp a$, for $a\in\mathfrak{a}^+$.

7.4. **Simple roots.** Observe that the order on $\mathcal{E}$ defined in subsection 7.3 is a total order; indeed, the kernel of every $\alpha\in\Sigma$ has empty intersection with the interior of $\mathfrak{a}^+$, so given distinct $u,v\in\mathcal{E}$, either $\alpha_{uv}\in\Sigma^+$ or $\alpha_{vu} = -\alpha_{uv}\in\Sigma^+$.

Consider pairwise distinct $u,v,w\in\mathcal{E}$ such that $u > v > w$. Note that if $a\in\mathfrak{a}^+\cap\ker\alpha_{uw}$ necessarily

$$a\in\ker\alpha_{uv}\cap\ker\alpha_{vw}.$$

A positive root α such that $\ker\alpha\cap\mathfrak{a}^+$ has maximal co-dimension is called a *simple root* associated to $\mathfrak{a}^+$. This corresponds to choosing two successive elements of $\mathcal{E}$. The set of simple roots is denoted by Π. Note that this is a basis of $\mathfrak{a}^*$.

From now on we will denote by $\mathcal{E} = \{u_1,\dots,u_d\}$ with $u_p > u_{p+1}$. Then $\mathfrak{n}^+$ can be interpreted as the space of upper triangular matrices on $\mathcal{E}$ (with 0's in the diagonal), and denoting by $a_i = \lambda_{u_i}(a)$ one has

$$\mathfrak{a}^+ = \{a\in\mathfrak{a} : a_1\geqslant\cdots\geqslant a_d\}.$$

We will use λ_{u_i} to introduce coordinates in $\mathfrak{a}$: if $(a_1,\dots,a_d)\in\mathbb{R}^d$ are such that $a_1+\cdots+a_d = 0$ then $(a_1,\dots,a_d)$ will denote the element $a\in\mathfrak{a}$ such that $\lambda_{u_i}(a) = a_i$. Finally, given $p\in\{1,\dots,d-1\}$ we will denote by α_p the simple root

$$\alpha_p(a) = \alpha_{u_pu_{p+1}}(a) = a_p - a_{p+1}.$$

[13] The fact that $\mathfrak{sl}(d,\mathbb{R})_0 = \mathfrak{a}$ is particular of $\mathfrak{sl}(d,\mathbb{R})$.

[14] In the case of $\operatorname{PSL}(d,\mathbb{R})$ one usually applies the inverse procedure: 'let $\{e_1,\dots,e_d\}$ be the canonical basis of $\mathbb{R}^d$ and let $\mathfrak{a}^+$ be the set of (determinant one) diagonal matrices with decreasing eigenvalues'.

7.5. **Flags.** Denote by $N = \exp \mathfrak{n}^+$ and by M the centralizer of $\exp \mathfrak{a}$ in K^o. The group M consists on (the projectivisation of) diagonal matrices w.r.t $\mathcal{E}$ with eigenvalues 1's and -1's.

The group $P = M \exp \mathfrak{a} N$ is called a *Borel subgroup* of $\mathrm{PSL}(d, \mathbb{R})$ and N is called its *unipotent radical*.

Recall that a *complete flag* on $\mathbb{R}^d$ is a collection of subspaces $E = \{E_p\}_1^{d-1}$ such that $E_p \subset E_{p+1}$ and $\dim E_p = p$. The spaces of complete flags is denoted by $\mathscr{F}$. Observe that $\mathrm{PSL}(d, \mathbb{R})$ acts transitively (i.e. has only one orbit) on $\mathscr{F}$ and that the group P is the stabilizer of

$$\{u_1 \oplus \cdots \oplus u_p\}_{p=1}^{d-1};$$

thus we obtain an equivariant identification $\mathscr{F} = \mathrm{PSL}(d, \mathbb{R})/P$.

Two complete flags E and F are in *general position* if for all $p = 1, \ldots, d-1$ one has

$$E_p \cap F_{d-p} = \{0\}.$$

Denote by $\mathscr{F}^{(2)}$ the space of pairs of flags in general position.

The same procedure applied to the Weyl chamber $-\mathfrak{a}^+$, provides the group $\check{P}$ that stabilizes the complete flag

$$\{u_d \oplus \cdots \oplus u_{d-p+1}\}_{p=1}^{d-1}.$$

Observe that the flags $\{u_1 \oplus \cdots \oplus u_p\}_{p=1}^{d-1}$ and $\{u_d \oplus \cdots \oplus u_{d-p+1}\}_{p=1}^{d-1}$ are in general position and that the stabilizer in $\mathrm{PSL}(d, \mathbb{R})$ of the pair is the group $M \exp \mathfrak{a} = P \cap \check{P}$.

7.6. **Flags and singular value decomposition.** If $\langle \cdot, \cdot \rangle \in o$ is an inner product with induced norm $\| \cdot \|$ on $\mathbb{R}^d$, note that the operator norm of $g \in \mathrm{GL}(d, \mathbb{R})$,

$$\sigma_1^o(g) = \|g\|_o = \sup \left\{ \frac{\|gv\|}{\|v\|} : v \in \mathbb{R}^d - \{0\} \right\},$$

only depends on o. The same holds for the other singular values, defined in subsection 2.2. Throughout this section, the choice of the inner product o is important, so we will stress the fact that the singular values depend on o, by denoting them by $\sigma_i^o(g)$.

The singular value decomposition provides a map $a : \mathrm{PSL}(d, \mathbb{R}) \to \mathfrak{a}^+$, called the *Cartan projection*, such that for every $g \in \mathrm{PSL}(d, \mathbb{R})$ there exist $k_g, l_g \in K^o$ such that

$$g = k_g \exp(a(g)) l_g.$$

More precisely, one has

$$a(g) = (\log \sigma_1^o(g), \ldots, \log \sigma_d^o(g)).$$

Recall from Section 2.2 that g has a gap of index p if $\sigma_p^o(g) > \sigma_{p+1}^o(g)$. If this is the case then

$$U_p^o(g) = k_g(u_1 \oplus \cdots \oplus u_p), \tag{7.2}$$

(note again the dependence on o).

Given $\alpha_p \in \Pi$ denote by $K^o(\{\alpha_p\})$ the stabilizer in K^o of the vector space $u_1 \oplus \cdots \oplus u_p$. Moreover, given a subset $\theta \subset \Pi$, denote by

$$K^o(\theta) = \bigcap_{\alpha_p \in \theta} K^o(\{\alpha_p\}).$$

If for some $p \in \{1, \dots, d\}$ and $g \in \mathrm{PSL}(d, \mathbb{R})$ one has $a_1(g) = a_p(g) > a_{p+1}(g)$, then any element of $k_g K^o(\{\alpha_p\})$ can be chosen in a Cartan decomposition of g. If all the gaps of g are indexed on a subset $\theta \subset \Pi$, then k_g is only defined modulo $K^o(\theta)$ and one has the *partial flag*

$$U^o(g) = \{U^o_p(g) : \alpha_p \in \theta\}.$$

Note that the Cartan projection of g^{-1} is simply $a(g^{-1}) = (-a_d(g), \dots, -a_1(g))$. The linear transformation $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ defined by

$$\mathrm{i}(a_1, \dots, a_d) = (-a_d, \dots, -a_1)$$

is called *the opposition involution*. If g has gaps indexed by θ then g^{-1} has gaps indexed by $\mathrm{i}\,\theta = \{\alpha \circ \mathrm{i} : \alpha \in \theta\}$. Denote by $S^o(g) = U^o(g^{-1})$.

7.7. **The symmetric space.** Recall that fixing a class o defines a splitting

$$\mathfrak{sl}(d, \mathbb{R}) = \mathfrak{p}^o \oplus \mathfrak{k}^o.$$

This splitting is orthogonal with respect to the *Killing form*, the symmetric bilinear form κ on $\mathfrak{sl}(d, \mathbb{R})$ defined by

$$\kappa(A, B) = \mathrm{Trace}(AB).$$

This linear form is related to adjoint involution ${}^{\mathtt{t}}$ in the following sense: the linear form $\kappa(\cdot, \cdot^{\mathtt{t}})$ is positive definite.

Since $\mathfrak{p}^o$ consists on fixed point for ${}^{\mathtt{t}}$, the restriction of κ to $\mathfrak{p}^o$, denoted by $(\cdot, \cdot)_o : \mathfrak{p}^o \times \mathfrak{p}^o \to \mathbb{R}$, is positive definite. Explicitly, if $v \in \mathfrak{p}^o$ then v is diagonalizable and

$$|v|^2_o := (v, v)_o$$

equals the sum of squared eigenvalues of v.

The space

$$X_d = \{\text{inner products on } \mathbb{R}^d\}/\mathbb{R}_+$$

is a contractible $\mathrm{PSL}(d, \mathbb{R})$-homogenous space, the action being given by

$$g \cdot \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle' \quad \text{where } \langle v, v \rangle' = \langle g^{-1} v, g^{-1} w \rangle.$$

The stabilizer of o is the group K^o and thus the orbit map $(g, o) \mapsto g \cdot o$ identifies the tangent space $\mathsf{T}_o X_d$ with the vector space $\mathfrak{p}^o$. A direct computation shows that the Riemannian metric $o \mapsto (\cdot, \cdot)_o$ is $\mathrm{PSL}(d, \mathbb{R})$-invariant.

The space $(X_d, (\cdot, \cdot)_o)$ is, by definition, *the symmetric space* of $\mathrm{PSL}(d, \mathbb{R})$.

7.8. **Maximal flats.** A direct computation shows that the orbit $\exp \mathfrak{a} \cdot o \subset X_d$ is isometric to $(\mathfrak{a}, (\cdot, \cdot)_o)$. Moreover, one can show that $\exp \mathfrak{a} \cdot o$ is a *maximal totally geodesic flat* (flat as in 'isometric to a Euclidean space', maximal with respect to dimension, see for example [Lang, Section XII.3]). For every $g \in \mathrm{PSL}(d, \mathbb{R})$ one has

$$d_{X_d}(o, g \cdot o) = d_{X_d}(o, \exp a(g) \cdot o) = |a(g)|_o = \sqrt{\sum_i (\log \sigma^o_i(g))^2}, \tag{7.3}$$

where d_{X_d} is the distance on X_d induced by the Riemannian metric $(\cdot, \cdot)_o$.

The orbit $g \exp \mathfrak{a} \cdot o$ is again a maximal totally geodesic flat (through $g \cdot o$) and hence, all geodesics of X_d are of the form

$$t \mapsto g \exp(ta) \cdot o$$

for a given $g \in \mathrm{PSL}(d, \mathbb{R})$ and $a \in \mathfrak{a}^+$.

In other words, a *maximal flat* in X_d consists on fixing a set $\mathcal{L}$ of d lines that span $\mathbb{R}^d$ and considering the space of inner products, up to homothety, that make $\mathcal{L}$ an orthogonal set.

The following lemma is simple but extremely useful for estimations:

Lemma 7.1. *Consider $\varphi \in \mathfrak{a}^*$ such that $\varphi|\mathfrak{a}^+ - \{0\} > 0$. Then there exists $c > 1$ such that for all $g \in \mathrm{PSL}(d,\mathbb{R})$ one has*

$$\frac{1}{c}\varphi(a(g)) \leqslant d_{X_d}(o, g\cdot o) \leqslant c\varphi(a(g)).$$

Proof. Since $\mathfrak{a}^+$ is closed and $\ker\varphi \cap \mathfrak{a}^+ = \{0\}$ the function

$$a \mapsto \frac{|a|_o}{\varphi(a)}$$

is invariant under multiplication by scalars and bounded on $\mathfrak{a}^+ - \{0\}$. Equation (7.3) completes the proof. □

For example, $\log\sigma_1^o(g)$ is comparable to $d_{X_d}(o, g\cdot o)$.

7.9. **The Furstenberg boundary and parallel sets.** A *parametrized flat* is a function $\mathsf{f} : \mathfrak{a} \to X_d$ of the form

$$\mathsf{f}(a) = g\exp a \cdot o$$

for some $g \in \mathrm{PSL}(d,\mathbb{R})$. A maximal flat is thus a subset of the form $\mathsf{f}(\mathfrak{a}) \subset X_d$ for some parametrized flat f.

Observe that $\mathrm{PSL}(d,\mathbb{R})$ acts transitively on the set of parametrized flats and that the stabilizer of $\mathsf{f}_0 : a \mapsto \exp a\cdot o$ is the group M. We will hence identify the space of parametrized flats with $\mathrm{PSL}(d,\mathbb{R})/M$.

Two parametrized flats f, g are *equivalent* if the function $\mathfrak{a} \to \mathbb{R}$ defined by

$$a \mapsto d_{X_d}(\mathsf{f}(a), \mathsf{g}(a))$$

is bounded on $\mathfrak{a}^+$.

The *Furstenberg boundary* of X_d is the space of equivalence classes of parametrized flats. Note that, *by definition* of $N = \exp\mathfrak{n}^+$ one has that the distance function

$$a \mapsto d_{X_d}(n\exp a\cdot o, \exp a\cdot o)$$

is bounded on $a \in \mathfrak{a}^+$ only if $n \in M\exp\mathfrak{a}N = P$[15]. Thus, the equivalence class of the flat f_0 is $P\cdot\mathsf{f}_0$. Hence, the Furstenberg boundary is $\mathrm{PSL}(d,\mathbb{R})$-equivariantly identified with the space of complete flags $\mathscr{F} = \mathrm{PSL}(d,\mathbb{R})/P$.

Given a parametrized flat f denote by $\mathsf{Z}(\mathsf{f}) \in \mathscr{F}$ its equivalence class in the Furstenberg boundary. Also, denote by $\check{\mathsf{Z}}(\mathsf{f}) \in \mathscr{F}$ the class of the parametrized flat[16]

$$a \mapsto \mathsf{f}(-a).$$

This last identifications can be seen directly: a parametrized flat f consists on fixing an *ordered*[17] set $\{\ell_1, \ldots, \ell_d\}$ of d lines that span $\mathbb{R}^d$ and considering all inner products (up to homothety) that make this set an orthogonal set, and the choice of

[15] The ij entry on $\mathcal{E}$ of $\exp(-ta)n\exp(ta)$ is $\exp(t(a_j - a_i))n_{ij}$, in order to have this entry bounded for all $t > 0$ one must have $n_{ij} = 0$ for all $j < i$.

[16] This is still a parametrized flat.

[17] Recall $\mathfrak{a}^+$ is fixed beforehand.

one of these inner products. The associated point 'at infinity' in the Furstenberg boundary of this parametrized flat is the complete flag

$$\mathsf{Z}(\mathsf{f})_p = \ell_1 \oplus \cdots \oplus \ell_p.$$

Moreover, $\check{\mathsf{Z}}(\mathsf{f})_p = \ell_d \oplus \cdots \oplus \ell_{d-p+1}$.

