Tangential polynomials and matrix KdV elliptic solitons

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1 Introduction

In the mid 1970's, following earlier work by several people in former Soviet Union, I.M. Krichever developed the Theory of scalar and vector Baker-Akhiezer functions. Given a d-marked compact Riemann Surface $(\Gamma, \{p_1, \cdots, p_d\})$ of genus $g > 0$, equipped with an effective divisor D of degree $(g+d-1)$, he constructed a meromorphic vector function $\psi_D(x, y, t; p) : \mathbb{C}^3 \times (\Gamma \setminus \{p_i\}) \to \mathbb{P}^1$ and two differential operators

$$L := \partial_x^2 + U(x, y, t) \quad \text{and} \quad M := \partial_x^3 + \frac{3}{2}U(x, y, t)\partial_x + W(x, y, t)$$

with $d \times d$-matrix valued coefficients, satisfying the system of equations:

$$\begin{cases} (\partial_y - L)\psi_D = 0 \\ (\partial_t - M)\psi_D = 0 \end{cases}$$

The corresponding compatibility equation $[\partial_y - L, \partial_t - M] = 0$ is equivalent to the matrix KP equation, a system of partial derivative equations satisfied by the $d \times d$-matrix functions U and W (cf. [4], p. 21-22 or [2] p. 86, 2.2 & 2.3).

Moreover, if there exists a meromorphic function $f : \Gamma \to \mathbb{P}^1$, with a double pole at each p_i, the B-A function ψ_D satisfies the system ([4],3.5 p. 21-22).

$$\begin{cases} (L - f)\psi_D &= 0 \\ (\partial_t - M)\psi_D &= 0 \end{cases}$$

implying that U is independent of y and solves the simpler matrix Korteweg-deVries equation

$$(KdV) \qquad U_t = \frac{1}{4}(3UU_x + 3U_xU + U_{xxx})$$

On the other hand, the former matrix K solutions are doubly periodic in x, and called elliptic KP solitons, if the spectral data satisfies the *elliptic criterion* presented in [5], Assertion p.289.

Both elliptic scalar cases plus the matrix KP one have been extensively studied (see [5] and [2], as well as [3], [6], [1] and [7]-[10]) but, to our knowledge, the matrix KdV elliptic issue has been left beside.

Given $d > 0$ and an elliptic curve $(X, q) := (\mathbb{C}/\Lambda, 0)$ our purpose in this article is manifold:

1. to present simple polynomial equations defining spectral curves of matrix KP elliptic solitons;

2. to give an effective construction of the corresponding polynomials;

3. to deduce arbitrarily high genus spectral curves of matrix KdV elliptic solitons.

We proceed as follows :

Section 2: given any cover $\pi : \Gamma \to X$, marked at d points $\{p_1, \cdots, p_d\}$ of the fibre $\pi^{-1}(q) \subset \Gamma$, we identify X and the smooth subset of Γ with their canonical images in the Jacobian of Γ. We call π *d-tangential* if the tangent to X at q is contained in $\sum_i T_{\Gamma,p_i}$, the subspace generated by the tangents to Γ at the d points $\{p_i\}$. Moreover, we call it *hyperelliptic d-tangential* if $(\Gamma, \{p_i\})$ is a hyperelliptic curve marked at d Weierstrass points. We prove they give rise to $d \times d$ matrix KP and KdV elliptic solitons respectively

Section 3: we associate to any such cover a d-tangential polynomial and a curve in a particular ruled surface $\mathcal{S} \to X$, through which the cover factors. We give a recursive construction of all d-tangential polynomials and deduce simple equations for a family of d-tangential covers already considered in [2] (see also [8]).

Section 4: we construct all *d-tangential polynomials* in terms of the B-A function of (X, q).

Section 5: given any $\mu \in \mathbb{N}^4$ satisfying $\mu_\circ + 1 \equiv \mu_j \pmod 2$ for $j = 1, 2, 3$, we construct a $(3 + \sum_i \mu_i)$-dimensional family of 2×2 matrix KdV elliptic solitons.

2 d-tangential covers and $d \times d$ matrix KP elliptic solitons

We fix hereafter a lattice $\Lambda \subset \mathbb{C}$ and a local coordinate, say z, at the origin of the elliptic curve $(X, q) := (\mathbb{C}/\Lambda, 0)$. To any projection $\pi : \Gamma \to X$ we associate the Abel embedding $\Gamma \to Jac\Gamma$ into its generalized Jacobian and dual morphism $\pi^* : X \to Jac\Gamma$, $q' \mapsto \mathcal{O}_\Gamma\big(\pi^*(q' - q)\big)$. The tangent space to Γ at any smooth point p, denoted $T_{\Gamma,p}$, can therefore be identified with its image in $\mathrm{H}^1(\Gamma, \mathcal{O}_\Gamma)$, the tangent space to $Jac\Gamma$ at its origin. We will also let $K(\Gamma)$ and $K(X)$ denote the corresponding fields of meromorphic functions.

Lemma 1 *([7], 1.4, p. 613)*
Given any projection $\pi : \Gamma \to X$, the derivative of its dual morphism $\pi^* : JacX \to Jac\Gamma$ injects $T_{X,q} = \mathrm{H}^1(X, \mathcal{O}_X)$ into $\mathrm{H}^1(\Gamma, \mathcal{O}_\Gamma)$.

Proof : The Albanese morphism $Alb(\pi) : Jac\Gamma \to JacX$, $M \mapsto det(\pi_*(M)) \otimes det(\pi_*(\mathcal{O}_\Gamma))^{-1}$ composed with π^* is the multiplication by $deg(\pi)$. Hence $Ker(\pi^*)$ is finite and $d(\pi^*)$ injective. ■

Definition 2
Let $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to X$ be a projection marked at d points of the fibre $\pi^{-1}(q)$.

1. *We will call π a d-tangential cover if and only if it satisfies the following conditions:*
   (a) $d(\pi^*)(T_{X,q}) \subset \sum_{i=1}^{d} T_{\Gamma,p_i} \subset \mathrm{H}^1(\Gamma, \mathcal{O}_\Gamma)$;
   (b) $d(\pi^*)(T_{X,q}) \nsubseteq \sum_{i \neq j} T_{\Gamma,p_i}$ for any $1 \leq j \leq d$.

