# ROTATION INTERVALS AND ENTROPY ON ATTRACTING ANNULAR CONTINUA

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Abstract. We show that if $f$ is an annular homeomorphism admitting an attractor which is an irreducible annular continua with two different rotation numbers, then the entropy of $f$ is positive. Further, the entropy is shown to be associated to a $C^0$-robust *rotational horseshoe*. On the other hand, we construct examples of annular homeomorphisms with such attractors so that the rotation interval is uniformly large but the entropy approaches zero as much as desired.

The developed techniques allow us to obtain similar results dealing with *Birkhoff attractors*.

## 1. Introduction

The rotation set is an invariant for many dynamical systems which has shown to contain the essential information of the dynamics when the underlying space has low dimension, in particular in dimensions one and two.

Poincare's theory for orientation preserving homeomorphisms on the circle is the paradigmatic case: this invariant turns out to be a number which provides a complete description of the underlying dynamics (see for example [KH, Chapter 11]). Still in dimension one, a natural generalization was considered for degree one endomorphisms of the circle, where the rotation set turns out to be a (possibily trivial) interval, and contains again crucial information of the dynamics, as the existence of periodic orbits with certain relative displacements, among other interesting properties (see [ALMM] and references therein).

In dimension two, the dynamics of certain surface homeomorphisms homotopic to the identity is usually described by means of this topological invariant. In particular, the annulus $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$ and the two-torus $\mathbb{T}^2 = \mathbb{R}^2/_{\mathbb{Z}^2}$ are contexts in which, it can be said, a theory has been build supported on this invariant. In these contexts, for a dynamics $f$ given by a homeomorphisms in the homotopy class of the identity, the rotation set is associated to a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$, as given by the set

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$$\rho(F) = \left\{ \lim_k \frac{\pi_1(F^{n_k}(x_k) - x_k)}{n_k} \mid x_k \in \mathbb{R}^2,\ n_k \nearrow +\infty \right\} \text{ , and}$$

$$\rho(F) = \left\{ \lim_k \frac{F^{n_k}(x_k) - x_k}{n_k} \mid x_k \in \mathbb{R}^2,\ n_k \nearrow +\infty \right\},$$

respectively, where $\pi_1$ is the projection over the first coordinate.

The shape of this set is given by a possibly degenerated interval in the annular case, and, as consequence of a foundational result by Misiurewicz and Zieman [MZ], by a possibly degenerated compact convex set in the toral case. From these facts, there exists a vast list of interesting results, where assuming possible geometries for the rotation set, descriptions of the underlying dynamics are obtained. We refer the interested reader to [Be, Pass] and references therein for a more complete account on this theory.

When the dimension of the rotation set equals the dimension of the space where the considered dynamics acts, it has been shown the existence of relations between the *geometry and arithmetics* of the rotation set, and the topological entropy of the system. For instance, for degree one maps on the circle, the topological entropy is bounded below by an explicit (and optimal) function of the extremal points of the rotation set as shown in [ALMM]. In the toral case, the quantity considered for such a lower bound is less explicit and as far as the authors are aware, not optimal. See [LlM, Kw, LCT].

In the annulus $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$, a large rotation set is not necessarily associated with large entropy. An integrable twist map preserving a foliation of essential circles has zero entropy but may have rotation sets of arbitrarily large size. One can look at the rotation set restricted to smaller invariant regions of the annulus and hope to draw better conclusions. This motivates the definition of the rotation set on a compact $f$-invariant set $K \subset \mathbb{A}$, and a lift $F$ of $f$, as

$$\rho_K(F) = \left\{ \lim_k \frac{\pi_1(F^{n_k}(x_k) - x_k)}{n_k} \mid \pi(x_k) \in K,\ n_k \nearrow +\infty \right\}$$

where $\pi : \mathbb{R}^2 \to \mathbb{A}$ is the canonical covering.

In this new context, one could try to find a relation between the entropy and the size of the rotation set. For this, the class of invariant sets to look at are the essential annular continuum; that is, $K \subset \mathbb{A}$ so that $\mathbb{A} \setminus K$ is given by exactly two unbounded connected components. These sets, are a natural object in surface dynamics, as they occur as boundaries of open invariant sets, and have been object of several works in the field. The topology can be very simple, as for the circle or the closed annulus itself, and very complex, as it is the case of the pseudo-circle.

We mentioned above that for the case where $K \subset \mathbb{A}$ is a closed essential annulus, there is no relation between the length of the rotation interval and the topological entropy. As a next step, one can look at those annular continua containing no essential annulus. For this class of continua, there exists an interesting example by Walker [Wal], in which an invariant annular continua having empty interior $K$, is constructed having zero entropy, and arbitrary large rotation set. Nevertheless, this continuum contains a essential circle inside, that is, $K$ is not irreducible. Irreducible annular continua, often called *circloids*, with non-trivial rotation sets are known as interesting examples, and it is possible to construct them so that they are robust in the $C^0$ topology (see [Bo, LeC]). Further, these kind of dynamics occur as global attractors of dissipative twist maps, given by the so called *Birkhoff attractors* [LeC].

In this article we show the following complementary facts. For an orientation preserving homeomorphisms $f$ and an attracting invariant circloid $\mathcal{C}$:

- We show in Theorem A that if $\mathcal{C}$ has a non-trivial rotation set, then its entropy is positive.
- In Theorem B, we show that there is no relation between the entropy and the length of the rotation set.

The first result answers positively a folklore problem (see for example [Ko]), assuming the circloid is a global attractor. The second result is quite surprising, as the known examples and their constructions show the same relation between the rotation intervals and the topological entropy as occurs for degree one maps in the circle, and this relation could be expected to be true in general for this context. Next, we give precise statement of the results.

**Theorem A.** *Assume that $f \in Homeo_+(\mathbb{A})$ has a global attractor $\mathcal{C}$ given by a circloid with empty interior, for which $\rho_{\mathcal{C}}(F)$ is a non-trivial interval. Then, $f$ has positive entropy. Moreover, there exist $\varepsilon > 0$ and a $C^0$-neighborhood $\mathcal{N}$ of $f$, so that $h_{top}(g) > \varepsilon$ for all $g \in \mathcal{N}$.*

*Remark* 1.1. The empty *interior hypothesis* is only used at a step which make use of a realization theorem by Barge and Guillette ([BG]) which has been recently improved by Koropecki ([Ko]) to the case with interior. This last result does not ensure that these realizations can be considered at the boundary of the circloid, property which would allow us to suppress the mentioned hypothesis.
In a work in progress by the first author and Koropecki ([KP]), the last result is being improved, so that the realizations can be considered at the boundary of he circloid. This, allows to remove the *empty interior* hypothesis in Theorem B.

*Remark* 1.2. Actually, we prove that the topological entropy of $g \in \mathcal{N}$ is given by a $C^0$-*rotational horseshoe* associated to some uniform power

$g^n$. This gives the existence of a semi-conjugacy between the full-shift on $\{0,1\}^{\mathbb{N}}$ and some $g^n$-invariant set $\Lambda$, where the symbols are related with the displacement of the orbits in the lift. See Section 3.6 for a detailed description.

The complementary result is given by the following.

**Theorem B.** *Given $\varepsilon > 0$ there exists a smooth diffeomorphisms $f \in$ Homeo$_+(\mathbb{A})$ admitting a global attractor $\mathcal{C}$ which is a circloid, such that $\rho_f(\mathcal{C}) \supset [0,1]$ while $h_{top}(f) < \varepsilon$.*

We have the following results as consequences of the developed techniques in the proofs of Theorems A and B.

Recall that a diffeomorphism $f : M \to M$ is said to be *dissipative*, whenever there exists $\varepsilon > 0$ such that $\det(Df_x) < 1 - \varepsilon$ for every $x \in M$. Further, recall that a diffeomprohism $f : \mathbb{A} \to \mathbb{A}$ is said to be a *twist map* if for some lift $F$ of $f$ there is $\varepsilon > 0$ so that $DF_x((0,1)) = (a(x), b(x))$ with $\varepsilon < a(x) < \frac{1}{\varepsilon}$. Given a dissipative twist maps of the annulus which maps an essential closed annulus in its interior one can associate a global attractor, the intersection of the iterates of the annulus, which given by an annular continua with empty interior. This annular continua, contains a unique circloid which is the so called Birkhoff attractor (see [LeC]). The arguments used in Theorem A, allow us to obtain the following result.

**Theorem C.** *Assume that $f : \mathbb{A} \to \mathbb{A}$ is an orientation preserving diffeomorphisms, which is dissipative, verifies the twist condition, and $f(\mathcal{A}) \subset \mathcal{A}$ for some essential annulus $\mathcal{A} \subset \mathbb{A}$. Further, assume that $\rho_{\mathcal{C}}(F)$ is a non-trivial interval, were $\mathcal{C}$ is the Birkhoff attractor of $f$. Then, $h_{top}(f\ |_{\mathcal{C}}) > 0$. Moreover, there exist $\varepsilon > 0$ and a $C^0$-neighborhood $\mathcal{N}$ of $f$, so that $h_{top}(g) > \varepsilon$ for all $g \in \mathcal{U}$.*

The proof for this case is given in Section 3.5, where the scope of the techniques used in Theorem A are discussed. Further, Remark 1.2 also applies to this situation.

We finish showing how to addapt the proof of Theorem B to show that the topological entropy and the length of rotation intervals are again not related for Birkhoff attractors.

**Theorem D.** *For every $\varepsilon > 0$ there exists a dissipative twist smooth diffeomorphisms $f : \mathbb{A} \to \mathbb{A}$ having an Birkhoff attractor $\mathcal{C}$ with $\rho_{\mathcal{C}}(F) \supset [0,1]$ and $h_{top}(f|_{\mathcal{C}}) < \varepsilon$.*

*Remark* 1.3. There is a certain analogy between Birkhoff attractors and *regions of instability* of conservative annulus homeomorphisms (see for example [FLC]). Recall that an *instability region $R$* for an area-preserving annular homeomorphisms, is an invariant closed essential annulus bounded by the essential curves $C_-$ and $C_+$, having a point with $\alpha$-limit in $C_-$ and $\omega$-limit in $C_+$, and a point with $\omega$-limit in $C_-$

and $\alpha$-limit in $C_+$. In a recent article P. Le Calvez and F. Tal [LCT] (see also [FH]) have shown that whenever an instability region has a non-trivial interval as rotation set, then the map has positive entropy. In the process of proving Theorem B and D we must construct an instablity region (of a smooth twist map) with rotation set containing $[0, 1]$ and arbitrarily small entropy, showing that in this context again, there is no relation between the size of the rotation interval and the topological entropy of the map.

1.1. **The techniques.** In order to construct the examples of Theorem B, the idea is to work with $C^1$-perturbations of a twist-map, which are based on the $C^1$-connecting lemma for pseudo orbits in the conservative setting, due to M.C. Arnaud, C. Bonatti and S. Crovisier ([ABC]). The applications of this result in this case is not completely straight-forward, as it is a result of generic nature, and we need to take care of some non-generic properties of our examples. However, by an inspection of the proof of this result in [Cr], one can state a suitable version in order to obtain our desired perturbations. We remark that this kind of perturbations was already considered in [Gi]. Using these perturbations one can construct a smooth diffeomorphism of the closed annulus which is conservative and for which points in each of the boundary components are homoclinically related (and have different rotation numbers). A further perturbation allows to destroy the annulus and force the appeareance of an attracting circloid which still has the same rotation set. As the derivative of the original map had small growth, the same holds for the perturbations which ensures the small entropy.

For proving Theorem A we propose the following way: create forward-invariant continua through periodic points, which acts as stable manifolds of a hyperbolic basic piece. For suitable periodic points in the circloid, these continua will intersect both boundary components of an annular neighborhood of the circloid. Working with two periodic points having different rotation vectors, we find a *rectangle* whose iteration in the lift by some big power of the map, will meets two copies of its self in a *Markovian* way. This serves a $C^0$-*horseshoe* which gives us the positive entropy stated in Theorem A, and turns to be $C^0$-*stable*.

Theorem B shows that the usual arguments dealing with Nielsen-Thurston theory as used for instance in [LlM] and [Kw] do not work for this case. On the other hand, recently Le Calvez and Tal [LCT] developed a *forcing* technique based in Le Calvez's foliation by Brower lines ([LeC$_2$]), which could provide an alternative proof of the positive entropy in Theorem A.

Let us end this introduction by mentioning that Crovisier, Kocsard, Koropecki and Pujals have announced progress in the study of a particular family of diffeomorphisms of the annulus which they call *strongly* *dissipative*. In this class, they are able, among other things, to prove positive entropy if there are two rotation vectors and the maximal invariant set is transitive. We notice that even if our proof does not give lower bounds on the entropy in all generality (and it cannot give one because of Theorem B), it is possible that for some families such a lower bound exists. In particular, we emphasize that our method does give a lower bound after some configuration is attained (see Lemma 3.1).