One easily concludes the following properties

- Given $x \in X_d$ and a complete flag F there exists a unique maximal flat $\mathsf{f}(\mathfrak{a})$ containing x such that $\mathsf{Z}(\mathsf{f}) = F$: apply the Gram-Schmidt process to flag F and any inner product in the class x.
- Given two flags in general position $(E, F) \in \mathscr{F}^{(2)}$ there exists a unique maximal flat $\mathsf{f}(\mathfrak{a})$ such that $\mathsf{Z}(\mathsf{f}) = E$ and $\check{\mathsf{Z}}(\mathsf{f}) = F$: it suffices to consider the ordered set $\ell_p = E_p \cap F_{d-p+1}$.

Recalling that $M \exp \mathfrak{a} = P \cap \check{P}$, observe that the maps $\check{\mathsf{Z}}$ and Z are exactly the canonical quotient projections

$$(\check{\mathsf{Z}}, \mathsf{Z}) : \mathrm{PSL}(d, \mathbb{R})/M \to \mathscr{F}^{(2)} = \mathrm{PSL}(d, \mathbb{R})/M \exp \mathfrak{a}.$$

Given a subset $\theta \subset \Pi$, denote by $\mathscr{F}_\theta$ the space of *partial flags* $E = \{E_p : \alpha_p \in \theta\}$ and denote by $\mathrm{proj}_\theta : \mathscr{F} \to \mathscr{F}_\theta$ the projection consisting on forgetting the irrelevant subspaces of a complete flag. Finally, denote by $\mathsf{Z}_\theta = \mathrm{proj}_\theta \mathsf{Z}$ and by $\check{\mathsf{Z}}_\theta = \mathsf{Z}_{\mathrm{i}\,\theta}$.

Given a pair of partial flags in general position $E \in \mathscr{F}_\theta$ and $F \in \mathscr{F}_{\mathrm{i}\,\theta}$ and a point $x \in X_d$, we define:

- The *Weyl cone* $V(x, E)$ determined by x and E is
$$\bigcup_{\mathsf{f}} \mathsf{f}(\mathfrak{a}^+),$$
where the union is indexed on all parametrized flats f with $\mathsf{f}(0) = x$ and $\mathsf{Z}_\theta(\mathsf{f}) = E$.
- The *parallel set* $P(F, E)$ determined F and E is
$$\bigcup_{\mathsf{f}} \mathsf{f}(\mathfrak{a}),$$
where the union is indexed on all parametrized flats f with $\check{\mathsf{Z}}_\theta(\mathsf{f}) = F$ and $\mathsf{Z}_\theta(\mathsf{f}) = E$.

7.10. **Parametrized flats through** o **and** $g \cdot o$**.** Consider $g \in \mathrm{PSL}(d, \mathbb{R})$ and $g = k_g \exp a(g) l_g$ a Cartan decomposition of g. Observe that the set of d lines

$$k_g \mathcal{E} = \{k_g u_1, \ldots, k_g u_d\}$$

is simultaneously o-orthogonal and $g \cdot o$-orthogonal. The set of classes of inner products that make this set orthogonal is hence a maximal flat through o and $g \cdot o$.

If g has gaps of certain indices, indexed by $\theta \subset \Pi$, then every element of $k_g K_o(\theta)$ can be chosen in a Cartan decomposition of g. Thus, the set of maximal flats through o and $g \cdot o$ is the $K_o(\theta)$-orbit $k_g K_o(\theta)\mathcal{E}$.

All parametrized flats f with $\mathsf{f}(0) = o$ such that $g \cdot o \in \mathsf{f}(\mathfrak{a}^+)$, have the flag $U^o(g)$ as a partial subflag of their corresponding flag at infinity $\mathsf{Z}(\mathsf{f})$ (recall equation (7.2)), see Figure 5.

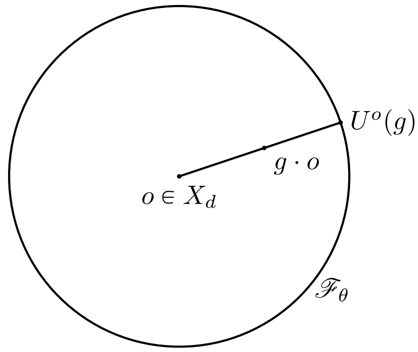


FIGURE 5. All flats through o and $g \cdot o$ have $U^o(g)$ as a partial flag at infinity.

7.11. **$\mathfrak{a}^+$-valued distance.** Recall that X_d is $\mathrm{PSL}(d, \mathbb{R})$-homogeneous and consider the map $\underline{a} : X_d \times X_d \to \mathfrak{a}^+$ defined by

$$\underline{a}(g \cdot o, h \cdot o) = a(g^{-1}h).$$

Note that $\underline{a}$ is $\mathrm{PSL}(d, \mathbb{R})$-invariant for the diagonal action of $\mathrm{PSL}(d, \mathbb{R})$ on $X_d \times X_d$, that

$$d_{X_d}(x, y) = |\underline{a}(x, y)|_o, \tag{7.4}$$

(this is due to equation (7.3)) and that $\mathrm{i}(\underline{a}(x, y)) = \underline{a}(y, x)$.

The *Weyl cone* $V(x, y)$ through x and y is defined by $\bigcup_{\mathsf{f}} \mathsf{f}(\mathfrak{a}^+)$, where the union is indexed on all parametrized flats f with $\mathsf{f}(0) = x$ and $\mathsf{f}(\underline{a}(x, y)) = y$. If one considers the subset of simple roots defined by

$$\theta(x, y) = \{\alpha \in \Pi : \alpha(\underline{a}(x, y)) \neq 0\},$$

then $V(x, y)$ is the Weyl cone (as defined in 7.9) through x and the partial flag

$$U(x, y) = \{gU^o_\alpha(g^{-1}h)\}_{\alpha \in \theta(x,y)}.$$

We will usually say that this flag is the *associated (partial) flag at infinity* to the Weyl cone $V(x, y)$.

Finally, the *diamond* between x and y is the subset

$$\diamondsuit(x, y) = V(x, y) \cap V(y, x).$$

This diamond is contained in the parallel set $P(U(x, y), S(x, y))$, where $S(x, y) = hS^o(g^{-1}h)$.

For example, consider $x = o$. Any $y \in X_d$ can be written as $y = g \exp a \cdot o$ with $a \in \mathfrak{a}^+$ and $g \cdot o = o$. Then:

$$V(o, y) = \{g \exp v \cdot o : v \in \mathfrak{a}^+\},$$
$$\diamondsuit(o, y) = \{g \exp v \cdot o : v \in \mathfrak{a}^+ \cap (a - \mathfrak{a}^+)\}.$$

7.12. **Angles and distances to parallel sets.** The purpose of this subsection and the next one is to relate the distance from a given point o to a parallel set $P(E, F)$, for two partial flags in general position, with the angle between E and F for an inner product in the class o (Proposition 7.2).

Fix an inner product $\langle \cdot, \cdot \rangle \in o$ and denote by $\| \cdot \|$ the induced norm on $\mathbb{R}^d$. The *o-angle* between non-zero vectors v, $w \in \mathbb{R}^d$ is defined as the unique number $\measuredangle_o(v, w)$

in $[0,\pi]$ whose cosine is $\langle v,w\rangle/(\|v\|\,\|w\|)$. If $E,\,F\subset\mathbb{R}^d$ are nonzero subspaces then we define their *o-angle* as:

$$\measuredangle_o(E,F):=\min_{v\in E^\times}\min_{w\in F^\times}\measuredangle(v,w)\,, \tag{7.5}$$

where $E^\times:=E\smallsetminus\{0\}$. We also write $\measuredangle_o(v,F)$ instead of $\measuredangle_o(\mathbb{R}v,F)$, if v is a nonzero vector. Observe that $\measuredangle_o(\cdot,\cdot)$ is independent on $\langle\cdot,\cdot\rangle\in o$.

We haven't found a precise reference for the following proposition, we will hence provide a proof. See Figure 6.

Proposition 7.2. *Given $\theta\subset\Pi$ there exist $c>1$ and $c'>0$, only depending on θ and the group $\mathrm{PSL}(d,\mathbb{R})$, such that if $(E,F)\in\mathscr{F}_\theta^{(2)}$, then*

$$\frac{-1}{c}\log\sin\min_{\alpha_p\in\theta}\measuredangle_o(E_p,F_{d-p})\leqslant d_{X_d}(o,P(F,E))\leqslant c'-c\log\sin\min_{\alpha_p\in\theta}\measuredangle_o(E_p,F_{d-p}).$$

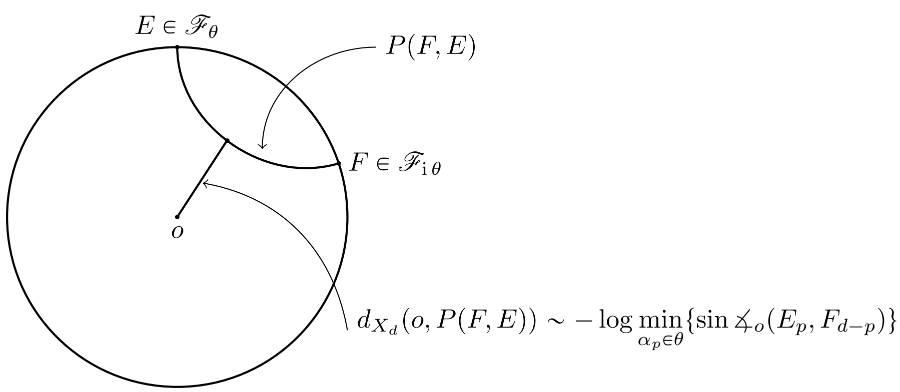


FIGURE 6. The statement of Proposition 7.2

Proof. Let us prove first the second inequality.

Denote by $G_{\mathcal{E}}(E,F)$ those elements in $\mathrm{PSL}(d,\mathbb{R})$ such that for all $\alpha_p\in\theta$ one has $g(u_1\oplus\cdots\oplus u_p)=E_p$ and $g(u_{p+1}\oplus\cdots\oplus u_d)=F_{d-p}$. Then, the parallel set

$$P(F,E)=\{g\cdot o:g\in G_{\mathcal{E}}(E,F)\}.$$

Write the set $\{p:\alpha_p\in\theta\}\cup\{0,d\}$ as $\{0=p_0<p_1<p_2<\cdots<p_k=d\}$. Denote by

$$H_i^0=u_{p_{i-1}}\oplus\cdots\oplus u_{p_i}.$$

Without loss of generality, we may assume that for all α_{d-p_i} one has $F_{d-p_i}=u_{d-p_i}\oplus\cdots\oplus u_d$. Denote by $H_i=E_{p_i}\cap F_{d-p_{i+1}}$. For all $i\in\{1,\ldots,k\}$ one has

$$H_1^0\oplus\cdots\oplus H_i^0=H_1\oplus\cdots\oplus H_i.$$

If $g\in\mathrm{PSL}(d,\mathbb{R})$ is such that $g(H_i^0)=H_i$ then $g\in G_{\mathcal{E}}(E,F)$ and hence $g\cdot o\in P(F,E)$. We will define a suited such g and estimate its operator norm for the class o.

We imitate the Gram–Schmidt procedure and take g as the unique element of $\mathrm{PSL}(d,\mathbb{R})$ such that, for each $i=1,\ldots,k$, the restriction of g^{-1} to H_i coincides with the orthogonal projection on H_i^0.

We proceed to estimate $\|g\|$. Given $v \in \mathbb{R}^d$, write it as $v = v_1 + \cdots + v_k$ with $v_i \in H_i^0$. Then

$$\|gv\| \leqslant \sum_{i=1}^{k} \|gv_i\| = \sum_{i=1}^{k} \frac{\|v_i\|}{\sin \measuredangle_o(g(v_i), H_1^0 \oplus \cdots \oplus H_{i-1}^0)} .$$

For each i, orthogonality yields $\|v_i\| \leqslant \|v\|$; moreover,

$$\begin{aligned} \measuredangle_o(g(v_i), H_1^0 \oplus \cdots \oplus H_{i-1}^0) & \\ \geqslant \measuredangle_o(H_i \oplus \cdots \oplus H_d, H_1^0 \oplus \cdots \oplus H_{i-1}^0) &= \measuredangle_o(E_{p_i}, F_{d-p_i}) \\ &\geqslant \min_{\alpha_p \in \theta} \measuredangle_o(E_p, F_{d-p}) . \end{aligned}$$

Therefore we obtain

$$\|g\| \leqslant \frac{d}{\min_{\alpha_p \in \theta} \sin \measuredangle_o(E_p, F_{d-p})} .$$

Taking log, Lemma 7.1 completes the proof. The first inequality is direct. □

7.13. **Regular quasi-geodesics and the Morse Lemma of** [KLP$_2$]**.** Let $I \subset \mathbb{Z}$ be an interval and let μ, c be positive numbers. A (μ, c)-*quasi-geodesic* is a map $x : I \to X_d$ (also denoted by $\{x_n\}_{n \in I}$) such that for all $n, m \in I$ one has

$$\frac{1}{\mu}|n-m| - c \leqslant d_{X_d}(x_n, x_m) \leqslant \mu|n-m| + c.$$

Let $\mathscr{C} \subset \mathfrak{a}^+$ be a closed cone. Following [KLP$_2$] we will say that a quasi-geodesic segment $\{x_n\}$ is $\mathscr{C}$-*regular* if for all $n < m \in I$ one has $\underline{a}(x_n, x_m) \in \mathscr{C}$. Note that $\{x_{-n}\}_{n \in -I}$ is $\mathrm{i}\mathscr{C}$-regular. Denote by

$$\theta_{\mathscr{C}} = \{\alpha \in \Pi : \ker \alpha \cap \mathscr{C} = \{0\}\}.$$

One has the following version of the Morse Lemma:

Theorem 7.3 ([KLP$_2$, Theorem 1.3])**.** *Let μ, c be positive numbers and $\mathscr{C} \subset \mathfrak{a}^+$ a closed cone, then there exists $\ell \in \mathbb{N}$ and $C > 0$ such that if $\{x_n\}_{n \in I}$ is a $\mathscr{C}$-regular (μ, c)-quasi-geodesic segment, then*

- *If I is finite and $|I| \geqslant \ell$ then $\{x_n\}$ is at distance at most C from the diamond $\diamondsuit(x_{\min I}, x_{\max I})$.*
- *If $I = \mathbb{N}$ then there exists $F \in \mathscr{F}_{\theta_{\mathscr{C}}}$ such that $\{x_n\}$ is contained in a C-neighborhood from the Weyl cone $V(x_{\min I}, F)$.*
- *If $I = \mathbb{Z}$ then there exists $(E, F) \in \mathscr{F}^{(2)}_{\theta_{\mathscr{C}}}$ such that $\{x_n\}$ is contained in a C-neighborhood from the union $V(z, E) \cup V(z, F)$ for some $z \in P(E, F)$ at uniform distance from $\{x_n\}$.*