2. *If Γ is a hyperelliptic curve and any p_i a Weierstrass point, we will say π is hyperelliptic d-tangential. In the latter case there exists a unique involution, denoted $\tau_\Gamma : \Gamma \to \Gamma$, fixing $\{p_i\}$ and with quotient curve isomorphic to $\mathbb{P}^1$.*

Remark 1

1. *The above condition $1(b)$ is equivalent to $\mathrm{h}^0(\Gamma, \mathcal{O}_\Gamma(\sum_i p_i)) = 1$ and always true if $d = 1$. Skipping it when $d > 1$ could give us superfluous marked points, meaning that π could be a d'-tangential cover for some $1 \leq d' < d$. This weaker notion still gives rise to $d \times d$ matrix KP elliptic solitons as was shown in [2] (see also [1] and [8]-1.13).*

2. *The equality $\mathrm{h}^0(\Gamma, \mathcal{O}_\Gamma(\sum_i p_i)) = 1$ is also true as long as Γ is a hyperelliptic curve of genus $g \geq d$ and p_i a Weierstrass point for any $i = 1, \cdots, d$.*

Theorem 3 (*d-tangency criterion [8]-1.8*)

A d-marked cover $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to X$ is d-tangential if and only if $\mathrm{h}^0(\Gamma, \mathcal{O}_\Gamma(\sum_i p_i)) = 1$, $\{p_i\} \subset \pi^{-1}(q)$ and there exists a morphism $\kappa : \Gamma \to \mathbb{P}^1$, called henceforth d-tangential, such that:

1. *κ is holomorphic outside $\pi^{-1}(q)$;*

2. *over a neighbourhood of $\pi^{-1}(q)$, the divisor of poles of $\kappa + \pi^*(\frac{1}{z})$ is equal to $\sum_i p_i$.*

Lemma 4 (*[5] Assertion, p.289*)

Let $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to X$ be a d-tangential cover, equipped with a d-tangential function $\kappa : \Gamma \to \mathbb{P}^1$ and a local coordinate at any p_i, say λ_i, such that $\kappa + \pi^*(\frac{1}{z}) = \frac{1}{\lambda_i} + O(\lambda_i)$.

Then, for any $\omega \in \Lambda$ there exists a holomorphic function $\varphi_\omega : \Gamma \setminus \{p_i\} \to \mathbb{C}$ with the following essential singularity at any p_i :

$$\varphi_\omega(\lambda_i) = exp(\frac{\omega}{\lambda_i})(1 + O(\lambda_i)).$$

Lemma 5

Let $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to X$ be a hyperelliptic d-tangential cover. Then:

1. *there exists a unique d-tangential function $\kappa : \Gamma \to \mathbb{P}^1$ satisfying $\kappa \circ \tau_\Gamma = -\kappa$;*

2. *there exists a projection $f : \Gamma \to \mathbb{P}^1$ with pole divisor $(f)_\infty = \sum_i 2p_i$ and same principal part as $\left(\kappa + \pi^*(\frac{1}{z})\right)^2$ at each Weierstrass point p_i.*

Proof :

1. Let $\kappa : \Gamma \to \mathbb{P}^1$ be the unique d-tangential function, up to an additive constant. One can first check that $-\tau_\Gamma^*(\kappa)$ is also d-tangential and has same principal parts as κ at $\{p_i\}$. Hence $\kappa + \tau_\Gamma^*(\kappa)$ is constant, say $c \in \mathbb{C}$. It follows that $\kappa + \frac{c}{2}$ is τ_Γ-anti-invariant.

2. Pick any $i = 1, \cdots, d$ and let λ_i and $f_i : \Gamma \to \mathbb{P}^1$ denote, respectively, the τ_Γ-anti-invariant local coordinate at p_i such that $\kappa + \pi^*(\frac{1}{z}) = \frac{1}{\lambda_i}$ and a degree-2 projection with principal part $\frac{1}{\lambda_i^2}$ at p_i. Then $f := \sum_{i=1}^d f_i$ has the required properties.■

Theorem 6

Let π be a d-tangential cover equipped with data $\left(\kappa, \{(p_i, \lambda_i)\}\right)$ as in Lemma 4. Then, the corresponding $d \times d$-matrix KP solutions are Λ-periodic in x. Analogously, if π is hyperelliptic d-tangential we obtain a family of $d \times d$-matrix KdV solutions Λ-periodic in x.

Proof : Let g denote the arithmetic genus of Γ and choose, at each p_i, a local coordinate λ_i satisfying $\kappa+\pi^*(\frac{1}{z}) = \frac{1}{\lambda_i}+O(\lambda_i)$. Given any $(x,y,t)\in\mathbb{C}^3$ and non-special degree $d+g$-1 effective divisor D, with support disjoint with $\{p_i\}$, we will denote $\psi_D(x,y,t)$ the vector Baker-Akhiezer function associated to the data $(\Gamma,\{(p_i,\lambda_i)\},D)$ (cf. [4]). It is the unique meromorphic function on $(\Gamma\setminus\{p_i\})$ such that:

1. its divisor of poles is bounded by D ;
2. in a neighbourhood of each p_i it has an essential singularity of the following type

   $$\psi_D(x,y,t)(\lambda_i) = e^{\frac{x}{\lambda_i}+\frac{y}{\lambda_i^2}+\frac{t}{\lambda_i^3}}\left(\vec{e}_i+\vec{\xi}_1^{\,i}(x,y,t)\lambda_i+O(\lambda_i^2)\right)\,, \tag{9}$$

   where $\vec{e}_i\in\mathbb{C}^d$ is the vector having a 1 at the *i-th* place and 0 everywhere else.

Recall also, or any $\omega\in\Lambda$, the holomorphic function $\varphi_\omega:\Gamma\setminus\{p_i\}\to\mathbb{C}$ constructed in Lemma 4 and having the following essential singularity at each p_i: $\varphi_\omega(\lambda_i) = e^{\frac{\omega}{\lambda_i}}(1+O(\lambda_i))$.

The uniqueness of ψ_D implies that for any $\omega\in\Lambda$ and $(x,y,t)\in\mathbb{C}^3$ we must have:

$$\psi_D(x+\omega,y,t) = \varphi_\omega\psi_D(x,y,t).$$

Comparing their developments around p_i we deduce that $\vec{\xi}_1^{\,i}(x,y,t)$ is Λ-additive in x, i.e.:

$$\forall i,\ \forall\omega\in\Lambda, \exists a\in\mathbb{C}\quad\text{such that}\quad \forall x,y,t\in\mathbb{C},\ \vec{\xi}_1^{\,i}(x+\omega,y,t) = \vec{\xi}_1^{\,i}(x,y,t)+a\vec{e}_i\,.$$

In particular the $d\times d$ matrix KP soliton

$$U(x,y,t) = -2\frac{\partial}{\partial x}\left(\vec{\xi}_1^{\,1}\cdots\vec{\xi}_1^{\,d}\right)$$

associated to $(\Gamma,\{(p_1,\lambda_1),\cdots,(p_d,\lambda_d)\},D)$ is Λ-periodic with respect to x.

At last, if π is hyperelliptic d-tangential we choose a τ_Γ-anti-invariant d-tangential function κ, local coordinates λ_i $(i=1,\cdots,d)$ and a projection $f:=\sum_i f_i:\Gamma\to\mathbb{P}^1$ as in Lemma 5. In the latter case $e^{yf(p)}$ is holomorphic outside the d marked points and has the following essential singularity at each p_i: $e^{yf(p)} = e^{y\frac{1}{\lambda_i^2}}$. We still get $d\times d$ matrix KP elliptic solitons but now ψ_D also satisfies $\psi_D(x,y,t) = \psi_D(x,0,t)e^{yf}$ as well as $\partial_y\psi_D = f\psi_D$, implying that $U = -2\frac{\partial}{\partial x}\left(\vec{\xi}_1^{\,i}\right)$ is independent of y and solves the KdV equation as explained in the Introduction. ∎

3 d-tangential covers and polynomials

Let z denote the canonical coordinate of $X=\mathbb{C}/\Lambda$ at its origin $q=0$, and let U and $\overline{U}$ denote $U:=X\setminus\{q\}$ and some neighbourhood of q. We start constructing a ruled surface through which any d-tangential cover factors.