1.2. **Organization of the paper.** The structure of the article is the following. We start with some preliminaries in the Section 2. From those, subsections 2.1 and 2.2 are used in the proof of Theorem B while subsections 2.3 and 2.4 is used for the proof of Theorem A.

Theorem A and B have independent proofs and can be read in any order. Theorem A and Theorem C are proved in section 3, whereas Theorem B and D are proved in section 4.

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## 2. General Preliminaries

We introduce in this section some preliminary well known results which will be used later, some results hold in higher dimensions too but we will always restrict to the surface case. The reader can safely skip this section and come back when results are referenced to.

2.1. **A remark on continuity of entropy in the $C^1$-topology.** For a surface map $f : M \to M$ there is a bound on the topological entropy given by

$$h_{top}(f) \leq 2 \log \ \sup_{x\in M} \|Df_x\| \ = 2 \log \|Df\| \ .$$

See for example [KH, Corollary 3.2.10]. Since $h_{\text{top}}(f) \leq \frac{1}{n} h_{\text{top}}(f^n)$, we have

$$h_{\text{top}}(f) \leq \frac{2}{n} \log \|Df^n\| \text{ for all } n \in \mathbb{N} \ .$$

We deduce the following:

**Proposition 2.1.** *Let $f : M \to M$ be a $C^1$-surface map such that $\lim_{n\to\infty} \frac{2}{n} \log \|Df^n\| = 0$. Then, for every $\varepsilon > 0$ there exists a $C^1$-neighborhood $\mathcal{N}$ of $f$ such that if $g \in \mathcal{U}$ one has that $h_{top}(g) < \varepsilon$.*

*Proof.* Fix $\varepsilon > 0$ and choose $n > 0$ such that $\frac{2}{n} \log \|Df^n\| < \varepsilon$. Choose a $C^1$-neighborhood $\mathcal{N}$ of $f$ so that for every $g \in \mathcal{N}$ one has $\frac{2}{n} \log \|Dg^n\| < \varepsilon$. By the estimate above, it follows that for every $g \in \mathcal{N}$ one has that $h_{top}(g) < \varepsilon$. □

2.2. **Connecting lemma for pseudo-orbits.** In this section we state a $C^1$-perturbation lemma for pseudo-orbits in the conservative setting in the spirit of the well known pseudo-orbit connecting lemma ([BC, ABC]).

Let $M$ be a surface, $\nu$ an area form in $M$ and let $\mathrm{Diff}^1_\nu(M)$ be the space of $C^1$ area preserving diffeomorphisms, with the $C^1$ topology. We recall that given $\varepsilon$, a finite sequence $(z_k)_{k=0}^n$ is a *$\varepsilon$-pseudo-orbit* (or *$\varepsilon$-chain*) from $p \in M$ to $q \in M$ when $z_0 = p,\ z_n = q$ and

$$d(f(z_k), z_{k+1}) < \varepsilon,\ \text{for all } k = 0, \dots, n-1\ .$$

Consider a compact set $K \subset M$. For $x,\ y \in M$ we denote $x \dashv_K y$ if for every $\varepsilon > 0$ there exists a $\varepsilon$ pseudo-orbit $(z_k)_{k=0}^m$ with $z_0 = x$, $z_n = y$ and

$$f(z_k),\ z_{k+1} \in K \text{ whenever } f(z_k) \neq z_{k+1}\ .$$

Denote by $\mathrm{Diff}^1_{\nu,per}(M)$ the set of those $f \in \mathrm{Diff}^1_\nu(M)$ for which the set of periodic points of period $k$ is finite, for all $k \in \mathbb{N}$. Recall that the *support* of a perturbation $g$ of $f$ is the set of points $x \in M$ where $g(x) \neq f(x)$.

**Theorem 2.2** (A version of the $C^1$-connecting lemma for pseudo-orbits [Cr])**.** *Let $M$ be a compact surface possibly with boundary and $f \in \mathit{Diff}^1_{\nu,per}(M)$. Given $\mathcal{N} \subset \mathit{Diff}^1_\nu$ a neighborhood of $f$, there exists $N = N(f, \mathcal{N})$ such that:*

- *if $K$ is a compact set disjoint from the boundary,*
- *$U$ is an arbitrary small neighborhood of $K \cup \dots \cup f^{N-1}(K)$,*
- *and $p,\ q \in M$ with $p \dashv_K q$,*

*then, there exist a perturbation $g \in \mathcal{N}$ of $f$ supported in $U$ and $n > 0$ such that $g^n(p) = q$.*

This result follows with the same proof of Theorem III.1 presented in [Cr] via [Cr, Theorem III.4] where the choice of $N$ appears. The difference is that in [Cr] the statement requires the complete pseudo-orbit to be contained in $K$ while here we demand only the jumps to be contained there. By inspection of the proofs in [Cr] one can see that one only performs perturbations when the pseudo-orbit has jumps and so our statement also holds with only minor modifications.

*Remark* 2.3. The diffeomorphism $g$ can be considered to be as smooth as $f$ since it is obtained by composing a finite number of elementary perturbations with small support, all of which are smooth (though their $C^r$-size with $r > 1$ might be huge).

2.3. **Some properties of separating continua.** We first recall some basic facts about continua and separation properties in surfaces. We refer the reader to [BG] and references therein for more information.

After this, we will show a property of irreducible annular continua that will be useful in the proof of Theorem A.

Throughout this article we consider the annulus $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$ and $\pi : \mathbb{R}^2 \to \mathbb{A}$ the usual covering map. Further, we will fix a two-point compactification of $\mathbb{A}$ given by the sphere $S^2$ and two different points $+\infty, -\infty \in S^2$.

Recall that a continuum is a compact connected metric space. We say a continuum $E \subset \mathbb{A}$ is *essential*, whenever there are two unbounded connected components in $\mathbb{A} \setminus E$. This connected components are denoted in general by $\mathcal{U}^+$ and $\mathcal{U}^-$ where the first one accumulates in $+\infty$ and the second one in $-\infty$, when considered in $S^2$. Notice that there could be also several bounded connected components in $E^c$. Non essential continuum in $\mathbb{A}$ are called *inessential*, and can be characterized as those continua contained in some topological disk in $\mathbb{A}$.

A annular continuum $K \subset \mathbb{A}$ is an essential continuum so that $K^c$ contains no bounded connected components. Finally, an irreducible *annular continua* or *circloid* $\mathcal{C}$, is an annular continuum which does not contain properly any other annular continua. As it is well know, the topology of these continua can very simple as the one of the circle, or extremely complicated as the case of the pseudo-circle. It could also be the case where $\mathcal{C}$ is a circloid with non-empty interior.

The are two big interesting classes for these continua: *decomposable* and *indecomposable*. A circloid $\mathcal{C}$ is decomposable, whenever there exists a compact connected set in the plane $\hat{\mathcal{C}}$ so that $\pi(\hat{\mathcal{C}}) = \mathcal{C}$, otherwise it is said that $\mathcal{C}$ is indecomposable. In this article we deal with the last class of circloids as decomposable circloid does not support two rotation vectors for a given dynamics ([BG], see also [JP]). When a circloid has empty interior, it is called *cofrontier*, as it coincides with the boundary of $\mathcal{U}^+$ and of $\mathcal{U}^-$. In general, for a circloid $\mathcal{C}$ it holds $\partial\mathcal{C} = \partial\mathcal{U}^+ \cap \partial\mathcal{U}^-$, and every annular continua having this property turns out to be a cricloid (see [Ja]).

Given a point $x$ in some continuum $\mathcal{X}$, the *composent* $C_x$ of $x$ is given by the union of every proper sub-continua of $\mathcal{X}$ containing $x$. If we have an indecomposable circloid $\mathcal{C} \subset \mathbb{A}$, the composent $C_x$ is given by a union of inessential continua properly contained in $\mathcal{C}$, as $\mathcal{C}$ can not contain properly an essential continuum. This means that whenever $y$ is a point in $C_x$, then there exists an inessential continuum $C_{x,y}$ containing both points.

We are interested in studying inessential continua intersecting an indecomposable circloid $\mathcal{C}$, which does no meet one of the unbounded components in the complement of the circloid. Fix a circloid $\mathcal{C}$ and let $K \not\subset \mathcal{C}$ be an inessential continuum in $\mathbb{A}$, so that $K \cap \mathcal{U}^- = \emptyset$. Everything we show for this situation also holds for the complementary case where $K \cap \mathcal{U}^+ = \emptyset$

In general for a continuum $C \subset \mathbb{A}$ we say that an injective curve $\gamma : [0, +\infty) \to \mathbb{A}$ lands at $z \in C$ from $+\infty$ if $\gamma(t) \in C^c$ for all $t \neq 0$, $\gamma(0) = z$, and $\lim_{t\to+\infty} \gamma(t) = +\infty$ when viewed in $S^2$. When $C$ is an essential continua, the points $z$ for which it is possible to consider such a curve, are called *accesible points*, and it is a well know fact, that they are a dense set in $C \cap \partial\mathcal{U}^+$. Thus in our situation we can consider a curve $\gamma$ as before, so that

- $\gamma \cap K = \emptyset$[1],
- $\gamma$ lands at $z \in \mathcal{C}$.

Let $\hat{A}$ as to be a connected component of $\pi^{-1}(\mathcal{U}^+ \setminus \gamma)$. Our main goal is to show the following property which is important to prove Theorem A.

**Proposition 2.4.** *It holds that $\pi^{-1}(K) \cap \hat{A}$ is bounded.*

Consider $\tilde{\mathcal{C}} = \pi^{-1}(\mathcal{C})$, $\tilde{\mathcal{U}}^+ = \pi^{-1}(\mathcal{U})$ and $\tilde{\mathcal{U}}^- = \pi^{-1}(\mathcal{U}^-)$. Fix a lift $\hat{K}$ of $K$ which intersects $\hat{A}$. In order to prove Proposition 2.4, it is enough to show that only finitely many horizontal integer translations of $\hat{A}$ meets $\hat{K}$.

For any lift $\hat{z}$ of $z$ contained in $\tilde{\mathcal{C}}$, let $\hat{C}_{\hat{z}}$ be the connected component of $\pi^{-1}(C_z)$ containing $\hat{z}$. As the circloid $\mathcal{C}$ is irreducible, every continuum contained in $C_z$ has a unique lift in $\hat{C}_{\hat{z}}$, which is again a continuum. This implies that $\hat{C}_{\hat{z}}$ can be recovered as the union of every continuum in $\tilde{C}$ containing $\hat{z}$. We prove the following lemma.

**Lemma 2.5.** *Fix $\hat{z} \in \pi^{-1}(z)$. If $\hat{K}$ intersects $\hat{C}_{\hat{z}} + k$ and $\hat{C}_{\hat{z}} + k'$ then $|k - k'| \leq 1$.*

*Proof.* Assume otherwise. Without loss of generality we can assume that $k' > k$.

Due to the fact that $\mathcal{C}$ is irreducible, we can consider continua $\Lambda_k \subset \hat{C}_{\hat{z}} + k$ containing $\hat{z} + k$ and intersecting $\hat{K}$ and $\Lambda_{k'} \subset \hat{C}_{\hat{z}} + k'$ containing $\hat{z} + k'$ and intersecting $\hat{K}$ .

Let $\hat{\gamma}$ the lift of $\gamma$ containing $\hat{z}$. We have that $\Lambda_k \cap (\hat{\gamma} + k) = \{\hat{z} + k\}$ and $\Lambda_k \cap (\hat{\gamma} + j) = \emptyset$ for every $j \in \mathbb{Z} \setminus \{k\}$, and the symmetric conditions hold for $\Lambda_{k'}$. See Figure 1.

Let $\Gamma = (\hat{\gamma} + k) \cup \Lambda_k \cup (\hat{\gamma} + k') \cup \Lambda_{k'} \cup \hat{K}$, which is a closed and connected set. Further, consider an horizontal segment $H \subset \tilde{\mathcal{U}}^+$ whose endpoints are contained one in $\hat{\gamma} + k$, the other one in $\hat{\gamma} + k'$, and there are no other intersection between these curves and $H$. Notice that this can be easily constructed since the vertical coordinate of points in $\tilde{\mathcal{C}}$ are uniformly bounded.

As $\Gamma \cap \tilde{\mathcal{U}}^- = \emptyset$, we have that $\tilde{\mathcal{U}}^-$ is contained in one connected component of $\Gamma^c$, that we call $U^-$. Moreover, $H$ must be contained

[1] We here abuse notation by identifying the curve and its image by the same name.

