ON THE BERNOULLI PROPERTY OF PLANAR HYPERBOLIC BILLIARDS

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Abstract. We consider billiards in non-polygonal domains of the plane with boundary consisting of curves of three different types: straight segments, strictly convex inward curves and strictly convex outward curves of a special kind. The map of these billiards is known to have non-vanishing Lyapunov exponents a.e. provided that the distance between the curved components of the boundary is sufficiently large, and the set of orbits having collisions only with the flat part of the boundary has zero measure. Under a few additional conditions, we prove that there exists a full measure set of the billiard phase space such that each of its points has a neighborhood contained up to a zero measure set in one Bernoulli component of the billiard map. Using this result, we show that there exists a large class of planar hyperbolic billiards that have the Bernoulli property. This class includes the billiards in convex domains bounded by straight segments and strictly convex inward arcs constructed by Donnay.

1. Introduction

A planar billiard is the mechanical system consisting of a point-particle moving freely inside a bounded domain $\Omega \subset \mathbb{R}^2$ with piecewise differentiable boundary, and being reflected off $\partial\Omega$ so that the angle of reflection equals the angle of incidence. This paper concerns hyperbolic billiards, i.e., billiards for which the corresponding map has no vanishing Lyapunov exponents. Maps with this property are not necessarily uniformly hyperbolic, but exhibit a weak form of hyperbolicity called non-uniform hyperbolicity [27].

The study of hyperbolic billiards was initiated by Sinai. In his seminal paper [28], he proved that billiards in 2-dimensional toral domains containing finitely many obstacles with strictly convex outward boundary are hyperbolic and K-mixing. In fact, Sinai billiards enjoy the Bernoulli property as well [17].

Later on, Bunimovich proved that also billiards in domains with boundary formed by strictly convex inward curves and straight segments can be hyperbolic [1, 2]. The most celebrated example of such

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a domain is the stadium, the region bounded by two semi-circles connected by two parallel segments. The only strictly convex inward curves allowed in Bunimovich billiards were arcs of circles. This limitation was eventually overcome by several researchers. Using new techniques for establishing the positivity of Lyapunov exponents [23, 32], Wojtkowski, Markarian and Donnay proved independently that besides arcs of circles many other strictly convex inward arcs can be used to construct hyperbolic billiards [16, 23, 25, 32]. Similar results were obtained by Bunimovich for a class of strictly convex arcs related to those of Donnay [5]. All these results showed that billiards in non-polygonal planar domains are hyperbolic if three conditions are fulfilled: B1) the strictly convex boundary components of the domain are of a special type, B2) the distance between these components and the other curved boundary components is sufficiently large, and B3) orbits having only collisions with straight segments of the boundary form a set of zero measure. A precise formulation of these conditions is given in Section 5.

In this paper, we address the question whether a hyperbolic billiard has the Bernoulli property (for short, 'it is Bernoulli'), i.e., whether it is isomorphic to a Bernoulli shift. The Bernoulli property is the strongest among the ergodic properties: it implies K-property, mixing and ergodicity. The Bernoulli property was proved for several billiards, including the Sinai billiards, the Bunimovich billiards, the Wojtkowski billiards and other special hyperbolic billiards [4, 11, 12, 13, 22, 24, 30]. Despite that, there remain many planar hyperbolic billiards for which not even the ergodicity has been proved. Notably, among them, there are the billiards constructed by Donnay [16]. The goal of this paper is to fill this gap: we show that there exists a large class of hyperbolic billiards, including the Donnay billiards, that have the Bernoulli property.

The key ingredient in the proof of this result is a local ergodic theorem for hyperbolic planar billiards, which is also the main result of this paper. Roughly speaking, our local ergodic theorem states that if a planar billiard satisfies conditions B1-B3 and the extra condition B4 (see Section 5), then there exists a full measure set H in the billiard phase space with the property that each element of H has a neighborhood contained (mod 0) in one ergodic component of the billiard.

Condition B4 regards the singular set of the billiard, the set formed by the elements where the billiard map is not defined or is not of twice-differentiable. This set corresponds to the trajectories that hit a corner of the billiard domain or have a tangential collision with the its boundary. Condition B4 requires the elements of the singular set whose trajectories have eventually collisions only with straight segments to form a negligible subset (in the the measure theoretical sense) of the singular set.

As a matter of fact, the neighborhood in the conclusion of the local ergodic theorem belongs (mod 0) to a single Bernoulli component of the billiard map (for the definition of a Bernoulli component, see Theorem A.6 in the appendix). As a consequence, every Bernoulli component of a billiard satisfying B1-B4 is open (mod 0). We stress, that although this is a remarkable property, it is not enough to yield the Bernoulli property of a billiard.

Our local ergodic theorem for billiards is derived from a similar result for general hyperbolic symplectomorphisms with singularities [14], which generalizes work of Liverani and Wojtkowski [22]. The proof of both [14] and [22] rely on the method developed by Sinai to prove the ergodicity of dispersing billiards [28]. Refinements of Sinai's method were obtained in [9, 21, 22, 29].

This paper is organized as follows. In Section 2, we recall the definition of billiard map and its main properties. In Section 3, we give the definition of the focusing time of a family of billiard trajectories. We also give the definition of a focusing arc introduced by Donnay, and collect the main properties of these arcs. Section 4 is concerned with the theory of invariant cone fields, and their construction for billiards. In this section, we also recall further results on focusing arcs. In Section 5, we give a detailed description of the hyperbolic billiards considered, and state the main results of this paper. In Section 6, we prove Conditions L1-L3 of the local ergodic theorem. Section 7 is entirely devoted to the proof of Condition L4. In Section 8, we introduce the class of billiards in polygons with pockets and bumps, and prove that they have the Bernoulli property. As a corollary, we obtain that the Donnay billiards have the Bernoulli property. In the Appendix, we state the local ergodic theorem for general hyperbolic symplectomorphisms with singularities from [14] together with relevant concepts and observations.

Sections 2, 3 and the beginning of Section 4 are meant to be an introduction to basic concepts on billiards and geometric optics, Donnay arcs and the theory of invariant cone fields. The reader already familiar with these notions may read quickly or skip these sections, and move directly to the second part of Section 4.

2. Generalities on billiards

A billiard system can be described either by a flow or a map. In this paper, we focus on the billiard map. For the relation between the billiard map and billiard flow, we refer the reader to the book [10]. In this section, we define the billiard map for a 2-dimensional domain, and single out its basic properties: regularity, natural invariant probability and singular sets.

2.1. **Billiard domain.** Let $0 \leq k \leq \infty$. A subset $\Gamma \subset \mathbb{R}^2$ is called an *arc of class* C^k if Γ is the image of a C^k embedding $\gamma \colon [0,1] \to \mathbb{R}^2$. The boundary of an arc Γ is given by $\partial\Gamma = \gamma(0) \cup \gamma(1)$. A subset $\Gamma \subset \mathbb{R}^2$ is called a *closed curve of class* C^k if Γ is C^k diffeomorphic to the unit circle S^1. Of course, $\partial\Gamma = \emptyset$ if Γ is a closed curve.

Let Ω be an open bounded connected subset of $\mathbb{R}^2$ such that $\partial\Omega$ is an union of finitely many disjoint closed curves of class C^0. We also assume that $\partial\Omega$ is an union of $n > 0$ arcs and closed curves $\Gamma_1, \ldots, \Gamma_n$ of class C^3. The set Ω defines the most general table of the billiards considered in this paper. The conditions imposed on Ω imply that the billiard tables may contain finitely many obstacles. Denote by ℓ_i the length of Γ_i, and consider the parametrization $\gamma_i : [0, \ell_i] \to \mathbb{R}^2$ of Γ_i by arc-length with the property that the interior of Ω remains on the left of the tangent vector $\gamma_i'(s)$ for $s \in [0, \ell_i]$. We assume that i) the curvature of each Γ_i computed with respect to the parametrization γ_i is either strictly negative, or strictly positive, or identically equal to zero, and ii) $\Gamma_i \cap \Gamma_j \subset \partial\Gamma_i \cap \partial\Gamma_j$ for $i \neq j$. Note that an arc Γ_i with zero curvature is just a straight segment.

The arcs and closed curves $\Gamma_1, \ldots, \Gamma_n$ are called the *components* of $\partial\Omega$. The union of all components with positive curvature (focusing components), negative curvature (dispersing components) and zero curvature (flat components) are denoted by Γ^+, Γ^-, Γ^0, respectively. A point of $\bigcup_{i=1}^n \partial\Gamma_i$ is called a *corner* of $\partial\Omega$.

2.2. **Billiard phase space.** For each $i = 1, \ldots, n$, define $M_i = [0, \ell_i] \times [0, \pi] \subset \mathbb{R}^2$ with the elements $(0, \theta)$ and (ℓ_i, θ) identified if Γ_i is a closed curve. Hence M_i is either a rectangle or a cylinder. Let M be the disjoint union of $M_1, \ldots, M_n$. An element $x \in M$ is therefore an ordered pair $(i, (s, \theta))$, and is called a *collision*. We define $i(x) = i$, $s(x) = s$ and $\theta(x) = \theta$ for $x = (i, (s, \theta)) \in M$. To simplify the notation, we identify each $x \in M$ with the corresponding pair (s, θ) by dropping the index i from the representation $(i, (s, \theta))$. When we need to specify i, we will write '$x \in M_i$'. We denote by M^+, M^-, M^0 the subsets of M obtained by taking the disjoint unions of sets M_i with Γ_i belonging to Γ^+, Γ^-, Γ^0, respectively.

The set M is a smooth manifold with boundary $\partial M = \bigsqcup_{i=1}^n \partial M_i$. We equip M with the Riemannian metric $g = \{g_x\}_{x \in M}$ and the symplectic form $\omega = \{\omega_x\}_{x \in M}$ given by $g_x = ds^2 + d\theta^2$ and $\omega_x = \sin\theta(x) ds \wedge d\theta$ for $x \in M$. The norm generated by g is denoted by $\|\cdot\|$. The Riemannian metric g induces in the usual way a distance d on each M_i, which can be extended to the entire set M by setting $d(x, y) = 1$ whenever $x \in M_i$ and $y \in M_j$ with $i \neq j$. We denote by m the volume generated by g. Then $\mu = (2\ell)^{-1} \sin\theta(x) m$ is the probability measure generated by ω with $\ell := \sum_{i=1}^n \ell_i$ being the total length of $\partial\Omega$.

There exists a natural involution $\mathcal{I}\colon M \to M$ given by $\mathcal{I}(s,\theta) = (s, \pi - \theta)$ whenever $(s,\theta) \in M_i$. For notational purposes, we write $-x$ instead of $\mathcal{I}(x)$. If B is a subset of M, then $-B$ will denote the set $\{-x \colon x \in B\}$.

2.3. **Billiard map.** Given $x = (s,\theta) \in M_i$, define $q(x) = \gamma_i(s) \in \Gamma_i$ and $v(x)$ to be the unit vector of $\mathbb{R}^2$ forming an angle θ with $\gamma_i'(s)$. We use the notation $(q(x), q(x) + \tau v(x))$ to denote the open segment of $\mathbb{R}^2$ with endpoints $q(x)$ and $q(x) + \tau v(x)$. Consider the set $\rho(x) = \{\tau > 0 \colon (q(x), q(x) + \tau v(x)) \subset Q\}$, and define $t(x) = 0$ if $\rho(x)$ is empty, and $t(x) = \sup \rho(x)$ otherwise. Also, define $q_1(x) = q(x) + t(x)v(x)$. Let

$$M' = \{x \in M \colon q_1(x) \text{ is not a corner of } \partial\Omega\}\,.$$

If $x \in M'$, then there exists a unique $i_1(x)$ such that $q_1(x)$ belongs to the interior of $\Gamma_{i_1(x)}$. Hence, we set $s_1(x) = \gamma_{i_1(x)}^{-1}(q_1(x))$. Now, let

$$v_1(x) = -v(x) + 2\langle \gamma_{i_1(x)}'(s_1(x)), v(x)\rangle \gamma_{i_1(x)}'(s_1(x)),$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^2$. Then, let $\theta_1(x) \in [0,\pi]$ be the oriented angle between $\gamma_{i_1(x)}'(s_1(x))$ and $v_1(x)$. The *billiard map* for the domain Ω is the transformation $T\colon M' \to M$ given by

$$Tx = (i_1(x), (s_1(x), \theta_1(x))) \qquad \text{for } x \in M'.$$

The regularity (continuity, differentiability, etc.) of T depends on the regularity of $\partial\Omega$. To clarify this point, we define:

$$\begin{aligned}
A_1 &= \{x \in M \colon q(x) \text{ is a corner of } \partial\Omega\}\,,\\
A_2 &= \{x \in M \colon \theta(x) \in \{0,\pi\}\}\,,\\
A_3 &= \{x \in M \setminus \partial M \colon q_1(x) \text{ is a corner of } \partial\Omega\}\,,\\
A_4 &= \{x \in M \setminus (\partial M \cup A_3) \colon Tx \in A_2\}\,.
\end{aligned}$$

Note that ∂M is equal to $A_1 \cup A_2$. Now, let S_1^+ be the closure of $A_3 \cup A_4$, and let $S_1^- = -S_1^+$. Then, define

$$R_1^+ = \partial M \cup S_1^+,$$

and for every $j \geq 1$, define iteratively

$$R_{j+1}^+ = R_j^+ \cup T^{-1} R_j^+ \quad \text{and} \quad R_j^- = -R_j^+.$$

The transformation T is a local C^{k-1} diffeomorphism at $x \in M \setminus R_1^+$ [20, Theorem 4.1]. Thus, the points where T is not continuous or more generally C^{k-1} are contained in R_1^+. Analogously, R_j^+ (resp. R_j^-) contains the points where T^j (resp. T^{-j}) is not C^{k-1}. We call R_j^+ and R_j^- the *singular set* of the billiard map T.