Proof. We can assume that $0 \in I$ and that $x_0 = o$. Consider then a sequence $\{h_n\}_{n \in I} \subset \mathrm{PSL}(d, \mathbb{R})$ such that $h_n \cdot o = x_n$. Since $\{x_n\}$ is a quasi-geodesic, equation 7.4 implies

$$|a(h_{n+1}^{-1} h_n)|_o = |\underline{a}(h_n \cdot o, h_{n+1} \cdot o)|_o = d_{X_d}(x_n, x_{n+1}) \leqslant \mu + c.$$

If we denote by $g_n = h_{n+1}^{-1} h_n$, then the last equation implies that $\{g_n\}$ lies in a compact subset of $\mathrm{PGL}(d, \mathbb{R})$. Moreover, if $m \geqslant n$ then

$$a(g_m \cdots g_n) = a(h_m^{-1} h_n) = \underline{a}(x_m, x_n).$$

One has the following:

1. The sequence $\{g_n\}_{n\in I}$ is α-dominated for all $\alpha \in \theta_{\mathscr{C}}$: indeed, since $\mathscr{C}$ is closed and does not intersect $\ker\alpha - \{0\}$, there exists $\delta > 0$ such that for all $a \in \mathscr{C} - \{0\}$ and $\alpha \in \theta_{\mathscr{C}}$ one has
$$\alpha(a) > \delta |a|_o.$$
Thus, since $\{x_n\}$ is $\mathscr{C}$-regular one concludes that
$$\alpha(a(g_m \cdots g_n)) = \alpha(\underline{a}(x_n, x_m)) > \delta |a(x_n, x_m)|_o > (\delta/\mu)|n-m| - \delta c.$$
In other words, the sequence $\{g_n\}_{n\in I}$ belongs to the space
$$\mathcal{D}^o(\mu + c, p, \delta\mu, c/\delta, I)$$
for all p such that $\alpha_p \in \theta_{\mathscr{C}}$.
2. There exists $\tau > 0$ such that for all $m > n$ and $\alpha \in \theta_{\mathscr{C}}$ one has
$$\frac{1}{\tau}(m-n) - c < \alpha\left(\sum_{i=n}^{i=m} \underline{a}(x_i, x_{i+1}) - \underline{a}(x_n, x_m)\right) < \tau(m-n) + c.$$
Hence, if $|m-n|$ is long enough (only dependent on μ, c and the cone $\mathscr{C}$) then $\sum_n^m \underline{a}(x_i, x_{i+1}) - \underline{a}(x_n, x_m)$ does not intersect $\ker\alpha$ for all $\alpha \in \theta_{\mathscr{C}}$.
3. Subsection 7.10 implies that the (partial) flag at infinity associated to the Weyl cone $V(o, x_m)$ for $m \geqslant 0$ is
$$U(o, x_m) = \{U^o_\alpha(g_m \cdots g_0)\}_{\alpha\in\theta_{\mathscr{C}}}.$$

Consider ℓ_1 given simultaneously by Lemmas 2.5 and A.10 for all $\mathcal{D}(\mu+c, p, \delta\mu, c/\delta, I)$ such that $\alpha_p \in \theta_{\mathscr{C}}$, and suppose that the interval of integers $[-\ell_1, \ell_1] \subset I$ (i.e. I is long enough). Item 3 and Lemma 2.5 imply the existsence of δ_0 such that if $m, n \in I$ are such that $m \geqslant \ell_1$ and $n \leqslant -\ell_1$ such that
$$\measuredangle_o(U(o, x_m), S(o, x_n)) > \delta_0.$$
Moreover, $\measuredangle_o(U(o, x_{\ell_1}), U(o, x_m)) < \varepsilon$ and the same occurs with $S(o, x_{-\ell_1})$ and $S(o, x_n)$.

Proposition 7.2 implies then that the distance between o and the parallel set $P(U(o, x_m), S(o, x_n))$ is bounded above by a number C, depending on μ, c, the cone $\mathscr{C}$ and *a priori* the point o, but independent of $\{x_n\}$. Item 2 implies moreover that o is at distance at most C from the diamond $\diamondsuit(x_n, x_m)$.

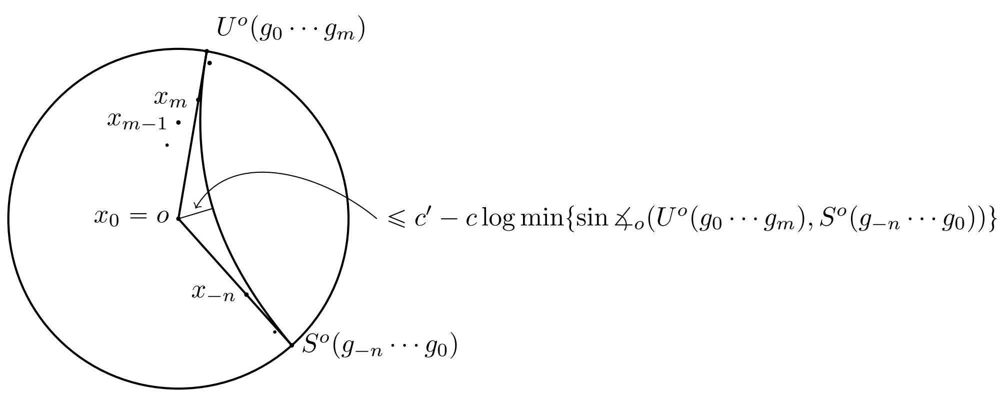


FIGURE 7. The flag at infinity associated to the Weyl cone $V(o, x_m)$ corresponds to the U^o flag of a dominated sequence.

Since X_d is $\mathrm{PSL}(d,\mathbb{R})$-homogeneous and this action is by isometries one concludes the last subsection for any $k \in I$ such that $[k-\ell_1, k+\ell_1] \subset I$.

We now split between the three cases of the statement.

If I is finite the remaining points of I are a finite amount (only dependent on the quasi-geodesic constants and $\mathscr{C}$), the upper bound can then be found explicitly using the fact

$$\measuredangle_o(U(o,x_{\ell_1}), U(o,x_m)) < \varepsilon$$

and that the same occurs for $S(o,x_{-\ell_1})$ and $S(o,x_n)$.

If $I = \mathbb{N}$ then for any $k \in I$ such that $[k-\ell_1, k] \subset I$ one has that the flag $U(x_k, x_m)$ is convergent, as $m \to \infty$, to the flag $E^{cs}(\{g_m\})$ and hence the the diamond $\diamondsuit(x_k, x_m)$ becomes, as $m \to \infty$ the Weyl cone $V(x_k, E^{cs}(\{g_m\}))$. Again, the first ℓ_1 elements of I can be controlled explicitly.

If $I = \mathbb{Z}$ the result follows from the last subsection. □

8. When the target group is a semi-simple Lie group

The purpose of this section is the extend the main results in the previous sections to the situation where the target group is a non-compact real-algebraic semi-simple Lie group.

We will begin by recalling the general structure theory of these groups, needed to define concepts such as domination, Anosov representation, regular quasi-geodesic... This basic structure theory can be found in [Hum_1], [Ebe], [Hel], [Lang].

We will then explain how the representation theory of these groups is used to reduce the general case to the $\mathrm{PSL}(d,\mathbb{R})$ case, for a well chosen d. Section 7 mimics, for $\mathrm{PSL}(d,\mathbb{R})$, the general structure presented here.

The first main goal is subsection 8.5, where Theorem 3.2 and Proposition 4.9 are extended to the general setting. This general case is reduced to the actual statement of 3.2 and 4.9 using Tits representations.

The remainder of the section is devoted to a new proof of the Morse Lemma of [KLP_2] for symmetric spaces of non-compact type. In contrast with subsection 8.5, a simple reduction to the $\mathrm{PSL}(d,\mathbb{R})$ case is not sufficient, one needs to have a finer control of distances to parallel sets when embedding symmetric spaces. This is achieved in Corollary 8.7.

If G is a Lie group with Lie algebra $\mathfrak{g}$, the *Killing form* of $\mathfrak{g}$ is the symmetric bilinear form defined by

$$\kappa(v,w) = \mathrm{Trace}(\mathrm{ad}_v \,\mathrm{ad}_w).$$

The group G is *semi-simple* if κ is non-degenerate.

We will assume from now on that G is semi-simple, real-algebraic (i.e. defined by polynomial equations with real coefficients) and has no compact factors.

8.1. **Root system.** A *Cartan involution* of $\mathfrak{g}$ is an involution $\tau : \mathfrak{g} \to \mathfrak{g}$ such that the bilinear form $(v,w) \mapsto -\kappa(v,\tau(w))$ is positive definite. The fixed point set

$$\mathfrak{k}^\tau = \{v \in \mathfrak{g} : \tau v = v\}$$

is the Lie algebra of a maximal compact subgroup K^τ. Consider $\mathfrak{p}^\tau = \{v \in \mathfrak{g} : \tau v = -v\}$ and note that

$$\mathfrak{g} = \mathfrak{k}^\tau \oplus \mathfrak{p}^\tau.$$

A computation shows that $[\mathfrak{p}^\tau, \mathfrak{p}^\tau] \subset \mathfrak{k}^\tau$ and hence any subalgebra of $\mathfrak{p}^\tau$ is necessarily abelian. Let $\mathfrak{a} \subset \mathfrak{p}^\tau$ be a maximal abelian subalgebra.

Denote by Σ the set of *roots* of $\mathfrak{a}$ on $\mathfrak{g}$. By definition,

$$\Sigma = \{\alpha \in \mathfrak{a}^* - \{0\} : \mathfrak{g}_\alpha \neq 0\}$$

where

$$\mathfrak{g}_\alpha = \{w \in \mathfrak{g} : [a, w] = \alpha(a)w \ \forall a \in \mathfrak{a}\}.$$

The closure of a connected component of

$$\mathfrak{a} - \bigcup_{\alpha\in\Sigma} \ker \alpha$$

is called *a closed Weyl chamber*. Fix a closed Weyl chamber $\mathfrak{a}^+$ and let $\Sigma^+ = \{\alpha \in \Sigma : \alpha|\mathfrak{a}^+ \geqslant 0\}$ be the set of *positive roots* associated to $\mathfrak{a}^+$. The set Σ^+ contains a subset Π that verifies

- Π is a basis of $\mathfrak{a}$ as a vector space,
- every element of Σ^+ has non-negative coefficients in the basis Π.

The set Π is called the *set of simple roots* determined by Σ^+, the sets $\ker \alpha \cap \mathfrak{a}^+$ for $\alpha \in \Pi$, are the *walls* of the chamber $\mathfrak{a}^+$.

The *Weyl group* W of Σ is defined as the group generated by the orthogonal reflections on the subspaces $\{\ker \alpha : \alpha \in \Sigma\}$.

The reflections associated to elements of Π span W. With respect to the word-length on this generating set, there exists a unique longest element in W, denoted by $u_0 : \mathfrak{a} \to \mathfrak{a}$. This is the unique element in W that sends $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. The *opposition involution* $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ is defined by $\mathrm{i} = -u_0$. If we denote by $(\cdot,\cdot)$ the bilinear form on $\mathfrak{a}^*$ dual to the Killing form, define

$$\langle \chi, \psi \rangle = \frac{2(\chi, \psi)}{(\psi, \psi)}$$

and let $\{\omega_\alpha\}_{\alpha\in\Pi}$ be the *dual basis* of Π, i.e. $\langle \omega_\alpha, \beta \rangle = \delta_{\alpha\beta}$. The linear form ω_α is *the fundamental weight* associated to α. Note that for every $\chi \in \mathfrak{a}^*$ one has

$$\chi = \sum_{\alpha\in\Pi} \langle \chi, \alpha \rangle \omega_\alpha. \tag{8.1}$$

Denote by $a = a_G : G \to \mathfrak{a}^+$ the *Cartan projection* of G. By definition, for every $g \in G$ one has $g \in K \exp a(g) K$ and $a(g^{-1}) = \mathrm{i}\, a(g)$.

8.2. **Parabolic subgroups.** Denote by M the centralizer of $\exp \mathfrak{a}$ in K and by $N = \exp \mathfrak{n}^+$ where $\mathfrak{n}^+ = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$. The group $P_\Pi = M \exp \mathfrak{a} N$ is called a *minimal parabolic subgroup* and its Lie algebra is $\mathfrak{p}_\Pi = \mathfrak{g}_0 \oplus \mathfrak{n}^+$. A *parabolic subgroup* of G is a subgroup that contains a conjugate of P_Π. Two parabolic subgroups are *opposite* if their intersection is a reductive group.[18]

To each subset θ of Π one associates two opposite parabolic subgroups of G, P_θ and $\check{P}_\theta$, whose Lie algebras are, by definition,

$$\mathfrak{p}_\theta = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in\langle\Pi-\theta\rangle} \mathfrak{g}_{-\alpha},$$

and

$$\check{\mathfrak{p}}_\theta = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_{-\alpha} \oplus \bigoplus_{\alpha\in\langle\Pi-\theta\rangle} \mathfrak{g}_\alpha,$$

[18]Recall that a Lie group is *reductive* if its Lie algebra splits as a semi-simple algebra and an abelian algebra.

where $\langle\theta\rangle$ is the set of positive roots generated by θ. Every pair of opposite parabolic subgroups of G is conjugate to $(P_\theta, \check{P}_\theta)$ for a unique θ, and every opposite parabolic subgroup of P_θ is conjugate to $P_{\mathrm{i}\,\theta}$: the parabolic group associated to

$$\mathrm{i}\,\theta = \{\alpha \circ \mathrm{i} : \alpha \in \theta\}.$$

The quotient space $\mathscr{F} = \mathscr{F}_\Pi = G/P_\Pi$ is called *the flag space* of G and if $\theta \subset \Pi$ then $\mathscr{F}_\theta$ is called *a partial flag space* of G. Denote by $\mathrm{proj}_\theta : \mathscr{F}_\Pi \to \mathscr{F}_\theta$ the canonical projection.

8.3. **Representations of G.** Let $\Lambda : G \to \mathrm{PSL}(V)$ be a finite dimensional rational[19] irreducible representation and denote by $\phi_\Lambda : \mathfrak{g} \to \mathfrak{sl}(V)$ the Lie algebra homomorphism associated to Λ. Then $\chi \in \mathfrak{a}^*$ is a *restricted weight* of Λ if the vector space

$$V_\chi = \{v \in V : \phi_\Lambda(a)v = \chi(a)v \ \forall a \in \mathfrak{a}\}$$

is non zero. Theorem 7.2 of Tits [Ti] states that the set of weights has a unique maximal element with respect to the order $\chi \geqslant \psi$ if $\chi - \psi$ is positive on $\mathfrak{a}^+$. This is called *the highest weight* of Λ and denoted by χ_Λ.