Definition 7

1. *We define the ruled surface $\pi_{\mathcal{S}}:\mathcal{S}\to X$ by glueing the fibers of $\mathbb{P}^1\times U$ and $\mathbb{P}^1\times\overline{U}$ over each $q'\in U\cap\overline{U}$ by means of a translation as follows:*

   $$\forall q'\in U\cap\overline{U}\quad\text{we identify}\quad (T\,,q')\in\mathbb{P}^1\times U\quad\text{with}\quad (\overline{T}\text{-}\tfrac{1}{z(q')}\,,q')\in\mathbb{P}^1\times\overline{U}.$$

2. *The infinity sections* $q' \in U \mapsto (\infty, q') \in \mathbb{P}^1 \times U$ *and* $q' \in \overline{U} \mapsto (\infty, q') \in \mathbb{P}^1 \times \overline{U}$ *get glued together defining a particular one denoted by* $C_o \subset S$.

3. *Given any* $Q(T) \in K(X)[T]$*, considered as a rational morphism* $\mathbb{P}^1 \times U \subset \mathcal{S} \to \mathbb{P}^1$*, the zero-divisors* $\{Q(T) = 0\} \subset \mathbb{P}^1 \times U$ *and* $\{Q(\overline{T}\text{-}\frac{1}{z}) = 0\} \subset \mathbb{P}^1 \times \overline{U}$ *get glued over* $U \cap \overline{U}$*, defining one in* $\mathcal{S}$ *denoted* D_Q.

Remark 2
Choose (T^{-1}, z) *as couple of local coordinates at* $p_o := (\infty, q) \in \mathbb{P}^1 \times X$ *and let* p_1 *denote the point infinitely near* p_o *corresponding to the tangent direction -1. By blowing up* p_o*, then* p_1*, and contracting the strict transform of* $\mathbb{P}^1 \times \{q\}$*, we obtain a ruled surface isomorphic to* $\mathcal{S}$.

Proposition 8
Let $\kappa_{\mathcal{S}} : \mathcal{S} \to \mathbb{P}^1$ *correspond to the first projection* $T : \mathbb{P}^1 \times X \to \mathbb{P}^1$*. Then:*

1. *the divisor of zeroes and poles of* $\kappa_{\mathcal{S}}$ *is equal to* $D_T\text{ - }(C_o + \mathcal{S}_q)$;
2. *the restriction of* $\kappa_{\mathcal{S}} + \pi_{\mathcal{S}}^*(\frac{1}{z})$ *to* $\mathbb{P}^1 \times \overline{U}$ *has a simple pole along* C_o;
3. C_o *has* 0 *self-intersection and* $K_{\mathcal{S}}$*, the canonical divisor of* S*, is linearly equivalent to* -2C_o.

Proof.

1. $\kappa_{\mathcal{S}}$ restricts over the open subsets $\mathbb{P}^1 \times U$ and $\mathbb{P}^1 \times \overline{U}$ to T and $\overline{T}\text{-}\frac{1}{z}$, respectively. Hence, its divisor of zeros and poles is $D_T\text{ - }(C_o + \mathcal{S}_q)$.
2. It also follows that $\kappa_{\mathcal{S}} + \pi_{\mathcal{S}}^*(\frac{1}{z})$ is given by $\overline{T}$ over $\mathbb{P}^1 \times \overline{U}$ and has a simple pole along C_o.
3. The section C_o having genus 1, the adjunction formula gives $1 = 1 + \frac{1}{2}C_o.(\text{-}\,C_o)$, implying $C_o.C_o = 0$. The wedge products $dT \wedge dz$ (on $\mathbb{P}^1 \times U$) and $d\overline{T} \wedge dz$ (on $\mathbb{P}^1 \times \overline{U}$) get glued over $U \cap \overline{U}$, defining a meromorphic differential with divisor class -2C_o as announced. ■

Lemma 9
Let $\pi : (\Gamma, \{p_i\}) \to X$ *be a d-tangential cover of degree n equipped with a d-tangential function* $\kappa : \Gamma \to \mathbb{P}^1$*. Then, its characteristic polynomial with respect to the degree-n algebraic extension* $K(\Gamma)/\pi^*\big(K(X)\big)$*, say* $P_\kappa(T) = T^n + \sum_{j=1}^n \alpha_{j,\kappa} T^{n\text{-}j} \in K(X)[T]$*, has the following properties:*

1. *any coefficient* $\alpha_{j,\kappa}$ *is holomorphic outside* q *and* $(\alpha_{j,\kappa})_\infty \leq jq$*, i.e.:* $\alpha_{j,\kappa} \in \mathrm{H}^0\big(X, \mathcal{O}_X(jq)\big)$;
2. *all coefficients of* $z^d P_\kappa(T\text{-}\frac{1}{z}) =: z^d\, T^n + \sum_{j=1}^n a_{j,\kappa} T^{n\text{-}j}$ *are holomorphic at* q;
3. $a_{j,\kappa}$ *vanishes to order* $\geq (d\text{-}j)$ *at* q *for all* $j < d$ *and there exists* $l \geq d$ *such that* $a_{l,\kappa}(q) \neq 0$.

Proof

1. Up to a sign $\alpha_{j,\kappa}$ is the j-th symmetric function of κ with respect to π. Recall also that κ is holomorphic outside $\pi^{-1}(q)$ and has, at any point $p \in \pi^{-1}(q)$, a pole of order bounded by $ind_\pi(p)$, the ramification index of π at p. Hence $\alpha_{j,\kappa}$ is holomorphic outside q while having at q a pole of order bounded by j.
2. Analogously, up to a sign $a_{j,\kappa} z^{-d}$ is the j-th symmetric function of $\kappa + \pi^*(\frac{1}{z})$. The latter has a simple pole at any marked point p_i and is holomorphic elsewhere in $\pi^{-1}(q)$. Hence $a_{j,\kappa} z^{-d}$ must have a pole at q of order bounded by $min\{d, j\}$.

3. One can check that $a_{l,\kappa}z^{-d}$ has order d (at least) for $l = \sum_{i=1}^{d} ind_\pi(p_i)$. In other words, $z^d P_\kappa(T - \frac{1}{z})$ has the announced properties. ■

Definition 10

A monic polynomial $P(T) = T^n + \sum_{j=1}^{n} \alpha_j T^{n-j} \in K(X)[T]$ will be called d-tangential if and only if it satisfies the following conditions:

1. *$\forall j = 1, \cdots, n$ the function α_j is holomorphic outside q and has a pole of order $\leq j$ at q;*
2. *all coefficients of $z^d P(T - \frac{1}{z}) =: z^d T^n + \sum_{j=1}^{n} a_j T^{n-j}$ are holomorphic at q;*
3. *d is the least positive integer satisfying the above property (i.e.: $\exists j \leq n$ such that $a_j(0) \neq 0$).*

We will let $\theta_{d,n}(X,z)$ denote the subset of d-tangential polynomials of degree n. The affine subspace cut out in $K(X)[T]$ by the first two conditions is the union $\Theta_{d,n}(X,z) := \cup_{i=1}^{d} \theta_{i,n}(X,z)$.