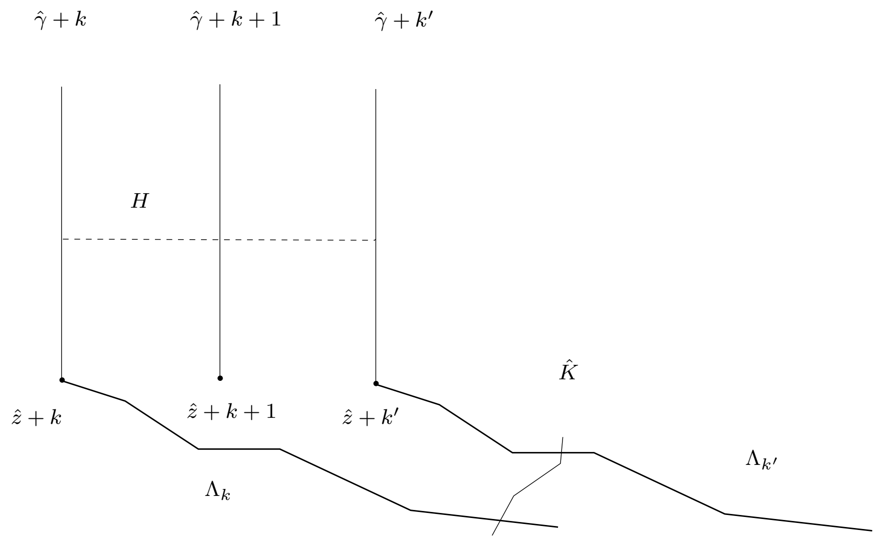


FIGURE 1. Proof of Lemma 2.5.

in a different connected component of $U^-$, as any curve from $H$ to $-\infty$ which does not intersect $\Gamma$, would allow to separate $\Gamma$ into two connected components, one containing $\hat\gamma + k$ and another one containing $\hat\gamma + k'$. We call the connected component containing $H$ by $U^+$.

Due to our assumption, we have that $\hat\gamma + k + 1$ intersect $H$. Therefore, $\hat z + k + 1$ is in the interior of $U^+$ and therefore is not accumulated by $\tilde{\mathcal{U}}^-$ which contradicts that $\tilde{\mathcal{C}}$ is the lift of a circloid.

□

Now we are ready to prove Proposition 2.4.

*Proof of Proposition 2.4.* Working with $\hat\gamma$ as before, we can assume without loss of generality that the closure of $\hat A$ contains both $\hat\gamma$ and $\hat\gamma + 1$.

We will show that if a connected component $\hat K$ of $\pi^{-1}(K)$ intersects $\hat A + k$ and $\hat A + k'$ for some $k \neq k'$ then it must intersect either $\hat C_{\hat z} + k$ or $\hat C_{\hat z} + k + 1$. Thus, Lemma 2.5 implies that $\hat K$ meets only finitely many lifts of $\hat A$ (in fact, at most three consecutive lifts), which implies that $\pi^{-1}(\hat K) \cap \hat A$ is bounded.

Without loss of generality, we assume that $\hat K$ intersects $\hat A$ and $\hat A + k$ for some $k \neq 0$ and assume by contradiction that $\hat K$ does not intersect $\hat C_{\hat z}$ nor $\hat C_{\hat z} + 1$. Choose $\eta$ a curve contained in $\hat A + k$ landing at point $y \in \hat K$ (recall that $\hat K \cap \gamma = \emptyset$). Choose also a point $x \in \hat K \cap \hat A$.

Recall that we denoted by $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ the projection onto the first coordinate. Since $\mathcal{C}$ is indecomposable, one has that $\pi_1(\hat C_{\hat z})$ is unbounded, and we assume without loss of generality that it is unbounded to the right (i.e. it accumulates in $+\infty$). Notice that $\pi_1$ is bounded both on $\hat{\gamma}$ and $\eta$.

Choose a very large $r > 0$ and consider a vertical line $v_r = \pi_1^{-1}(r)$ which intersects $\hat{C}_{\hat{z}}$ and $\hat{C}_{\hat{z}}+1$ in points $w_0^r$ and $w_1^r$ respectively. Choose a non-separating continua $\Lambda_0^r$ in $\hat{C}_{\hat{z}}$ containing $\hat{z}$ and $w_0^r$ and similarly consider $\Lambda_1^r \subset \hat{C}_{\hat{z}} + 1$ containing $\hat{z} + 1$ and $w_1^r$, which can be done due to the arguments we did before. Define $I_r \subset v_r$ as the segment joining $w_0^r$ with $w_1^r$.

Let $\Gamma_r = \hat{\gamma} \cup \Lambda_0^r \cup (\hat{\gamma} + 1) \cup \Lambda_1^r \cup I_r$. Then, by the same argument we did before, one can consider $U^+(r)$ as the connected component of $\Gamma_r^c$ containing an horizontal segment $H$ joining $\hat{\gamma}$ and $\hat{\gamma} + 1$. It is easy to see, that for every $r$ big enough, we find a point of $\hat{K} \cap \hat{A}$ in $U^+(r)$. Therefore, as $\hat{K}$ is compact and connected, there exits $r_0 \in \mathbb{R}$, so that $\hat{K} \subset U^+(r)$ for every $r > r_0$. Notice, that we do not claim that $U^+(r) \subset \hat{A}$, which is false in general.

On the other hand, as $\eta$ can not intersect $H$, one can see that $\eta$ meets $U^+(r)^c$ for all $r > r_0$. Thus, if one considers $r' > r_0$ so that $\eta \cap v_{r'} = \emptyset$ (which can be done as $\pi_1(\eta)$ is bounded), we have that $U^+(r') \cap \eta = \emptyset$, otherwise $\eta$ intersects $\Gamma_{r'} \setminus v_{r'}$, which is imposible by construction. This is a contradiction as $\eta \cap \hat{K} \neq \emptyset$ and $\hat{K} \subset U^+(r')$.

□

*Remark* 2.6. We remark that the proof does not say that $\pi(\hat{C}_{\hat{z}})$ is the unique accessible composant of $\mathcal{C}$ as it is possible that $\mathcal{U}^-$ intersects $v_r$ and therefore accumulate in both connected components of the complement of $\Gamma_r \cup v_r$.

2.4. **Prime-end compactification.** Consider a homeomorphism $f : \mathbb{A} \to \mathbb{A}$ which we can compactify to a homeomorphism $\hat{f} : S^2 \to S^2$ by adding two fixed points at infinity. In our context, there is a global attractor $\mathcal{C}$ in $\mathbb{A}$ which is a circloid, this implies that the points at infinity are sources for $\hat{f}$ and the boundary of their basins coincide.

Let $\mathcal{U}^+$ and $\mathcal{U}^-$ the unbounded connected components of $\mathbb{A} \setminus \mathcal{C}$. Denote as $\tilde{\mathcal{U}}^{\pm}$ their lifts to the universal cover $\mathbb{R}^2$ which are connected sets. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of $f$ to $\mathbb{R}^2$, it follows that $F \circ T = T \circ F$ where $T$ is any integer translation in the first coordinate.

We denote as $\hat{\mathcal{U}}^{\pm} = \mathcal{U}^{\pm} \cup \{\pm\infty\}$ the corresponding components in $S^2$. These are $f$ (resp. $\hat{f}$) invariant simply connected open sets and the dynamics is the dynamics of the basin of a source in each $\hat{\mathcal{U}}^{\pm}$. We introduce here some very basic facts from prime-end theory used in this paper and refer to the reader to [Mat, KLCN, Mats] or [Ko, Section 2.2] for more details and references.

The prime end compactification of $\hat{\mathcal{U}}^{\pm}$ is a closed topological disk $U^{\pm} \cong \mathbb{D}^2$ obtained as a disjoint union of $\hat{\mathcal{U}}^{+}$ and a circle with an appropriate topology (see [Mat]).

If one lifts the inclusion $\mathcal{U}^{\pm} \hookrightarrow U^{\pm} \setminus \{\pm\infty\}$ one obtains a homeomorphism $p^{\pm} : \tilde{\mathcal{U}}^{\pm} \to \mathbb{H}^2$, and by considering $\hat{F}^{\pm}$ the homeomorphism of $\mathbb{H}^2$ induced by $F$ on $\tilde{\mathcal{U}}^{\pm}$ (i.e. such that $p^{\pm} \circ F = \hat{F}^{\pm} \circ p^{\pm}$) one sees that $\hat{F}^{\pm}$ extends to a homeomorphism of the closure cl$[\mathbb{H}^2]$ in $\mathbb{R}^2$ and still commutes with horizontal integer translations. This allows one to compute the *upper and lower prime end rotation numbers* of $\mathcal{C}$ (see [Ko] for more details). However, we shall not use this, but just use the following facts about $\hat{F}^{\pm}$ and its relation with $F$.

- The map $\hat{F}^{\pm}$ restricted to $\partial\mathbb{H}^2 \cong \mathbb{R}$ is the lift of a circle homeomorphism where the horizontal integer translations act as deck transformations.

We finish with a last topological property for the Prime-end compactification. Let $\mathcal{U}$ be a topological disk bounded by a continuum $\mathcal{C}$ contained in some surface. For any curve $\gamma : [0,1] \to \mathcal{U} \cup \mathcal{C}$, with $\gamma(t) \in \mathcal{C}$ iff $t = 0$, we have that the corresponding curve $\eta : (0,1] \to \mathbb{D}$ of $\gamma|_{(0,1]}$ admits a unique continuous extension to a curve $\overline{\eta} : [0,1] \to \mathbb{D}$, with $\eta(1) \in \partial\mathbb{D}$.

## 3. Attracting circloids and entropy

In this section we give a proof of Theorem A, stating that an attracting circloid with two different rotation numbers has positive entropy. We first present a proof of the first part of Theorem A. Then, in Section 3.5 we show how the hypothesis in Theorem A can be relaxed, to obtain further applications as the positive entropy announced in Theorem C. The $C^0$-stability of the entropy is discussed in 3.6.

To fix the context, we introduce the following hypothesis:

(GA) $f : \mathbb{A} \to \mathbb{A}$ is an orientation preserving homeomorphism of the infinite annulus $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$ such that it has a global attractor $\mathcal{C}$ which is a circloid and the rotation set of $f$ restricted to $\mathcal{C}$ is a non-trivial interval.

Theorem A states that if $f$ verifies (GA) then it has positive entropy.

### 3.1. Some previous definitions.

Chose $\mathcal{A} \subset \mathbb{A}$ any annular neighborhood of $\mathcal{C}$ (i.e. homeomorphic to $\mathbb{S}^1 \times [-1,1]$ containing $\mathcal{C}$ in its interior) so that $f(\mathcal{A}) \subset \mathcal{A}$. Since $\mathcal{C}$ is global attractor, we have $\mathcal{C} = \bigcap_{n\in\mathbb{N}} f^n(\mathcal{A})$.

Denote by $\mathcal{U}^+$ and $\mathcal{U}^-$ to the connected components of $\mathbb{A} \setminus \mathcal{C}$, accumulating in $+\infty$ and $-\infty$ respectively and by $\partial^+\mathcal{A}$ and $\partial^-\mathcal{A}$ the connected components of $\partial\mathcal{A}$, contained in $\mathcal{U}^+$ and $\mathcal{U}^-$ respectively.

Given any essential annulus $\mathcal{A}$ in $\mathbb{A}$, with boundary components $\partial^-\mathcal{A}$ and $\partial\mathcal{A}^+$, we say that a continuum $D$ *joins the boundaries of* $\mathcal{A}$ if it verifies the following conditions:

(1) $D \subset \mathcal{A}$ and it intersects both boundaries, i.e. $D \cap \partial^+\mathcal{A} \neq \emptyset$, $D \cap \partial^-\mathcal{A} \neq \emptyset$.
(2) $D$ is inessential (i.e, it is contained in a topological disk).

Let $D_0$ and $D_1$ be two disjoint continua in $\mathcal{A}$ joining the boundaries. It follows that $\mathcal{A} \setminus (D_0 \cup D_1)$ has at least one connected component $R$ which contains a curve joining the boundaries of $\mathcal{A}$. Such a component must verify that its closure intersects both $D_0$ and $D_1$ and it will be called a *rectangle addapted* to $D_0$ and $D_1$. It is easy to show that it is an open connected subset of $\mathcal{A}$ whose boundary (relative to $\mathcal{A}$) is contained in $D_0 \cup D_1$.

Recall that we have considered the projection $\pi : \mathbb{R}\times\mathbb{R} \to \mathbb{A} = \mathbb{S}^1\times\mathbb{R}$ the cannonical projection where $\mathbb{S}^1$ is identified with $\mathbb{R}/_{\mathbb{Z}}$. Given an inessential continuum $D \subset \mathcal{A}$ which joins the boundaries of $\mathcal{A}$, one considers $\hat{D}$ to be a connected component [2]of $\pi^{-1}(D)$ in $\hat{\mathcal{A}} = \pi^{-1}(\mathcal{A})$. One defines *the right* of $\hat{D}$ to be the (unique) unbounded component of $\hat{\mathcal{A}} \setminus \hat{D}$ accumulating in $+\infty$ in the first coordinate. One defines *the left* of $\hat{D}$ symmetrically.

Notice that if $D_0$ and $D_1$ are two disjoint continua joining the boundaries of $\mathcal{A}$, and $R$ is a rectangle adapted to $D_0$ and $D_1$ then, if $\hat{R}$ is a connected component of the lift of $R$, there is a unique connected component of the lift of $D_0$ (resp. $D_1$) such that it intersects the closure of $\hat{R}$. Call these components by $\hat{D}_0$ and $\hat{D}_1$. One has that either $\hat{D}_0$ is at the left of $\hat{D}_1$ or it is at the right of $\hat{D}_1$. Assume, for convenience, that $\hat{D}_0$ is at the left of $\hat{D}_1$. It follows that if a point $x \in \hat{\mathcal{A}}$ is at the left (resp. right) of $\hat{D}_0$ (resp. $\hat{D}_1$) then $x$ is at the left (resp. right) of $\hat{R}$.