The map $T\colon M \setminus R_1^+ \to M \setminus R_1^-$ is a C^{k-1} diffeomorphism preserving the symplectic form ω and the probability measure μ provided that $k > 1$ (see [20, Corollaries 4.1 and 4.4, Part V]). Under proper conditions on the components $\Gamma_1, \ldots, \Gamma_n$, which are satisfied by the billiards considered in this paper (more precisely, for billiards satisfying Conditions B1 and B2 in Section 5), the sets R_j^+ and R_j^- are union of finitely many arcs of class C^2 [15]. It follows that $\mu(R_j^+) = \mu(R_j^-) = 0$ for every $j \geq 1$. Finally, we observe that T is time-reversible; this means that $\mathcal{I} \circ T = T^{-1} \circ \mathcal{I}$ on $M \setminus R_1^+$.

3. Focusing times and focusing arcs

We now introduce the concept of focusing times of an infinitesimal family of trajectories. This notion is borrowed from geometrical optics, and permits to obtain an intuitive description of the action of the derivative of the billiard map on the projective line.

3.1. **Focusing times.** Given a tangent vector $0 \neq u \in T_x M$ with $x \in M$, let $(-\delta, \delta) \ni a \mapsto \varphi(a) \in M$ be a differentiable curve for some $\delta > 0$ such that $\varphi(0) = x$ and $\varphi'(0) = u$. Next, consider the parametrization

$$(-\delta, \delta) \times \mathbb{R} \ni (a, t) \mapsto \ell_a^+(t) := q(\varphi(a)) + t \cdot v(\varphi(a)).$$

The set $\ell_a^+ := \{\ell_a^+(t) \colon t \in \mathbb{R}\}$ is a straight line for every $a \in \mathbb{R}$, and the family $\{\ell_a^+\}$ of lines forms a variations of the line ℓ_0^+. We obtain a second family $\{\ell_a^-\}$ of lines by replacing the curve φ in the definition of $\ell_a^+(t)$ with the curve $-\varphi$. In geometrical terms, each line ℓ_a^- is obtained by reflecting ℓ_a^+ about the tangent of $\partial\Omega$ at the point $q(\varphi(a))$.

All lines $\{\ell_a^+\}$ intersect in linear approximation at a point $q_+ \in \ell_0^+$. However, if the derivative of $v \circ \varphi$ vanishes at $a = 0$, then all lines $\{\ell_a^+\}$ are parallel in linear approximation, and we define q_+ to be the point at infinity. Similarly, all lines $\{\ell_a^-\}$ intersect in linear approximation at a point $q_- \in \ell_0^-$, but if the derivative of $v \circ \mathcal{I} \circ \varphi$ vanishes at $a = 0$, then all lines $\{\ell_a^-\}$ are parallel in linear approximation, and we define q_- to be the point at infinity. Note that q_+ and q_- depend only on the vector u and not on the choice of the curve φ. The points q_+ and q_- are called the *forward focal point* of u and the *backward focal point* of u, respectively.

The distances between $q(x)$ and q_+ and between $q(x)$ and q_- are denoted by $\tau^+(x, u)$ and $\tau^-(x, u)$, respectively. The first distance is called the *forward focusing time* of u, whereas the second distance is called the *backward focusing time* of u. Let $u = (u_s, u_\theta)$ in coordinates (s, θ), and define

$$m(u) = \frac{u_\theta}{u_s} \in \hat{\mathbb{R}},$$

where $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}$. Let $\kappa(x)$ be the curvature of $\Gamma_{i(x)}$ at $q(x)$. A straightforward computation (for example, see [32, Section 2]) shows that

$$\tau^{\pm}(x,u) = \begin{cases} \frac{\sin\theta(x)}{\kappa(x)\pm m(u)} & \text{if } m(u) \neq \mp\kappa(x), \\ \infty & \text{otherwise.} \end{cases} \tag{1}$$

Let $d(x) = \sin\theta(x)/\kappa(x)$. From (1), one can easily derive the well known *Mirror Equation* of geometrical optics, relating the focusing times $\tau^+(x,u)$ and $\tau^-(x,u)$:

$$\frac{1}{\tau^+(x,u)} + \frac{1}{\tau^-(x,u)} = \frac{2}{d(x)}. \tag{2}$$

Definition 3.1. *Let $x \in M$ and $u \in T_xM \setminus \{0\}$, and suppose that T^j is a local diffeomorphism at x for some $j \in \mathbb{Z}$. Then define $\tau_j^{\pm}(x,u) = \tau^{\pm}(T^jx, D_xT^ju)$.*

3.2. **Focusing arcs.** We now recall the notion of a focusing arc introduced by Donnay [16].

We start by assuming that

$$\int_0^{\ell_i} \kappa(s(x))ds \leq \pi \tag{3}$$

for every component $\Gamma_i \subset \Gamma^+$. The geometrical meaning of this condition is that the tangents to Γ_i at its endpoints form an angle that is not larger than π. This implies immediately that no components of positive curvature can be a closed curve. Condition (3) has another important consequence: if a billiard orbit is trapped by a component $\Gamma_i \subset \Gamma^+$, i.e. it does not leave M_i, then it is a periodic orbit of period 2, and its trajectory coincides with the segment joining the endpoints of Γ_i [16, Lemma 1.1]. Examples of components with such orbits are half-ellipses: the trapped orbits correspond to the semi-axis along which the ellipse is cut in half.

Definition 3.2. *Let $n(x) = \sup\{j \geq 0 \colon x \in M \backslash R_j^+$ and $x, Tx, \ldots, T^jx \in M_{i(x)}\}$ for $x \in M^+ \setminus A_2$, where $R_0^+ := \emptyset$.*

The number $n(x)$ is always finite, and represents the number of consecutive collisions of the trajectory of x with the component $\Gamma_{i(x)} \subset \Gamma^+$ before leaving it.

Definition 3.3. *Given $\Gamma_i \subset \Gamma^+$, let $E_i = \{x \in M_i \setminus A_2 \colon n(-x) = 0\}$. Also, let E^+ be the union of all the E_i's.*

Note that if $x \in E_i$, then $\{x, Tx, \ldots, T^{n(x)}x\}$ is the longest sequence of consecutive collisions with Γ_i; any other collision preceding x has to belong to M_j with $j \neq i$.

Now, suppose that $\Gamma_i \subset \Gamma^+$. Let $x \in E_i$, and denote by $u_x \in T_xM$ the tangent vector such that $\tau^-(x, u_x) = \infty$. In other words, the variation associated to u_x consists of lines parallel to $v(x)$. The vector $v(x)$ and the number $t(x)$ are defined in Section 2.

Definition 3.4. *Suppose that* $\Gamma_i \subset \Gamma^+$. *We say that* $x \in E_i$ *is* focused *by* Γ_i *if*

(1) $0 < \tau_i^+(x, u_x) < t(T^i x)$ *for* $0 \leq i < n(x)$,
(2) $0 < \tau_{n(x)}^+(x, u_x) < +\infty$.

The concept of a focused collision is key in the definition below of a focusing arc. In words, a collision is focused by Γ_i if x has a finite number $n(x)$ of consecutive collisions with Γ_i before leaving it, and if the infinitesimal family of trajectories associated to u_x focuses between every two consecutive collisions with Γ_i and after the last one.

Definition 3.5. *A component* $\Gamma_i \subset \Gamma^+$ *is called* focusing *if i)* Γ_i *is an arc of class* C^∞, *ii)* Γ_i *satisfies Condition* (3), *and iii) if every* $x \in E_i$ *is focused by* Γ_i.

Bunimovich introduced the notion of an *absolutely focusing component* [3, 5], which is similar to that of a focusing component. The relation between these two notions is discussed in [5].

The following results concern the existence of focusing components, their robustness under perturbations, and use in constructing hyperbolic billiards. They are proved in [16, Theorems 2-4].

Theorem 3.6. *Given an arc* Γ *of class* C^∞ *with positive curvature and a point* $q \in \Gamma$, *there exists a neighborhood* U *of* q *in* Γ *such that* U *is focusing.*

A similar conclusion when Γ is only of class C^4, but satisfies also the condition $d^2(\kappa^{-1/3})/ds^2 > 0$ was obtained by Markarian [23].

Theorem 3.7. *Suppose that* Γ *is a focusing. Then there exists* $\epsilon = \epsilon(\Gamma) > 0$ *such that every arc* Γ' *of class* C^∞ *with positive curvature, of the same length as* Γ, *and* ϵ*-close to* Γ *in the* C^6 *topology is focusing.*

Theorem 3.8. *The billiard map of a convex region* Ω *with boundary consisting of focusing components connected by straight segments sufficiently long has non-zero Lyapunov exponents almost everywhere.*

Remark 3.9. *We believe that the original proof of Theorems 3.6-3.8, 4.2 and Proposition 4.3 remains essentially valid for arcs of class* C^6 *rather than of class* C^∞. *Since the explanation of this claim would require some time, and this is not the main purpose of this paper, we opted to stick to the original formulation of the results of* [16].

Remark 3.10. *The important property of a focusing component* Γ_i *that makes it suitable for constructing hyperbolic billiards is that the focusing time* $\tau_{n(x)}^+(x, u_x)$ *is uniformly bounded for* $x \in E_i$ [16, Theorem 4.4]. *For this property to hold, an arc with positive curvature need not be of class* C^6. *Indeed, Wojtkowski and Markarian both discovered families of arcs of class* C^4 *with positive curvature that have the previous property, and used them to construct billiards with non-zero*

Lyapunov exponents [23, 32]*. However, the curvature of their arcs have to satisfy some additional conditions.*

Examples of focusing arcs are arcs of circles, arcs of cardioids, arcs of logarithmic spirals and elliptical arcs. An example of a focusing arc not belonging to the classes of Wojtkowski arcs and Markarian arcs is the half-ellipse $\{(x,y) \in \mathbb{R}^2 \colon x^2/a^2 + y^2/b^2 = 1 \text{ and } x \geq 0\}$ with $a/b < \sqrt{2}$ [16, Theorem 7.1].

In this paper, we consider billiard domains that are more general than those considered in Theorem 3.8. Indeed, the billiard domains considered here are not necessarily convex, and their boundaries are allowed to have components with negative curvature.

4. Cone fields for billiards

In this section, we first recall the concepts of an invariant cone field and a monotone quadratic form. Since we are dealing with planar billiards, we will restrict ourselves to present definitions in the 2-dimensional setting [32]. Then, we introduce a family of cone fields for the billiard map T, each of which is eventually strictly invariant for the billiard domains considered in this paper. This family of cone fields plays a fundamental role in our proof of the Bernoulli property of T.