Example 1

Let $\wp \in K(X)$ denote the unique meromorphic function with a double pole at q and local development $\wp(z) = \frac{1}{z^2} + O(z^2)$, and $\wp'$ its derivative. Then $P(T) = T^3 - 3\wp T + \wp' + b\wp$ belongs to $\theta_{2,3}(X,z)$ for any $b \neq 0$, and $R(T) = T^3 - (2c+1)\wp T + c\wp'$ belongs to $\theta_{2,3}(X,z)$ for any $c \neq 1$.

One can also check that for any $d, n \in \mathbb{N}^*$, $\Theta_{d,d}(X,z) = T^d + \sum_{j=1}^{d} \mathrm{H}^0\big(X, \mathcal{O}_X(jq)\big) T^{d-j}$ and $dim\, \Theta_{d,d}(X,z) = \sum_{j=1}^{d} j = \frac{1}{2} d(d+1)$ while $\Theta_{0,n}(X,z)$ is empty.

Lemma 11

Let $\Delta, \Delta^{-1} : K(X)[T] \to K(X)[T]$ denote the $K(X)$-lineal morphisms such that

$$\forall m \geq 0, \ \ \Delta(T^m) = mT^{m-1} \quad and \quad \Delta^{-1}(T^m) = \frac{1}{m+1} T^{m+1}.$$

For any $P \in K(X)[T]$ they satisfy:

1. *$\Delta \circ \Delta^{-1}(P) = P$ and $\Delta^{-1} \circ \Delta(P) = P - P(0)$;*
2. *$\forall n > d, \ \ P \in \Theta_{d,n}(X,z)$ implies $\frac{1}{n}\Delta(P) \in \Theta_{d,n-1}(X,z)$;*
3. *if $\Theta_{d,n}(X,z) \neq \emptyset$ the map $\frac{1}{n}\Delta : \Theta_{d,n}(X,z) \to \Theta_{d,n-1}(X,z)$ has kernel $\mathrm{H}^0\big(X, \mathcal{O}_X(dq)\big)$.*

Theorem 12 - *Recursive formula*

For any $0 < d < j \leq n$ and $P \in \Theta_{d,j-1}(X,z)$ there exists $\alpha \in \mathrm{H}^0\big(X, \mathcal{O}_X(jq)\big)$, unique modulo $\mathrm{H}^0\big(X, \mathcal{O}_X(dq)\big)$, such that $j\Delta^{-1}(P) + \alpha$ belongs to $\Theta_{d,j}(X,z)$. It follows that $\Theta_{d,n}(X,z)$ is not empty and has dimension $(n-d)d + dim\big(\Theta_{d,d}(X,z)\big) = (n-d)d + \frac{1}{2}d(d+1) = nd - \frac{1}{2}d(d-1)$.

Proof: The function $j\Delta^{-1}(P)(\frac{1}{z})$ has a pole of order j at q, because $j\Delta^{-1}(P) \in K(X)[T]$ is monic of degree j. Hence, there exists $\alpha \in \mathrm{H}^0\big(X, \mathcal{O}_X(jq)\big)$ such that $z^d\big(\alpha + j\Delta^{-1}(P)(\frac{1}{z})\big)$ is holomorphic at q, implying that $j\Delta^{-1}(P) + \alpha \in \Theta_{d,j}(X,z)$, as well as the other properties. ■

Lemma 13 - *Reducibility criterion*

The subset

$$\bigcup \theta_{d',n'}(X,z)\theta_{d-d',n-n'}(X,z) \ \subset \theta_{d,n}(X,z),$$

with the union taken over all d', n' such that $0 < d' < d$ and $0 < n' < n$, contains all reducible elements. In other words $P \in \theta_{d,n}(X,z)$ is reducible in $K(X)[T]$ if and only if it factors as $P = QR$, with $Q \in \theta_{d',n'}(X,z)$ and $R \in \theta_{d-d',n-n'}(X,z)$ for some $0 < d' < d$ and $0 < n' < n$.

Proof : If $P \in \theta_{d,n}(X,z)$ is reducible we can assume it factors as as product $P = QR$ of two monic polynomials with coefficients holomorphic outside $q \in X$. A straightforward verification confirms they must satisfy property 11.1) above, as well as 11.2) for some $d', d'' \in \mathbb{N}^*$. In particular all coefficients of $z^{d'+d''}P(T\text{-}\frac{1}{z}) = z^{d'}Q(T\text{-}\frac{1}{z})z^{d''}R(T\text{-}\frac{1}{z})$ must be holomorphic at q and its restriction to $z = 0$ can not vanish, implying $d' + d'' = d$ as asserted. ■

Theorem 14

For any $1 \le d \le n$ $\theta_{d,n}(X,z)$ *is an open dense subset of* $\Theta_{d,n}(X,z)$*, with irreducible generic element.*

Proof : The complement of $\theta_{d,n}(X,z) \subset \Theta_{d,n}(X,z)$ is the affine subspace $\Theta_{d\text{-}1,n}(X,z)$, which has positive codimension (equal to $n\text{-}d+1$). Hence, $dim\big(\theta_{d,n}(X,z)\big) = nd\text{-}\frac{1}{2}d(d\text{-}1)$, bigger than the dimension of the reducible ones, and its generic element must be irreducible. ■

Remark 3

For any $P(T) \in \Theta_{d,n}(X,z)$ *the coefficients of* $z^dP(T\text{-}\frac{1}{z}) = z^dT^n + \sum_{j=1}^n a_jT^{n\text{-}j}$ *are holomorphic at* q*. We also know that* $a_j(z) = z^{d\text{-}j}b_j(z)$*, for any* $j = 1,\cdots,d$*, with* b_j *holomorphic at* q *and* $b_1 = \text{-}n$ *(cf. proof of Lemma 9.3)). Thus, we are naturally lead to the following definitions.*

Definition 15

To any $P(T) \in \Theta_{d,n}(X,z)$ *we associate*

$$V_d(P) := z^dP(T\text{-}\frac{1}{z})_{|z=0} = \sum_{j=d}^{n} a_j(0)T^{n\text{-}j} \quad \textit{as well as} \quad M_d(P) := w^d\,\text{-}nw^{d\text{-}1} + \sum_{j=2}^{d} b_j(0)w^{d\text{-}j}$$

and let

$$V_d : \Theta_{d,n}(X,z) \to \mathbb{C}_{n\text{-}d}[T] \quad \textit{and} \quad M_d : \Theta_{d,n}(X,z) \to \mathbb{C}_d[w]$$

denote the corresponding (affine) linear maps.