3.2. **A criteria for positive entropy.** We start with a Lemma inspired in [Gr, Section 3] which guarantees positive entropy under certain conditions.

**Lemma 3.1.** *Let $\mathcal{A} \subset \mathbb{A}$ be an essential annulus as before, and $h : \mathcal{A} \to \mathcal{A}$ be a continuous map with $h(\mathcal{A}) \subset int(\mathcal{A})$. Assume we have two disjoint continua $D_0$ and $D_1$ joining the boundaries of $\mathcal{A}$ such that for some rectangle $R$ adapted to $D_0$ and $D_1$ and some connected component $\hat{R}$ of the lift of $R$ there is a pair of positive integers $n$, $j$ with the following properties:*

- *if $\hat{D}_0$ and $\hat{D}_1$ denote the connected components of the lift of $D_0$ and $D_1$ intersecting the closure of $\hat{R}$ we have that $\tilde{h}^n(\hat{D}_0)$ is at the left of the closure of $\hat{R}$,*
- *$\tilde{h}^n(\tilde{D}_1)$ is at the right of the closure of $\hat{R} + j$.*

[2]Notice that since $\mathcal{A}$ is essential, one has that $\pi^{-1}(\mathcal{A})$ is connected.

*Then, there exists a $C^0$-neighborhood $\mathcal{N}$ of $h$ such that every $g \in \mathcal{N}$ has entropy larger or equal to $\frac{1}{n}\log(j+1)$.*

*Proof.* We start by noticing that the hypothesis are $C^0$-robust, so it is enough to work with $h$.

We assume for simplicity that $\hat{D}_0$ is at the left of $\hat{D}_1$, the other case being analogous. Notice that the closure of $\hat{R}$ projects injectively to the closure of $R$.

Consider the space $Q$ obtained by identifying the complement of $R$ in $\mathcal{A}$ (containing the boundaries) to a point. This topological space is homeomorphic to a rectangle with the two vertical sides identified to a point, which we denote as $q$, and therefore is homotopy equivalent to a circle. To see this, just notice that the interior of $R$ is diffeomorphic to the interior of a true rectangle and $R$ without the boundaries is then homeomorphic to a rectangle without two opposite sides, since we are making the quotient of these boundaries to a point one obtains the desired claim.

Let $\tilde{h}$ the lift of $h$ to the universal cover given by the hypothesis of the lemma and $\hat{R}$ as above which we identify with $R$ via the projection.

We define a continuous map $\varphi : Q \to Q$ as follows: We demand that $\varphi(q) = q$. If $x \in \hat{R}$ is any point we define $\varphi(x)$ as follows:

- if $\tilde{h}^n(x) \in \bigcup_{i=0}^{j}(\hat{R}+i)$ we define $\varphi(x) = h^n(x)$,
- otherwise $\varphi(x) = q$.

It is easy to check that $\varphi$ is continuous and has degree $j+1$ because $\tilde{h}^n(\hat{D}_0)$ is at the left of $\hat{D}_0$ and $\tilde{h}^n(\hat{D}_1)$ at the right of $\hat{D}_1 + j$. It follows by Manning's estimate [KH, Section 8.1] that $\varphi$ has entropy bounded from below by $\log(j+1)$. Since $h^n$ is semiconjugated to $\varphi$ one deduces that $h^n$ has entropy bounded from below by $\log(j+1)$ and therefore $h_{top}(h) \geq \frac{1}{n}\log(j+1)$.

□

The following is an important Remark, concerning the used mechanism in order to obtain the positive entropy.

*Remark* 3.2. The conclusion of the last theorem can be improved to the existence of a semi-conjugacy between $h^n$ restricted to some set $\Lambda$ and a full shift $\{0, 1, \ldots, j-1\}^{\mathbb{Z}}$, where the symbol 0 can be associated to none rotation, and the symbol $j-1$ to a rotation of angle $j$ (see the definition of rotational Markov partition in [Pass$_2$] for a precise description). The set $\Lambda$ turns out to be the $C^0$-horseshoe mentioned in the introduction, which further turns to exists for every homeomorphism in a $C^0$-neighborhood of $f$. We avoid the construction of $\Lambda$ in the article in sake of brevity, as it is a folklore construction. The rectangle $R$ in Figure 2 defines $\Lambda$ as in a usual construction of a horseshoe.

In order to prove Theorem A, the crucial idea is the following: using the fact that the dynamics is given on a circloid, and that it is an attractor, we will construct a sort of stable manifolds for some periodic points $p_0$ and $p_1$, given by two continua $C_0$ and $C_1$, so that they have to intersects both components $\mathcal{U}^+$ and $\mathcal{U}^-$. These continua will play the role of those continua in the hypothesis of the last lemma, and this will provide positive entropy.

We see in the next lema how the existence of such continua considered above allow us to use the previous lemma.

**Lemma 3.3.** *Let $f : \mathbb{A} \to \mathbb{A}$ so that verifies* (GA) *and assume that there exist two periodic points $p_0$, $p_1$ with different rotation numbers and two contractible continua $C_0$, $C_1$ containing $p_0$, $p_1$ respectively, such that for $i = 0, 1$*

1. *$f^{n_i}(C_i) \subset C_i$ where $n_i$ is the period of $p_i$.*
2. *$C_i$ is inessential and intersects both boundaries of $\mathcal{A}$.*

*Then, $f$ has positive topological entropy.*

We remark that we are not assuming that the sets $C_i$ are contained in $\mathcal{A}$, so we can not consider them as joining boundary componentes of $\mathcal{A}$.

*Proof.* Consider an iterate $g$ of $f$ and a lift $G$ to the the universal cover $\tilde{\mathcal{A}}$ so that both $p_0$ and $p_1$ are fixed and their lifts $\tilde{p}_0$ and $\tilde{p}_1$ verify $G(\tilde{p}_0) = \tilde{p}_0 - j$ and $G(\tilde{p}_1) = \tilde{p}_1 + l$ for some positive integers $j$, $l$ (i.e., $p_0$ rotates negatively and $p_1$ rotates positively).

As $C_i$ and $\mathcal{A}$ are forward invariant by $f$ (and therefore also for $g$) we have for $i = 0, 1$ that $g(C_i \cap \mathcal{A}) \subset C_i \cap \mathcal{A}$. Further, as $C_i$ intersects both boundaries of $\mathcal{A}$ and $p_i$ are contained in the interior of $\mathcal{A}$ there exist some continua $D_i \subset C_i$ in $\mathcal{A}$ for $i = 0, 1$, joining the boundary componentes of $\mathcal{A}$, see figure 2 (for a proof of this folklore topological fact, see for instance Theorem 14.3 in [New]).

We pick now some rectangle $R$ adapted to $D_0$, $D_1$, and a $\hat{R}$ a connected component of the lift of $R$. Let $\hat{C}_0$ and $\hat{C}_1$ be the lifts of $C_0$ and $C_1$, containing $\hat{D}_0$ and $\hat{D}_1$ as defined above.

As both $\hat{C}_0$ and $\hat{C}_1$ have bounded diameter, and rotate negatively and positively, we must have for some sufficiently large $n \in \mathbb{N}$ that

- $G^n(\hat{D}_0)$ is at the left of $\hat{D}_0$,
- $G^n(\hat{D}_1)$ is at the right of $\hat{D}_1 + 1$

Lemma 3.1 now implies that $g$ has positive entropy, and therefore, so does $f$. □

3.3. **A first reduction.** The next result, whose importance we believe transcends the context, will be proved in the next subsection. We will use it here in order to complete the proof of Theorem A.

**Theorem 3.4.** *Let $f : \mathbb{A} \to \mathbb{A}$ so that verifies* (GA) *and let $p \in \partial C$ be a periodic point. Then, there exist an innessential continuum $C_p$*

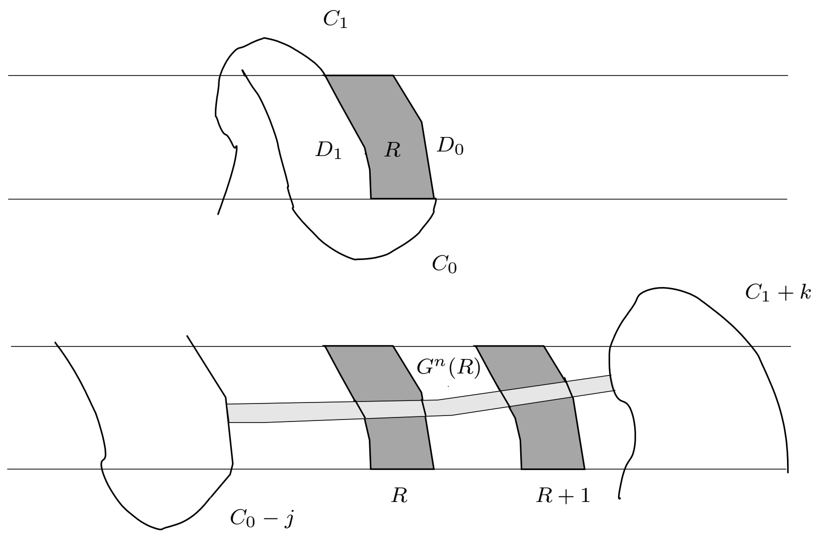


Figure 2. The rotational entropy.

*containing $p$ such that $f^{n_p}(C_p) \subset C_p$ where $n_p$ is the period of $p$ and $C_p \cap \partial\mathcal{A} \neq \emptyset$.*

Notice that the continuum $C_p$ might not intersect a priori both boundary components of $\mathcal{A}$. Moreover, although $C_p$ meets $\mathcal{C}^c$, it may happens that $C_p$ intersects only one of the unbounded connected components $\mathcal{U}^+$ and $\mathcal{U}^-$, that is, $C_p \subset (\mathcal{U}^-)^c$ or $C_p \subset (\mathcal{U}^+)^c$.

We now proceed with the proof of Theorem A assuming Theorem 3.4.

By Lemma 3.3 it is enough to find two periodic points $p_0$ and $p_1$ with different rotation vectors for which $C_{p_0}$ and $C_{p_1}$ intersect both boundary components of $\mathcal{A}$. Since $\mathcal{C}$ is a global attractor, it is enough to find $p_0$ and $p_1$ with different rotation number so that:

(1) $C_{p_0} \cap \mathcal{U}^+ \neq \emptyset$ and $C_{p_0} \cap \mathcal{U}^- \neq \emptyset$ , $C_{p_1} \cap \mathcal{U}^+ \neq \emptyset$ and $C_{p_1} \cap \mathcal{U}^- \neq \emptyset$ .

We conclude the section by proving the existence of periodic points $p_0$ and $p_1$ so that (1) holds.

Let us state the following realization theorem of [Ko] generalizing [BG]. Here is one of the essential points were we use that $\mathcal{C}$ is irreducible (see [Wal]).

**Theorem 3.5** (Theorem B of [Ko])**.** *Let $h : \mathbb{A} \to \mathbb{A}$ be a homeomorphism of the annulus preserving a circloid $\mathcal{C}$, then, every rational point in the rotation set of $h$ restricted to $\mathcal{C}$ is realized by a periodic orbit.*

*Remark* 3.6. In a work in progress between Koropecki and the first author of this paper it is shown that the periodic point can be considered to be contained in the boundary of $\mathcal{C}$ which is important for our proof. We remark however that if one assumes $\mathcal{C}$ to be a cofrontier, then $\mathcal{C} = \partial\mathcal{C}$ and therefore this is immediate and does not require this work in progress. In the cofrontier case, see also [BG].

The idea is to use points which are in $\partial\mathcal{C}$ but are not accessible from $\mathcal{U}^+$ and $\mathcal{U}^-$ in such a way that a connected set which intersects the boundary of $\mathcal{A}$ will necessarily intersect both boundaries. Recall that a point $x \in \partial\mathcal{C}$ is accessible if there exists a continuous arc $\gamma : [0,1] \to \mathcal{A}$ such that $\gamma([0,1)) \subset \mathcal{A} \setminus \mathcal{C}$ and $\gamma(1) = x$.

Here we shall use a weaker form of accessibility which, moreover, involves the dynamics of $f$ in the annulus. We will say that a periodic point $p \in \mathcal{C}$ is *dynamically continuum accessible from above* (resp. *dynamically continuum accessible from below*) if there exist a continuum $C_p$ such that:

- $p \in C_p$
- $C_p \setminus \mathcal{C}$ is non empty and contained in $\mathcal{U}^+$ (resp. $\mathcal{U}^-$).
- $C_p$ is inessential in $\mathbb{A}$.
- $f^{n_p}(C_p) \subset C_p$ for $n_p$ the period of $p$.