4.1. **Cone fields.** Let V be Consider a two-dimensional vector space V. Given two linear independent vectors X_1 and X_2 of V, we say that the set $C(X_1, X_2) := \{a_1X_1 + a_2X_2 \colon a_1a_2 \geq 0\} \subset V$ is the *cone generated* by X_1 and X_2. We also define $\operatorname{int} C = \{a_1X_1 + a_2X_2 \colon a_1a_2 > 0\} \cup \{0\}$ and $C'(X_1, X_2) = C(X_1, -X_2)$. The latter set is called the *interior* of $C(X_1, X_2)$, whereas the former set, which is in fact a cone, is called the *complementary cone* of $C(X_1, X_2)$

Now, let U be an open set of M, and suppose that X_1 and X_2 are two measurable vector fields on U such that $X_1(x)$ and $X_2(x)$ are linear independent for all $x \in U$. A *cone field* C on U, denoted by (U, C), is a family of cones $\{C(x)\}_{x\in U}$ given by $C(x) = C(X_1(x), X_2(x)) \subset T_xM$ for every $x \in U$. A cone field (U, C) is called *continuous* if the vector fields X_1 and X_2 are continuous on U. A cone field (U, C) is called *invariant* (resp. *strictly invariant*) if $x \in U$ and $T^kx \in U$ with $k > 0$ implies that $D_xTC(x) \subset C(T^kx)$ (resp. $D_xT^kC(x) \subset \operatorname{int} C(T^kx)$). A cone field (U, C) is called *eventually strictly invariant* if it is invariant, and for almost every $x \in U$, there exists an integer $k(x) > 0$ such that $T^{k(x)}x \in U$ and $D_xT^{k(x)}C(x) \subset \operatorname{int} C(T^{k(x)}(x))$.

Remark 4.1. *A 2-dimensional cone C may be naturally identified with a closed interval I of the projective space $\mathbb{P}(V)$. Accordingly, a cone field (U, C) can be written as follows:*

$$C(x) = \{u \in T_xM \setminus \{0\} \colon m(u) \in I(x)\} \cup \{0\} \quad \textit{for } x \in U,$$

where $\{I(x) \subset \hat{\mathbb{R}} \colon x \in U\}$ *is a proper family of closed intervals. Since the focusing times* $\tau^+(x,\cdot)$ *and* $\tau^-(x,\cdot)$ *are projective transformations, the cone field* (U, C) *has two alternative representations:*

$$C(x) = \{u \in T_x M \setminus \{0\} \colon \tau^{\pm}(x,u) \in I_{\pm}(x)\} \cup \{0\} \quad \text{for } x \in U,$$

where $\{I_{\pm}(x) \subset \hat{\mathbb{R}} \colon x \in U\}$ *are proper families of closed intervals.*

4.2. **Quadratic forms.** Consider a cone field (U, C) generated by the vector fields X_1 and X_2. The quadratic form $Q_C = \{Q_C(x,\cdot) \colon x \in U\}$ associated to (U, C) is defined by

$$Q_C(x,u) = \omega_x(u_1,u_2) \qquad \text{for } x \in U,$$

where u_1 and u_2 are vectors of subspaces generated by $X_1(x)$ and $X_2(x)$, respectively, such that $u = u_1 + u_2$. The form Q_C is called *monotone* (resp. strictly monotone) if $Q_C(T^k x, D_x T^k u) \geq Q_C(x,u)$ (resp. $Q_C(T^k x, D_x T^k u) > Q_C(x,u)$) for all $0 \neq u \in T_x M$ whenever $x \in U$ and $T^k x \in U$ with $k > 0$. The form Q_C is called *eventually strictly monotone* if it is monotone, and for almost every $x \in U$, there exists an integer $k(x) > 0$ such that $Q_C(T^{k(x)} x, D_x T^k u) > Q_C(x,u)$ for all $0 \neq u \in T_x M$.

Following [22], to measure the expansion generated by the action of DT^k with $k > 0$ on vectors in (U, C) with respect to Q_C, we define

$$\sigma_C(D_x T^k) = \inf_{u \in \operatorname{int} C(x) \setminus \{0\}} \sqrt{\frac{Q_C(T^k x, D_x T^k u)}{Q_C(x,u)}},$$

and

$$\sigma^*_C(D_x T^k) = \inf_{u \in \operatorname{int} C(x) \setminus \{0\}} \frac{\sqrt{Q_C(T^k x, D_x T^k u)}}{\|u\|}.$$

For $k < 0$, we define σ_C and σ^*_C similarly by replacing $C(x)$ and Q_C in the definition above with the complementary cone $C'(x)$ and $-Q_C$, respectively.

The cone field (U, C) is invariant (resp. strictly invariant) if and only if the quadratic form Q_C is monotone (resp. strictly monotone), and (U, C) is eventually strictly invariant if and only if Q_C is eventually strictly monotone. Furthermore, $D_x T^k C(x) \subset C(T^k x)$ (resp. $D_x T^k C(x) \subset \operatorname{int} C(T^k x)$) for $x \in U$ such that $T^k x \in U$ with $k > 0$ is equivalent to $\sigma_C(D_x T^k) \geq 1$ (resp. $\sigma_C(D_x T^k) > 1$). See [22], for a detailed proof of these properties. From the definitions of strict invariance of a cone field, σ_C and σ^*_C, one can easily deduce that if $k_1, k_2, n \in \mathbb{Z}$ with $k_1 \cdot k_2 \geq 0$, then

$$\sigma_C(D_x T^n) > 1 \quad \Longrightarrow \quad \sigma^*_C(D_x T^n) > 0, \tag{4}$$

and

$$\sigma^*_C(D_x T^{k_1+k_2}) \geq \sigma^*_C(D_x T^{k_1}) \cdot \sigma_C(D_{T^{k_1} x} T^{k_2}). \tag{5}$$

4.3. **Cone fields for billiards.** Previous approaches to the study of the ergodicity of billiards used a single continuous invariant cone field. Here we take a slightly different approach. Since cone fields for focusing arcs are piecewise continuous in general, we find it more convenient to work with a family of continuous cone fields $\{(U_x, C_x)\colon x \in E\}$ with U_x being a neighborhood of x. We stress that C_x is not a single cone at the point x, but the family of cones $\{C_x(y)\colon y \in U_x\}$ on the neighborhood U_x of x. Note that we restrict ourselves to consider cone fields defined on the subset E rather than on the entire domain M of T. Since $x \in E$, we will be able to apply the local ergodic theorem stated in the Appendix A only to neighborhoods of points of E, and so to prove the local ergodicity of T only on E. However, since in a hyperbolic billiard, almost every orbit of a hyperbolic billiard visits E infinitely many times, the previous conclusion will be enough to obtain the local ergodicity on the entire domain of T.

We define the cone fields (U_x, C_x) separately for $x \in E^+$ and for $x \in E^-$, i.e., for focusing components and components with negative curvature, which from now on we will call *dispersing* components.

4.3.1. *Focusing components.* For $x \in E^+$, the cone field (U_x, C_x) is given by $C_x(y) = C(y, g_x(y)) \cup \{0\}$ for all $y \in U_x$, where

$$C(y, g_x(y)) := \{u \in T_yM \setminus \{0\}\colon g_x(y) \le m(u) \le \kappa(y)\},$$

and $g_x\colon U_x \to \mathbb{R}$ is a continuos function such that $-\kappa(y) < g_x(y) \le \kappa(y)$ for all $y \in U_x$. The functions g_x are chosen in Theorem 4.2.

Let κ_i be the maximum of the curvature of the focusing component Γ_i.

Theorem 4.2. *Given a focusing component Γ_i, there exist constants $t_i^\pm, a_i, \theta_i > 0$, $0 < m_i \le \kappa_i$ and continuous functions $g_x\colon U_x \to \mathbb{R}$ with $U_x \subset M_i \setminus A_2$ being a neighborhood of x for all $x \in E_i$ such that if $y \in U_x$, $u \in C(y, g_x(y))$ and $0 \le k \le n(y)$, then*

(1) $-\kappa(T^k y) + m_i \le m(D_yT^k u) \le \kappa(T^k y)$,
(2) $|m(D_yT^k u)| \le a_i \cdot \min\{\theta(y), \pi - \theta(y)\}$ *whenever* $\theta(y) \in (0, \theta_i) \cup (\pi - \theta_i, \pi)$,
(3) $d(T^k y)/2 \le \tau_k^+(y, u) < t(T^k y) - d(T^{k+1} y)/2$ *whenever* $k < n(y)$,
(4) $\inf_{v \in C(y, g_x(y))} \tau^-(y, v) \le t_i^-$ *and* $\sup_{v \in C(y, g_x(y))} \tau^+_{n(y)}(y, v) \le t_i^+$.

Proof. For the proof of Parts (1) and (2), see the proofs of [16, Theorems 4.4 and 5.6], whereas for the proof of Parts (3) and (4), see [16, Proposition 4.1 and Theorem 4.4]. □

Theorem 4.2 has the following geometrical interpretation. Part (1) states that the lower edge of the cone $C_x(y)$ and its iterates along consecutive collisions of x with the focusing component Γ_i are uniformly bounded from $-\kappa(y)$ for all $x \in E_i$ and all $y \in U_x$. This conclusion is strengthen in Part (2) when $\theta(y)$ is uniformly close to 0 or π. In this case, the iterates of the lower edge of the cone $C_x(y)$ remains uniformly close to the horizontal direction. Parts (3) and (4) implies the every $y \in U_x$ is focused (c.f. 3.4), and that the backward and forward focusing times of $C_x(y)$ are uniformly bounded by constants that depend only on Γ_i. In fact, the constants $t_i^\pm$ are continuous functions of Γ_i in the C^6 topology [16, Theorem 4.4].

The following proposition provides an upper bound on the total number of consecutive collisions along a focusing arc depending on the angle formed by the initial collision with the arc. For its proof see [16, Corollary 5.3 and Part (1) of Proposition 6.1].

Proposition 4.3. *Let Γ_i be a focusing component, and let θ_i be the constant in Theorem 4.2. Then there exist a constant $c_i > 0$ and a function $N_i\colon (0,\pi/2) \to \mathbb{N}$ such that if $x \in M_i \backslash A_2$, then $n(x) \leq c_i/\theta(x)$ whenever $\theta(x) \in (0,\theta_i) \cup (\pi - \theta_i, \pi)$, and $n(x) < N(\theta_i)$ whenever $\theta(x) \in [\theta_i, \pi - \theta_i]$.*

4.3.2. *Dispersing components.* For convenience, we extend Definition 3.3 to dispersing components. Given $\Gamma_i \subset \Gamma^-$, define $E_i = M_i$. Also, define E^- to be the union of all E_i such that $\Gamma_i \subset \Gamma^-$, and $E = E^- \cup E^+$.

Suppose that $\Gamma_i \subset \Gamma^-$. For every $x \in E_i$, choose $U_x = M_i$ and

$$C_x(y) = \{u \in T_y M_i \setminus \{0\} \colon m(u) \leq \kappa(x)\} \cup \{0\} \qquad \text{for } y \in U_x.$$

This cone field is clearly continuous. It was introduced by Wotjkowski [32]. In geometrical terms, $C_x(y)$ consists of tangent vectors focusing inside the osculating disk tangent to Γ_i at $q(y)$ and of radius $(4|\kappa(y)|)^{-1}$. Indeed, using (1), we see that if $0 \neq u \in C_x(y)$, then $d(y)/2 \leq \tau^+(y,u) \leq 0$. Note also that $0 \leq \tau^-(y,u) \leq +\infty$.

It is useful to introduce a new quantity $G_x^\pm$ associated to the cone field (U_x, C_x) for all $x \in E$.

Definition 4.4. *For every $x \in E$ and every $y \in U_x$, let*

$$G_x^\pm(y) = \begin{cases} \frac{\sin\theta(y)}{\kappa(y) \pm g_x(y)} & \text{if } x \in E^+, \\ 0 & \text{if } x \in E^-. \end{cases}$$

It is easy to check that each cone $C_x(y)$ can be written in terms of the projective coordinates τ^+ and τ^- as follows:

$$\begin{aligned} C_x(y) &= \big\{u \in T_y M_i \setminus \{0\} \colon d(y)/2 \leq \tau^+(y,u) \leq G_x^+(y)\big\} \cup \{0\} \\ &= \big\{u \in T_y M_i \setminus \{0\} \colon G_x^-(y) \leq \tau^-(y,u) \leq +\infty\big\} \cup \{0\}. \end{aligned}$$

Note that for $x \in E^+$, the numbers $G_x^\pm(y)$ are just the forward and backward focusing times of the vectors contained in $C_x(y)$ with slope $g_x(y)$.