Note that if χ is a restricted weight and $v \in V_\chi$ then, for $n \in \mathfrak{g}_\alpha$ with $\alpha \in \Sigma$ one has that $\phi_\Lambda(n)v$ is an eigenvector of $\phi_\Lambda(\mathfrak{a})$ of eigenvalue $\chi+\alpha$, [20] *unless* $\phi_\Lambda(n)v = 0$. Since for all $\beta \in \Sigma^+$ one has $\chi_\Lambda + \beta \geqslant \chi_\Lambda$ and χ_Λ is maximal, one concludes that $\chi_\Lambda + \beta$ is not a weight, i.e. for all $n \in \mathfrak{g}_\beta$ and $v \in V_{\chi_\Lambda}$ one has $\phi_\Lambda(n)v = 0$.

Let $\theta_\Lambda \in \Pi$ be the set of simple roots α such that $\chi_\Lambda - \alpha$ is still a weight of Λ.

Remark 8.1. The subset θ_Λ is the smallest subset of simple roots such that the following holds: Consider $\alpha \in \Sigma^+$, $n \in \mathfrak{g}_{-\alpha}$ and $v \in \chi_\Lambda$, then $\phi_\Lambda(n)v = 0$ if and only if $\alpha \in \langle \Pi - \theta_\Lambda \rangle$.

Observe that any other weight of Λ is hence of the form

$$\chi_\Lambda - \sum_{\alpha \in \Pi} k_\alpha \alpha, \tag{8.2}$$

where $k_\alpha \geqslant 0$ and $\prod_{\alpha\in\theta} k_\alpha \neq 0$ (i.e. the numbers k_α, for $\alpha \in \theta_\Lambda$, do not simultaneously vanish). Thus, if we denote by $p = \dim V_{\chi_\Lambda}$ then for all $g \in G$ one has

$$\alpha_p(a(\Lambda g)) = a_p(\Lambda g) - a_{p+1}(\Lambda g) = \sum_{\alpha\in\Pi} k_\alpha \alpha(a(g)) \geqslant k_\beta \beta(a(g)) \tag{8.3}$$

for some $\beta \in \theta_\Lambda$.

Consider an inner product on V invariant under ΛK, such that $\Lambda \exp \mathfrak{a}$ is symmetric. Then, for the Euclidean norm $\| \ \|$ induced by this scalar product, one has

$$\log \|\Lambda g\| = \chi_\Lambda(a(g)). \tag{8.4}$$

If $g = k(\exp \sigma(g))l$ with $k, l \in K$, then for all $v \in kV_{\chi_\Lambda}$ one has

$$\|\Lambda g(v)\| = \|\Lambda g\| \|v\|.$$

We will denote by Λo this homothety class of inner product.

---

[19] i.e. a rational map between algebraic varieties.

[20] Indeed, this follows from $\phi_\Lambda([a,n])v = \alpha(a)\phi_\Lambda(n)v$.

Denote by W_{χ_Λ} the $\Lambda(\exp \mathfrak{a})$-invariant complement of V_{χ_Λ}. Note that the stabilizer in G of W_{χ_Λ} is $\check{P}_\theta$, which is conjugated to $P_{\mathrm{i}\,\theta_\Lambda}$, and thus one has a map of flag spaces

$$(\xi_\Lambda, \xi_\Lambda^*) : \mathscr{F}^{(2)}_{\theta_\Lambda} \to \mathscr{G}^{(2)}_{\dim V_{\chi_\Lambda}}(V). \tag{8.5}$$

This is a proper embedding which is an homeomorphism onto its image.

8.4. **A set of representations defined by Tits.** One has the following proposition by Tits (see also Humphreys [Hum$_2$, Chapter XI]).

Proposition 8.2 (Tits [Ti])**.** *For each $\alpha \in \Pi$ there exists a finite dimensional rational irreducible representation $\Lambda_\alpha : G \to \mathrm{PSL}(V_\alpha)$, such that χ_{Λ_α} is an integer multiple of the fundamental weight ω_α and $\dim V_{\chi_{\Lambda_\alpha}} = 1$. All other weights of Λ_α are of the form*

$$\chi_\alpha - \alpha - \sum_{\beta \in \Pi} n_\beta \beta,$$

where $n_\beta \in \mathbb{N}$.

Such a set of representations is not necessarily unique (e.g. if G does not split), we will fix from now on such a set of representations and say that Λ_α is the *Tits representation* of G associated to α. Observe that for all $g \in G$ one has

$$a_1(\Lambda_\alpha g) - a_2(\Lambda_\alpha g) = \alpha(a(g)) + \sum_{\beta \in \Pi} n_\beta \beta(\sigma(g)) \geqslant \alpha(\sigma(g)). \tag{8.6}$$

8.5. **Dominated representations: reduction to the $\mathrm{GL}(d, \mathbb{R})$ case:** A representation $\rho : \Gamma \to G$ is *θ-dominated* if there exist positive constants μ and c such that for all $\alpha \in \theta$ and $\gamma \in \Gamma$ one has

$$\alpha(a(\rho\gamma)) \geqslant \mu|\gamma| - c.$$

Assume that ρ is a θ-dominated representation. Then for all $\alpha \in \theta$, the representation $\Lambda_\alpha \rho : \Gamma \to \mathrm{PSL}(V_\alpha)$ is 1-dominated in the sense of Subsection 3.1. Indeed, Equation (8.6) implies that

$$\log \sigma_1(\Lambda_\alpha \rho(\gamma)) - \log \sigma_2(\Lambda_\alpha \rho(\gamma)) \geqslant \alpha(a(\rho\gamma)) \geqslant \mu|\gamma| - c,$$

or equivalently

$$\frac{\sigma_2(\Lambda_\alpha \rho(\gamma))}{\sigma_1(\Lambda_\alpha \rho(\gamma))} \leqslant e^c e^{-\mu|\gamma|}.$$

Thus, Theorem 3.2 implies that Γ is word-hyperbolic. Moreover, Proposition 4.9, together with Guichard-Wienhard [GW, Proposition 4.3], implies that ρ is P_θ-Anosov[21].

Thus, Theorem 3.2, Proposition 4.9 together with [GW, Proposition 4.3] and Tits representations 8.2 prove the following.

Theorem 8.3. *Let $\rho : \Gamma \to G$ be a θ-dominated representation, then Γ is word-hyperbolic and ρ is P_θ-Anosov.*

[21] See [Lab] or [GW] for a precise definition.

8.6. **A Plücker representation.** Given $\theta \subset \Pi$ one can construct a rational irreducible representation of G such that P_θ is the stabilizer of a line, and hence $\check{P}_\theta$ will be the stabilizer of a hyperplane.[22] More precisely, one has the following result from representation theory (see Guichard-Wienhard [GW, §4]).

Proposition 8.4. *Given $\theta \subset \Pi$, there exists a finite dimensional rational irreducible representation $\Lambda : G \to \mathrm{PSL}(V)$ and $\ell \in \mathbb{P}(V)$ such that*

$$P_\theta = \{g \in G : \Lambda\, g(\ell) = \ell\}.$$

Such a representation can be defined as follows: if we denote by $k = \dim \mathfrak{p}_\theta$ then the composition $\Lambda^k \operatorname{Ad} : G \to \mathrm{PSL}(\Lambda^k \mathfrak{g})$ verifies the desired conditions, except (maybe) irreducibility, this is fixed by considering the vector space V spanned by the G-orbit of the line $\ell = \Lambda^k \mathfrak{p}_\theta$,

$$V = \langle \Lambda^k \operatorname{Ad} G \cdot \ell \rangle,$$

and considering the restriction of $\Lambda^k \operatorname{Ad} G$ to V.

A representation verifying the statement of Proposition 8.4 will be called a *Plücker representation* of G associated to θ. For such a representation one has a continuous equivariant map $(\xi, \xi^*) : \mathscr{F}_\theta^{(2)} \to \mathbb{P}^{(2)}(V)$ homeomorphic to its image.

8.7. **The symmetric space.** The *symmetric space of G* is the space of Cartan involutions on $\mathfrak{g}$, and is denoted by X. This a G-homogeneous space, and the stabilizer of $o \in X$ is the compact group K^o, whose lie algebra is $\mathfrak{k}^o$. The tangent space $\mathsf{T}_o X$ is hence identified with $\mathfrak{p}^o$. The G-invariant Riemannian metric on X is the restriction of the Killing form κ to $\mathfrak{p}^o \times \mathfrak{p}^o$.

If d_X is the distance on X induced by $\kappa|\mathfrak{p}^o \times \mathfrak{p}^o$, then the Euclidean norm $\| \ \|_o$ induced on $\mathfrak{a}$ is invariant under the Weyl group, and for all $a \in \mathfrak{a}$ one has $d_X(o, (\exp a)\cdot o) = \|a\|_o$. The Cartan decomposition of G implies hence that for all $g \in G$ one has

$$d_X(o, g\cdot o) = \|a(g)\|_o.$$

Consider the map $\underline{a} : X \times X \to \mathfrak{a}^+$ defined by $\underline{a}(g\cdot o, h\cdot o) = a(g^{-1}h)$. Note that $\underline{a}$ is G-invariant for the diagonal action of G on $X \times X$, that

$$d_X(p,q) = \|\underline{a}(p,q)\|_o \tag{8.7}$$

and that $\mathrm{i}(\underline{a}(p,q)) = \underline{a}(q,p)$.

8.8. **Flats.** A *parametrized flat* is a map $\mathsf{f} : \mathfrak{a} \to X$ of the form $\mathsf{f}(v) = g\exp(v)\cdot o$ for some $g \in G$. Observe that G acts transitively on the set of parametrized flats and that the stabilizer of $\mathsf{f}_0 : v \mapsto \exp(v)\cdot o$ is the group M of elements in K commuting with $\exp(\mathfrak{a})$. We will hence identify the space of parametrized flats with G/M.

Two parametrized flats f, g are *equivalent* if the function $\mathfrak{a} \to \mathbb{R}$, defined by

$$v \mapsto d_X(\mathsf{f}(v), \mathsf{g}(v)),$$

is bounded on $\mathfrak{a}^+$. The *Furstenberg boundary* $\mathscr{F}$ of X is the space of the equivalent parametrized flats. It is a standard fact that $\mathscr{F}$ is G-equivariantly identified with G/P_Π. Denote by Z the canonical projection and by $\check{\mathsf{Z}}(\mathsf{f}) = \mathsf{Z}(\mathsf{f} \circ \mathrm{i})$.The pair $(\check{\mathsf{Z}}(\mathsf{f}), \mathsf{Z}(\mathsf{f}))$ belongs to $\mathscr{F}^{(2)}$, the unique open G-orbit on $\mathscr{F} \times \mathscr{F}$.

The following proposition is standard.

[22] There are actually infinitely many such representations.

Proposition 8.5 (see [GJT, Chapter III]).

(i) *A pair* $(p, x) \in X \times \mathscr{F}$ *determines a unique parametrized flat* f *such that* $\mathsf{f}(0) = p$ *and* $\mathsf{Z}(\mathsf{f}) = x$.

(ii) *A point* $(x, y) \in \mathscr{F}^{(2)}$ *determines a unique maximal flat* $\mathsf{f}_{xy}(\mathfrak{a})$ *such that* $\check{\mathsf{Z}}(\mathsf{f}_{xy}) = x$ *and* $\mathsf{Z}(\mathsf{f}_{xy}) = y$.

Given a subset of simple roots $\theta \in \Pi$, a pair of partial flags in general position $(x, y) \in \mathscr{F}_\theta^{(2)}$ and a point $p \in X$ then the *parallel set* $P(x, y)$ through x and y is

$$\bigcup_{\mathsf{f}} \mathsf{f}(\mathfrak{a}),$$

where the union is indexed on all parametrized flats f with $\check{\mathsf{Z}}_\theta(\mathsf{f}) = x$ and $\mathsf{Z}_\theta(\mathsf{f}) = y$.

8.9. **Representations and distances to parallel sets.** The purpose of this Subsection are the following proposition and corollary. Only the statement of Corollary 8.7 (and not its proof) will be needed in the sequel. If $\Lambda : G \to \mathrm{PSL}(V)$ is a finite dimensional rational irreducible representation, denote by $D\Lambda : \mathsf{T}X \to \mathsf{T}X_V$ the differential mapping of $\Lambda : X \to X_V$. If $p \in X$, the map $D_p\Lambda$ is nothing but $\phi_\Lambda|\mathfrak{p}^p : \mathfrak{p}^p \to \mathfrak{p}^{\Lambda p}$.

Proposition 8.6. *Let* $\Lambda : G \to \mathrm{PSL}(V)$ *be a finite dimensional rational irreducible representation. Then, there exists a constant* $\delta > 0$ *such that if* $(x, y) \in \mathscr{F}_{\theta_\Lambda}^{(2)}$ *and* $p \in P(x, y)$ *then*

$$\measuredangle(D_p\Lambda((\mathsf{T}_pP(x, y))^\perp), \mathsf{T}_{\Lambda p}P(\xi_\Lambda x, \xi_\Lambda^* y)) > \delta,$$

where $\measuredangle$ *denotes the angle on* $\mathsf{T}_{\Lambda p}X_V$.

Corollary 8.7. *Let* $\Lambda : G \to \mathrm{PSL}(V)$ *be a finite dimensional irreducible representation. Then there exists* $c > 0$ *such that if* $o \in X$ *and* $(x, y) \in \mathscr{F}_{\theta_\Lambda}^{(2)}$, *then*

$$\frac{1}{c} d_{X_V}(\Lambda o, P(\xi_\Lambda x, \xi_\Lambda^* y)) \leqslant d_X(o, P(x, y)) \leqslant c d_{X_V}(\Lambda o, P(\xi_\Lambda x, \xi_\Lambda^* y)).$$

Let us show how Proposition 8.6 implies the corollary.

Proof. Since X is non-positively curved and $P(x, y)$ is totally geodesic, the distance from o to $P(x, y)$ is attained at a unique point $p \in P(x, y)$. Moreover, the geodesic segment $\sigma : I \to X$, from p to o is orthogonal to $P(x, y)$ at p, i.e. $\dot\sigma(0) \in \mathsf{T}_pP(x, y)^\perp$. Similarly, the distance from Λo to $P(\xi_\Lambda x, \xi_\Lambda^* y)$ is also attained at a unique point q and the geodesic segment from q to Λo is perpendicular to $P(\xi_\Lambda x, \xi_\Lambda^* y)$.

Since ΛX is totally geodesic in X_V, we can estimate the angle at the vertex Λp of the geodesic triangle $\{\Lambda o, \Lambda p, q\}$. Indeed, Proposition 8.6 implies that this angle is bounded below by $\delta > 0$, independently of o and (x, y). Since the angle at the vertex q is $\pi/2$, standard trigonometry completes the proof. □

The remainder of the subsection is devoted to the proof of Proposition 8.6.