Using the prime-end theory and the result stated in the paragraph 2.3, one can show the following result.

**Proposition 3.7.** *Let $p$ and $q$ in $\partial\mathcal{C}$ be periodic points of $f$ which are both dynamically continuum accessible from above (resp. from below). Then, for any lift of $f$ to $\mathbb{R}^2$ both $p$ and $q$ have the same rotation number.*

*Proof.* Assume by contradiction that $p$ and $q$ have different rotation numbers for some lift. Considering an iterate $f^j$ and a suitable lift $F$ of $f^j$ to $\mathbb{R}^2$ we can assume that $F(\tilde{p}) = \tilde{p}$ and $F(\tilde{q}) = \tilde{q} + k$ with $k \neq 0$.

Let $C_p$ and $C_q$ be given by the fact that $p$ and $q$ are continuum accessible from above. By definition, we have that they are disjoint and inessential. Thus, we can consider a proper arc $\gamma : [0, +\infty) \to \mathcal{U}^+$ so that $\gamma(0) = z \in \mathcal{C}$ and $\gamma(t) \in (\mathcal{C} \cup C_p \cup C_q)^c$ for every $t \in (0, +\infty)$ and $\lim_{t\to+\infty} = +\infty$ (see subsection 2.3).

Let $\tilde{\mathcal{U}}^+ = \pi^{-1}(\mathcal{U}^+)$, $\hat{\gamma}$ a lift of $\gamma$, and $\hat{A}$ the lift of $\pi^{-1}(\mathcal{U}^+ \setminus \gamma)$ containing $\hat{\gamma}$ and $\hat{\gamma} + 1$ in its boundary. Further, consider $\hat{C}_p$ and $\hat{C}_q$ the connected components of $\pi^{-1}(C_p)$ and $\pi^{-1}(C_q)$ intersecting $\hat{A}$ respectively, $\hat{K}_p = \hat{C}_p \cap \tilde{\mathcal{U}}^+$ and $\hat{K}_q = \hat{C}_q \cap \tilde{\mathcal{U}}^-$. We have that $F(\hat{K}_p) \subset \hat{K}_p$ and $F(\hat{K}_q) \subset \hat{K}_q + k$.

As $\hat{C}_p$ and $\hat{A}$ are in the situation of Proposition 2.4, we have that $\hat{C}_p$ intersects only finitely many of the sets $\hat{A} + j,\ j \in \mathbb{Z}$, and the same holds for $\hat{C}_q$.

Consider the map $G : \mathbb{H}^2 \to \mathbb{H}^2$ induced by $F$ and the prime-end compactification of $\mathcal{U}^+$ as stated in sub-section 2.4, and let $p : \tilde{\mathcal{U}}^+ \to \mathbb{H}^2$ be the induced conjugacy between $F|_{\tilde{\mathcal{U}}^+}$ and $\hat{G}|_{\mathbb{H}^2}$. As $\gamma$ lands at an accesible point $z$, we have that $\eta = p(\hat{\gamma} \setminus \hat{\gamma}(0))$ can be extended continuously in $t = 0$, so that $\eta(0) \in \partial\mathbb{H}^2$ with respect to the usual topology of $\mathbb{R}^2$ (see 2.4).

Then, we have that the sets $\overline{K}_p = \mathrm{cl}[p(\hat{K}_p)]$ and $\overline{K}_q = \mathrm{cl}[p(\hat{K}_q)]$ are contained in a region of $\mathrm{cl}[\mathbb{H}^2]$ between the extended curves $\eta - j_0$ and $\eta + j_1$ for some $j_0, j_1 \in \mathbb{N}$. Furthermore, if $\hat{G}$ is the continuos extension of $G$ to $\mathrm{cl}[\mathbb{H}^2]$, we can assume with out lose of generality, that $\hat{G}^n(\overline{K}_p) \subset \overline{K}_p$ and $\hat{G}^n(\overline{K}_q) \subset \overline{K}_q + nk$ for all $n \in \mathbb{N}$. Let $g$ be the restriction of $\hat{G}$ to $\partial\mathbb{H}^2$, which is know to lift an orientation preserving circle homeomorphism as stated in 2.4.

Thus we obtain two compact sets $L_p = \overline{K}_p \cap \partial\mathbb{H}^2$ and $L_q = \overline{K}_q \cap \partial\mathbb{H}^2$, so that $g^n(L_p) \subset L_p$ and $g^n(L_q) \subset L_q + kn$ for all $n \in \mathbb{N}$, which is impossible, as $g$ lifts an orientation preserving circle homeomorphism. □

*Remark* 3.8. We remark that we have used strongly in the proof the fact that $\mathcal{C}$ is an indecomposable circloid in an annulus. The proof indeed gives that the rotation number of a dynamically continuum accessible periodic point from above equals the upper prime-end rotation number (see section 2.4). Notice also that in our application, we do not claim that there exist dynamically continuum accessible periodic points, we just proved that in case they exist, they have two possible rotation vector (one from above and one from below), which implies the existence of periodic points which are not dynamically continuum accessible.

We are now ready to complete the proof of Theorem A by showing:

**Proposition 3.9.** *There exists two periodic points $p_0, p_1$ in $\partial\mathcal{C}$ with different rotation numbers so that $C_{p_0}$ and $C_{p_1}$ satisfy* (1)*.*

*Proof.* Pick four rational points $r_0, r_1, r_2, r_3 \in \rho_{\mathcal{C}}(F)$ with different denominators (in particular, different from each other). Using Theorem 3.5 we know that all four are realized by periodic points $p_i$, and using Proposition 3.7 we know that at least two of them, say $p_0$ and $p_1$ are not continuum accesible. By remark 3.6 we can assume that both $p_0$ and $p_1$ belong to $\partial\mathcal{C}$.

Consider the compact connected sets $C_{p_0}$ and $C_{p_1}$ given by Theorem 3.4, since $p_0$ and $p_1$ are not continuum accessible, it follows directly that equation (1) is verified as desired. □

3.4. **Proof of Theorem 3.4.** Let $\partial^+\mathcal{A}$ and $\partial^-\mathcal{A}$ be the two boundaries of $\mathcal{A}$. Let $\mathcal{F}_0^+$ be a foliation by essential simple closed curves in the upper connected component of $\overline{\mathcal{A} \setminus f(\mathcal{A})}$ such that they coincide in the boundary with $\partial^+\mathcal{A}$ and $f(\partial^+\mathcal{A})$ and let

$$\mathcal{F}^+ = \bigcup_{n\geq 0} f^n(\mathcal{F}_0^+).$$

In a similar way we define $\mathcal{F}^-$. Notice that any annulus $\mathcal{A}_1$ bounded by a leaf of $\mathcal{F}^+$ and a leaf of $\mathcal{F}^-$ satisfies $f(\mathcal{A}_1) \subset \text{int}(\mathcal{A}_1)$.

From now on we fix a periodic point $p \in \partial\mathcal{C}$ as in Theorem 3.4. Replacing $f$ by an iterate and choosing an appropriate lift $F$ we may assume that $p$ is fixed and rotates zero. Let $q$ be another periodic point with different rotational speed. We may assume without loss of generality that $q$ is fixed and rotates one.

**Lemma 3.10.** *There exist $\eta > 0$, an annulus $\mathcal{A}_1$ bounded by leaf of $\mathcal{F}^+$ and a leaf of $\mathcal{F}^-$ and an arc $I_q \subset \mathcal{A}_1$ containing $q$ and joining both boundaries of $\mathcal{A}_1$ such that, if $g$ is $\eta$-$C^0$-close to $f$ and $G$ is the lift $\eta$-close to $F$ we have that $G(\hat{I}_q)$ is to the right of $\hat{I}_q$ and $G^2(\hat{I}_q)$ is to the right of $\hat{I}_q + 1$ in $\pi^{-1}(\mathcal{A}_1)$.*

*Proof.* Let $\epsilon > 0$ be such that $B(p,\epsilon) \cap B(q,\epsilon) = \emptyset$. Let $\delta$ be small enough such that $f(B(q,\delta))$ and $f^2(B(q,\delta))$ are contained in $B(q,\epsilon/2)$ (recall $f(q) = q$). Denote $B = B(q,\delta)$

One can choose unique leaves $\mathcal{F}_\delta^+$ and $\mathcal{F}_\delta^-$ of $\mathcal{F}^+$ and $\mathcal{F}^-$ which are tangent to $\partial B$, and do not intersect $B$.

We may assume (reducing $\delta$ if necessary) that both leaves also intersect $B(p,\epsilon)$ and consider the annulus $\mathcal{A}_1$ determined by $\mathcal{F}_\delta^+$ and $\mathcal{F}_\delta^-$. Denote by $K$ the connected component of $B(q,\epsilon) \cap \mathcal{A}_1$ that contains $B$. Notice that $K$ is inessential in $\mathcal{A}_1$ since it is disjoint from $B(p,\epsilon)$ (and there is an arc in $B(p,\epsilon)$ joining the two boundaries of $\mathcal{A}_1$).

Let $\eta > 0$ be small enough such that if $g$ is $\eta$-$C^0$ close to $f$ in $\mathcal{A}$ then:

- $g(\mathcal{A}_1) \subset \text{int}(\mathcal{A}_1)$.
- $g(B(q,\delta))$ and $g^2(B(q,\delta))$ are contained in $B(q,\epsilon)$
- $g(B(q,\delta)) \cap B(q,\delta) \neq \emptyset$ and $g^2(B(q,\delta)) \cap B(q,\delta) \neq \emptyset$.

Let $I_q$ be an arc inside $B(q,\delta)$ joining the two boundaries of $\mathcal{A}_1$. Notice that $g(I_q)$ and $g^2(I_q)$ are both contained in $K$. Now, fix a lift $\hat{q}$ of $q$ and $\hat{I}_q$ a lift of $I_q$ containing $\hat{q}$ and let $\hat{K}$ the connected component of $\pi^{-1}(K)$ that contains $\hat{I}_q$. Let $G$ be the lift of $g$ which is $\eta$-close to the lift $F$ of $f$. Since $F(\hat{q}) = \hat{q} + 1$ we have that $F(\hat{I}_q) \subset \hat{K} + 1$ and $F^2(\hat{I}_q) \subset \hat{K} + 2$, and the same holds for $G$ which completes the proof. □

From now on we fix the annulus $\mathcal{A}_1$ given by the previous lemma. The idea will be to approach $f$ by homeomorphisms presenting a stable manifold of $p$ escaping $\mathcal{A}_1$ and not intersecting $I_q$ so that we will control its convergence in the limit.

**Lemma 3.11.** *There exists a sequence of homemorphisms $f_n$ converging to $f$ in the $C^0$ topology such that:*

1. *$p$ is a hyperbolic fixed point of $f_n$.*
2. *$W^s(p, f_n)$ intersects the boundary of $\mathcal{A}_1$.*

*Proof.* Let $\epsilon_n$ be a positive sequence converging to zero. We may assume that $B(p, 2\epsilon_n) \subset \mathcal{A}_1$ for every $n$. Let $\mathcal{F}^+_{\epsilon_n}$ and $\mathcal{F}^-_{\epsilon_n}$ be the unique leaves of the foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ which are tangent to $\partial B(p, \epsilon_n)$, and do not intersect $B(p, \epsilon_n)$. Let $\mathcal{A}_{\epsilon_n}$ be the annulus determined by those leaves. Now consider $g_n$ such that $g_n = f$ outside $B(p, \epsilon_n/2)$ and $p$ is hyperbolic fixed point of $g_n$. The $C^0$ distance between $g_n$ and $f$ is bounded by $\epsilon_n/2$.

Fix a fundamental domain $D^s$ of $W^s(p, g_n)$ inside $B(p, \epsilon_n/2)$ and join an interior point $z$ of $D$ with a point $y$ in $\mathcal{F}^+_{\epsilon_n} \cap \partial B(p, \epsilon_n)$ by a poligonal arc inside $B(p, \epsilon_n)$, see figure 3. Let $U$ be a neighborhood of this arc such that does not intersects the iterates $g_n^m(D^s), m \geq 1$ and such that $\overline{U}$ is contained in the interior of $g_n^{-1}(\mathcal{A}_{\epsilon_n})$, which is equal to $f^{-1}(\mathcal{A}_{\epsilon_n})$. We may assume that $U \subset B(p, 2\epsilon_n)$ as well. See figure 3.

Consider $\varphi : \mathcal{A} \to \mathcal{A}$ such that $\varphi \equiv \mathrm{id}$ outside $U$ and $\varphi(y) = z$. The $C^0$ distance between $\varphi$ and the identity is bounded by $2\epsilon_n$. Let $f_n = g_n \circ \varphi$. We have that $y \in W^s(p, f_n)$ and $f_n^{-1}(y)$ belongs to the boundary of $f^{-1}(\mathcal{A}_{\epsilon_n})$. Since $g_n = f_n$ outside $U$ and $g_n = f$ outside $B(p, \epsilon_n/2)$, iterating backwards we eventually have that $W^s(p, f_n)$ intersects the boundary of $\mathcal{A}_1$.