5. Results

In this section, we give a detailed description of the billiards that we want to study, and formulate the main results of the paper.

The billiards in question are characterized by four conditions called B1-B4. It is well known that B1-B3 are sufficient to guarantee the hyperbolicity of the billiard map T (see Proposition 5.3). This together with the Spectral Theorem implies that T has at most countably many ergodic components of positive measure with respect to μ, with each ergodic component further decomposed into finitely many Bernoulli components cyclically permuted by T (see Theorem A.6).

The main result of this paper is the following: if in addition to B1-B3, a billiard satisfies also Condition B4, then there exists a measurable set $H \subset M$ of full measure such that for every $x \in H$, there is a neighborhood of x in M contained up to a set of zero measure in a single Bernoulli component of T (see Theorem 5.6). This result implies immediately that every Bernoulli component of T is open up to a set of zero measure. Results of this type are often called Local Ergodic Theorems.

Local ergodicity alone is not enough to conclude that T is Bernoulli. This is obtained by imposing on T some extra conditions. It is not an easy task to formulate these conditions for the generality of the billiards considered in this paper. Therefore, rather than trying to formulate the optimal condition for the Bernoulli property of hyperbolic billiards, we limit ourselves here to give a simple condition, called B5, that yields the Bernoulli property of interesting subclasses of hyperbolic billiards (i.e., billiards with domains without straight boundaries, i.e., $\Gamma^0 = \emptyset$). It is unfortunate that B5 does not hold for Donnay billiards. Nevertheless, in Section 8, we prove that these billiards and some generalizations (billiards with pockets and bumps) are Bernoulli, using a proof that does not require B5.

5.1. **Important sets.** Next, we introduce several sets involved in the formulation of Conditions B1-B5. Recall that $R_k^{\pm}$ are the singular sets defined in Section 2. Define

- $R_\infty^{\pm} = \bigcup_{k \geq 1} R_k^{\pm}$,
- $R = R_\infty^- \cap R_\infty^+$,
- $N^{\pm} = \left\{ x \in M \setminus R_\infty^{\pm} \colon \exists k > 0 \text{ s.t. } T^{(\pm)n} x \in M_0 \text{ for all } n \geq k \right\}$,
- $N = N^- \cap N^+$,
- $N' = (R_\infty^- \cap N^+) \cup (R_\infty^+ \cap N^-)$,
- $H = M \setminus (R \cup N \cup N')$.

The geometric meaning of these sets is the following: R_∞^+ (resp. R_∞^-) is the set of collisions with finite positive (resp. negative) semi-orbit; R is the set of collisions with finite orbit; N^+ (resp. N^-) is the set of collisions with positive (resp. negative) semi-orbit visiting eventually only flat components of $\partial\Omega$; N is the set of collisions with both semi-orbits visiting eventually only flat components of $\partial\Omega$; N' is the set of collisions with one semi-orbit being finite and the other semi-orbit visiting eventually only flat components of $\partial\Omega$; H is the set of collisions with one semi-orbit visiting the curved components of $\partial\Omega$ infinitely many times.

5.2. **Hyperbolic billiards.** We are ready to formulate Conditions B1-B5.

B1 (Non-polygonal domain): The domain Ω is not a polygon, and its boundary components can only be of the following type: straight segments, dispersing arcs of class C^3 and focusing arcs.

B2 (Distance between boundary components): For each curved component Γ_i, we define $\lambda_i^\pm = 0$ if Γ_i is dispersing, and $\lambda_i^\pm = t_i^\pm$ with $t_i^\pm$ as in Theorem 4.2 if Γ_i is focusing. Given two curved components Γ_i and Γ_j, denote by $t_{ij} \geq 0$ the infimum of the Euclidean length of all finite billiard orbits $\{x_0, \dots, x_n\}$ with $n > 0$ such that $x_0 \in M_i$ and $x_n \in M_j$. We assume that

(1) there exists $\lambda > 0$ such that if Γ_i or Γ_j is focusing, then
$$t_{ij} \geq \lambda_i^- + \lambda_j^+ + \lambda,$$
(2) the distance between each focusing component Γ_i and the set of corners of $\partial\Omega$ formed by two straight segments is greater than λ_i^-.

B3 (Neutral orbits): We assume that $\mu(N^-) = 0$.

B4 (Singular-Neutral orbits): We assume $m_-(S_1^- \cap N^+) = 0$, where m_- is the measure induced by the Riemann metric g on S_1^-.

B5 (Connectedness): The set $H \cap M_i$ is connected for every component Γ_i of $\partial\Omega$.

Condition B2 has a couple of obvious consequences for the geometry of Ω: i) the internal angle between a focusing component and an adjacent curved component is greater than π, and ii) the internal angle between a focusing component and an adjacent flat component is greater than $\pi/2$. Also, note that Conditions B1-B4 allow $\partial\Omega$ to have cusps formed by two dispersing components or a dispersing and a flat component.

Remark 5.1. *Since $N^+ = -N^-$ and $S_1^+ \cap N^- = -(S_1^- \cap N^+)$, Conditions B3 and B4 imply that $\mu(N^+) = 0$ and $m_+(S_1^+ \cap N^-) = 0$, where m_+ is the measure induced by the Riemann metric g on S_1^+.*

From Conditions B1 and B2, it follows that cone field $\{(U_x, C_x)\}_{x\in E}$ is strictly invariant along a piece of an orbit connecting two elements of E. The next lemma is proved in [15, Lemma 5.2].

Lemma 5.2. *Suppose that the billiard in Ω satisfies Conditions B1 and B2. Also, suppose that there exist $x_1, x_2 \in E$ and $y \in E \cap U_{x_1} \setminus R_k^+$ for some $k > 0$ such that $T^k y \in E \cap U_{x_2}$. Then*

$$D_y T^k C_{x_1}(y) \subset \operatorname{int} C_{x_2}(T^k y).$$

From the previous lemma, one obtains the hyperbolicity of the billiard map T provided that Conditions B1-B3 are satisfied. This is well known fact, but we give its proof for completeness.

Proposition 5.3. *If a billiard in a domain Ω satisfies Conditions B1-B3, then the Lyapunov exponents of the map T are non-zero a.e. on M.*

Proof. Let E' be the subset of $E \setminus (R_\infty^+ \cup N^+)$ defined by

$$E' = \left\{ x \in E \setminus (R_\infty^+ \cup N^+) \colon \exists n_k \nearrow +\infty \quad \text{s.t.} \quad T^{n_k} x \in V_x \quad \forall k > 0 \right\},$$

where (U_x, C_x) is the cone field associated to x. Since (U_x, C_x) is strictly invariant for every $x \in E'$ by Lemma 5.2, results of Wojtkowski [32] imply that the Lyapunov exponents of T are non-vanishing at every point of E'. By $\mu(R_\infty^+) = \mu(N^+) = 0$ and the Poincaré Recurrence Theorem, we obtain that $\mu(E') = \mu(E) > 0$. This fact together with $\mu(R_\infty^+) = \mu(N^+) = 0$ gives that the orbit of a.e. point of M visits E'. Since the Lyapunov exponents are constant along orbits, we can finally conclude that the Lyapunov exponents of T are non-vanishing at a.e. point of M. □

Conditions B1-B3 are sufficient for the hyperbolicity of the map T. In fact, even part (2) of B2 can be dropped if we are only interested in the hyperbolicity of T. The extra Condition B4 is required to prove that T is locally ergodic. This condition is related to the Sinai-Chernov Ansatz (c.f. Condition L3 of Theorem A.13). Note also that B3 is a necessary condition for the hyperbolicity of T.

Remark 5.4. *We do not know whether or not, for a domain Ω satisfying B1 and B2, the conditions B3 or B4 is automatically satisfied. We also do not know whether B3 and B4 are independent. These questions are strictly related to the problem of understanding the distribution of orbits in polygonal billiards.*

Lemma 5.5. *We have $\mu(H) = 1$ provided that B1 and B3 and are satisfied.*

Proof. Since $\mu(R_1^+) = 0$ for billiards satisfying B1 (see the end of Subsection 2.3), we trivially obtain $\mu(R) = \mu(N') = 0$. From B3, we obtain immediately $\mu(N) = 0$. □

5.3. **Main results.** The central result of this paper is the following theorem. Its proof is given in Section 6.

Theorem 5.6. *If a billiard in a domain Ω satisfies Conditions B1-B4, then every point of H has a neighborhood contained (mod 0) in a Bernoulli component of T.*

We now prove that the map T is Bernoulli if it also satisfies Condition B5. As already explained in the introduction to this section, B5 applies only to a small subclass of billiards satisfying B1-B4 (see Theorem 8.7). We could have weaken considerable B5 for it to include many more hyperbolic billiards, but at the price of a much more technical formulation. Instead of attempting to give the weakest formulation of B5, we opted for a strong condition but with a simple formulation that allows for a relatively simple proof of the Bernoulli property for billiards.

Corollary 5.7. *If a billiard in a domain Ω satisfies Conditions B1-B4, then every Bernoulli component of T is open (mod 0).*

Proof. Let B be a Bernoulli component. Since $\mu(B) > 0$, we have $\mu(B \cap H) > 0$. Let $x \in B \cap H$, and let U be the neighborhood of x as in Theorem 5.6. The set $V := \bigcup_{n\in\mathbb{Z}} T^n U$ is open. Moreover, since V is invariant and contained (mod 0) in B, it follows that $B = V$ (mod 0). □

Corollary 5.8. *If the billiard in a domain Ω satisfies Conditions B1-B5, then the map T is Bernoulli.*

Proof. By Theorem 5.6, every point of H has a neighborhood contained up to a set of zero measure in a Bernoulli component of T. The same is true for every connected component of H, and so for every $M_i \cap H$ such that $M_i \subset M^- \cup M^+$ by the first part of Condition B5. Since $\mu(H) = 1$ (see Remark 5.1), we conclude that every set $M_i \subset M^- \cup M^+$ is contained (mod 0) in a single Bernoulli component of T.

We now show that if Γ_i and Γ_j intersect, then M_i and M_j are contained in the same Bernoulli component of T. First, note that $S_1^- \setminus (R_\infty^+ \cup N^+)$ is contained in H. Next, since $S_1^- \cap R_k^+$ is finite for every $k > 0$ (see [15, Propositions 6.17-6.19]), it follows that $S_1^- \cap R_\infty^+$ is countable. This together with B4 implies that m_--a.e. element of S_1^- is contained in H. Hence, if $p \in \Gamma_i \cap \Gamma_j$ is a corner of $\partial\Omega$, then we can find $x \in H \cap S_1^-$ such that the ray emerging from $-x$ is arbitrarily close to ℓ, the line bisecting p. Now, let U be the neighborhood of x contained (mod 0) in one Bernoulli component of T as stated by Theorem 5.6. It is clear that $T^{-1}U$ is contained (mod 0) in the same Bernoulli component. But, if $-x$ is sufficiently close to ℓ, then $M_i \cap T^{-1}U$ and $M_j \cap T^{-1}U$ are non-empty open sets so that M_i and M_j must belong (mod 0) to the same Bernoulli component.

The previous conclusion implies that $q^{-1}(\Sigma)$ is contained (mod 0) in a Bernoulli component for every connected component Σ of $\partial\Omega$. Since Ω is connected, it follows that all sets M_i belongs to the same Bernoulli component, i.e., T is Bernoulli. □

6. Local ergodicity

In this section, we prove Theorem 5.6. This is accomplished by applying Theorem A.13 to the billiard map T. Because of its length, Theorem A.13 is stated in the Appendix together with all the notions required for its formulation. This theorem is the version for planar billiards of a Local Ergodic Theorem for hyperbolic symplectomorphisms with singularities in any dimension proved in [14]. Theorem A.13 has four main hypotheses called Conditions L1-L4. In this section, we prove Conditions L1-L3. Section 7 is entirely devoted to the proof of Conditions L4.

Proof of Theorem 5.6. The wanted conclusion follows at once by applying Theorem A.13 to points of H. To do that, we need to show that every point of H is sufficient (see Definition A.1), and that for each of these points, Conditions L1-L4 of Theorem A.13 are satisfied. The first fact is proved in Corollary 6.3, whereas the second fact is proved in Propositions 6.4, 6.5, 6.6 and 7.29. □

6.1. **Sufficient points.** We now prove that every point of H is sufficient (see Definition A.1) and that Conditions L1-L3 are satisfied. As already mentioned, Condition L4 is proved in Section 7. We start with some remarks concerning the cone fields used in our arguments.