Proof. Let $L(x, y)$ be the stabilizer in G of $(x, y) \in \mathscr{F}_{\theta_\Lambda}^{(2)}$ and let $\mathfrak{l}$ be its Lie algebra. Moreover, let $L(\xi_\Lambda x, \xi_\Lambda^* y)$ be the stabilizer in $\mathrm{PSL}(V)$ of $(\xi_\Lambda x, \xi_\Lambda^* y)$ and $\mathfrak{l}' \subset \mathfrak{sl}(V)$ its Lie algebra.

If $\mathfrak{l}^\perp$ denotes the orthogonal of $\mathfrak{l}$ with respect to the Killing form κ, then we will show that

$$\phi_\Lambda(\mathfrak{l}^\perp) \cap \mathfrak{l}' = \{0\}. \tag{8.8}$$

This will imply the proposition since if $p \in P(x,y)$ then $\mathsf{T}_p P(x,y) = \mathfrak{p}^p \cap \mathfrak{l}$, its orthogonal in $\mathsf{T}_p X$ is hence $\mathfrak{p}^p \cap \mathfrak{l}^\perp$ and given $q \in P(x,y)$ there exists $g \in L(x,y)$ such that $gp = q$, thus the angle

$$\measuredangle(\mathfrak{p}^{\Lambda p} \cap \phi_\Lambda(\mathfrak{l}^\perp), \mathfrak{p}^{\Lambda p} \cap \mathfrak{l}') \geqslant \measuredangle(\phi_\Lambda(\mathfrak{p}^p \cap \mathfrak{l}^\perp), \mathfrak{p}^{\Lambda p} \cap \mathfrak{l}')$$

is independent of p in $P(x,y)$.

We will hence show that equation (8.8) holds. Note that, by homogeneity, we can assume that the stabilizer of x is P_{θ_Λ} and that the stabilizer of y is $\check{P}_{\theta_\Lambda}$. The parallel set $P(x,y)$ is hence the orbit $L_{\theta_\Lambda} \cdot o$, where L_{θ_Λ} is the Levi group $P_\theta \cap \check{P}_\theta$.

An explicit computation shows that for $\alpha, \beta \in \Sigma$, the eigenspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ (recall Subsection 7.1) are orthogonal with respect to the Killing form κ, whenever $\alpha \neq -\beta$. If we denote by $\mathfrak{g}_{\{\alpha,-\alpha\}} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ then the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{\{\alpha,-\alpha\}}$$

is orthogonal with respect to κ.

Note that the Cartan involution o sends $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$ and hence $\mathfrak{g}_{\{\alpha,-\alpha\}}$ is o-invariant. Since $-\kappa(\cdot, o(\cdot))$ is positive definite and $\kappa|\mathfrak{g}_\alpha = 0$ one concludes that $\mathfrak{p}^o \cap \mathfrak{g}_{\{\alpha,-\alpha\}} \neq \{0\}$. One finds then a *orthogonal decomposition* of the tangent space to X at o,

$$\mathsf{T}_o X = \mathfrak{p}^o = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{p}^o \cap \mathfrak{g}_{\{\alpha,-\alpha\}}. \tag{8.9}$$

The Lie algebra of L_{θ_Λ} is

$$\mathfrak{l}_{\theta_\Lambda} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \langle \Pi - \theta_\Lambda \rangle} \mathfrak{g}_{\{\alpha,-\alpha\}}$$

and hence the decomposition

$$\mathfrak{g} = \mathfrak{l}_{\theta_\Lambda} \oplus \bigoplus_{\alpha \in \Sigma^+ - \langle \Pi - \theta_\Lambda \rangle} \mathfrak{g}_{\{\alpha,-\alpha\}}$$

is orthogonal with respect to κ. This is to say

$$\mathfrak{l}_{\theta_\Lambda}^\perp = \bigoplus_{\alpha \in \Sigma^+ - \langle \Pi - \theta_\Lambda \rangle} \mathfrak{g}_{\{\alpha,-\alpha\}}. \tag{8.10}$$

The following lemma implies the proposition, denote by $\ell = \dim V_{\chi_\Lambda}$.

Lemma 8.8. *The subspace* $\phi_\Lambda(\bigoplus_{\alpha \in \Sigma^+ - \langle \Pi - \theta_\Lambda \rangle} \mathfrak{g}_{-\alpha})$ *has trivial intersection with* $\mathfrak{p}_{\{\alpha_\ell\}}$.

Proof. Recall that

$$\mathfrak{p}_{\{\alpha_p\}} = \mathfrak{a}_V \oplus \bigoplus_{\beta \in \Sigma_V^+} \mathfrak{sl}(V)_\beta \oplus \bigoplus_{\beta \in \langle \Pi_V - \{\alpha_p\} \rangle} \mathfrak{sl}(V)_{-\beta},$$

where $\mathfrak{a}_V$ is a maximal abelian subalgebra of $\mathfrak{p}^{\Lambda o} \subset \mathfrak{sl}(V)$ containing $\phi_\Lambda(\mathfrak{a})$, Σ_V is the set of roots of $\mathfrak{a}_V$ and Π_V is a set of simple roots.

Consider then $\alpha \in \Sigma^+ - \langle \Pi - \theta_\Lambda \rangle$ and $n \in \mathfrak{g}_{-\alpha}$ and recall from [Hum$_1$, Section 6.4] that $\phi_\Lambda(n)$ is nilpotent. Remark 8.1 implies that if $v \in V_{\chi_\Lambda}$ then $\phi_\Lambda(n)v \neq 0$, Hence,

$$\phi_\Lambda(n) \notin \mathfrak{a}_V \oplus \bigoplus_{\beta \in \Sigma_V^+} \mathfrak{sl}(V)_\beta \oplus \bigoplus_{\beta \in \langle \Pi_V - \{\alpha_p\} \rangle} \mathfrak{sl}(V)_{-\beta}.$$

Consider $\{\gamma_i\}_1^k \subset \Sigma^+ - \langle \Pi - \theta\rangle$ pairwise distinct and let $n_{\gamma_i} \in \mathfrak{g}_{-\gamma_i} - \{0\}$. Then $\phi_\Lambda(n_{\gamma_i})v$ is a non-zero eigenvector of $\mathfrak{a}$ of eigenvalue $\chi - \gamma_i$, hence

$$\phi_\Lambda(n_{\gamma_1} + \cdots + n_{\gamma_k})v = \phi_\Lambda(n_{\gamma_1})v + \cdots + \phi_\Lambda(n_{\gamma_k})v \neq 0.$$

This proves the lemma. □

As an example, consider the irreducible representation (unique up to conjugation) $\Lambda : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(3,\mathbb{R})$. Explicitely Λ is the action of $\mathrm{PSL}(2,\mathbb{R})$ on the symmetric power $\mathbf{S}^2(\mathbb{R}^2)$, this is a 3-dimensional space spanned by

$$\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_2\},$$

where $e_1 = (1,0)$ and $e_2 = (0,1)$.

The group $L(\langle e_1\rangle, \langle e_2\rangle)$ has Lie algebra $\mathfrak{l} = \mathfrak{a} = \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\rangle$, its orthogonal with respect to the Killing form of $\mathrm{PSL}(2,\mathbb{R})$ is

$$\mathfrak{a}^\perp = \langle \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\rangle.$$

The group $L(\langle e_1 \otimes e_1\rangle, \langle e_1 \otimes e_2, e_2 \otimes e_2\rangle)$ has Lie algebra

$$\mathfrak{l}' = \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\rangle.$$

The orthogonal to $\mathfrak{l}'$ with respect to the Killing form of $\mathrm{PSL}(3,\mathbb{R})$ is

$$\mathfrak{l}'^\perp = \langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\rangle.$$

An explicit computation shows that $\Lambda \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) = \begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$, hence

$$\phi_\Lambda \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which does *not* belong to $\mathfrak{l}'^\perp$ (this is what one would have hoped for). Nevertheless, it does not belong to $\mathfrak{l}'$ either, wich gives the definite angle of Proposition 8.6.

8.10. **Gromov product and representations.** Consider $\theta \subset \Pi$ and denote by

$$\mathfrak{a}_\theta = \bigcap_{\alpha \in \Pi - \theta} \ker \alpha$$

the Lie algebra of the center of the reductive group $L_\theta = P_\theta \cap \check{P}_\theta$. Recall from [Sa] that the *Gromov product*[23] based on $o \in X$ is the map

$$(\cdot|\cdot)_o : \mathscr{F}_\theta^{(2)} \to \mathfrak{a}_\theta$$

defined by the unique vector $(x|y)_o \in \mathfrak{a}_\theta$ such that

$$\omega_\alpha((x|y)_o) = -\log \sin \measuredangle_{\Lambda_\alpha o}(\xi_{\Lambda_\alpha} x, \xi^*_{\Lambda_\alpha} y)$$

for all $\alpha \in \theta$, where ω_α is the fundamental weight associated to the Tits representation Λ_α of α. Note that

$$\max_{\alpha\in\theta} \omega_\alpha((x|y)_o) = \max_{\alpha\in\theta} |\omega_\alpha((x|y)_o)| = -\log \min_{\alpha\in\theta} \sin \measuredangle_{\Lambda_\alpha o}(\xi_{\Lambda_\alpha} x, \xi^*_{\Lambda_\alpha} y). \tag{8.11}$$

Note also that, since $\{\omega_p|\mathfrak{a}_\theta\}_{\alpha_p\in\theta}$ is a basis of $\mathfrak{a}_\theta$, the right hand side of equation (8.11) is comparable to the norm $\|(x|y)_o\|_o$.

This vector-valued Gromov product keeps track of Gromov products on all irreducible representations of G, indeed one has the following consequence of equation (8.4).

[23] This is the negative of the defined in [Sa].

Remark 8.9. Let $\Lambda : G \to \mathrm{PSL}(V)$ be a finite dimensional rational irreducible representation, if $(x,y) \in \mathscr{F}^{(2)}_{\theta_\Lambda}$ then

$$(\xi_\Lambda x|\xi^*_\Lambda y)_{\Lambda o} = \chi_\Lambda((x|y)_o) = \sum_{\alpha\in\theta_\Lambda} \langle \chi_\Lambda, \alpha\rangle \omega_\alpha((x|y)_o),$$

where Λo is the homothety class of the inner product on V such that ΛK is orthogonal. Note that, by definition of θ_Λ (recall subsection8.3), the coefficients in the last equation are all strictly positive.

8.11. **Gromov product and distances to parallel sets.** The aim of this subsection is to prove the following.

Proposition 8.10. *Given $\theta \subset \Pi$ there exist $c > 1$ and $c' > 0$ only depending on G such that for all $(x,y) \in \mathscr{F}^{(2)}_\theta$ one has*

$$\frac{1}{c}\|(x|y)_o\|_o \leqslant d_X(o, P(x,y)) \leqslant c\|(x|y)_o\|_o + c'.$$

Proof. As for $\mathrm{PSL}(d,\mathbb{R})$, the first inequality follows easily. Let us show the second one. Let $\Lambda : G \to \mathrm{PSL}(V)$ be a Plücker representation of G associated to the set θ, (recall Subsection 8.6). Corollary 8.7 implies that there exists c_0 such that for all $o \in X$ and $(x,y) \in \mathscr{F}^{(2)}_{\theta_\Lambda}$ one has

$$d_X(o, P(x,y)) \leqslant c_0 d_{X_V}(\Lambda o, P(\xi_\Lambda x, \xi^*_\Lambda y)).$$

Note that $\xi_\Lambda x \in \mathbb{P}(V)$ and that $\xi^*_\Lambda y \in \mathbb{P}(V^*)$, Proposition 7.2 applied to the set $\{\alpha_1\} \subset \Pi_V$ implies that

$$d_{X_V}(\Lambda o, P(\xi_\Lambda x, \xi^*_\Lambda y)) \leqslant c(\xi_\Lambda x|\xi^*_\Lambda y)_{\Lambda o} + c'.$$

Finally, Remark 8.9 states that

$$(\xi_\Lambda x|\xi^*_\Lambda y)_{\Lambda o} = \sum_{\alpha\in\theta} \langle \chi_\Lambda, \alpha\rangle \omega_\alpha((x|y)_o),$$

where $\langle \chi_\Lambda, \alpha\rangle > 0$ for all $\alpha \in \theta$, thus, this last quantity is comparable to $\|(x|y)_o\|$. This completes the proof. □

8.12. **A Morse Lemma due to [KLP$_2$].** Let us begin with some definitions. For $p, q \in X$, $\theta \subset \Pi$ and $(x,y) \in \mathscr{F}^{(2)}_\theta$ then

- the *Weyl cone* $V(p,x)$ through p and x is $\bigcup_{\mathsf{f}} \mathsf{f}(\mathfrak{a}^+)$, where the union is indexed on all parametrized flats f with $\mathsf{f}(0) = p$ and $\mathsf{Z}_\theta(\mathsf{f}) = x$.
- the *Weyl cone* $V(p,q)$ through p and q (in that order) is $\bigcup_{\mathsf{f}} \mathsf{f}(\mathfrak{a}^+)$, where the union is indexed on all parametrized flats f with $\mathsf{f}(0) = p$ and such that $q \in \mathsf{f}(\mathfrak{a}^+)$.
- Finally, the *diamond* between p and q is the subset

$$\diamondsuit(p,q) = V(p,q) \cap V(q,p).$$

If $\mathscr{C} \subset \mathfrak{a}^+$ is a closed cone, consider the subset

$$\theta_{\mathscr{C}} = \{\alpha \in \Pi : \ker\alpha \cap \mathscr{C} = \{0\}\}.$$

Following [KLP$_2$] we will say that a quasi-geodesic segment $\{p_n\}_0^N \subset X$ is *$\mathscr{C}$-regular* if for all $n < m$ one has

$$\underline{a}(p_n, p_m) \in \mathscr{C}.$$

One has the following version of the Morse Lemma.

Theorem 8.11 ([KLP$_2$, Theorem 1.3])**.** *Let μ, c be positive numbers and $\mathscr{C} \subset \mathfrak{a}^+$ a closed cone, then there exists $\ell \in \mathbb{N}$ and $C > 0$ such that if $\{p_n\}_{n\in I}$ is a $\mathscr{C}$-regular (μ, c)-quasi-geodesic segment, then*

- *If I is finite and $|I| \geqslant \ell$ then $\{p_n\}$ is at distance at most C from the diamond $\diamondsuit(p_{\min I}, p_{\max I})$.*
- *If $I = \mathbb{N}$ then there exists $x \in \theta_{\mathscr{C}}$ such that $\{p_n\}$ is contained in a C-neighborhood from the Weyl cone $V(p_{\min I}, x)$.*
- *If $I = \mathbb{Z}$ then there exist two partial flags in general position $(x, y) \in \mathscr{F}^{(2)}_{\theta_{\mathscr{C}}}$ such that $\{p_n\}$ is contained in a C-neighborhood from the union $V(z, x) \cup V(z, y)$ for some $z \in P(x, y)$ at uniform distance from $\{p_n\}$.*

Proof. Let $\Lambda : G \to \mathrm{PSL}(V)$ be a Plücker representation associated to $\theta_{\mathscr{C}}$. If $\{p_n\}$ is a $\mathscr{C}$-regular quasi geodesic then $\{\Lambda p_n\}$ is a $\Lambda\mathscr{C}$-regular quasi-geodesic. Moreover, equation (8.3) implies that the cone $\Lambda\mathscr{C}$ does not intersect the wall $\ker \alpha_1$.