Finally, it is clear that the $C^0$ distance from $f_n$ to $f$ goes to zero with $\epsilon_n$ as desired. □

Denote by $W^s_1(p, f_n)$ the connected component of $W^s(p, f_n) \cap \mathcal{A}_1$ that contains $p$.

*Remark* 3.12. The set $W^s_1(p, f_n)$ verifies that $f_n(W^s_1(p, f_n)) \subset W^s_1(p, f_n)$. Indeed, $f_n(\mathcal{A}_1) \subset \mathcal{A}_1$ and $W^s(p, f_n)$ is also $f_n$-invariant.

We now use Lemma 3.10 to control the diameter of $W^s_1(p, f_n)$ in order to be able to consider a limit continuum through $p$ which will be invariant by $f$.

**Lemma 3.13.** *Let $\mathcal{A}_1$ and $I_q$ be as in Lemma 3.10. Then, it holds that $W^s_1(p, f_n) \cap I_q = \emptyset$ for every large enough $n$.*

*Proof.* In the lift $\tilde{\mathcal{A}}_1$ of $\mathcal{A}_1$, we choose $\hat{p}$ in the fundamental domain $D$ determined by a connected component $\hat{I}_q$ of the lift of $I_q$ and $\hat{I}_q - 1$.

Consider a lift $\hat{W}^s_1(\hat{p}, f_n)$ of $W^s_1(p, f_n)$ through $\hat{p}$. Let $W$ be the connected component of $\hat{W}^s_1(\hat{p}, f_n) \cap D$ that contains $\hat{p}$ and let $F_n$ be a lift of $f_n$ close to the lift $F$ of $f$. Notice that $F_n(W) \subset W$. We may assume that $f_n$ is $\eta$-close to $f$ where $\eta$ is as in Lemma 3.10.

Assume that $\hat{W}^s_1(\hat{p}, f_n) \cap \hat{I}_q \neq \emptyset$. Then $W \cap \hat{I}_q \neq \emptyset$. But then $F_n(W) \subset W \subset D$. Since $F_n(\hat{I}_q)$ is to the right of $\hat{I}_q$ by Lemma 3.10 we have that

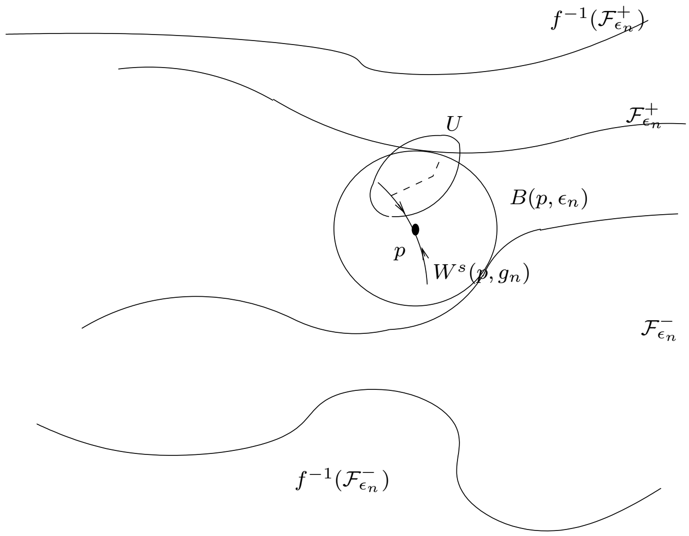


FIGURE 3. Construction of small perturbations having fixed hyperbolic saddles with stable manifolds accumulating at $-\infty$ or $+\infty$.

$F_n(W)$ is not contained in $D$, a contradiction. If $\hat{W}^s_1(\hat{p}, f_n)\cap(\hat{I}_q-1) \neq \emptyset$ we arrive to a contradiction as well, since then $F_n^2(W)$ is contained in $W$ and contains a point in $F_n^2(\hat{I}_q-1)$ which is to the right of $\hat{I}_q$ and so it must intersect $\hat{I}_q$. $\square$

**End of proof of Theorem 3.4:** We say that a set $S \subset \mathbb{R}^2$ has bounded horizontal diameter, and denoted by $\mathrm{diam}_H(S)$ is bounded, if its projection to the first coordinate is bounded. We consider the lift $\hat{\mathcal{A}}$ of $\mathcal{A}$. Let $\mathcal{A}_1$ be as in Lemma 3.10 and let $\hat{\mathcal{A}}_1$ be its lift inside $\hat{\mathcal{A}}$.

In this context, we have that the fundamental domain in $\hat{\mathcal{A}}_1$ determined by $\hat{I}_q-1$ and $\hat{I}_q$ has bounded horizontal diameter, say by $a>0$. This implies, by Lemma 3.13 that $\mathrm{diam}_H(\hat{W}^s_1(p_n,f_n))$ is also bounded by $a$.

Let $m$ be the first positive integer such that $f^m(\mathcal{A}) \subset \mathcal{A}_1$. Notice that $f_n^m(\mathcal{A}) \subset \mathcal{A}_1$ by construction. Let $F$ be the lift of $f$ and $F_n$ the lift of $f_n$. Then $F_n^{-m}(\hat{W}^s_1(p,f_n))$ has bounded diameter in $\mathbb{R}^2$. Let $\hat{C}_n = F_n^{-m}(\hat{W}^s_1(p,f_n))$. We have that:

(1) $\hat{C}_n$ is a continuum containing $\hat{p}$.
(2) $\hat{C}_n$ is forward invariant by $F_n$(c.f. remark 3.12).
(3) $\hat{C}_n$ intersects the boundary of $\hat{\mathcal{A}}$.
(4) $\hat{C}_n$ has uniformly bounded diameter.
(5) $F_n \rightrightarrows F$.

Then, by taking the Hausdorff limit $\hat{C}_p$ of $(\hat{C}_n)_{n\in\mathbb{N}}$ we have a continuum which is forward invariant under $F$ and contains $\hat{p}$. Moreover, it intersects $\partial\hat{\mathcal{A}}$, and its projection into $\mathbb{A}$ must be inessential, since otherwise it would intersect $I_q$ which is not possible. Taking $C_p = \pi(\hat{C}_p)$ we are done.

□

3.5. **Proof of Theorem C.** In this section we comment on the proof of Theorem A to see that weaker hypothesis are enough to obtain positive entropy. These remarks will show that under the hypothesis of Theorem C the same proof can be carried out.

Consider an annulus homeomorphism $f : \mathbb{A} \to \mathbb{A}$ which leaves an essential continuum $\mathcal{C}$ invariant. Assume moreover that $\rho_{\mathcal{C}}(F)$ is a non-trivial interval for some lift $F$ of $f$ to $\mathbb{R}^2$.

The proof of Theorem 3.4 extends to the following with the same proof:

**Theorem 3.14.** *Assume that $\mathcal{A}$ is a compact annulus and let $p, q \in \mathcal{C}$ be periodic points with different rotation number such that $f^n(\partial^+\mathcal{A})$ and $f^n(\partial^-\mathcal{A})$ accumulate on $p$ and $q$. Then, there exists a compact continuum $C_p$ such that $f^k(C_p) \subset C_p$ where $k$ is the period of $p$ and such that $C_p \cap \partial\mathcal{A} \neq \emptyset$.*

Assume that $f$ as above verifies that there exists a compact essential annulus $\mathcal{A} \subset \mathbb{A}$ such that $f(\mathcal{A}) \subset \text{int}(\mathcal{A})$. It follows that there exists an essential invariant subset $\mathcal{K} \subset \text{int}(\mathcal{A})$. Assume that $\mathcal{K}$ has empty interior[3]. Therefore, if $\mathcal{V}^{\pm}$ denote the connected components of $\mathbb{A} \setminus \mathcal{K}$ we have that $\mathcal{C} = \partial\mathcal{V}^+ \cap \partial\mathcal{V}^-$ is an essential continuum (which in fact is a cofrontier). It follows that our proof extends to:

**Theorem 3.15.** *In the situation above, if $\mathcal{C}$ has a non-trivial rotation interval for some lift, then $f$ has a $C^0$-neighborhood $\mathcal{U}$ for which every $g \in \mathcal{U}$ has positive entropy.*

Notice that Theorem C is a direct consequence of this Theorem as Birkhoff attractors verify these conditions (see [LeC]). We remark that the twist condition is irrelevant.

*Indication of the proof.* As in the proof of Theorem A, in this situation one can find four periodic points with different rotation numbers in $\mathcal{C}$. One can apply Theorem 3.14 to find compact connected sets on each one. Notice that the iterates of both boundaries of $\mathcal{A}$ must accumulate every point in $\mathcal{C}$ as $\mathcal{K}$ has empty interior.

At most two of these periodic points can be dynamically continuum accessible (c.f Proposition 3.9), which implies that the other two verify that their continua given by Theorem 3.14 intersects both boundaries of the annulus after backward iteration (c.f. Lemma 3.3) and so Lemma

[3]This is the case for example when one assumes a (weak) type of dissipation, for example the non-existence of invariant bounded open sets.

3.1 can be applied. Notice that the fact that $f(\mathcal{A}) \subset \text{int}(\mathcal{A})$ is important for this.

□

3.6. **$C^0$-rotational Horseshoes.** We have proved that a map $f$ verifying (GA), presents a region $R$ as in figure 2 for some power $f^n$. As claimed in Remark 3.2, it is possible to construct for $f^n$ an invariant compact set $\Lambda \subset R$, in the same way the classical Smale horseshoe is constructed for smooth maps. From a dynamical point of view, there is a natural partition for this set given by two topological rectangles $R_0$ and $R_1$ joining boundaries[4] in $\mathcal{A}$, which turns out to be a rotational Markov partition in the sense of [Pass$_2$].

It is clear that this construction is $C^0$ robust, that is, there exists a $C^0$-neighborhood $\mathcal{N}$ of $f$ so that for every $g \in \mathcal{N}$ and $n \in \mathbb{N}$ as before, a set $\Lambda_g$ can be considered with the same dynamical properties as for $\Lambda$.

Thus, we obtain in particular uniform positive entropy in $\mathcal{N}$. Furthermore, for elements in $\mathcal{N}$ we find exponential growths of periodic orbits of fixed periods, whose rotation vectors are given by the itinerary with respect the considered partitions.

In the context of Theorem C, the same arguments we do here work due to the facts stated in section 3.5.

## 4. Entropy versus rotation set for circloids.

Let $\mathbb{A}$ be the annulus $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$ where we identify $\mathbb{S}^1 = \mathbb{R}/_{\mathbb{Z}}$. We consider the usual covering $\pi : \mathbb{R}^2 \to \mathbb{A}$ given by $\pi(x, y) = (x \pmod{\mathbb{Z}}, y)$.

We consider a diffeomorphism $\tau : \mathbb{A} \to \mathbb{A}$ the integrable twist map, given by the lift:

$$T(x, y) = (x + y, y)$$

If we denote by $\mathcal{F} = \{C_y\}_{y\in\mathbb{R}}$ the foliation of $\mathbb{A}$ by essential circles given by $C_y = \pi(\mathbb{R} \times \{y\})$, we have that $\tau|_{C_y}$ is a rotation of angle $y$.

*Remark* 4.1. A simple computation gives that $\frac{1}{n} \log \|D\tau^n\| = 0$. This can be combined with Proposition 2.1 to get that given $\varepsilon > 0$ there is a $C^1$-neighborhood $\mathcal{N}_\varepsilon$ of $\tau$ such that if $f \in \mathcal{N}_\varepsilon$ then $h_{top}(f) < \varepsilon$.

We will prove the following theorem.

**Theorem 4.2.** *For every $C^1$-neighborhood $\mathcal{N}$ of $\tau$ there exists $f \in \mathcal{N}$ so that $f$ has a global attractor given by an essential circloid $\mathcal{C}$ with $\rho_{\mathcal{C}}(F) \supset [0, 1]$ for some lift $F$ of $f$.*

Combining this with remark 4.1, we show that there are circloids with rotation set $[0, 1]$ whose entropy approaches zero as much as desired, therefore establishing Theorem B. Notice that the twist condition is

[4]As it is considered in the smooth case.

$C^1$-open so that we can assume also that the obtained diffeomorphism verifies the twist condition. To obtain a dissipation hypothesis one has to perform a slightly different perturbation which is explained at the end of this section.

We fix $\mathcal{N}$ and construct $f \in \mathcal{N}$ by means of a series of $C^1$ perturbations of $\tau$. We remark that all the perturbations are just $C^1$ small, but the map itself can be considered to be smooth (see remark 2.3).

4.1. **First perturbation.** We first fix some notation. For $y < y'$ we denote by $[C_y, C_{y'}]$ to the compact region between these two circles, and by $(C_y, C_{y'})$ its interior.