For technical reasons, which will be clear from the proof of Proposition 6.6, we extend the family of cone fields $\{(U_x, C_x)\}_{x\in E}$ introduced in Subsection 4.3 to $S_1^- \setminus (R_\infty^+ \cup N^+)$ and $S_1^+ \setminus (R_\infty^- \cup N^-)$. We explain how to define (U_x, C_x) only for $x \in S_1^- \setminus (R_\infty^+ \cup N^+)$, the definition for $x \in S_1^+ \setminus (R_\infty^- \cup N^-)$ being similar. Since $x \notin R_\infty^+ \cup N^+$, there exists $n_k \nearrow \infty$ such that $T^{n_k}x \in E$ for every $k > 0$. Write $x_1 = T^{n_1}x$, and consider the cone field (U_{x_1}, C_{x_1}). Then choose U_x to be a neighborhood of x contained in $T^{-n_1}U_{x_1}$ such that $U_x \cap R_{n_1}^+ = \emptyset$, and define $C_x(y) = D_{x_1}T^{-n_1}C_{x_1}(y)$ for all $y \in T^{n_1}U_x$.

Given $x \in M \setminus R_m^+$ for some $m > 0$, we denote by $l(x, T^m x)$ the sum of the length of the segments $[q(T^k x), q(T^{k+1}x)]$ for $0 \le k \le m-1$. Also, let $\lambda > 0$ be the constant in Condition B2.

Given $x, T^k x \in E$ for some $k \in \mathbb{N}$, define

$$\tilde{\sigma}(D_x T^k) = \inf_{u \in \operatorname{int} C_x(x)\setminus\{0\}} \sqrt{\frac{Q_{C_{T^k x}}(T^k x, D_x T^k u)}{Q_{C_x}(x, u)}},$$

and

$$\tilde{\sigma}^*(D_xT^k) = \inf_{u\in\operatorname{int} C_x(x)\setminus\{0\}} \frac{\sqrt{Q_{C_{T^kx}}(T^kx, D_xT^ku)}}{\|u\|}.$$

Note the two cone fields C_x and C_{T^kx} entering into the previous definition. This is the only difference between $\tilde{\sigma}(D_xT^k)$ and $\tilde{\sigma}^*(D_xT^k)$ and the analogous quantities introduced in Subsection 4.2.

Lemma 6.1. *Suppose that $x \in E_j \setminus R^+_m$ for some $m \geq 1$, $T^mx \in E_k$ and $T^ix \notin E$ for every $1 \leq i < m$. Then there exists a constant c independent of x such that*

$$\tilde{\sigma}(D_xT^m) \geq \sqrt{1+c\delta} + \sqrt{c\delta},$$

where $\delta = \lambda$ if Γ_j or Γ_k is focusing, and $\delta = l(x, T^mx)$ if Γ_j and Γ_k are dispersing.

Proof. Note that if $m > n(x) + 1$, then we must have $T^ix \in M^0$ for $n(x) < i < m$. For this reason, the parameter m does not play any role in the computation of $\tilde{\sigma}(D_xT^m)$, only the parameter $l = l(x, T^mx)$ matters. We can therefore assume without loss of generality that $m = n(x) + 1$.

By the definition of $\{(U_x, C_x)\}_{x\in E}$, we can write

$$\begin{aligned} C_{T^mx}(T^mx) &= \left\{u \in T_{T^mx}M \setminus \{0\} \colon a \leq \tau^-(T^mx, u) \leq b\right\} \cup \{0\}, \\ D_xT^mC_x(x) &= \left\{u \in T_{T^mx}M \setminus \{0\} \colon \bar{a} \leq \tau^-(T^mx, u) \leq \bar{b}\right\} \cup \{0\}, \end{aligned}$$

where

$$\begin{aligned} a &= G^-_{T^mx}(T^mx), \quad b = +\infty, \\ \bar{a} &= l - \sup\left\{\tau^+_{m-1}(x,u) \colon 0 \neq u \in C_x(x)\right\}, \\ \bar{b} &= l - \inf\left\{\tau^+_{m-1}(x,u \colon 0 \neq u \in C_x(x)\right\}. \end{aligned}$$

Since $m - 1 = n(x)$, Condition B2 implies that

$$\sup\left\{\tau^+_{m-1}(x,u) \colon 0 \neq u \in C_x(x)\right\} \leq \lambda^+_j.$$

By the definition of $C_x(x)$ when Γ_j is dispersing and Part (3) of Theorem 4.2 when Γ_j is focusing, we deduce that

$$\frac{d(T^{m-1}x)}{2} \leq \inf\left\{\tau^+_{m-1}(x,u) \colon 0 \neq u \in C_x(x)\right\}.$$

Also, note that $G^-_{T^mx}(T^mx) \leq \lambda^-_k$. Therefore,

$$a \leq \lambda^-_k, \qquad l - \lambda^+_j \leq \bar{a}, \qquad \bar{b} \leq l - \frac{d(T^{m-1}x)}{2}. \tag{6}$$

Since $D_xT^mC_x(x) \subset \operatorname{int} C_{T^mx}(T^mx)$ by Lemma 5.2, we can use a formula for $\tilde{\sigma}(D_xT^m)$ proved by Wojtkowski [32, Lemma A.4 and Appendix B], and obtain[1]

$$\tilde{\sigma}(D_xT^m) = \sqrt{1+w} + \sqrt{w}, \qquad w = \frac{\bar{a}-a}{\bar{b}-\bar{a}}. \tag{7}$$

From (6), it follows that

$$w \geq \frac{l - \lambda_j^+ - \lambda_k^-}{\lambda_j^+ - d(T^{m-1}x)/2}.$$

It is easy to see that $d(T^{m-1}x)/2 < \lambda_j^+$ and $d(T^{m-1}x)/2 \leq 1/c$ for some constant c depending only on Ω. Hence,

$$w \geq c(l - \lambda_j^+ - \lambda_k^-).$$

The wanted conclusion now follows from (7) once we have observed that $l - \lambda_j^+ - \lambda_k^- \geq \lambda$ by B2 if Γ_j or Γ_k is focusing, and $\lambda_j^+ = \lambda_k^- = 0$ if Γ_j and Γ_k are dispersing. □

Proposition 6.2. *Let $x \in E \setminus R_\infty^+$, and suppose that there exists a strictly increasing (decreasing) sequence of positive (negative) integers $\{n_k\}_{k\in\mathbb{N}}$ such that $T^{n_k}x \in E$ for every $k > 0$. Then*

$$\lim_{k\to+\infty} \tilde{\sigma}(D_xT^{n_k}) = +\infty.$$

Proof. We prove the proposition only for the case when $\{n_k\}_{k\in\mathbb{N}}$ is a strictly increasing sequence of positive integers. For the other case, the proof is similar in view of the fact that $\tilde{\sigma}(D_xT^{-n}) = \tilde{\sigma}(D_{T^{-n}x}T^n)$ for $n > 0$ (see [22, Section 6]).

Without any loss of generality, we can assume that $T^jx \notin E$ for all $n_k < j < n_{k+1}$. Set $n_0 = 0$, and define $x_k = T^{n_k}x$ and $m_k = n_{k+1} - n_k$ for $k \geq 0$. Also, let l_k be the distance between $q(T^{-1}x_{k+1})$ and $q(x_{k+1})$. By the supermultiplicativity of $\tilde{\sigma}$, we have

$$\tilde{\sigma}(D_xT^{n_k}) \geq \prod_{i=0}^{k-1} \tilde{\sigma}(D_{x_i}T^{m_i}). \tag{8}$$

By Lemma 6.1, we see that the sequence $\prod_{i=0}^{k-1} \tilde{\sigma}(D_{x_i}T^{m_i})$ is strictly increasing in k, and it does not diverge only if $T^{n_i}x \in M^-$ for i sufficiently large and $\lim_{i\to+\infty} l_i = 0$. This means that the positive semi-trajectory of x is trapped forever inside a cusp formed by two dispersing components or a dispersing and a flat component. But every trajectory entering a cusp leaves it after finitely many collisions (for instance, see Appendix A1.3 of [7]). Hence, we must have $\lim_{n\to+\infty} \prod_{i=0}^{k-1} \tilde{\sigma}(D_{x_i}T^{m_i}) = +\infty$, which implies the wanted conclusion by (8). □

[1] In general, we have $\rho = \frac{b-\bar{b}}{b-a} \cdot \frac{\bar{a}-a}{\bar{b}-\bar{a}}$. Here, we have used $b = +\infty$.

For the definitions of sufficient and essential points and relative notations, see Definitions A.1 and A.2.

Corollary 6.3. *Suppose that $x \in H$. Then there exists $l \in \mathbb{Z}$ such that $T^l x$ is essential, and so x is sufficient. If we further assume that x belongs to $S_1^- \cup S_1^+ \cup E$, then x is essential.*

Proof. Let $x \in H$. By the definition of H, we can find a sequence $n_k \nearrow \infty$ such that either $T^{n_k}x \in E$ or $T^{-n_k}x \in E$ for $k > 0$. We prove the corollary only for the case $T^{n_k}x \in E$ for $k > 0$, the proof for the other case being similar.

Let $x_k = T^{n_k}x$ for $k > 0$. By Lemma 5.2, we have $\tilde{\sigma}(D_{x_1}T^{n_2-n_1}) > 1$, and so $\tilde{\sigma}^*(D_{x_1}T^{n_2-n_1}) > 0$ by (4). Fix $\alpha > 0$. Proposition 6.2 allows us to find $k > 2$ such that $\tilde{\sigma}(D_{x_1}T^{n_k-n_1}) > \alpha/\tilde{\sigma}^*(D_{x_1}T^{n_2-n_1})$. Using (5), we then obtain $\tilde{\sigma}^*(D_{x_1}T^{n_k-n_2}) > \alpha$. Now, it is easy to see that there exists a neighborhood V of x_1 such that $V \subset U_{x_1}$, $V \cap R^+_{n_k-n_1} = \emptyset$ and $T^{n_k-n_1}V \subset U_{x_k}$. Since $\tilde{\sigma}^*(D.T^{n_k-n_2})$ is continuous at y, we can choose V so that $\tilde{\sigma}^*(D_z T^{n_k-n_2}) > \alpha$ for every $z \in V$. But this means that x_1 is u-essential with $n_{x_1,\alpha} = n_k - n_1$, $O_{x_1,\alpha} = V$ and the cone field $K_{x_1,\alpha}$ given by $K_{x_1,\alpha} = C_{x_1}$ on V, and $K_{x_1,\alpha} = C_{x_k}$ on $T^{n_k-n_1}V$. By choosing $\alpha = 3$, we see that x is sufficient with quadruple (l, N, O, K) such that $l = n_1$, $N = n_{x_1,3}$, $O = T^{n_{x_1,3}}O_{x_1,3}$ and $K = K_{x_1,3}$.

To prove the last part of the corollary, we observe that if we further assume that $x \in S_1^- \cup S_1^+ \cup E$, then (U_x, C_x) is defined (if $x \in H \cap S_1^\pm$, then $x \in S_1^\pm \setminus (R_\infty^\mp \cup N^\mp)$), and the previous argument can be repeated verbatim with x_1 and n_1 replaced by x and 0, respectively. ☐

We can now proceed to prove Conditions L1-L3.

6.2. **Proof of Conditions L1-L3.** Consider $x \in H$. By Corollary 6.3, it follows that x is sufficient. Let (l, N, O, K) be the quadruple associated to x. From the proof of Corollary 6.3, we see that if $z := T^l x \in E$, then $T^{-N}z \in E$, $O \subset U_z$, $T^{-N}O \subset U_{T^{-N}z}$ and the cone field $(O \cup T^{-N}O, K)$ is given by $K = C_z$ on O and $K = C_{T^{-N}z}$ on $T^{-N}O$. Also, note that by construction O and $T^{-N}O$ do not intersect ∂M. Since E is open (in the topology of M), by taking O sufficiently small, we can assume without loss of generality that O and $T^{-N}O$ are both contained in E.

To prove Conditions L1-L3, we use several results concerning the singulars sets R_k^- and R_k^+ proved in [15].