Lemma 16

For any $1 \le d \le n$ *and generic* $P \in \theta_{d,n}(X,z) \subset \Theta_{d,n}(X,z)$:

1. $V_d : \Theta_{d,n}(X,z) \to \mathbb{C}_{n\text{-}d}[T]$ *is surjective with kernel* $\Theta_{d\text{-}1,n}(X,z)$;
2. $V_d(P)$ *has degree* $n\text{-}d$ *and only simple roots*;
3. $\frac{d!}{n!}\Delta^{\circ(n\text{-}d)} : \Theta_{d,n}(X,z) \to \Theta_{d,d}(X,z)$ *is surjective*;
4. $M_d(P)$ *has* d *simple non-zero roots.*

Proof :

1. The first item implies the second one and can be proved by induction on n. Let us indeed assume $V_d : \Theta_{d,n\text{-}1}(X,z) \to \mathbb{C}_{n\text{-}1\text{-}d}[T]$ is surjective. The result follows by coupling the surjectivity of $\Delta := \partial_T$ with the fact that it commutes with V_d.
2. According to Lemma 11 and Theorem 12 the linear map $\frac{1}{j}\Delta : \Theta_{d,j}(X,z) \to \Theta_{d,j\text{-}1}(X,z)$ is surjective for any $d < j \le n$, and $\frac{d!}{n!}\Delta^{\circ(n\text{-}d)} : \Theta_{d,n}(X,z) \to \Theta_{d,d}(X,z)$ as well.
3. Pick any $P_o \in \Theta_{d,n}(X,z)$ such that $\frac{d!}{n!}\Delta^{n\text{-}d}(P_o) = T^d$. Then one can check that $M_d(P_o) = w^d + \sum_{j=1}^d(\text{-}1)^j\binom{n}{j}w^{d\text{-}j}$, which has d simple non-zero roots. The latter property being an open one the result follows. ■

Remark 4

Given any $P \in \theta_{d,n}(X,z) \subset K(X)[T]$ *we will consider its zero-divisor* $D_P \subset \mathcal{S}$ *(see Definition 8.3)) and let* $e : \hat{D}_P \subset \hat{\mathcal{S}} \to D_P \subset \mathcal{S}$ *denote its strict transform by the blowing up of* $p_{\mathcal{S}} \in \mathcal{S}$*. It comes with projections* $\pi := \pi_{\mathcal{S}} \circ e : \hat{D}_P \to X$ *and* $\kappa := \kappa_{\mathcal{S}} \circ e : \hat{D}_P \to \mathbb{P}^1$*.*

Lemma 17

The zero-divisor $D_P \subset \mathcal{S}$ *of any* $P \in \theta_{d,n}(X,z)$ *has the following local and global properties:*

1. D_P *is defined on the open subset* $\mathbb{P}^1 \times (X \setminus \{q\}) \subset \mathcal{S}$ *by the equation* $P(T,z) = 0$*;*
2. D_P *is defined over a neighbourhood of* $p_{\mathcal{S}} \in \mathcal{S}$ *by the equation* $z^d \overline{T}^{\,\text{-}n} P(\overline{T}\text{-}\frac{1}{z}, z) = 0$*;*
3. $D_P \cap C_o = \{p_{\mathcal{S}}\}$ *and* $D_P \cap S_q = \{p_{\mathcal{S}}\} \cup \{(\overline{T}, q),\ V_d(P)(\overline{T}) = 0\}$*;*
4. *its tangent cone at* $p_{\mathcal{S}}$ *is defined by the equation* $\overline{T}^{\,\text{-}d} M_d(P)(z\overline{T}) = 0$*;*
5. D_P *is linearly equivalent to* $nC_o + d\mathcal{S}_q$*.*

Proof : 1), 2) and 3) - The first two items follow from the construction of $\mathcal{S}$ (cf. Remark 3.2)), while the third one from the definition of $\theta_{d,n}(X,z)$.

4) - Over a neighbourhood of $\pi^{\text{-}1}(q)$, $\kappa + \pi^*(\frac{1}{z})$ has pole-divisor equal to $\sum_i p_i$, and characteristic polynomial with respect to π equal to $P_\kappa(\overline{T}\text{-}\frac{1}{z}) =: \overline{T}^n + \sum_{j=1}^n c_{j,\kappa} \overline{T}^{n\text{-}j}$. Up to a sign, its coefficients are the symmetric functions of $\kappa + \pi^*(\frac{1}{z})$ with respect to π and satisfy:

$$c_{j,\kappa} = \frac{1}{z^j} O(1) \quad \text{for any} \quad 1 \le j \le d \quad (\text{resp.: } c_{j,\kappa} = \frac{1}{z^d} O(1) \quad \text{for any} \quad d \le j \le n).$$

On the other hand D_P is given over a neighbourhood of $p_{\mathcal{S}} \in \mathcal{S}$ as zero-divisor of

$$\overline{T}^{\,\text{-}n} z^d P_\kappa(\overline{T}\text{-}\frac{1}{z}) =: z^d + \sum_{j=1}^n z^d c_{j,\kappa} \overline{T}^{\,\text{-}j} = z^d + \sum_{j=1}^d z^j c_{j,\kappa} z^{d\text{-}j} \overline{T}^{\,\text{-}j} + \sum_{j>d}^n z^d c_{j,\kappa} \overline{T}^{\,\text{-}j}.$$

Hence, its tangent cone at $p_{\mathcal{S}}$ is given by the equation $z^d + \sum_{j=1}^d (z^j c_{j,\kappa})_{|z=0} z^{d\text{-}j} \overline{T}^{\,\text{-}j} = 0$ and the assertion follows.

5) - Once we know that $D_P \cap C_o = \{p_{\mathcal{S}}\}$ and D_P has a singularity of multiplicity d at $p_{\mathcal{S}}$ transverse to C_o, we deduce $D_P.C_o = d$ and $D_P.\mathcal{S}_q = n$ implying $D_P \in |nC_o + d\mathcal{S}_q|$. ■

Proposition 18

For any $1 \le d \le n$ *and generic* $P \in \theta_{d,n}(X,z)$ *:*

1. D_P *has an ordinary singularity at* $p_{\mathcal{S}}$*, of multiplicity* d *and transverse to* $C_o + \mathcal{S}_q$*;*
2. D_P *is irreducible, smooth outside* $p_{\mathcal{S}}$ *and of arithmetic genus* $nd + 1 \text{-} d$*.*

Proof

1. The tangent cone of D_P at $p_{\mathcal{S}}$ is the zero-locus of the degree-d form $\overline{T}^{\,\text{-}d} M_d(P)(z\overline{T})$. For generic P it is the union of d lines transverse to $C_o + \mathcal{S}_q$.