As usual, given a map $f : M \to M$ and a point $x \in M$, we define for $\varepsilon > 0$ the local stable set of $x$ by $W^s_\varepsilon(x, f) = \{y \in M \mid d(f^n(y), f^n(x)) < \varepsilon \text{ for all } n \in \mathbb{N}\}$, and the stable set of $x$ by $W^s(x, f) = \{y \in M \mid \lim_n f^n(y) = x\}$. The local unstable and unstable sets are defined by considering $f^{-1}$ instead of $f$. When these sets are considered for hyperbolic periodic points, the local stable set is a sub-manifold tangent to the stable subspace at $x$, and it holds $W^s(x, f) = \bigcup_{n\in N} f^{-n}(W^s_\varepsilon(x, f))$ (see [KH, Section 6]). Analogous notation is used for the unstable sets changing $s$ for $u$ and considering iterates in the other sense the same results hold.

The first perturbation will be $f_1 \in \mathcal{N}$ so that:

(1) $f_1$ is conservative restricted to the annulus $[C_0, C_1]$.
(2) $f_1(C_r) = C_r$ for $r \in \{0, 1\}$.
(3) $f_1$ has a saddle $x_0 \in C_0$ and a saddle-node $p_0 \in C_0$, so that $W^u(x_0, f_1) = C_0 \setminus p_0$, which implies that $W^s(p_0, f_1) \supseteq C_0 \setminus \{x_0\}$.
(4) $f_1$ has a saddle $x_1 \in C_1$ and a saddle-node $p_1 \in C_1$, so that $W^u(x_1, f_1) = C_1 \setminus p_1$, which implies as before that $W^s(p_1, f_1) \supseteq C_1 \setminus \{x_1\}$.
(5) There is a forward invariant arc $I^s_0 \subset W^s(p_0, f_1) \cap (-\infty, C_0]$ with one endpoint at $p_0$, and a small backward invariant compact arc $I^u_0 \subset W^u(p_0, f_1) \cap [C_0, C_1]$ with one endpoint in $p_0$.
(6) There is a forward invariant arc $I^s_1 \subset W^s(p_1, f_1) \cap [C_1, +\infty)$ with one endpoint at $p_1$, and a small backward invariant compact arc $I^u_1 \subset W^u(p_1, f_1) \cap [C_0, C_1]$ with one endpoint in $p_1$.
(7) $[C_0, C_1]$ is a global attractor for $f_1$.
(8) For every $n \in \mathbb{N}$, $f_1$ has finitely many points of period $n$.

This can be done by $C^1$-small smooth perturbations around the circles $C_0$ and $C_1$ and the Franks' lemma [Fr] (see [BDP, Proposition 7.4] for the conservative version) for suitable perturbations of the derivative in the conservative setting. For obtaining point (7), one can just take a dissipative perturbation supported in $(C_0, C_1)^c$. The last point (8) can be achieved by means of usual arguments in generic dynamics: a simple Baire argument allows to find a smooth diffeomorphism nearby for which all periodic points in the interior of the annulus have no eigenvalues equal to $\pm 1$, and this implies that the set of those having period $n$ is finite for all $n \in \mathbb{N}$. This first perturbation is depicted in Figure 4.

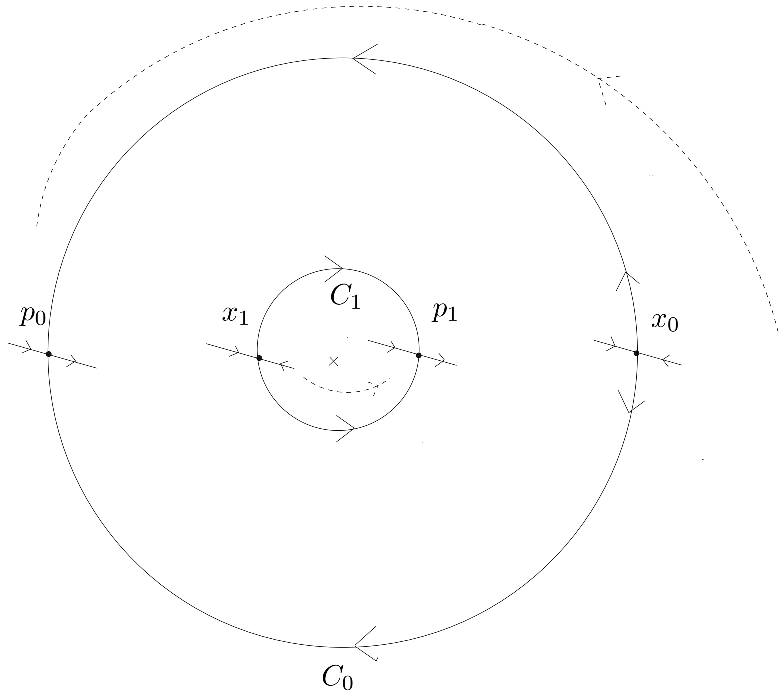


FIGURE 4. The map $f_1$.

4.2. **Second perturbation.** For the second perturbation, we make use of Theorem 2.2. We construct $f_2 \in \mathcal{N}$ so that:

(1) $f_2$ is conservative in $[C_0, C_1]$,
(2) $f_2(x) = f_1(x)$ outside $(C_{r_1}, C_{r_2})$ for values $0 < r_1 < r_2 < 1$,
(3) there is a transverse intersection between the connected component of $I_0^u \cap [C_0, C_{r_1}]$ containing $p_0$ and $W^s(x_0, f_2)$ and a transverse intersection between the connected component of $I_1^u \cap [C_{r_2}, C_1]$ containing $p_1$ and $W^s(x_1, f_2)$.

*Remark* 4.3. The diffeomorphism $f_2$ restricted to $[C_0, C_1]$ is a conservative annulus diffeomorphism which deviates the vertical and the whole annulus is an instability region. In particular, the rotation set in this instability region is $[0, 1]$ and the entropy can be chosen to be as small as desired.

In order to produce $f_2$ we just have to choose a perturbation of $f_1$ in $\mathcal{N}$ which is conservative in $[C_0, C_1]$, supported outside a neighborhood of $C_0$ and $C_1$ in $[C_0, C_1]$ and connects the forward orbit of a small arc in $I_0^u$ (inside the neighborhood where the perturbation is made) with the stable manifold of $x_1$ and symmetrically connects the forward orbit of $I_1^u$ with the stable manifold of $x_0$ . See figure 5.

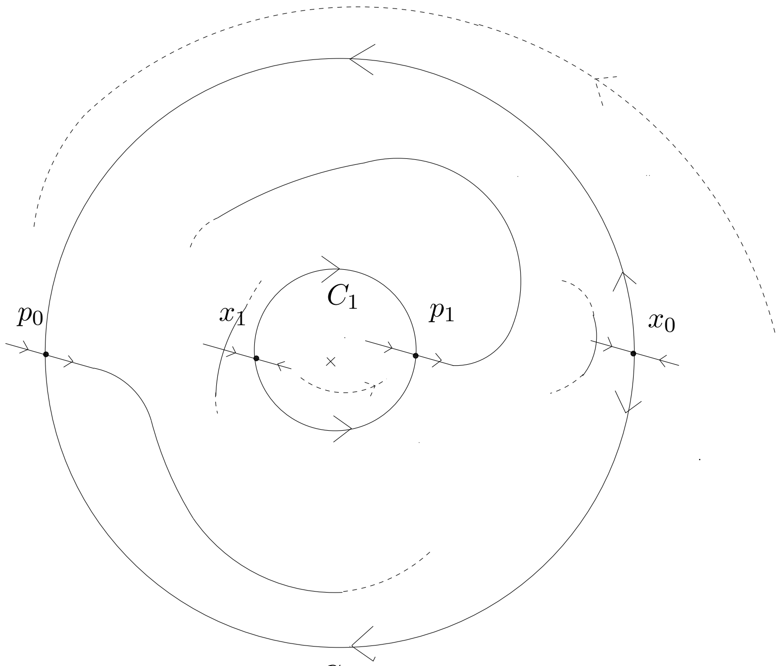


FIGURE 5. The map $f_2$.

This will be achieved by means of Theorem 2.2. But first we need to show an abstract lemma to put ourselves in the hypothesis of the theorem.

**Lemma 4.4.** *Assume $h : [C_0, C_1] \to [C_0, C_1]$ is an area preserving diffeomorphism and $D$ is a connected open subset whose closure is contained in $(C_0, C_1)$. Let $z, w$ be points in $(C_0, C_1)$ such that there are integers $n_z > 0$ and $n_w > 0$ so that $h^{n_z}(z)$ and $h^{-n_w}(w)$ are contained in $D$. Then $z \dashv_{cl[D]} w$.*

*Proof.* Notice that it is enough to show that for every pair of points $p$ and $q$ in $D$ and $\varepsilon > 0$ one can construct a pseudo-orbit with jumps in $D$ going from $p$ to $q$ since one can go without jumps from the interior of $D$ to the points $z$ and $w$.

We fix $p$ in the interior of $D$, and consider for every $\varepsilon > 0$ the set $P_\varepsilon$ of those points $q \in D$ so that there exists a $\varepsilon$ pseudo-orbit $(z_k)_{k=0}^n$ with $z_0 = p,\ z_n = q$ and $h(z_k),\ z_{k+1} \in D$ whenever $h(z_k) \neq z_{k+1}$. It is enough to prove that $P_\varepsilon$ is open, closed in $D$ and non-empty.

For $q \in P_\varepsilon$ we can consider a $\varepsilon$ pseudo-orbit $(z_k)_{k=0}^n$ as before. Then, $d(h(z_{n-1}), q) < \varepsilon' < \varepsilon$. Pick a neighborhood $V$ of $q$ in $D$, so that $V \subset B(q, \varepsilon - \varepsilon')$ and take $z \in V$.

- if $h(z_{n-1}) = q$, we have that $z_0, \dots, z_{n-1}, z$ is a $\varepsilon$ pseudo-orbit whose jumps are in $D$. Thus $z \in P_\varepsilon$, and $V \subset P_\varepsilon$, so $P_\varepsilon$.

- If $h(z_{n-1}) \neq q$, then both $h(z_{n-1}),\ z$ are contained in $D$. Thus the pseudo-orbit $z_0, \dots, z_{n-1}, z$ is a $\varepsilon$ pseudo-orbits who has it jumps in $D$. Thus, we have again $V \subset P_\varepsilon$.

Therefore, we can conclude that $P_\varepsilon$ is open. In order to check that it is also closed, we consider a sequence of points $q_n \in P_\varepsilon$ converging to a point $q$ in $D$. Fix $q_n$ so that $d(q_n, q) < \varepsilon$ and let $V$ be a neighborhood of $q_n$ in $D$, contained in $B(q, \varepsilon)$.

We consider a $\varepsilon$ pseudo-orbit $p = z_0, \dots, z_m = q_n$ with jumps inside $D$. Hence, $d(h(z_{m-1}), q_n) < \varepsilon$. Poincaré's recurrence Theorem (see [KH, Section 4.1]) implies that we can consider a recurrent point $r \in V$ so that $d(h(z_{m-1}), r) < \varepsilon$. Let $h^l(r) \in V$ and define the pseudo-orbit

$$p = z_0, \dots, z_{m-1}, r, h(r), \dots, h^{l-1}(r), q \ .$$

Then, we have a $\varepsilon$ pseudo-orbit from $p$ to $q$ whose jumps are all contained in $D$.

To show that $P_\varepsilon$ is non-empty notice that again by Poincaré's recurrence theorem, one has that $p \in P_\varepsilon$.

□

Now let us construct the desired perturbation of $f_1$.

As $f_1 \in \mathrm{Diff}^1_{\nu,per}(\mathbb{A})$ we can consider for the prescribed neighborhood $\mathcal{N}$, the positive integer $N$ of the connecting lemma (Theorem 2.2). We consider first the set $D \subset (C_0, C_1)$ given by $D_0 = (C_{a_0}, C_{b_0})$ so that the arc $I_0^u$ and the invariant manifold $W^s(x_1, f_1)$ intersects $D_0$. Choose $0 < a_1 < a_0$ and $b_0 < b_1 < 1$ so that $D_1 = (C_{a_1}, C_{b_1})$ contains $\mathrm{cl}[D_0 \cup \ldots \cup f_1^{N-1}(D_0))]$.

Choose a point $z \in I_0^u \setminus D_1$ and $w \in W^s(x_1, f_1) \setminus D_1$. It follows from Lemma 4.4 that one has $z \dashv_{D_0} w$. Theorem 2.2 implies that there exists $g \in \mathcal{U}$ such that $g^n(z) = w$ and such that $g = f_1$ outside $D_1$. Due to the way $w$ is chosen, and since $g = f_1$ outside $D_1$, it follows that $w$ still belongs to $W^s(x_1, g)$ after perturbation[5] and the same holds for $I_0^u$ so we deduce that $I_0^u$ intersects $W^s(x_1, g)$. A further small perturbation makes this intersection transversal.

Now, we do the same argument again but reducing further $a_1$ and $b_1$ so that we can connect $I_1^u$ with the stable manifold of $x_0$ and again make the intersection transversal. We can choose the perturbation small enough so that the intersection we had already created persists thanks to transversality. This concludes the proof that $f_2 \in \mathcal{N}$ can be constructed.