Proposition 6.4. *Condition L1 is satisfied.*

Proof. For billiards satisfying Conditions B1 and B2, the regularity of the singular sets R_k^- and R_k^+ was proved in [15, Theorem 2.2]. ☐

Proposition 6.5. *Every point $x \in H$ satisfies Condition L2.*

Proof. This proposition follows from [15, Proposition 6.2]. For the convenience of the reader, we give here a more direct proof. We prove only the first part of Condition L2, because the second one can be proved similarly. Let Σ be a component of R_k^- with $k > 0$. Since $\partial M \cap T^{-N}O = \emptyset$, we can assume without loss of generality that Σ is not contained in ∂M. It follows from the definition of R_k^- that there exists $0 \leq i < k$ such that the rays emerging from the points of $-T^{-i}\Sigma$ focus in linear approximation at a corner of ∂Q or a point lying on a dispersing component of ∂Q. Therefore, if $y \in \Sigma \cap T^{-N}O$, then we have $\tau_{-i}^-(y,u) = t(-T^{-i}y)$ for every $u \in T_y^*\Sigma$. Now, it is not difficult to see that the invariance of $\{(U_z, C_z)\}_{z\in E}$ and the second part of Condition B2 imply that $\tau_{-i}^-(y,v) < t(-T^{-i}y)$ whenever $0 \neq v \in K'(y)$. We conclude that $T_y\Sigma \subset \operatorname{int} K(y)$. □

Proposition 6.6. *Every point $x \in H$ satisfies Condition L3.*

Proof. The proofs of Condition L3 concerning S_1^- and S_1^+ are essentially the same: one is obtained from the other one by exchanging the symbols $+$ and $-$, and replacing T with T^{-1}. Thus, it is enough to prove the part concerning S_1^-.

The set $S_1^- \cap R_\infty^+$ is at most countable by [15, Propositions 6.17-6.19]. This fact and Condition B3 imply that $m_-(S_1^- \cap (N^+ \cup R_\infty^+)) = 0$. Therefore, it is enough to prove that each element of $S_1^- \setminus (N^+ \cup R_\infty^+)$ is u-essential. This is so because of Corollary 6.3. The second part of L3 is a direct consequence of Lemma 5.2 and the construction of the cone fields (O, K) for $x \in H$ and $(O_{y,\alpha} \cup T^{n_{y,\alpha}}O_{y,\alpha}, K_{y,\alpha})$ for $y \in S_1^- \setminus (N^+ \cup R_\infty^+)$ in the proof of Corollary 6.3. □

7. Proof of Condition L4

In this section, we prove Condition L4 for every $x \in H$. This condition requires the existence of stable and unstable manifolds a.e. on the neighborhood O of x, which is guaranteed by Proposition A.3. We derive L4 from the *non-contraction property*: there exists $\beta' > 0$ such that if $z \in E \setminus R_m^+$ and $T^m z \in E \cup -E$ with $m > 0$, then

$$\|D_z T^m v\| \geq \beta' \|v\| \qquad \text{for } v \in C_z(z), \tag{9}$$

where $\{(U_z, C_z)\}_{z\in E}$ is the cone field introduced in Subsection 4.3. To prove that this property holds true we decompose the billiard orbits into special blocks, and study (9) separately for each block. This analysis will be carried out using certain semi-norms defined in terms of transversal Jacobi fields along billiard orbits.

7.1. **Jacobi fields and semi-norms.** Let $y \in M$. There is a bijective correspondence between a vector $u \in T_y M$ and a vector $(J, J') \in \mathbb{R}^2$ such that $\langle J, v(y)\rangle = \langle J', v(y)\rangle = 0$. The pair (J, J') defines a Jacobi field along the segment $[q(y), q_1(y)]$ of the billiard trajectory of y given by $J(s) = J + sJ'$ for $0 \leq s \leq t(y)$. The property of (J, J') implies that $\langle J(s), v(y)\rangle = 0$ for $0 \leq s \leq t(y)$. Such a Jacobi field is called *transversal*. For more on transversal Jacobi fields and billiards, see [10, 33].

In fact that the pair (J, J') forms a system of coordinates for the tangent space T_yM. The change of coordinates $(u_s, u_\theta) \mapsto (J, J')$ is given by

$$J = \sin\theta(y)u_s \qquad \text{and} \qquad J' = -\kappa(y)u_s - u_\theta,$$

where we have chosen the orientation so that the line orthogonal to $v(y)$ makes with the positive tangent line to $\partial\Omega$ an angle less than $\pi/2$. To underline the dependence of J and J' on the vector u, we will write $J(u)$ and $J'(u)$. When no collision with $\partial\Omega$ occurs during an interval of length $t \in \mathbb{R}$, the evolution of (J, J') is given by a linear transformation $F(t)$. At a collision $y \in M \setminus \partial M$, the pair (J, J') is transformed according to a linear transformation $R(y)$. The maps $F(t)$ and $R(y)$ are as follows:

$$F(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R(y) = \begin{pmatrix} -1 & 0 \\ \frac{2}{d(y)} & -1 \end{pmatrix}, \tag{10}$$

where the entry $2/d(y)$ has to be understood as zero when $y \in M^0$. Hence, the matrix of D_yT in coordinates J and J' is given by

$$D_yT = R(Ty)F(t(y)) = \begin{pmatrix} -1 & -t(y) \\ \frac{2}{d(Ty)} & -1 + 2\frac{t(y)}{d(Ty)} \end{pmatrix}. \tag{11}$$

Note that all the matrices (10) and (11) have determinant equal to ± 1. This means that in coordinates J and J', the map D_yT preserves the standard symplectic form $J \wedge J'$. Finally, we observe that the forward focusing time of a tangent vector $u \in T_yM$ with $y \in M$ in terms of $J(u)$ and $J'(u)$ is given by

$$\tau^+(y, u) = \begin{cases} -\frac{J(u)}{J'(u)} & \text{if } J'(u) \neq 0, \\ \infty & \text{if } J'(u) = 0. \end{cases} \tag{12}$$

Definition 7.1. *For every $y \in M \setminus \partial M$ and every $u \in T_yM$, define*

$$\|u\|_J = \sqrt{J^2(u) + J'^2(u)} \quad \text{and} \quad |u|_{J'} = |J'(u)|.$$

In the rest of this subsection, we prove several relations involving the semi-norms $\|\cdot\|$ and $\|\cdot\|_J$. The goal here is to show that $\|\cdot\|$ and $\|\cdot\|_J$ are equivalent on a certain subset of the tangent bundle TM. The cone fields $\{(U_y, C_y)\}_{y\in E}$ used in the propositions below are those introduced in Subsection 6.2.

Lemma 7.2. *There exists a constant $\alpha_1 > 1$ depending only on Ω and the family of cone fields $\{(U_y, C_y)\}_{y\in E^+}$ such that if $y \in E^+$, $z \in U_y$ and $0 \leq k \leq n(z)$, then*

$$|D_zT^k u|_{J'} \leq \|D_zT^k u\|_J \leq \alpha_1 |D_zT^k u|_{J'} \qquad \textit{for } u \in C_y(z).$$

Proof. Let y, z, k, u be as in the hypotheses of the lemma. It is clear that $|D_zT^ku|_{J'} \leq \|D_zT^ku\|_J$. By Part (3) of Theorem 4.2, there exists a constant $f > 0$ depending only on $\{(U_y, C_y)\}_{y\in E}$ such that

$$\frac{d(T^kz)}{2} \leq \tau_k^+(z,u) \leq f.$$

By (12), we then have $|J(D_zT^ku)| \leq f|J'(D_zT^ku)|$, which gives the remaining inequality with $\alpha_1 = (1+f^2)^{1/2}$. □

Lemma 7.3. *There exists $\alpha_2 > 1$ such that if $y \in M \setminus \partial M$, then*

$$\|u\|_J \leq \alpha_2\|u\| \qquad \textit{for } u \in T_yM.$$

Proof. A straightforward computation gives the wanted inequality with $\alpha_2 = (2+\kappa_1^2)^{1/2}$, where $\kappa_1 = \max_{z\in M}|\kappa(z)|$. □

Lemma 7.4. *There exists $\alpha_3 > 0$ such that if $y \in M^-$ and $z \in U_y$, then*

$$\|u\| \leq \alpha_3\|u\|_J \qquad \textit{for } u \in C_y(z).$$

Proof. Let $\kappa_2 = \min_{z\in M^-}|\kappa|$. By the definition of $C_y(z)$, we have $u_\theta/u_s \leq \kappa(z) < 0$ for all $0 \neq u \in C_y(z)$. Hence,

$$\begin{aligned}\|u\|_J^2 \geq |J(u)|^2 &= \kappa^2(z)u_s^2 + u_\theta^2 + 2\kappa(z)u_su_\theta \\ &\geq \kappa_2^2u_s^2 + u_\theta^2 \geq \frac{\kappa_2^2}{1+\kappa_2^2}(u_s^2+u_\theta^2) = \frac{\kappa_2^2}{1+\kappa_2^2}\|u\|^2.\end{aligned}$$

□

Lemma 7.5. *There exists $\alpha_4 > 0$ such that if $y \in E^+$, $z \in U_y$ and $u \in C_y(z)$, then*

$$\|D_zT^ku\| \leq \alpha_4\|D_zT^ku\|_J \qquad \textit{for } 0 \leq k \leq n(z).$$

Proof. Denote by κ_3 the maximum of κ on M^+, and denote by $\overline{m}$ the smallest m_i associated to focusing components of $\partial\Omega$ as in Theorem 4.2. Let $0 \neq u \in C_y(z)$, and write $(u_{k,s}, u_{k,\theta})$ for the vector D_zT^ku. From Theorem 4.2, we know that if $z \in M_i \subset M^+$, then $-\kappa(T^kz) + m_i \leq m(u_k) \leq \kappa(T^kz)$ for $0 \leq k \leq n(z)$. In particular, $|u_{k,\theta}| \leq \kappa(T^kz)|u_{k,s}|$. Therefore,

$$\begin{aligned}\|D_zT^ku\|_J^2 &\geq \left(\kappa(T^kz)u_{k,s} + u_{k,\theta}\right)^2 = \left(\kappa(T^kz) + m(u_k)\right)^2 u_{k,s}^2 \\ &\geq \left(\kappa(T^kz) - \kappa(T^kz) + m_i\right)^2 u_{k,s}^2 \geq \frac{m_i^2}{1+\kappa^2(T^kz)}\|D_zT^ku\|^2 \\ &\geq \frac{\overline{m}^2}{1+\kappa_3^2}\|D_zT^ku\|^2.\end{aligned}$$

□

Lemma 7.6. *There exist $\epsilon_0 > 0$ and $0 < \theta_0 < \pi/2$ such that $\theta(M^0 \cap S_1^-(\epsilon_0)) \in (\theta_0, \pi - \theta_0)$.*

Proof. The lemma is an immediate consequence of the following easy-to-check fact. Consider a flat component Γ_i of $\partial\Omega$. If the line containing Γ_i contains also a corner p or is tangent to a dispersing component Γ_j, then no ray emerging from the elements of $M_i \cap S_1^+$ contains p or is tangent to Γ_j. □

Lemma 7.7. *Let ϵ_0 be the constant in Lemma 7.6. There exists $\alpha_5 > 0$ such that if $y \in M^0 \cap S_1^-(\epsilon_0)$, then*

$$\|u\| \leq \alpha_5 \|u\|_J \qquad \text{for } u \in T_y M.$$

Proof. Since $\kappa(y) = 0$, we have $\|u\|_J^2 \geq \sin^2\theta_0 u_s^2 + u_\theta^2 > \sin^2\theta_0 \|u\|^2$. □

Corollary 7.8. *There exist two constants $0 < A_1 < A_2$ such that if $y \in E$, $z \in U_y$, $u \in C_y(z)$ or $y \in M^0 \cap S_1^-(\epsilon_0)$, $u \in T_y M$, then*

$$A_1 \|u\|_J \leq \|u\| \leq A_2 \|u\|_J.$$

Proof. The claim follows from Lemmas 7.3-7.7. □

Remark 7.9. *It can be easily proved that Corollary 7.8 remains valid if $\|\cdot\|$ is replaced by $|\cdot|_{J'}$, thus showing that the semi-norms $\|\cdot\|$, $\|\cdot\|_J$, $|\cdot|_{J'}$ are equivalent in the sense specified in the corollary. However, this general equivalence is not needed for the proof of L4.*

7.2. **Non-contraction property and block decomposition.** The non-contraction property is just a slight modification of the condition with the same name introduced in [22]. Roughly speaking, this property states that the contraction of the iterations of vectors from the cones in $\{U_y, C_y\}_{y \in E}$ is uniformly bounded below along orbits starting at E and ending at $E \cup -E$. This property provides some control on the accumulation of the expansion along the unstable direction along billiard orbits. In particular, it implies no loss of expansion along arbitrarily long sequences of consecutive collisions with focusing components or between flat components.