2. According to the preceding results $K_{\mathcal{S}} = -2C_o$, the generic $P \in \theta_{d,n}(X,z)$ is irreducible and $D_P \in |nC_o + d\mathcal{S}_q|$. We deduce its arithmetic genus via the adjunction formula. Forcing D_P to be smooth at every point in $\mathcal{S}_q \setminus \{p_{\mathcal{S}}\}$ or outside $\mathcal{S}_q$ are open conditions on $\theta_{d,n}(X,z)$. The first one is true as soon as $V_d(P)$ has $n-d$ simple roots. As for the second one, one can check that for almost any $a \in \mathbb{C}$ the divisor D_{P+a} is smooth outside $\mathcal{S}_q$. ■

Theorem 19

For any $1 \leq d < n$ and generic $P \in \theta_{d,n}(X,z)$ as above:

1. *$\pi : \hat{D}_P \to X$ is non-ramified at the d pre-images $e^{-1}(p_S) \subset \pi^{-1}(q)$;*
2. *$\hat{D}_P$ is smooth of genus $nd - \frac{1}{2}d(d+1) + 1$;*
3. *κ is holomorphic outside $\pi^{-1}(q)$ and $\kappa + \pi^*(\frac{1}{z})$ has simple poles at $e^{-1}(p_S)$;*
4. *$\pi : (\hat{D}_P, e^{-1}(p_S)) \to X$ is a d-tangential cover.*

Proof: The first three items follow immediately from the preceding properties. As for the fourth one, assume that generically $\mathrm{h}^0(\hat{D}_P, \mathcal{O}_{\hat{D}_P}(\sum_i p_i)) > 1$ and recall the relation :

$$dim\big(\Theta_{d,n}(X,z)\big) - dim\big(\Theta_{d-1,n}(X,z)\big) = n - d + 1 \geq 2.$$

Then, for any non constant $h \in \mathrm{H}^0(\hat{D}_P, \mathcal{O}_{\hat{D}_P}(\sum_i p_i))$ there exists at least one $\lambda \in \mathbb{C}$ such that $\kappa + \lambda h + \pi^*(\frac{1}{z})$ has less than d poles. Hence, the characteristic polynomials of $\{\kappa + \lambda h, \lambda \in \mathbb{C}\}$ define a 1-dimensional family in $\Theta_{d,n}(X,z)$ intersecting $\Theta_{d-1,n}(X,z) = \bigcup_{j=1}^{d-1} \theta_{j,n}(X,z)$.

In particular, $dim\big(\Theta_{d,n}(X,z)\big)$ should be bounded by $dim\big(\Theta_{d-1,n}(X,z)\big) + 1$. Contradiction!. ■

Corollary 20 *([2], p.288; see also [8], p.546)*

For any $n > d \geq 1$ there exists a family of dimension $\frac{1}{2}d(2n-d+1)$, of smooth d-tangential covers of degree n over (X,q) and genus $\frac{1}{2}d(2n-d+1)-(d-1)$. They give rise to a $(2nd+1-d^2)$-dimensional family of $d \times d$ matrix KP elliptic solitons.

4 d-tangential polynomials in terms of the B-A function

Building upon classical properties of the Weierstrass Sigma and Zeta functions, $\sigma(z)$ and $\zeta(z) := \frac{\partial}{\partial z}\ln\sigma(z)$ (e.g.: [5] p. 283), it can be proved that for any $x \in \mathbb{C}$ the Baker-Akhiezer function $\psi_q(x)(z) := e^{x\zeta(z)}\frac{\sigma(z-x)}{\sigma(z)}$ is well defined on X and holomorphic outside q, where it has the local development $\psi_q(x)(z) = e^{\frac{x}{z}}\frac{1}{z}O(1)$. Once x is formally replaced by $\Delta := \partial_T$ in its Maclaurin development, it defines a linear application $\psi_q(\Delta)(z) : \mathbb{C}[T] \to K(X)[T]$. Given the fact that $e^{-\frac{\Delta}{z}}(P(T)) = P(T-\frac{1}{z})$ for any $P(T) \in K(X)[T]$, the outcome is an isomorphism between the subspaces $M_n \subset \mathbb{C}[T]$ of degree-n monic polynomials and the 1-tangential polynomials $\Theta_{1,n}(X,z) \subset K(X)[T]$ for any n (cf. [7]). In order to prove analogous characterizations for any $n > d \geq 1$, we will need the following properties of $\psi_q(x)(z)$ and its k-th partial derivatives as functions of the parameter x.

Lemma 21

The Maclaurin development in x of $F(x,z) := \sigma(z-x)$ and $\psi(x,z) := \psi_q(x)(z)$ satisfy:

1. *$\psi(x,z) := 1 + \sum_{m=2}^{\infty} \alpha_m x^m$, with α_m meromorphic on X;*

2. *for any $m \geq 2$ the coefficient α_m has pole-divisor $(\alpha_m)_\infty = mq$ and*

$$z^m \alpha_{m|z=0} = \frac{1\text{-}m}{m!} \;;$$

3. *for any $m \geq 1$ the m-th partial derivative $\psi^{(m)} := \partial_x^m \psi(x,z)$ is equal to*

$$\psi^{(m)} = \frac{e^{x\zeta(z)}}{m!F(0,z)}\Big(\sum_{j=0}^{m} \tbinom{m}{j} \zeta^{m\text{-}j} \partial_x^j F\Big) = m!\alpha_m + O(x)$$

and satisfies:

(a) *$z^{m+1}e^{\text{-}\frac{x}{z}}\psi^{(m)} = \sum_{j=0}^{\infty} \beta_j x^j$ has all its coefficients holomorphic at q;*

(b) *its restriction to $z = 0$ satisfies $z^{m+1}e^{\text{-}\frac{x}{z}}\psi^{(m)}_{|z=0} = \frac{\text{-}x}{m!}\big(1 + O(x)\big)$.*

Proof: 1) - Developping $\psi(x,z) := e^{x\zeta}\frac{F(x,z)}{F(0,z)} = \big(1+\sum_{j=1}^{\infty}\frac{1}{j!}\zeta^j x^j\big)\big(1+\sum_{i=1}^{\infty}\frac{\partial_x^i F}{F}(0,z)x^i\big)$ with respect to x we obtain the following formula:

$$m!\alpha_m = \zeta^m + \sum_{k=1}^{m} \tbinom{m}{k}\zeta^{m\text{-}k}\frac{\partial_x^k F}{F}(0,z).$$

2) - For any $m \geq 2$ a direct calculation gives $z^m m!\alpha_m(z) = 1\text{-}m + O(z)$. Hence $(\alpha_m)_\infty = mq$.

3) - Recall that $z\zeta(z)$, $\zeta(z)\text{-}\frac{1}{z}$ and $\frac{z}{F(0,z)} = \frac{z}{\sigma(z)}$ are holomorphic in a neighbourhood of q, with values at q equal to 1, 0 and 1 respectively. Hence the Maclaurin development in x of

$$z^{m+1}e^{\text{-}\frac{x}{z}}\psi^{(m)} = \frac{e^{x(\zeta\text{-}\frac{1}{z})}}{m!}\,\frac{z}{F(0,z)}\Big(\sum_{j=0}^{m}\tbinom{m}{j}(z\zeta)^{m\text{-}j}z^j\partial_x^j F\Big)$$

has all its coefficients holomorphic at q. It also follows that its restriction to $z = 0$ is equal to $\frac{1}{m!}F(x,0)$, and therefore to $\frac{\text{-}x}{m!}\big(1 + O(x)\big)$. ■

Theorem 22

For any $n \geq 1$ the linear map $\psi := \psi(\Delta, z) : \mathbb{C}[T] \to K(X)[T]$ restricts to an isomorphism from M_n onto $\theta_{1,n}(X,z) = \Theta_{1,n}(X,z)$ (i.e.: $\Theta_{1,n}(X,z) = \psi(M_n)$). Moreover, for any $n \geq d > 1$:

$$\Theta_{d,n}(X,z) = \psi(M_n) \oplus \bigoplus_{k=1}^{d\text{-}1} \psi^{(k)}(T\mathbb{C}_{n\text{-}1\text{-}k}[T]).$$

5 d-tangential polynomials and matrix KdV elliptic solitons

At last we consider the canonical involution $[\text{-}1] : (X,q) \to (X,q)$, fixing $\omega_o := q$ as well as the three other half-periods $\{\omega_j,\, j = 1,2,3\}$ and satisfying : $[\text{-}1]^*(z) = \text{-}z$.