[5]Technically one has to choose $z \neq p_0$ in the connected component of $I_0^u \setminus D_1$ containing $p_0$ and $w \neq x_1$ in the connected component of $W^s(x_1, f_1) \setminus D_1$ containing $x_1$.

4.3. **Final perturbation.** For our last move, we fix $z_0$ in one of the connected components of $C_0 \setminus \{x_0, p_0\}$ and $z_1$ in one of the connected components of $C_1 \setminus \{x_1, p_1\}$. Consider for $k = 0, 1$ a open ball $B(z_k, \delta)$ so that $B(z_k, \delta) \cap C_k = B(z_k, \delta) \cap C_k \setminus \{x_k, p_k\} = I_k$, where $I_k$ is a wandering interval, i.e., $I_k \cap \bigcup_{n \in \mathbb{Z} \setminus \{0\}} f_2^n(I_k) = \emptyset$.

We now take two $C^\infty$-diffeomorphisms $b_0$ and $b_1$ which are arbitrary $C^\infty$-close to the identity, supported in $B(z_0, \delta)$ and $B(z_1, \delta)$.

If we set for every $p \in \mathbb{R}^2$ the coordinates $\tilde{x} = \pi_1(p - z_0)$ and $\tilde{y} = \pi_2(p - z_0)$[6], the first map is given by

$$b_0(p) = (\tilde{x}, \tilde{y} + \mu(\tilde{x}, \tilde{y})) \ ,$$

where $\mu : \mathbb{R}^2 \to [0, 1]$ is some $C^\infty$ function which is zero outside $B(0, \delta)$. The function $\mu(x, \cdot)$ is a symmetric bump function for every $x \in \mathbb{R}$ which is zero as $x$ approaches $\delta$.

For $b_1$, if we now set for every $p \in \mathbb{R}^2$ the coordinates $\tilde{x} = \pi_1(p - z_1)$ and $\tilde{y} = \pi_2(p - z_1)$, we set

$$b_1(p) = (\tilde{x}, \tilde{y} - \mu(\tilde{x}, -\tilde{y})) \ .$$

We call by $L_0$ the open disk between $I_0$ and $b_0(I_0)$ and $L_1$ the open disk in-between $I_1$ and $b_1(I_1)$.

We are ready now to state or final perturbation. We consider $f \in \mathcal{N}$ so that

$$f = b_1 \circ b_0 \circ f_2 \ ,$$

where the following holds:

(1) Property (3) of the second perturbation $f_2$ still holds.
(2) $\lim_n f^{-n}(l) = -\infty$ for all $l \in L_0$,
(3) $\lim_n f^{-n}(l) = +\infty$ for all $l \in L_1$.

Indeed, the choice of $b_0$ and $b_1$ imply immediately the last two properties and if $b_0$, $b_1$ are considered small enough then the transverse intersections required in (3) of $f_2$ are still valid. Notice that $f = f_2$ in a neighborhood of $x_0$, $x_1$, $p_0$ and $p_1$. See Figure 6 for a schematic drawing.

4.4. **The perturbation verifies the announced properties.** We must now show that $f$ verifies our theorem 4.2. Consider the set

$$\mathcal{B} = \mathrm{cl}[W^u(x_0, f)]$$

Observe that it is a closed connected set. The next lemma shows that it coincides with $\mathrm{cl}[W^u(x_1, f)]$, and by construction $\mathcal{B} \subset [C_0, C_1]$. Thus, we actually have that $\mathcal{B}$ is an essential continuum with $\rho_{\mathcal{B}}(F) \supseteq [0, 1]$

[6]Here $\pi_1$ and $\pi_2$ stay for the projections over the first and second coordinate in $\mathbb{R}^2$.

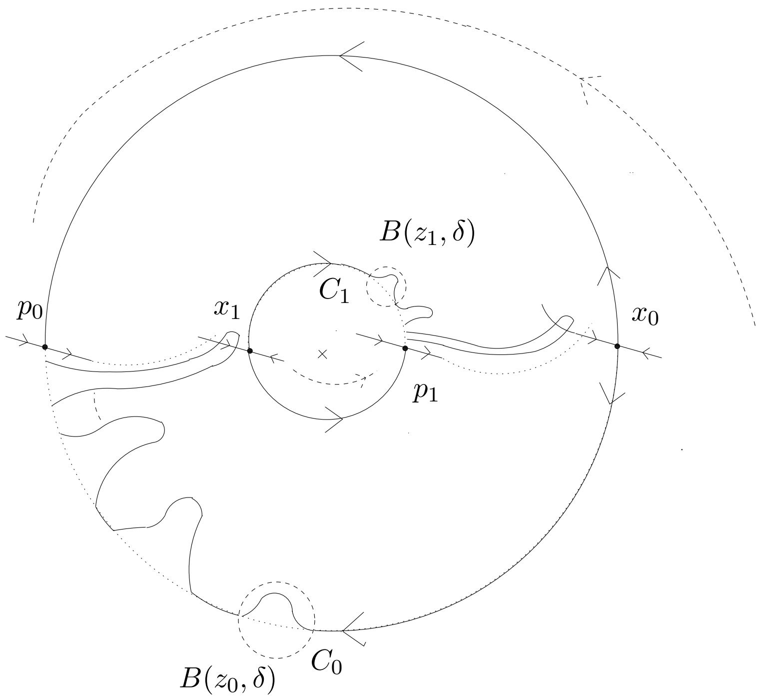


FIGURE 6. The finial map $f$. We perform a small perturbation near $z_0$, $z_1$ so that $x_0, p_0, x_1, p_1$ belong to the same homoclinic class. The closure of $W^u(x_0, f)$ will give our desired circloid.

for some suitable lift $F$ of $f$. Let us call $\mathcal{U}^-$ and $\mathcal{U}^+$ the two unbounded connected components of $\mathbb{A} \setminus \mathcal{B}$.

**Lemma 4.5.** *The points $x_0$ and $x_1$ are homoclinically related.*

*Proof.* This follows by applying a small variation of the $\lambda$-lemma [KH] in a neighborhood of $p_0$ (resp. $p_1$). Notice that the usual $\lambda$-lemma does not apply since $p_0$ is not hyperbolic but by looking at the local dynamics of $p_0$ and the way we have performed the perturbation $b_0$ (far from $p_0$) one has that the new unstable manifold of $x_0$ will approach for forward iterates the unstable manifold of $p_0$ which is connected to the stable manifold of $x_1$. The symmetric argument gives that the unstable manifold of $x_1$ must intersect transversally the stable manifold of $x_0$ which concludes. □

Furthermore, as we have one directrix of $W^s(x_0, f)$ contained in $\mathcal{U}^-$, using Lemma 4.5 shows that $\partial\mathcal{U}^- \supseteq \mathcal{B}$. On the other hand, as we have the saddle $x_1 \in H$ containing one of the directrix in $\mathcal{U}^+$, we have that $W^s(x_0, f)$ must intersect $\mathcal{U}^+$. Therefore, arguing with the $\lambda$-lemma, we find that $\partial\mathcal{U}^+ \supseteq \mathcal{B}$. Thus,

$$\mathcal{B} \subset \partial\mathcal{U}^- \cap \partial\mathcal{U}^+.$$

This implies, by means for instance of [Ja, Corollary 3.3], that $\mathcal{B}$ is the boundary of a circloid $\mathcal{C}$, with $\mathbb{A}\setminus\mathcal{C} = \mathcal{U}^-\cup\mathcal{U}^+$. In order to obtain 4.2, we need to prove that $\mathcal{C}$ is the global attractor of $f$.

For this, it is enough to show that every point $u\in\mathcal{U}^-$ has its $\alpha$-limit in $-\infty$ and that every point $v\in\mathcal{U}^+$ has it $\alpha$-limit in $+\infty$. We work with $\mathcal{U}^-$, as for the other case one can perform the same arguments. Recall the definition of the disk $L_0$ associated to the wandering interval $I_0$. We have by construction that $L_0$ is bounded by the concatenation of curves $j_1\subset I_0$ and $j_2 = f(j_1)$. Denote by $\tilde{j}_1$ the maximal open interval in $j_1$.

In order to show that $-\infty = \lim_n f^{-n}(u)$ for all $u\in\mathcal{U}^-$ is enough to show following lemma.

**Lemma 4.6.** *We have that* $\mathcal{U}^- = (-\infty, C_0)\cup\bigcup_{n\in\mathbb{N}} f^n(L_0\cup\tilde{j}_1)$.

*Proof.* It is easy to see that $(-\infty, C_0)\subset\mathcal{U}^-$. Further, as $L_0\cup\tilde{j}_1\subset\mathcal{U}^-$, we have that $\mathcal{U}^-\supseteq(-\infty, C_0]\cup\bigcup_{n\in\mathbb{N}} f^n(L_0)$. We must look now for the symmetric inclusion.

Observe that $f^n(j_1)\subset C_0$ for all $n\in\mathbb{N}$ and that $f^n(j_2)\subset[C_0,+\infty)\cap\mathcal{C}$ for all $n\in\mathbb{N}$.

We name by $W$ the connected component in $C_0\setminus\{x_0,p_0\}$ which contains $j_1$. Observe that the closure of the complementary connected component is contained in $\mathcal{C}$. Then it holds

$$C_0\setminus\mathcal{C} = C_0\setminus\left(\bigcup_{n\in\mathbb{N}} f^n(\tilde{j}_1)\right).$$

Assume $x\in\mathcal{U}^-\cap[C_0,+\infty)$, hence we can connect $x$ to $-\infty$ throughout simple curve $\Gamma'\subset\mathcal{U}^-$, which must contain a compact arc $\Gamma\subset[C_0,+\infty)$ from $x$ to certain point in $f^{n_0}(j_1)$. Thus, $\Gamma$ must be contained in a disk bounded by the concatenation of $f^{n_0}(j_1)$ and $f^{n_0}(j_2)$, otherwise $\Gamma$ meets $f^n(j_2)\subset\mathcal{C}$.

Therefore we get that $x\in f^{n_0}(L_0)$, and we have

$$\mathcal{U}^- = (-\infty, C_0]\cup\bigcup_{n\in\mathbb{N}} f^n(L_0\cup\tilde{j}_1).$$

□

We conclude that the non-wandering set of $f$ is contained in $\mathcal{C}$, so $\mathcal{C}$ must be a global attractor for $f$, and we are done with the proof of Theorem 4.2 (and consequently of Theorem B).

4.5. **Proof of Theorem D.** We here perform some modifications to the construction developed above to obtain a proof of Theorem D. In the construction of $f_2$, it is not hard to construct another pair of saddle periodic points inside $(C_0, C_1)$, so that they are homoclinically related and have different rotation numbers which are as close as desired to 0 and 1 respectively. This can be achieved using Theorem 2.2.

Then, for a fixed $\delta > 0$ we can choose $f_2$ so that there is a homoclinic class having periodic points as before, which rotate $\delta$ and $1-\delta$. Notice that $f_2$ verifies that $f_2(\mathbb{S}^1 \times [-1,2]) \subset \mathbb{S}^1 \times (-1,2)$ and we can assume that the determinant of the derivative of $f_2$ is everywhere smaller than $1-\delta$ outside $\mathbb{S}^1 \times [-1,2]$.

Now, instead of pushing the unstable manifolds of $x_0$ and $x_1$ we will consider smooth diffeomorphisms $h_n$ which coincide with the identity outside the region $(C_{-n}, C_{n+1})$, and having the form $h_n(x,y) = (x, \hat{h}_n(y))$ and $\hat{h}_n : \mathbb{R} \to \mathbb{R}$ is a function such that:

- $\hat{h}'_n(y) \in (1-1/n, 1+1/n)$ for every $y \in \mathbb{R}$ and $\hat{h}'_n(y) < 1-1/2n$ if $y \in [-1,2]$
- the $C^1$-distance between $h_n$ and the identity tends to 0 as $n \to \infty$.

We will consider the perturbations $g_n = h_n \circ f_2$.

Since $f_2$ has the considered homoclinic class, it follows that for large enough $n$, this class will persist and contain in its rotation set the interval $[\delta, 1-\delta]$. Moreover, for large enough $n$ there will still be a global attractor as one has $g_n([C_{-1}, C_2]) \subset (C_{-1}, C_2)$, and the dynamics is dissipative since the jacobian of $g_n$ in $[C_{-1}, C_2]$ is everywhere less than $1 - 1/2n < 1$. Thus $g_n$ presents Birkhoff Attractors for large $n \in \mathbb{N}$.

Since $f_2$ satisfies the twist condition, the same holds for $g_n$ when $n$ is large, as it is an open property. This implies that the homoclinic class with such rotation set is contained in a Birkhoff attractor. Further, $g_n$ has a rotation set larger than $[\delta, 1-\delta]$ but always contained in $[0,1]$. As $g_n$ can be considered in an arbitrary small $C^1$ neighborhood of $\tau$, the entropy of $g_n$ can be considered arbitrary small (Proposition 4.1), say smaller than $\varepsilon/3$, and then[7] choosing $g_n^3$ we obtain the proof of Theorem D.

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