We now show that every sequence of collisions $\{z, \ldots, T^m z\}$ as in the non-contraction property can be decomposed into a finite number of special subsequences called blocks.

Definition 7.10. *Let $z \in M \setminus R_m^+$ with $m > 0$. A sequence of consecutive collision $\{z, \ldots, T^m z\}$ is called a* block of type $i \in \{1, \ldots, 4\}$ *if the corresponding condition (i) below is satisfied:*

(1) $z \in E^+$, $\{T^{n(z)+1} z, \ldots, T^{m-1} z\} \subset M^0$ and $T^m z \in M^-$,
(2) $z \in M^-$, $\{Tz, \ldots, T^{m-1} z\} \subset M^0 \cup M^-$ and $T^m z \in M^-$,
(3) $z \in M^-$, $\{Tz, \ldots, T^{m-1} z\} \subset M^0$ and $T^m z \in E^+$,
(4) z and $T^m z$ belong to E^+.

Definition 7.11. *A block is called* minimal *if it does not contain any other block of the same type. A block included in a sequence of consecutive collisions φ is called* maximal in φ *if it is not contained in any other block of the same type in φ.*

We observe that blocks of type 1 and 3 are always maximal and minimal. Also, we observe that every block of type 2 and 4 is a union of finitely many minimal blocks of the type 2 and 4, respectively.

Proposition 7.12. *Let $\varphi = \{z, \dots, T^m z\}$ be a sequence of collisions such that both z and $T^m z$ belong to E. Then $\varphi = \varphi_1 \cup \cdots \cup \varphi_n$ with $n \leq 5$ and $\varphi_1, \dots, \varphi_n$ being maximal blocks of type 1-4. Moreover, this decomposition is unique.*

Proof. First, suppose that φ does not contain blocks of type 2 and 4. Then we see that either φ is a block of type 1 or 3, or φ is the union of two blocks: a block of type 3 followed by a block of type 1.

Now, suppose that φ contains blocks of type 2, but does not contain blocks of type 4. A maximal block of type 2 can only be preceded by a block of type 1 and followed by a block of type 3. Hence, φ can contain at most two maximal blocks of type 2. Moreover, if φ contains exactly two maximal blocks of type 2, then φ is the union of a maximal block of type 2, a block of type 1, a block of type 3 and a maximal block of type 2, following one another in this order. Instead, if φ_1 contains only a single maximal block of type 2, then there are three possibilities: φ is a block of type 2, φ is the union of a maximal block of type 2 and a block of type 1 or 3, and φ is the union of a block of type 1, a maximal block of type 2 and a block of type 3, following one another in this order.

Finally, suppose that φ contains blocks of type 4. From the definition of these blocks, we see that there is only one single maximal block of type 4 in φ. Such a block can only be preceded by a block of type 3 and followed by a block of type 1. Moreover, blocks of type 1 and 3 in φ can only be adjacent to blocks of type 2 or 4. Since there is only one maximal block of type 4 in φ, besides this block, φ can contain at most a block of type 1, at most a block of type 3 and at most two maximal blocks of type 2. The blocks of type 1 and 3 are attached to the block of type 4, whereas one block of type 2 is attached to the block of type 1, and the second block of type two is attached to the block of type 3.

It is not difficult to see that the decomposition of φ into blocks of type 1-4 we have just obtained is unique, because the blocks forming it are maximal. □

7.3. **Proving the non-contraction property.** It suffices to show that (9) holds along each block of type 1-4 with β' depending only on the block type.

We start with some preliminary results.

Lemma 7.13. *Let $0 \leq m_1 < m_2$, and suppose that $z \in E \setminus R^+_{m_2}$ and $0 \neq v \in C_z(z)$. Then*

$$\frac{|D_z T^{m_2} v|_{J'}}{|D_z T^{m_1} v|_{J'}} = \prod_{k=m_1+1}^{m_2} \left| \frac{\tau_k^-(z,v)}{\tau_k^+(z,v)} \right| .$$

Proof. Define $z_k = T^k z$ and $v_k = D_z T^k v$ for $0 \le k \le n(z)$. By the definition of C_z and its invariance, it is not difficult to see using Theorem 4.2 that $J'(v_k) \neq 0$ for $0 \le k \le n(z)$. Using (11) and Condition B2, one can further show that $J(v_k) \neq 0$ for $1 \le k \le n(z)$. Therefore,

$$\frac{|v_m|_{J'}}{|v_{m_1}|_{J'}} = \prod_{k=m_1+1}^{m} \left| \frac{J'(v_k)}{J(v_k)} \right| \cdot \left| \frac{J(v_k)}{J'(v_{k-1})} \right|.$$

Now, the wanted equality follows from $\tau^+(z_k, v_k) = -J(v_k)/J'(v_k)$ and $\tau^-(z_k, v_k) = J(v_k)/J'(v_{k-1})$. The first expression for $\tau^+(z_k, v_k)$ is just (12), whereas the one for $\tau^-(z_k, v_k)$ can be easily derived from (11) and (12). □

Proposition 7.14. *There exists $\gamma_1 > 0$ such that if $z \in E^+$ and $0 \le m_1 < m_2 \le n(z)$, then*

$$\|D_z T^{m_2} v\| \ge \gamma_1 \|D_z T^{m_1} v\| \qquad \text{for } v \in C_z(z).$$

Proof. In virtue of Lemmas 7.2, 7.3 and 7.5, we can prove the proposition with the norm $\|\cdot\|$ replaced by the semi-norm $|\cdot|_{J'}$. Let $z \in E_i \subset E^+$ for some i, and pick $v \in C_z(z)$. Of course, it is enough to prove the proposition with γ_1 depending on the focusing component Γ_i. By taking the minimum of such γ_1's over all focusing components, we obtain the proposition in its general form. Let θ_i, a_i, c_i, N_i be as in Theorem 4.2 and Proposition 4.3. We define $z_k = T^k z$, $v_k = D_z T^k v$ for $0 \le k \le n(z)$.

Suppose first that $\theta = \theta(z) \in (0, \theta_i) \cup (\pi - \theta_i, \pi)$. We consider only the case when $\theta \in (0, \theta_i)$, because the argument for $\theta \in (\pi - \theta_i, \pi)$ is similar. By Theorem 4.2, we have $m(v_k) \ge -a_i\theta$. Let $R_i = \max_{y \in M_i} 1/\kappa(y)$. Using (1), we easily obtain

$$\frac{\tau^-(z_k, v_k)}{\tau^+(z_k, v_k)} \ge \frac{1 - R_i a_i \theta}{1 + R_i a_i \theta}$$

By Proposition 4.3, we have $n(z) \le c_i/\theta$, and so Lemma 7.13 implies that

$$\frac{|v_{m_2}|_{J'}}{|v_{m_1}|_{J'}} \ge \left(\frac{1 - R_i a_i \theta}{1 + R_i a_i \theta} \right)^{\frac{c_i}{\theta}}.$$

Since the right hand-side of the previous inequality converges to $e^{-2a_i c_i R_i} > 0$ as $\theta \to 0^+$, there exists $0 < \bar{\theta} < \theta_i$ such that

$$\frac{|v_{m_2}|_{J'}}{|v_{m_1}|_{J'}} \ge \frac{1}{2} e^{-2a_i c_i R_i} \qquad \text{for } \theta \in (0, \bar{\theta}). \tag{13}$$

Now, suppose that $\theta(z) \in [\bar{\theta}, \pi - \bar{\theta}]$. By Part (3) of Theorem 4.2, there exists a constant $f_i > 0$ such that $d(z_k)/2 \le \tau^{\pm}(z_k, v_k) \le f_i$ for $1 \le k \le m_2$. Let $r_i = \min_{y \in M_i} 1/\kappa(y)$, and let $d_i = r_i \sin \theta_i$. Then

$$\frac{\tau^-(z_k, v_k)}{\tau^+(z_k, v_k)} \ge \frac{d(z_k)}{2f_i} \ge \frac{d_i}{2f_i} \qquad \text{for } 1 \le k \le m_2.$$

Using Lemma 7.13 and Proposition 4.3, we conclude that

$$\frac{|v_{m_2}|_{J'}}{|v_{m_1}|_{J'}} \geq \min\left\{1, \left(\frac{d_i}{2f_i}\right)^{N_i(\bar{\theta})}\right\}. \tag{14}$$

The wanted conclusion now follows from (13) and (14). □

Let λ be as in B2, and let f_i be as in the proof of Proposition 7.14. Define f to be the maximum of all f_i's.

Lemma 7.15. *Consider a sequence of collisions* $\{z, \dots, T^m z\}$ *with* $m > 0$ *such that* $z, T^m z \in E$, *and* $T^k z \in M^0$ *for* $n(z) < k < m$. *Then, for every* $0 \neq v \in C_z(z)$ *and every* $n(z) \leq k < m$, *we have*

$$|D_z T^m v|_{J'} > \delta |D_z T^k v|_{J'},$$

where $\delta = \lambda/f$ *if* $T^m z \in M^+$, *and* $\delta = 1$ *if* $T^m z \in M^-$.

Proof. Let $0 \neq v \in C_z(z)$, and define $z_k = T^k z$ and $v_k = D_z T^k v$ for $0 \leq k \leq m$. From $\{z_{n(z)+1}, \dots, z_{m-1}\} \subset M^0$, it follows that $|v_k|_{J'} = |v_{m-1}|_{J'}$ for all $n(z) \leq k < m$. Then, by Lemma 7.13, we have

$$\frac{|v_m|_{J'}}{|v_k|_{J'}} = \frac{|v_m|_{J'}}{|v_{m-1}|_{J'}} = \frac{\tau^-(z_m, v_m)}{\tau^+(z_m, v_m)} \qquad \text{for } n(z) \leq k < m.$$

If $z_m \in M^+$, then $\tau^-(z_m, v_m) = l(z_{n(z)}, z_m) - \tau^+(z_{n(z)}, v_{n(z)}) > \lambda$ by Condition B2. Since $0 < \tau^+(z_m, v_m) \leq f$ (see the proof of Proposition 7.14), we obtain $\tau^-(z_m, v_m)/\tau^+(z_m, v_m) > \lambda/f$. If $z_m \in M^-$, then we use the Mirror Formula (2) to relate the focusing times $\tau^-(z_m, v_m)$ and $\tau^+(z_m, v_m)$. A simple computation using $d(z_m) < 0$ shows that $\tau^-(z_m, v_m) < \tau^+(z_m, v_m) < 0$, and so $\tau^-(z_m, v_m)/\tau^+(z_m, v_m) > 1$. □

Lemma 7.16. *Consider a sequence of collisions* $\{z, \dots, T^m z\}$ *with* $m > 0$ *such that* $z \in M^-, T^m z \in E$ *and* $T^k z \in M^0$ *for* $n(z) < k < m$. *Then, for every* $0 \neq v \in C_z(z)$ *and every* $0 \leq k < m$, *we have*

$$|J(D_z T^m v)| > |J(D_z T^k v)|.$$

Proof. Let $0 \neq v \in C_z(z)$. By the definition of C_z (see Subsection 4.3), it is easy to see that $J(v)J'(v) \geq 0$ and $J'(v) \neq 0$. Since $\{Tz, \dots, T^{m-1}z\} \subset M^0$, we have $|J'(D_z T^k v)| = |J'(v)|$ for $0 < k < m$. Using (11), we obtain

$$|J(D_z T^k v)| = |J(v)| + l(z, T^k z)|J'(v)| \qquad \text{for } 0 \leq k \leq m,$$

where $l(z, T^k z)$ is the length of the piece of trajectory starting at z and ending at $T^k z$ (see the beginning of Subsection 6.1). The wanted conclusion follows immediately from the previous equality. □

Lemma 7.17. *There exists a constant* $\delta_1 > 0$ *such that for every sequence of consecutive collisions* $\{z, \dots, T^m z\}$ *with* $z \in E^+$, $T^m z \in E$ *and* $T^k z \in M^0$ *for* $n(z) < k < m$, *we have*

$$\|D_z T^m v\| \geq \delta_1 \|D_z T^{n(z)} v\| \qquad \textit{for } v \in C_z(z).$$