Recall also that given a hyperelliptic curve Γ there exists a unique involution $\tau_\Gamma : \Gamma \to \Gamma$ such that the quotient curve is isomorphic to $\mathbb{P}^1$. Its fixed points are the so-called Weierstrass points.

We gather hereafter the first basic definitions and results concerning hyperelliptic d-tangential covers (cf. [9] 4.1 p.457 and Definition 3.2)).

Definition 23

We let $\tau_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$ denote the involution defined by $(T,z) \mapsto (\text{-}T, \text{-}z)$ and $(\overline{T},z) \mapsto (\text{-}\overline{T}, \text{-}z)$ over each trivialization of $\pi_{\mathcal{S}}$ (see Definition 8.1)). It satisfies $\pi_{\mathcal{S}} \circ \tau_{\mathcal{S}} = [\,\text{-}1] \circ \pi_{\mathcal{S}}$ and has two fixed points over each half-period ω_i : one in C_o, denoted by s_i, and the other one denoted by r_i $(i = 0, .., 3)$. In particular $s_\circ = p_{\mathcal{S}} := C_\circ \cap \mathcal{S}_q$.

Proposition 24 *([8] 2.5)*

Any degree-n hyperelliptic d-tangential cover $\pi : (\Gamma, \{p_1, \cdots, p_d\}) \to X$ has unique d-tangential function $\kappa : \Gamma \to \mathbb{P}^1$ and associated morphism $\iota : \Gamma \to \mathcal{S}$ such that:

1. *$\kappa \circ \tau_\Gamma = \text{-}\kappa$ and its characteristic polynomial satisfies $P_\kappa(\text{-}T, \text{-}z) = (\text{-}1)^n P_\kappa(T, z)$;*
2. *π factors as $\pi = \pi_{\mathcal{S}} \circ \iota$, with $\iota(\Gamma) = D_{P_\kappa}$, the zero-divisor of $P_\kappa(T)$ (see Definition 7.3));*
3. *$\iota \circ \tau_\Gamma = \tau_{\mathcal{S}} \circ \iota$, hence $\iota(\Gamma)$ is $\tau_{\mathcal{S}}$-invariant and $\pi \circ \tau_\Gamma = [\,\text{-}1] \circ \pi$;*
4. *$\iota^{\text{-}1}(s_0) = \{p_1, \cdots, p_d\}$ while $\cup_{i=0}^{3} \iota^{\text{-}1}(r_i)$ is made of all other Weierstrass points.*

Definition 25

1. *We say that a $\tau_{\mathcal{S}}$-invariant effective divisor D of $\mathcal{S}$ has type $(\gamma_i) \in \mathbb{N}^4$ if and only if γ_i is the multiplicity of D at r_i for any $i = 0, \cdots, 3$.*
2. *For any $n \geq d \geq 1$ we let $\Theta^{\tau}_{d,n}(X, z) \subset \Theta_{d,n}(X, z)$ denote the affine subspace of so-called symmetric d-tangential polynomials $P(T, z)$ such that $P(\text{-}T, \text{-}z) = (\text{-}1)^n P(T, z)$.*

Theorem 26 *(cf.* [9] 6.2)

For any $\mu := (\mu_i) \in \mathbb{N}^4$ satisfying $\mu_0 + 1 \equiv \mu_1 \equiv \mu_2 \equiv \mu_3 (mod 2)$ and $n \in \mathbb{N}$ such that $2n + 1 = \sum_i \mu_i$ there exists a unique $\tau_{\mathcal{S}}$-invariant irreducible curve of type μ in $|nC_o + \mathcal{S}_q|$.

Lemma 27

Fix $\mu := (\mu_i) \in \mathbb{N}^4$ satisfying $\mu_0 + 1 \equiv \mu_1 \equiv \mu_2 \equiv \mu_3 (mod 2)$ and choose $\alpha, \beta \in \mathbb{N}^4$ equal, up to a common permutation of their coordinates, to $(2,0,0,0)$ and $(0,2,0,0)$. We then denote:

1. *$\mu^{(0)}, \mu^{(1)}, \mu^{(2)}, \mu^{(3)} \in \mathbb{N}^4$ the integer vectors μ, $\mu + \alpha$, $\mu + \beta$ and $\mu + \alpha + \beta$;*
2. *n_i such that $2n_i + 1 = \sum_{j=0}^{3} \left(\mu_j^{(i)}\right)^2$ for any $i = 0, \cdots, 3$;*
3. *n the common value $n := n_1 + n_2 = n_0 + n_3$ and $\gamma := \mu^{(1)} + \mu^{(2)} = \mu^{(0)} + \mu^{(3)} \in \mathbb{N}^4$;*
4. *$\Gamma_i \in |n_i C_o + \mathcal{S}_q|$ the unique $\tau_{\mathcal{S}}$-invariant curve of type $\mu^{(i)}$, for any $i = 0, \cdots, 3$.*

Then, any element D of the pencil generated by the divisors $\Gamma_1 + \Gamma_2$ and $\Gamma_0 + \Gamma_3$ satisfies:

1. *D is $\tau_{\mathcal{S}}$-invariant, has type γ and belongs to the linear system $|nC_o + 2\mathcal{S}_q|$;*
2. *generically, D is irreducible and has an ordinary singularity of multiplicity 2 at s_0.*

Proof: Let us only prove the last assertion. The tangent cones of $\Gamma_1 + \Gamma_2$ and $\Gamma_0 + \Gamma_3$ at s_0 are given by the equations

$$z^2 \text{-} nz\overline{T}^{\text{-}1} + n_1(n \text{-} n_1)\overline{T}^{\text{-}2} = 0 \quad \text{and} \quad z^2 \text{-} nz\overline{T}^{\text{-}1} + n_0(n \text{-} n_0)\overline{T}^{\text{-}2} = 0$$

and have no tangent line in common because $\{n_1, n \text{-} n_1\} \cap \{n_0, n \text{-} n_0\} = \emptyset$. Any reducible element of this pencil has tangent cone at s_0 given by $z^2 \text{-} nz\overline{T}^{\text{-}1} + m(n \text{-} m)\overline{T}^{\text{-}2} = 0$ for some integer $0 \leq m \leq n$. For a generic D it is given instead by $z^2 \text{-} nz\overline{T}^{\text{-}1} + a\overline{T}^{\text{-}2} = 0$ with an arbitrary coefficient $a \in \mathbb{C}$. Hence, it is irreducible for almost any a. ∎

Proposition 28

Let $e : \mathcal{S}^\perp \to \mathcal{S}$ *denote the blowing-up of* $\tau_{\mathcal{S}}$*'s fixed points* $\{s_i, r_i\}$, $\tau_{\mathcal{S}^\perp} : \mathcal{S}^\perp \to \mathcal{S}^\perp$ *the pull-back of the involution* $\tau_{\mathcal{S}}$ *and* $\Gamma_k^\perp$ *the strict transform of* Γ_k, *for any* $0 \le k \le 3$.