The proof now follows the same lines as the proof of Theorem 7.3, provided Proposition 8.10. □

Appendix A. Auxiliary technical results

In this appendix we collect a number of lemmas that are used elsewhere in the paper. These lemmas are either quantitative linear-algebraic facts or properties of dominated splittings. Certainly many of these results are known, but they do not necessarily appear in the literature in the exact form or setting that we need; therefore we include proofs for the reader's convenience.

A.1. **Angles, Grassmannians.** The *angle* between non-zero vectors v, $w \in \mathbb{R}^d$ is defined as the unique number $\measuredangle(v, w)$ in $[0, \pi]$ whose cosine is $\langle v, w\rangle/(\|v\|\,\|w\|)$. If P, $Q \subset \mathbb{R}^d$ are nonzero subspaces then we define their *angle* as:

$$\measuredangle(P, Q) := \min_{v\in P^\times} \min_{w\in Q^\times} \measuredangle(v, w)\,, \tag{A.1}$$

where $P^\times := P \smallsetminus \{0\}$. We also write $\measuredangle(v, Q)$ instead of $\measuredangle(\mathbb{R}v, Q)$, if v is a nonzero vector.

Given integers $1 \leqslant p < d$, we let $\mathscr{G}_p(\mathbb{R}^d)$ be the *Grassmaniann* formed by the p-dimensional subspaces of $\mathbb{R}^d$. Let us metrize $\mathscr{G}_p(\mathbb{R}^d)$ in a convenient way. If $p = 1$, the sine of the angle defines a distance. (We leave for the reader to check the triangle inequality.) In general, we may regard each element of $\mathscr{G}_p(\mathbb{R}^d)$ as a compact subset of $\mathscr{G}_1(\mathbb{R}^d)$, and then use the Hausdorff metric induced by the distance on $\mathscr{G}_1(\mathbb{R}^d)$. In other words, we define, for P, $Q \in \mathscr{G}_p(\mathbb{R}^d)$:

$$d(P, Q) := \max\left\{ \max_{v\in P^\times} \min_{w\in Q^\times} \sin\measuredangle(v, w),\ \max_{w\in Q^\times} \min_{v\in P^\times} \sin\measuredangle(v, w) \right\}. \tag{A.2}$$

Note the following trivial bound:

$$d(P, Q) \geqslant \sin\measuredangle(P, Q)\,. \tag{A.3}$$

Actually the two quantities between curly brackets in the formula (A.2) coincide, that is,

$$d(P, Q) = \max_{w\in Q^\times} \min_{v\in P^\times} \sin\measuredangle(v, w) = \max_{w\in Q^\times} \sin\measuredangle(w, P)\,. \tag{A.4}$$

This follows from the existence of a isometry of $\mathbb{R}^d$ that interchanges P and Q (see [Wo, Theorem 2]).

As the reader can easily check, relation (A.4) can be rewritten in the following ways:

$$d(P,Q) = \max_{w\in Q^\times} \max_{u\in (P^\perp)^\times} \cos\measuredangle(u,w)\,, \tag{A.5}$$

$$= \max_{w\in Q^\times} \min_{v\in P} \frac{\|v-w\|}{\|w\|}\,, \tag{A.6}$$

where $P^\perp$ denotes the orthogonal complement of P. As a consequence of (A.1) and (A.5), we have:

$$d(P,Q) = \cos\measuredangle(P^\perp,Q)\,. \tag{A.7}$$

Another expression for the distance $d(P,Q)$ is given below in (A.17). Let us mention that the distance can be also characterized as the sine of the largest *canonical angle* [Ste, § 4.5] between P and Q.

A.2. **More about singular values.** If $A\colon E\to F$ is a linear map between two inner product vector spaces then we define its *norm* and its *minimorm* respectively by:

$$\|A\| := \max_{v\in E^\times} \frac{\|Av\|}{\|v\|}\,, \qquad \mathbf{m}(A) := \min_{v\in E^\times} \frac{\|Av\|}{\|v\|}\,.$$

The following properties hold whenever they make sense:

$$\|AB\| \leqslant \|A\|\,\|B\|\,, \qquad \mathbf{m}(AB) \geqslant \mathbf{m}(A)\,\mathbf{m}(B)\,, \qquad \mathbf{m}(A) = \|A^{-1}\|^{-1}\,.$$

In terms of singular values, we have $\|A\| = \sigma_1(A)$ and $\mathbf{m}(A) = \mathbf{m}(A)$.

For convenience, let us assume that A is a linear map from $\mathbb{R}^d$ to $\mathbb{R}^d$. We have the following useful "minimax" characterization of the singular values:

$$\sigma_p(A) = \max_{P\in\mathscr{G}_p(\mathbb{R}^d)} \mathbf{m}(A|_P)\,, \tag{A.8}$$

$$\sigma_{p+1}(A) = \min_{Q\in\mathscr{G}_{d-p}(\mathbb{R}^d)} \|A|_Q\|\,; \tag{A.9}$$

see [Ste, Corol. 4.30]. Moreover, if A has a gap of index p (that is, $\sigma_p(A) > \sigma_{p+1}(A)$) then the maximum and the minimum above are respectively attained at the spaces $P = S_{d-p}(A)^\perp$ and $Q = S_{d-p}(A)$ (defined at § 2.2).

A.3. **Linear-algebraic lemmas.** In this subsection we collect a number of estimates that will be useful later. Fix integers $1 \leqslant p \leqslant d$.

Lemma A.1. *Let A, $B \in \mathrm{GL}(d,\mathbb{R})$. Then:*

$$\max\left\{\mathbf{m}(A)\sigma_p(B), \sigma_p(A)\mathbf{m}(B)\right\} \leqslant \sigma_p(AB) \leqslant \min\left\{\|A\|\sigma_p(B), \sigma_p(A)\|B\|\right\}.$$

Proof. The two inequalities follow from (A.8) and (A.9), respectively. □

Lemma A.2. *Let $A \in \mathrm{GL}(d,\mathbb{R})$ have a gap of index p. Then, for all unit vectors v, $w\in\mathbb{R}^d$, we have:*

$$\|Av\| \geqslant \sigma_p(A)\sin\measuredangle(v, S_{d-p}(A))\,, \tag{A.10}$$

$$\|A^{-1}w\| \geqslant \sigma_{p+1}(A)^{-1}\sin\measuredangle(w, U_p(A))\,. \tag{A.11}$$

Also, for all $Q\in\mathscr{G}_{d-p}(\mathbb{R}^d)$ and $P\in\mathscr{G}_p(\mathbb{R}^d)$, we have:

$$\|A|_Q\| \geqslant \sigma_p(A)\,d(Q, S_{d-p}(A))\,, \tag{A.12}$$

$$\|A^{-1}|_P\| \geqslant \sigma_{p+1}(A)^{-1}\,d(P, U_p(A))\,. \tag{A.13}$$

Proof. Given a unit vector $v \in \mathbb{R}^d$, decompose it as $v = s + u$ where $s \in S_{d-p}(A)$ and $u \in S_{d-p}(A)^\perp$; Then $\|u\| = \sin \measuredangle(v, S_{d-p}(A))$. Moroever, since As and Au are orthogonal, we have $\|Av\| \geqslant \|Au\| \geqslant \sigma_p(A)\|u\|$. This proves (A.10), from which (A.12) follows. The proofs of (A.11) and (A.13) are analogous. ☐

The next three lemmas should be thought of as follows: if A has a strong gap of index p and $\|B^{\pm 1}\|$ are not too large, then then $U_p(AB)$ is close to $U_p(A)$, and $U_p(BA)$ is close to $B(U_p(A))$; moreover, $A(P)$ is close to $U_p(A)$ for any $P \in \mathscr{G}_p(\mathbb{R}^d)$ whose angle with $S_{d-p}(A)$ is not too small.[24]

Lemma A.3. *Let A, $B \in \mathrm{GL}(d, \mathbb{R})$. If A and AB have a gap of index p then:*

$$d(U_p(A), U_p(AB)) \leqslant \|B\| \, \|B^{-1}\| \, \frac{\sigma_{p+1}(A)}{\sigma_p(A)}.$$

Proof. We have:

$$\begin{aligned} d(U_p(AB), U_p(A)) &\leqslant \sigma_{p+1}(A) \, \|A^{-1}|_{U_p(AB)}\| && \text{(by (A.13))} \\ &\leqslant \sigma_{p+1}(A) \, \|B\| \, \|B^{-1}A^{-1}|_{U_p(AB)}\| \\ &= \sigma_{p+1}(A) \, \|B\| \, \sigma_p(AB)^{-1} \\ &\leqslant \sigma_{p+1}(A) \, \|B\| \, \|B^{-1}\| \, \sigma_p(A)^{-1} && \text{(by Lemma A.1).} \end{aligned}$$ ☐

Lemma A.4. *Let A, $B \in \mathrm{GL}(d, \mathbb{R})$. If A and BA have a gap of index p then:*

$$d\big(B(U_p(A)), U_p(BA)\big) \leqslant \|B\| \, \|B^{-1}\| \, \frac{\sigma_{p+1}(A)}{\sigma_p(A)} \, .$$

Proof. We have:

$$\begin{aligned} d(B(U_p(A)), U_p(BA)) &\leqslant \sigma_{p+1}(BA) \, \|A^{-1}B^{-1}|_{B(U_p(A))}\| && \text{(by (A.13))} \\ &\leqslant \|B\| \, \sigma_{p+1}(A) \, \|A^{-1}|_{U_p(A)}\| \, \|B^{-1}\| && \text{(by Lemma A.1)} \\ &= \|B\| \, \sigma_{p+1}(A) \, \sigma_p(A)^{-1} \, \|B^{-1}\| \, . \end{aligned}$$ ☐

Lemma A.5. *Let $A \in \mathrm{GL}(d, \mathbb{R})$ have a gap of index p. Then, for all $P \in \mathscr{G}_p(\mathbb{R}^d)$ we have:*

$$d(A(P), U_p(A)) \leqslant \frac{\sigma_{p+1}(A)}{\sigma_p(A)} \, \frac{1}{\sin \measuredangle(P, S_{d-p}(A))} \, .$$

Proof. By (A.13),

$$d(A(P), U_p(A)) \leqslant \sigma_{p+1}(A) \, \|A^{-1}|_{A(P)}\| = \frac{\sigma_{p+1}(A)}{\mathbf{m}(A|_P)} \, .$$

By (A.10), $\mathbf{m}(A|_P) \geqslant \sigma_p(A) \sin \measuredangle(P, S_{d-p}(A))$, so the lemma is proved. ☐

The next lemma implies that the singular values of a product of matrices are approximately the products of the singular values, provided that certain angles are not too small:

---

[24] We remark that angle estimates of this flavor appear in the usual proofs os Oseledets theorem; see e.g. [Sim, p. 141–142]. We also note that [GGKW, Lemma 5.8] contains a generalization of Lemmas A.3 and A.5 to more general Lie groups.

Lemma A.6. *Let A, $B \in \mathrm{GL}(d,\mathbb{R})$. Suppose that A and AB have a gap of index p. Let $\alpha := \measuredangle\big(U_p(B), S_{d-p}(A)\big)$. Then:*

$$\sigma_p(AB) \geqslant (\sin\alpha)\,\sigma_p(A)\,\sigma_p(B)\,,$$
$$\sigma_{p+1}(AB) \leqslant (\sin\alpha)^{-1}\,\sigma_{p+1}(A)\,\sigma_{p+1}(B)\,.$$

Proof. By (A.8) we have:

$$\sigma_p(AB) \geqslant \mathbf{m}(AB|_{B^{-1}(U_p(B))}) \geqslant \mathbf{m}(A|_{U_p(B)})\,\mathbf{m}(B|_{B^{-1}(U_p(B))}) = \mathbf{m}(A|_{U_p(B)})\,\sigma_p(B)\,.$$

On the other hand, inequality (A.10) yields $\mathbf{m}(A|_{U_p(B)}) \geqslant (\sin\alpha)\,\sigma_p(A)$, and so we obtain the first inequality in the lemma. The second inequality follows from the first one, using the fact that $\sigma_{p+1}(A) = 1/\sigma_{d-p}(A)$. ☐

A.4. **A sketch of the proof of Theorem 2.2.** Note that due to the uniqueness property of dominated splittings (Proposition 2.1), it is sufficient to prove the theorem in the case $\mathbb{T} = \mathbb{Z}$, which is done in [BG]. For the convenience of the reader, let us include here a summary of this proof, using some lemmas that we have already proved in § A.3.

The "only if" part of the theorem is not difficult, so let us consider the "if" part. So assume that the gap between the p-th and the $(p+1)$-th singular values of ψ_x^n increases uniformly exponentially with time n. Fix any $x \in X$, and consider the sequence of spaces $U_p\big(\psi^n_{\phi^{-n}(x)}\big)$ in $\mathscr{G}_p(E_x)$. Using Lemma A.3, we see that the distance between consecutive elements of the sequence decreases exponentially fast, and in particular the sequence has a limit E^{cu}_x. Uniform control on the speed of convergence yields that E^{cu} is a continuous subbundle of E. Lemma A.4 implies that this subbundle is invariant.[25] Analogously we obtain the subbundle E^{cs}.

To conclude the proof, we need to show that the bundles E^{cu} and E^{cs} are transversal, and that the resulting splitting is indeed dominated. Here, [BG] uses an ergodic-theoretical argument: The gap between singular values implies that for any Lyapunov regular point x, the difference $\lambda_p(x) - \lambda_{p+1}(x)$ between the p-th and the $(p+1)$-th Lyapunov exponents is bigger than some constant $2\varepsilon > 0$. Moreover, Oseledets theorem implies that E^{cu}_x and E^{cs}_x are sums of Oseledets spaces, corresponding to Lyapunov exponents $\geqslant \lambda_p(x)$ and $\leqslant \lambda_{p+1}(x)$, respectively. Bearing these facts in mind, assume for a contradiction that E^{cu} does not dominate E^{cs}. Then there exist points $x_i \in X$, unit vectors $v_i \in E^{\mathrm{cs}}_{x_i}$ and $w_i \in E^{\mathrm{cu}}_{x_i}$, and times $n_i \to \infty$ such that

$$\frac{\|\psi^{t_i}(v_i)\|}{\|\psi^{t_i}(w_i)\|} > e^{-\varepsilon t_i}\,.$$

It follows from a Krylov–Bogoliubov argument[26] (making use of the continuity of the subbundles; see [BG] for details) that there exists a Lyapunov regular point x such that $\lambda_p(x) - \lambda_{p+1}(x) \leqslant \varepsilon$. This contradiction establishes domination.

A.5. **Expansion on the Grassmannian.** The aim of this subsection is to prove the following lemma, which is used in Subsection 3.6.

[25] This step is missing in [BG].

[26] i.e. convergence of measures supported in long segments of orbits in the weak star topology to an invariant measure.

Lemma A.7. *Given $\alpha > 0$, there exists $b > 0$ with the following properties. Let $A \in \mathrm{GL}(d, \mathbb{R})$. Suppose that $P \in \mathscr{G}_p(\mathbb{R}^d)$ and $Q \in \mathscr{G}_{d-p}(\mathbb{R}^d)$ satisfy*

$$\min\{\measuredangle(P,Q), \measuredangle(AP,AQ)\} \geqslant \alpha\,. \tag{A.14}$$

Then there exists $\delta > 0$ such that if P_1, $P_2 \in \mathscr{G}_p(\mathbb{R}^d)$ are δ-close to P then

$$d(A(P_1), A(P_2)) \geqslant b\, \frac{\mathbf{m}(A|_Q)}{\|A|_P\|}\, d(P_1, P_2)\,. \tag{A.15}$$

Before proving this lemma, we need still another characterization of the distance (A.4) in the Grassmannian. Suppose that P, $Q \in \mathscr{G}_p(\mathbb{R}^d)$ satisfy $d(P,Q) < 1$. Then $Q \cap P^\perp = \{0\}$, and so there exists a unique linear map

$$L_{Q,P} \colon P \to P^\perp \quad \text{such that} \quad Q = \big\{v + L_{Q,P}(v) : \ v \in P\big\}\,. \tag{A.16}$$

We have:

$$\|L_{Q,P}\| = \frac{d(P,Q)}{\sqrt{1 - d(P,Q)^2}}\,; \tag{A.17}$$

indeed, letting $\theta \in [0, \pi/2)$ be such that $\sin\theta = d(P,Q)$, using (A.6) we conclude that $\|L_{Q,P}\| = \tan\theta$.

Lemma A.8. *Let P, Q_1, $Q_2 \in \mathscr{G}_p(\mathbb{R}^d)$. For each $i = 1$, 2, assume that $d(Q_i, P) < 1/\sqrt{2}$, and let $L_i = L_{Q_i,P}$. Then:*

$$d(Q_1, Q_2) \leqslant \|L_1 - L_2\| \leqslant 4d(Q_1, Q_2)\,.$$

Proof. Consider an arbitrary $u_1 \in Q_1^\times$, and write it as $u_1 = v + L_1 v$ for some $v \in P^\times$. By orthogonality, $\|u_1\| \geqslant \|v\|$. Letting $u_2 := v + L_2 v$, we have:

$$\frac{\|u_1 - u_2\|}{\|u_1\|} \leqslant \frac{\|L_1 v - L_2 v\|}{\|v\|} \leqslant \|L_1 - L_2\|.$$

Using (A.6) we conclude that $d(Q_1, Q_2) \leqslant \|L_1 - L_2\|$, which is the first announced inequality.

For each $i = 1$, 2, we have $\|L_i\| \leqslant 1$, as a consequence of (A.17) and the hypothesis $d(Q_i, P) < 1/\sqrt{2}$. Consider an arbitrary unit vector $v_1 \in P$. Let $w_1 := v_1 + L_1 v_1$, so $\|w_1\| \leqslant 2$. By (A.6), there exists $w_2 \in Q_2^\times$ such that

$$\frac{\|w_2 - w_1\|}{\|w_1\|} \leqslant d(Q_1, Q_2)\,.$$

Let $v_2 \in P$ be such that $w_2 = v_2 + L_2 v_2$. By orthogonality,

$$\|w_1 - w_2\| \geqslant \max\{\|v_1 - v_2\|, \|L_1 v_1 - L_2 v_2\|\}\,,$$

so

$$\begin{aligned}
\|L_1 v_1 - L_2 v_1\| &\leqslant \|L_1 v_1 - L_2 v_2\| + \|L_2 v_2 - L_2 v_1\| \\
&\leqslant \|L_1 v_1 - L_2 v_2\| + \|v_2 - v_1\| \\
&\leqslant 2\|w_1 - w_2\| \\
&\leqslant 2\|w_1\| d(Q_1, Q_2) \\
&\leqslant 4d(Q_1, Q_2)\,.
\end{aligned}$$

Taking sup over unit vectors $v_1 \in Q_1$ we obtain $\|L_1 - L_2\| \leqslant 4d(Q_1, Q_2)$. $\square$

Proof of Lemma A.7. First consider the case $\alpha = \pi/2$, so (A.14) means that $Q = P^\perp$ and $AQ = (AP)^\perp$. Assume that P_1, P_2 are δ-close to P, for some small $\delta > 0$ to be chosen later. Recall notation (A.16) and, for each $i = 1, 2$, consider the linear maps $L_i := L_{Q_i,P}$ and $M_i := L_{AQ_i,AP}$, which are well-defined since $\delta < 1$ guarantees that $P_i \cap Q = \{0\}$. These maps are related by $L_i = (A^{-1}|_{AQ}) \circ M_i \circ (A|_P)$. As a consequence,

$$\|L_1 - L_2\| = \left\|(A^{-1}|_{AQ}) \circ (M_1 - M_2) \circ (A|_P)\right\| \leqslant \frac{\|A|_P\|}{\mathbf{m}(A|_Q)} \|M_1 - M_2\| .$$

On the other hand, by (A.17) we have $\|L_i\| \leqslant d(P_i, P) < \delta$, and therefore $\|M_i\| \leqslant \|A^{-1}\| \, \|L_i\| \, \|A\| < 1$, provided δ is chosen sufficiently small (depending on A). Using (A.17) again we guarantee that $d(AP_i, AP) < 1/\sqrt{2}$. This allows us to apply Lemma A.8 and conclude that:

$$d(AP_1, AP_2) \geqslant \frac{1}{4}\|M_1 - M_2\| \geqslant \frac{1}{4} \frac{\mathbf{m}(A|_Q)}{\|A|_P\|} \|L_1 - L_2\| \geqslant \frac{1}{4} \frac{\mathbf{m}(A|_Q)}{\|A|_P\|} d(P_1, P_2) .$$

So the lemma holds with $b = 1/4$ when $\alpha = \pi/2$.

The general case can be reduced to the previous one by changes of inner products, whose effect on all involved quantities can be bounded by a factor depending only on α. □

A.6. **Additional lemmas about dominated splittings.**

A.6.1. *Determinants.* If A is a linear map between two inner product vector spaces of same dimension, then $|\det A|$ is defined as the absolute value of the determinant of the matrix of A with respect to a pair of orthonormal bases.

Lemma A.9. *Let E be a vector bundle of fiber dimension d over a compact space X. Consider a linear flow $\{\psi^t : E \to E\}_{t \in \mathbb{T}}$ with a dominated splitting $E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$ where the dominating bundle E^{cu} has dimension p. Then there exist constants $\varepsilon > 0$, $\hat{c} > 0$ such that for all $x \in X$ and $t \geqslant 0$ we have*

$$\left|\det \psi^t|_{E_x^{\mathrm{cu}}}\right|^{1/p} \geqslant \hat{c} e^{\varepsilon t} \left|\det \psi^t|_{E_x}\right|^{1/d} .$$

Proof. Since $E_x^{\mathrm{cu}} \oplus E_x^{\mathrm{cs}} = E_x$, we have:

$$\left|\det \psi^t|_{E_x}\right| \leqslant \left|\det \psi^t|_{E_x^{\mathrm{cu}}}\right| \, \left|\det \psi^t|_{E_x^{\mathrm{cs}}}\right|$$

It follows that:

$$\frac{\left|\det \psi^t|_{E_x^{\mathrm{cu}}}\right|^{\frac{1}{p}}}{\left|\det \psi^t|_{E_x}\right|^{\frac{1}{d}}} \geqslant \frac{\left|\det \psi^t|_{E_x^{\mathrm{cu}}}\right|^{\frac{1}{p}-\frac{1}{d}}}{\left|\det \psi^t|_{E_x^{\mathrm{cs}}}\right|^{\frac{1}{d}}} \geqslant \left(\frac{\mathbf{m}(\psi^t|_{E_x^{\mathrm{cu}}})}{\|\psi^t|_{E_x^{\mathrm{cs}}}\|}\right)^{1-\frac{p}{d}} \geqslant \hat{c} e^{\varepsilon t} ,$$

for appropriate positive constants ε, $\hat{c}$. □

A.6.2. *More about domination of sequences of matrices.* Recall from § 2.3 the definition of the sets $\mathcal{D}(K, p, \mu, c, I)$.

Lemma A.10. *Given $K > 1$, $\mu > 0$, and $c > 0$, there exists $\ell \in \mathbb{N}$ and $\tilde{c} > c$ such that if $I \subset \mathbb{Z}$ is an interval*

and $(A_i)_{i \in I}$ is an element of $\mathcal{D}(K, p, \mu, c, I)$, then the following properties hold:

(i) *If $n' < n < k$ all belong to I and $k - n \geqslant \ell$ then:*

$$d\big(U_p(A_{k-1} \cdots A_{n+1} A_n), \, U_p(A_{k-1} \cdots A_{n'+1} A_{n'})\big) < \tilde{c} e^{-\mu(k-n)} .$$

(ii) *If $k < m < m'$ all belong to I and $k - j \geqslant \ell$ then:*

$$d\big(S_{d-p}(A_{m-1}\cdots A_{k+1}A_k),\, S_{d-p}(A_{m'-1}\cdots A_{k+1}A_k)\big) < \tilde{c}e^{-\mu(m-k)}\,.$$

Proof. Given K, μ, and c, let $\ell \in \mathbb{N}$ be such that $ce^{-\mu\ell} < 1$. Fix $(A_n)_{n\in I} \in \mathcal{D}(K,p,\mu,c,I)$ and $k \in I$. If $n \in I$ satisfies $n \leqslant k - \ell$ then the space $P_n := U_p(A_{k-1}\cdots A_{n+1}A_n)$ is well-defined. If $n-1 \in I$ then it follows from Lemma A.3 that

$$d(P_n, P_{n-1}) \leqslant K^2 c e^{-\mu(k-n)}\,.$$

Therefore, if $n' < n$ belongs to I then

$$d(P_n, P_{n'}) \leqslant d(P_n, P_{n-1}) + d(P_{n-1}, P_{n-2}) + \cdots + d(P_{n'+1}, P_{n'}) \leqslant \tilde{c}e^{-\mu(k-n)}\,.$$

where $\tilde{c} := K^2 c/(1-e^{-\mu})$. This proves part (i) of the lemma. An analogous argument yields part (ii). □

The following "extension lemma" is useful to deduce one-sided results from two-sided ones. Let $\mathbb{N}$ denote the set of nonnegative integers.

Lemma A.11. *Given $K > 1$, $\mu > 0$, and $c > 0$, there exists $c' > c$ such that every one-sided sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(K,p,\mu,c,\mathbb{N})$ can be extended to a two-sided sequence $(A_n)_{n\in\mathbb{Z}}$ in $\mathcal{D}(\mu,c',K,p,\mathbb{Z})$.*

Proof. Fix $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(K,p,\mu,c,\mathbb{N})$. Let $Q_n := S_{d-p}\big(A_{n-1}A_{n-2}\cdots A_0\big)$, which is defined for sufficiently large n. By Lemma A.10, the spaces Q_n converge to some $Q \in \mathscr{G}_p(\mathbb{R}^d)$; moreover, we can find some n_0 depending only on the constants K, μ, c (and not on the sequence of matrices) with the property that for all $n \geqslant n_0$ we have $d(Q_n, Q) < 1/\sqrt{2}$ or equivalently, by (A.7), $\measuredangle(Q_n, Q^\perp) < \pi/4$.

Let B be a matrix satisfying the following conditions:

$$\|B^{\pm 1}\| \leqslant K\,, \quad \frac{\sigma_{p+1}}{\sigma_p}(B) < e^{-\mu}\,, \quad B(Q) = Q = S_{d-p}(B)\,, \quad B(Q^\perp) = Q^\perp = U_p(B)\,.$$

In particular, for all $m \geqslant 0$ we have $\frac{\sigma_{p+1}}{\sigma_p}(B^m) < e^{-\mu m}$. Then, for all $n \geqslant n_0$, Lemma A.6 yields:

$$\frac{\sigma_{p+1}}{\sigma_p}(A_{n-1}\cdots A_0 B^m) \leqslant 2ce^{-\mu(n+m)}\,.$$

This implies that if we set $A_n = B$ for all $n < 0$ then the extended sequence $(A_n)_{n\in\mathbb{Z}}$ belongs to $\mathcal{D}(K,p,\mu,c',\mathbb{Z})$ for some suitable c' depending only on μ, c, and n_0. □

Combining the previous lemma with Proposition 2.4, we obtain:

Corollary A.12. *Given $K > 1$, $\mu > 0$, and $c > 0$, there exist $\tilde{c} > 0$, $\tilde{\mu} > 0$, and $\alpha > 0$ with the following properties. For every one-sided sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(K,p,\mu,c,\mathbb{N})$, the following limit exists:*

$$Q = \lim_{n\to\infty} S_{d-p}(A_{n-1}\cdots A_0)\,,$$

and moreover there exists $\tilde{Q} \in \mathscr{G}_p(\mathbb{R}^d)$ such that for every $n > 0$ we have:

$$\measuredangle\big(A_{n-1}\cdots A_0(\tilde{Q}), A_{n-1}\cdots A_0(Q)\big) \geqslant \alpha\,,$$

$$\frac{\|A_{n-1}\cdots A_0|_Q\|}{\mathbf{m}(A_{n-1}\cdots A_0|_{\tilde{Q}})} < \tilde{c}e^{-\tilde{\mu}n}\,.$$

A.6.3. *Hölder continuity of the bundles.*

Theorem A.13. *Let $\phi^t : X \to X$ a Lipschitz flow on a compact metric space X with $t \in \mathbb{T}$ and E a vector bundle over X. Consider ψ^t a be a β-Hölder linear flows over ϕ^t which admits a dominated splitting with constants $C, \lambda > 0$. Then, if $\alpha < \beta$ and $\lambda K^\alpha < 1$ where K is a Lipschitz constant for ϕ^1 and ϕ^{-1}, then the maps $x \mapsto E^{\mathrm{cu}}(x)$ and $x \mapsto E^{\mathrm{cs}}(x)$ are α-Hölder.*

Sketch of the proof. Choose Lipschitz approximations $\hat{E}^{\mathrm{cs}}$ and $\hat{E}^{\mathrm{cu}}$ of E^{cs} and E^{cu} respectively. One can define then the bundle $\mathcal{E}$ over X corresponding to the linear maps from $\hat{E}^{\mathrm{cu}}(x)$ to $\hat{E}^{\mathrm{cs}}(x)$, that is $\mathcal{E}(x) = \mathrm{Hom}(\hat{E}^{\mathrm{cu}}(x), \hat{E}^{\mathrm{cs}}(x))$. Let $\mathcal{T} \subset \mathcal{E}$ given as $\mathcal{T} = \{(x, L) \in \mathcal{E} \ : \ \|L\| \leqslant 1\}$. We consider the standard graph transform $H : \mathcal{T} \to \mathcal{E}$ given as:

$$H(x, L) = (\phi^1(x), H_x(L))$$

defined so that if $(w, v) \in \hat{E}^{\mathrm{cs}}(x) \oplus \hat{E}^{\mathrm{cu}}(x)$ is in the graph of L (i.e. $w = Lv$) then one has that $(\psi^1 w, \psi^1 v)$ is in the graph of $H_x(L)$. It follows that the map H is β-Hölder and a standard computation shows that it leaves invariant the set $\mathcal{T}$.