Then, the divisors $\Gamma_1^\perp + \Gamma_2^\perp$ *and* $\Gamma_0^\perp + \Gamma_3^\perp$ *are* $\tau_{\mathcal{S}^\perp}$*-invariant and have the following properties:*

1. *they are linearly equivalent and do not intersect each other;*

2. *they generate a pencil of divisors with smooth irreducible generic term of genus* $g := 2 + \sum_i \mu_i$.

Proof:

1. According to the adjunction formula, $\Gamma_1 + \Gamma_2$ and $\Gamma_0 + \Gamma_3$ have arithmetic genus $2n - 1$ and same multiplicities at all blown-up points $\{s_i, r_i\}$. Hence, their strict transforms remain linearly equivalent and have arithmetic genus $2n - 1 - 1 - \frac{1}{2}\sum_i(\gamma_i^2 - \gamma_i) = 2 + \sum_i \mu_i$. A direct calculation also shows they no longer intersect.

2. The latter lemma also implies they generate a pencil with irreducible generic element, smooth according to Bertini's Theorem, and $\tau_{\mathcal{S}^\perp}$-invariant just as $\Gamma_1^\perp + \Gamma_2^\perp$ and $\Gamma_0^\perp + \Gamma_3^\perp$ are. ■

Corollary 29

For any $\mu \in \mathbb{N}^4$ *as above and* $0 \le j < k \le 3$ *there exists a pencil of smooth hyperelliptic* 2*-tangential covers of degree* $n := \sum_i \mu_i^2 + 2(\mu_j + \mu_k) + 3$ *and genus* $g := 2 + \sum_i \mu_i$.

Proof: Up to a common permutation of their coordinates we can assume α, β in the latter Theorem so chosen that the scalar product $< \mu, \alpha + \beta >$ is equal to $2(\mu_j + \mu_k)$. For a generic D in the pencil generated by $\Gamma_1 + \Gamma_2$ and $\Gamma_0 + \Gamma_3$, $D^\perp \subset \mathcal{S}^\perp$ is a smooth irreducible $\tau_{\mathcal{S}^\perp}$-invariant curve of genus $g := 2 + \sum_i \mu_i$. Restricting $\pi_{\mathcal{S}} \circ e : \mathcal{S}^\perp \to X$ and $\kappa_{\mathcal{S}} \circ e : \mathcal{S}^\perp \to \mathbb{P}^1$ to $D^\perp$ equips it with the 2-marked projections $\pi : \left(D^\perp, e^{-1}(s_0)\right) \to X$ and $\kappa : \left(D^\perp, e^{-1}(s_0)\right) \to \mathbb{P}^1$.

Arguing as in the proof of Theorem 19 one can show that π is a smooth 2-tangential cover of type $\gamma := 2\mu + \alpha + \beta$. At last, it only remains to check that $\tau_{\mathcal{S}^\perp} : D^\perp \to D^\perp$ has $2g + 2$ fixed points, including $\{p_1, p_2\} := e^{-1}(s_0)$. It would then follow that $(D^\perp, \{p_1, p_2\})$ is a hyperelliptic curve marked at two Weierstrass points, but also that $\mathrm{h}^0\left(D^\perp, \mathcal{O}_{D^\perp}(p_1 + p_2)\right) = 1$ because $g \ge 2$.

Recall that the $\tau_{\mathcal{S}}$-invariant divisors $\Gamma_1 + \Gamma_2$ and $\Gamma_0 + \Gamma_3$ have singularities of same multiplicity at $\{s_0, r_0, \cdots, r_3\}$, but yet no common tangent line. Hence, D has ordinary singularities with $\tau_{\mathcal{S}}$-invariant tangent cones, implying that $\tau_{\mathcal{S}^\perp}$ inherits γ_i fixed points over each r_i. Adding $\{p_1, p_2\} := e^{-1}(s_0)$ they sum up to $2 + \sum_i \gamma_i = 6 + 2\sum_i \mu_i = 2g + 2$ as requested. ■

Example 2

The Λ*-periodic Weierstrass functions* $\wp : \mathbb{C}/\Lambda \to \mathbb{P}^1$ *and its derivative* $\wp'$, *satisfy the relation*

$$\wp'^2 = 4\Pi_{j=1}^3(\wp - e_j) = 4\wp^3 - g_2\wp - g_3$$

where $e_j := \wp(\omega_j)$, *the value of* $\wp$ *at the half-period* ω_j $(j = 1, 2, 3)$. *According to the latter corollary, the simplest family of hyperelliptic* 2*-tangential covers (i.e.: degree* $n = 4$ *and genus* $g = 3$*) is obtained by choosing* $(\mu, \alpha, \beta) = \left((1,0,0,0), (0,2,0,0), (0,0,2,0)\right)$, *in which case the pencil is generated by the zero-divisors of the following symmetric* 2*-tangential polynomials:*

$$(T^2 - \wp + e_1)(T^2 - \wp + e_2) \quad \text{and} \quad T^4 + 3(e_3 - 2\wp)T^2 + 4\wp' T - 3(\wp - e_1)(\wp - e_2).$$

The next simplest case (i.e.: genus $g = 5$ *and degree* $n = 8$*) corresponds to* $(\mu, \alpha, \beta) = \left((1,2,0,0), (0,0,2,0), (0,0,0,2)\right)$ *and the pencil generated by the* 2*-tangential polynomials:*

$$\left(T^4 + 3(e_3 - 2\wp)T^2 + 4\wp' T - 3(\wp - e_1)(\wp - e_2)\right)\left(T^4 + 3(e_2 - 2\wp)T^2 + 4\wp' T - 3(\wp - e_1)(\wp - e_3)\right)$$

and

$$\big(T^2 - \wp + e_1\big)\big(T^6 - 15\wp T^4 + 20\wp' T^3 - \frac{9}{4}(20\wp^2 - 3g_2)T^2 + 12\wp\wp' T - \frac{5}{4}\wp'^2\big).$$

In order to obtain higher degree and genus examples one needs more τ_S-invariant curves associated to other cases $(\mu, \mu+\alpha, \mu+\beta, \mu+\alpha+\beta)$ as above. The latter can indeed be done by means of A.O.Smirnov's algorithm (cf.[6]). The corresponding 34 first polynomials have been presented with his permission in Bobenko-Enolskii's fine encyclopedic survey [3].

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