Given two sections σ_0 and σ_1 of $\mathcal{E}$ such that $\sigma_i(x) \in \mathcal{T}$ for every $x \in X$, one shows that the α-Hölder distance of $H \circ \sigma_0$ and $H \circ \sigma_1$ is uniformly contracted[27] if $\alpha < \beta$ and $\lambda k^{-\alpha} < 1$ where $k = \min_{x \neq y}\{\frac{d(\phi^1(x), \phi^1(y))}{d(x,y)}\}$. Indeed, the graphs are getting contracted at a rate similar to λ while points cannot approach faster than k which gives that the α-Hölder distance contracts[28].

As a consequence, one obtains that there is a unique H-invariant section σ of this bundle which, moreover, it is α-Hölder.

It is direct to show that this section corresponds to the bundle E^{cu}. A symmetric argument shows that E^{cs} is α-Hölder, proving the result. □

Corollary A.14. *If $u \mapsto \psi_u^t$ is a β-Hölder family of linear flows over ϕ^t a Lipschitz flow and ψ_0^t admits a dominated splitting, then there exists a neighborhood D of 0 such that the maps $(u, x) \mapsto E_u^{\mathrm{cs}}(x)$ and $(u, x) \mapsto E_u^{\mathrm{cu}}(x)$ are α-Hölder.*

Proof. Fix α as in the previous theorem. There exists a neighborhood D of 0 for which $\lambda K^\alpha < 1$ (here λ denotes the strength of the domination for ψ_u^t with $u \in D$).

Now one applies the previous theorem to the linear flow $\hat{\psi}^t$ over $\phi^t \times \mathrm{id} : X \times D \to X \times D$. Note that the Lipschitz constants of $\phi^1 \times \mathrm{id}$ and $\phi^{-1} \times \mathrm{id}$ are the same as the ones of ϕ^1 and ϕ^{-1}. The result follows. □

References

[ABY] A. Avila, J. Bochi, J.-C. Yoccoz. Uniformly hyperbolic finite-valued SL(2, ℝ)-cocycles. *Comment. Math. Helv.* 85 (2010), 813–884. (Cited on pages 1, 2, 21, and 22.)

[Be] Y. Benoist. Sous-groupes discrets des groupes de Lie. European Summer School in Group Theory, Luminy, 1997. (Cited on page 10.)

[BM] A. Blumenthal, I.D. Morris. Characterization of dominated splittings for operator cocycles acting on Banach spaces. Preprint arXiv:1512.07602 (Cited on page 4.)

[27] To show that this metric is contracted, one needs to assume that the constant C appearing in the dominated splitting is equal to 1. Otherwise, one can argue for an iterate and the same will hold.

[28] The need for $\alpha < \beta$ is evident since the section cannot be more regular than the cocycle. In the computation this appears because an error term of the form $d(x, y)^{\alpha-\beta}$ appears which will then be negligeable as $d(x, y) \to 0$ and gives the desired statement. See [CP, Section 4.4] for a similar computation.

[BG] J. Bochi, N. Gourmelon. Some characterizations of domination. *Math. Z.* 263 (2009), 221–231. (Cited on pages 1, 2, 3, 4, 5, 21, 22, and 50.)

[Bog] O. Bogopolski. Infinite commensurable hyperbolic groups are bi-Lipschitz equivalent. *Algebra and Logic* 36 (1997), no. 3, 155–163. (Cited on page 23.)

[BDV] C. Bonatti, L.J. Díaz, M. Viana. *Dynamics beyond uniform hyperbolicity: a global geometric and probabilistic perspective.* Encyclopaedia of Mathematical Sciences, 102. Mathematical Physics, III. Springer-Verlag, Berlin, 2005. (Cited on page 2.)

[BDP] C. Bonatti, L.J. Díaz, E. Pujals. A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources. *Ann. of Math.* 158 (2003), no. 2, 355–418. (Cited on pages 4 and 20.)

[Bow] B.H. Bowditch. A topological characterisation of hyperbolic groups. *J. Amer. Math. Soc.* 11 (1998), 643–667. (Cited on page 8.)

[BCS] M. Bridgeman, R. Canary, A. Sambarino. An introduction to pressure metrics on higher Teichmüller spaces. Preprint arXiv:1507.03894. (Cited on page 2.)

[BCLS] M. Bridgeman, R. Canary, F. Labourie, A. Sambarino. The pressure metric for Anosov representations. *Geom. Funct. Anal.* 25 (2015), 1089–1179. (Cited on pages 2, 17, 25, and 26.)

[BH] M.R. Bridson, A. Haefliger. *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999. (Cited on pages 7, 16, and 23.)

[Ca] J.W. Cannon. The combinatorial structure of cocompact discrete hyperbolic groups. *Geom. Dedicata* 16 (1984), no. 2, 123–148. (Cited on page 22.)

[Ch] C. Champetier. Petite simplification dans les groupes hyperboliques. *Ann. Fac. Sci. Toulouse Math.* (6) 3 (1994), no. 2, 161–221. (Cited on pages 15 and 16.)

[CK] F. Colonius, W. Kliemann. *The dynamics of control.* Birkhäuser Boston, Inc., Boston, MA, 2000. (Cited on pages 3 and 4.)

[CDP] M. Coornaert, T. Delzant, A. Papadopoulos. *Géométrie et théorie des groupes: Les groupes hyperboliques de Gromov.* Lecture Notes in Mathematics, 1441. Springer-Verlag, Berlin, 1990. (Cited on pages 7, 8, 20, 23, and 26.)

[CP] S. Crovisier, R. Potrie. *Introduction to partially hyperbolic dynamics.* School on Dynamical Systems, ICTP, Trieste, 2015. (Cited on pages 3, 4, 26, and 54.)

[DT] B. Deroin, N. Tholozan. Dominating surface group representations by Fuchsian ones. Preprint arXiv:1311.2919 (Cited on page 7.)

[Ebe] P. Eberlein. *Geometry of nonpositevly curved manifolds.* Chicago Lectures in Mathematics. The University of Chicago Press, 1996. (Cited on pages 28 and 38.)

[EHY] H. Eliasson, H. Hofer and J.-C. Yoccoz (organizers). Dynamische Systeme. Abstracts from the workshop held July 19–25, 2015. *Oberwolfach Rep.* (2015). (Cited on page 3.)

[GH] E. Ghys, P. de la Harpe (editors). *Sur les groupes hyperboliques d'après Mikhael Gromov.* Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988. Progress in Mathematics, 83. Birkhäuser Boston, Inc., Boston, MA, 1990. (Cited on pages 7 and 8.)

[Go] A. Gogolev. Diffeomorphisms Hölder conjugate to Anosov diffeomorphisms. *Ergodic Theory DynamṠystems* 30 (2010), no. 2, 441–456. (Cited on page 20.)

[Gr] M. Gromov. "Hyperbolic groups," in *Essays in Group Theory*, MSRI Publ. 8 (1987), 75–263. (Cited on pages 7 and 15.)

[GGKW] F. Guéritaud, O. Guichard, F. Kassel, A. Wienhard. Anosov representations and proper actions. To appear in *Geometry and Topology.* Preprint arXiv:1502.03811. (Cited on pages 1, 2, 5, 6, 10, 15, 19, 20, and 49.)

[GKW] F. Guéritaud, F. Kassel, M. Wolff. Compact anti-de Sitter 3-manifolds and folded hyperbolic structures on surfaces. *Pacific J. Math.* 275 (2015), no. 2, 325–359. (Cited on page 7.)

[GW] O. Guichard, A. Wienhard. Anosov representations: domains of discontinuity and applications. *Invent. Math.* 190 (2012), 357–438. (Cited on pages 2, 15, 17, 24, 41, and 42.)

[GJT] Y. Guivarc'h, L. Ji, J.C. Taylor. *Compactifications of symmetric spaces*, Progress in Mathematics, 156 (1998), Birkhäuser, Boston, MA. (Cited on pages 28 and 43.)

[Hel] S. Helgason. *Differential geometry and symmetric spaces.* Academic Press, New York–London, 1962. (Cited on pages 28 and 38.)

[HPS] M. Hirsch, C. Pugh, M. Shub. *Invariant manifolds.* Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977. (Cited on page 25.)

[Hit] N.J. Hitchin. Lie groups and Teichmüller space. *Topology* 31 (1992), no. 3, 449–473. (Cited on page 2.)

[Hö] L. Hörmander. *An introduction to complex analysis in several variables.* 3rd ed. North-Holland Publishing Co., Amsterdam, 1990. (Cited on page 27.)

[HJ] R.A. Horn, C.R. Johnson. *Matrix analysis.* 2nd ed. Cambridge University Press, Cambridge, 2013. (Cited on page 4.)

[Hub] J. Hubbard. *Teichmüller theory.* Matrix Editions, Ithaca, NY, 2006. (Cited on page 26.)

[Hum_1] J. Humphreys. *Introduction to Lie algebras and representation theory.* 2nd printing, revised. Graduate Texts in Mathematics, 9. Springer-Verlag, New York–Berlin, 1978. (Cited on pages 38 and 44.)

[Hum_2] J. Humphreys. *Linear algebraic groups.* Graduate Texts in Mathematics, 21. Springer-Verlag, New York–Heidelberg, 1975. (Cited on page 41.)

[Kap] M. Kapovich. Representations of polygons of finite groups. *Geom. Topol.* 9 (2005), 1915–1951. (Cited on page 8.)

[KLP_1] M. Kapovich, B. Leeb, J. Porti. Morse actions of discrete groups on symmetric space. Preprint arXiv:1403.7671 (Cited on pages 2, 15, and 21.)

[KLP_2] M. Kapovich, B. Leeb, J. Porti. A Morse Lemma for quasigeodesics in symmetric spaces and euclidean buildings. Preprint arXiv:1411.4176 (Cited on pages 1, 2, 3, 7, 15, 27, 28, 36, 38, 46, and 47.)

[KLP_3] M. Kapovich, B. Leeb, J. Porti. Some recent results on Anosov representations. Preprint arXiv:1511.08940 (Cited on pages 2 and 15.)

[Lab] F. Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.* 165 (2006), 51–114. (Cited on pages 2, 15, 24, and 41.)

[Lang] S. Lang. *Fundamentals of Differential Geometry.* Graduate Texts in Mathematics,191. Springer, New Haven, 1999. (Cited on pages 28, 31, and 38.)

[Led] F. Ledrappier. Structure au bord des variétés à courbure négative. *Sémin. Théor. Spectr. Géom.*, 13, 97–122. Univ. Grenoble I, Saint-Martin-d'Hères, 1995. (Cited on page 16.)

[Len] D. Lenz. Existence of non-uniform cocycles on uniquely ergodic systems. *Ann. Inst. H. Poincaré Probab. Statist.* 40 (2004), no. 2, 197–206. (Cited on pages 4 and 27.)

[LM] D. Lind, R. Marcus. *An introduction to symbolic dynamics and coding.* Cambridge University Press, Cambridge, 1995. (Cited on page 21.)

[Mat] F. Mathéus. Flot géodésique et groupes hyperboliques d'après M. Gromov. *Sémin. Théor. Spectr. Géom.*, 9, 67–87. Univ. Grenoble I, Saint-Martin-d'Hères, 1991. (Cited on pages 15 and 16.)

[Mi] I. Mineyev. Flows and joins of metric spaces. *Geom. Topol.* 9 (2005), 403–482. (Cited on pages 15 and 16.)

[Pa] K.J. Palmer. Exponential separation, exponential dichotomy and spectral theory for linear systems of ordinary differential equations. *J. Differ. Eq.* 46 (1982), no. 3, 324–345. (Cited on page 3.)

[PS] R. Potrie, A. Sambarino. Eigenvalues and entropy of a Hitchin representation. Preprint arXiv:1411.5405. (Cited on page 25.)

[Ru] D. Ruelle. Analyticity properties of the characteristic exponents of random matrix products. *Adv. Math.* 32 (1979), no. 1, 68–80. (Cited on page 26.)

[Sa] A. Sambarino. The orbital counting problem for hyperconvex representations. *Ann. Inst. Fourier* 65 (2015), no. 4, 1755–1797. (Cited on page 45.)

[Sam] M. Sambarino. A (short) survey on dominated splittings. *Mathematical Congress of the Americas.* J.A. de la Peña, J.A. López-Mimbela, M. Nakamura, J. Petean, eds. Contemporary Mathematics 656 (2016), 149–183. (Cited on page 3.)

[Sim] B. Simon. *A Comprehensive Course in Analysis, Part 3: Harmonic analysis.* AMS, Providence, RI, 2015. (Cited on page 49.)

[Sma] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.* 73 (1967), 747–817. (Cited on page 2.)

[Ste] G.W. Stewart. *Matrix algorithms. Vol. I: Basic decompositions.* SIAM, Philadelphia, PA, 1998. (Cited on page 48.)

[Ti] J. TITS. Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque. *J. reine angew. Math.* 247 (1971), 196–220. (Cited on pages 40 and 41.)

[Wie] A. WIENHARD. Representations and Geometric Structures. Preprint arXiv:1602.03480. (Cited on page 2.)

[Wo] Y.-C. WONG. Differential geometry of Grassmann manifolds. *Proc. Nat. Acad. Sci. U.S.A.* 57 (1967), 589–594. (Cited on page 47.)

[Yo] J.-C. YOCCOZ. Some questions and remarks about $\mathrm{SL}(2,\mathbb{R})$ cocycles. *Modern dynamical systems and applications*, 447–458, Cambridge Univ. Press, Cambridge, 2004. (Cited on page 4.)

Jairo Bochi
Facultad de Matemáticas
Pontificia Universidad Católica de Chile
Av. Vicuña Mackenna 4860 Santiago Chile
`jairo.bochi@mat.uc.cl`

Rafael Potrie
CMAT Facultad de Ciencias
Universidad de la República
Iguá 4225 Montevideo Uruguay
`rpotrie@cmat.edu.uy`

Andrés Sambarino
IMJ-PRG (CNRS UMR 7586)
Université Pierre et Marie Curie (Paris VI)
4 place Jussieu 75005 Paris France
`andres.sambarino@imj-prg.fr`