Proof. If we replace $\|\cdot\|$ with $|\cdot|_{J'}$, then the wanted conclusion with $\delta_1 = \delta$ follows from Lemma 7.15. To obtain the actual conclusion, use the obvious fact that $\|\cdot\|_J \geq |\cdot|_{J'}$, apply Lemma 7.3 to $\|D_z T^m v\|_J$, and finally apply Lemmas 7.2 and 7.5 to $|D_z T^{n(z)} v|_{J'}$ and $\|D_z T^{n(z)} v\|_J$, respectively. □

Lemma 7.18. *There exists $\beta_1' > 0$ such that every sequence of consecutive collisions $\{z, \ldots, T^m z\}$ such that $z \in E^+$, $\{T^{n(z)+1} z, \ldots, T^{m-1} z\} \subset M^0$ and $T^m z \in E$ satisfies (9) with $\beta' = \beta_1'$. In particular, the previous conclusion is true for every block of type 1.*

Proof. Let $\varphi = \{z, \ldots, T^m z\}$ be a block of type 1, and choose $0 \neq v \in C_z(z)$. By Proposition 7.14, we have $\|D_z T^{n(z)} v\| \geq \gamma_1 \|v\|$. To complete the proof, use Lemma 7.17. □

Lemma 7.19. *There exists $\beta_2' > 0$ such that every block of type 2 satisfies (9) with $\beta' = \beta_2'$.*

Proof. Note first that every block of type 2 consists of finitely many minimal blocks of type 2. Next, suppose that $\{z, \ldots, T^m z\}$ is a minimal block of type 2, and let $0 \neq v \in C_z(z)$. In this case, we have $T^k z \in M^0$ for $1 \leq k \leq m-1$. It follows that we have $|J'(D_z T^m v)| > |J'(v)|$ by Lemma 7.15, and $|J(D_z T^m v)| > |J(v)|$ by Lemma 7.16. Therefore, $\|D_z T^m v\|_J > \|v\|_J$. Of course, the same conclusion extends to a general block of type 2. To complete the proof, apply Lemmas 7.3 and 7.4 to $\|D_z T^m v\|_J$ and $\|v\|_J$, respectively. □

Lemma 7.20. *There exists $\beta_3' > 0$ such that every block of type 3 satisfies (9) with $\beta' = \beta_3'$.*

Proof. Suppose that $\{z, \ldots, T^m z\}$ is a block of type 3, and let $0 \neq v \in C_z(z)$. Since $T^k z \in M^0$ for $1 \leq k < m$, we have $|J'(D_z T^m v)| > |J'(v)|\lambda/f$ by Lemma 7.15, and $|J(D_z T^m v)| > |J(v)|$ by Lemma 7.16. Hence, $\|D_z T^m v\|_J > \min\{\lambda/f, 1\}\|v\|_J$. To complete the proof, apply Lemmas 7.3 and 7.4 to $\|D_z T^m v\|_J$ and $\|v\|_J$, respectively. □

Let $\{z, \ldots, T^m z\}$ be a block of type 4. As before, we define $z_j = T^j z$ for all $0 \leq j \leq m$. We will use this notation throughout the rest of this section. Note that every block of type 4 consists of finitely many minimal blocks of type 4. In other words, there exist $N > 0$ integers $0 = i_0 < \cdots < i_N = m$ such that $\{z_{i_k}, \ldots, z_{i_{k+1}}\}$ is a minimal block of type 4 for each $0 \leq k \leq N-1$ (see Fig. 1). Now, recall that $n(z_{i_k})$ denotes the number of consecutive collisions of z_{i_k} with the focusing component Γ_i before leaving it (see Definition 3.2). Define $j_k = n(z_{i_k}) + i_k$. Then

$$D_z T^m = D_{z_{j_{N-1}}} T^{i_N - j_{N-1}} \circ D_{z_{j_0}} T^{j_{N-1} - j_0} \circ D_{z_0} T^{j_0}.$$

Proposition 7.21. *If $N > 1$, then the matrix of $D_{z_{j_0}} T^{j_{N-1}-j_0}$ in coordinates (J, J') is given by*

$$D_{z_0} T^{j_{N-1}-j_0} = F_{N-1}^{-1} A_{N-1} B_{N-2} \cdots B_1 A_1 B_0 F_0,$$

where A_k, B_k, F_k are matrices of the form

$$A_k = \pm \begin{pmatrix} 1/\zeta_k & 0 \\ \eta_k & \zeta_k \end{pmatrix}, \quad B_k = \pm \begin{pmatrix} 1 + a_k & b_k \\ c_k & 1 + d_k \end{pmatrix}, \quad F_k = \begin{pmatrix} 1 & f_k \\ 0 & 1 \end{pmatrix} \tag{15}$$

with $\zeta_k, \eta_k, f_k > 0$, $a_k, c_k, d_k \geq 0$ and $b_k > \lambda$.

Proof. For every $0 \leq k \leq N-1$, define

$$s_k^- = G_{z_{i_k}}^-(z_{i_k}) \quad \text{and} \quad s_k^+ = \sup \left\{ \tau_{j_k - i_k}^+(z_{i_k}, v) \colon 0 \neq v \in C_{z_{i_k}}(z_{i_k}) \right\},$$

where $G_{z_{i_k}}^{\pm}$ are defined in Definition 4.4. Recall the definitions of the matrices F and R in (10). If we define

$$\begin{aligned} A_k &= F(s_k^+) D_{z_{i_k}} T^{j_k - i_k} R(z_{i_k}) F(s_k^-), \\ B_k &= F^{-1}(s_k^-) R^{-1}(z_{i_{k+1}}) D_{z_{j_k}} T^{i_{k+1} - j_k} F^{-1}(s_k^+), \\ F_k &= F(s_k^+), \end{aligned}$$

then we easily see that

$$D_{z_{j_k}} T^{j_{k+1} - j_k} = F_{k+1}^{-1} A_{k+1} B_k F_k.$$

Therefore,

$$D_{z_0} T^{j_{N-1} - j_0} = \prod_{k=0}^{N-2} D_{z_{j_k}} T^{j_{k+1} - j_k} = F_{N-1}^{-1} A_{N-1} B_{N-2} \cdots B_1 A_1 B_0 F_0.$$

The form of the matrices F_k can be immediately derived from (10). Since $f_k = s_k^+$, the positivity of f_k follows from $s_k^+ > 0$. We now determine the form of the matrices A_k and B_k. Let $v_k \in T_{z_{i_k}} M$ such that $\tau^+(z_{i_k}, v_k) = G_{z_{i_k}}^+(z_{i_k})$, and denote by q_k^- and q_k^+ the points along the trajectory of z_{i_k} where the vector v_k focuses backward, and the vector $D_{z_{i_k}} T^{j_k - i_k} v_k$ focuses forward, respectively. The matrix A_k describes the dynamics of transversal Jacobi fields along the trajectory starting at q_k^- and ending at q_k^+ (see Fig. 1). In other words, if (J_0, J_0') is a transversal Jacobi field at q_k^-, then $(J_1, J_1') := A_k(J_0, J_0')$ is its evolution at q_k^+.

By construction the property of the billiard cone field (see Theorem 4.2), every transversal Jacobi field focusing at q_k^- focuses again at q_k^+. Since by the Mirror Formula, τ^+ is a strictly decreasing function of τ^-, it follows that every transversal Jacobi field consisting of rays parallel to the billiard trajectory passing through q_k^- focuses before reaching q_k^+. Thus, there exist $\xi_k, \zeta_k, \eta_k > 0$ such that

$$A_k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \pm \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix}, \qquad A_k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \pm \begin{pmatrix} 0 \\ \zeta_k \end{pmatrix}.$$

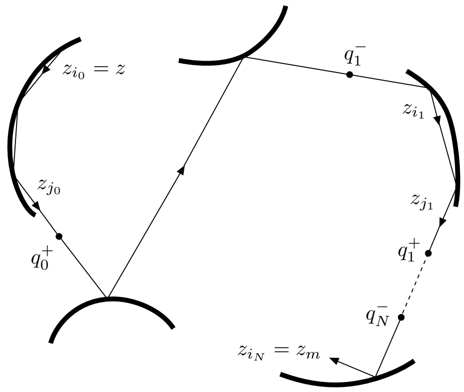

FIGURE 1. Decomposition of a block of type 4. The collisions $z_{i_k} \in E^+$ and $z_{j_k} \in -E^+$ are the first and the last in a sequence of consecutive collisions with a focusing arc, respectively. The sequence of collisions $\{z_{i_k}, \ldots, z_{i_{k+1}}\}$ is a minimal block of type 4. The dashed piece of trajectory between the points q_1^+ and q_N^- represents a sequence of collisions with finitely many boundary components of the billiard domain.

Since A_k is product of finitely many matrices as in (10), we have $\det A_k = 1$, and so $\xi_k = \zeta_k^{-1}$. We conclude that

$$A_k = \pm \begin{pmatrix} 1/\zeta_k & 0 \\ \eta_k & \zeta_k \end{pmatrix}.$$

To derive the form of B_k, we argue similarly. The matrix B_k describes the dynamics of transversal Jacobi fields along the piece of the billiard trajectories starting at q_k^+ and ending at q_{k+1}^- (see Fig. 1). By the definition of $q_k^\pm$, we see that along this piece of trajectory, there are only collisions with flat or dispersing components. Hence, it easily follows from (11) that B_k is a product of finitely many matrices of the form

$$P = \pm \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix} \tag{16}$$

with $a, b, c, d \geq 0$ and $\det P = 1$. All the matrices P forms a semigroup with respect to the standard matrix multiplication. For this reason, there are $a_k, b_k, c_k, d_k \geq 0$ such that

$$B_k = \pm \begin{pmatrix} 1+a_k & b_k \\ c_k & 1+d_k \end{pmatrix}.$$

Since F and R are upper and lower triangular, respectively, we see that the entry b_k cannot be smaller than the length of the piece of the trajectory connecting q_k^+ and q_{k+1}^-. This observation combined with Condition B2 gives $b_k > \lambda$. $\square$

The notation used in the following proposition is as in Proposition 7.21 and the paragraph before it.

Proposition 7.22. *There exists* $\beta_4'' > 0$ *such that*

$$|F_{N-1}D_zT^{j_{N-1}}v|_{J'} \geq \beta_4''|F_0D_zT^{j_0}v|_{J'} \qquad \text{for } v \in C_z(z).$$

Proof. The inequality holds trivially if $N = 1$. Therefore, we assume that $N > 1$. Given a matrix L, denote by $|L|$ the matrix obtained by replacing each entry of L with its absolute value. Given two square matrices L_1 and L_2 of the same order, we write $L_1 \geq L_2$ if each entry of L_1 is greater than or equal to the corresponding entry of L_2. Let $\zeta = \zeta_1 \cdots \zeta_{N-1} > 0$. By the properties the matrices A_k and B_k, it follows that

$$\begin{aligned}|A_{N-1}B_{N-2}\cdots A_1B_0| &= |A_{N-1}||B_{N-2}|\cdots|A_1||B_0|\\ &\geq |A_{N-1}||A_{N-2}|\cdots|A_1||B_0|\\ &\geq \begin{pmatrix}1/\zeta_{N-1} & 0\\ 0 & \zeta_{N-1}\end{pmatrix}\cdots\begin{pmatrix}1/\zeta_1 & 0\\ 0 & \zeta_1\end{pmatrix}\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}1/\zeta & \lambda/\zeta\\ 0 & \zeta\end{pmatrix}.\end{aligned}$$

Let $v \in C_z(z)$. Also, define $v_k = D_zT^kv$ for $0 \leq k \leq m$, $w_0 = F_0v_{j_0}$ and $w_1 = F_{N-1}v_{j_{N-1}}$. By Proposition 7.21, we have

$$w_1 = A_{N-1}B_{N-2}\cdots B_1A_1B_0w_0.$$

From the construction of the cone field $\{(U_y, C_y)\}_{y\in E}$ and the definition of F_k, we easily see that $0 \leq J(w_i)/J'(w_i) \leq f$ for $i = 0, 1$, where f is the positive constant in the proof of Lemma 7.2. The fact that $J(w_i)$ and $J'(w_i)$ have the same sign implies that

$$|J(w_1)| \geq \frac{1}{\zeta}|J(w_0)| + \frac{\lambda}{\zeta}|J'(w_0)| \quad \text{and} \quad |J'(w_1)| \geq \zeta|J'(w_0)|. \tag{17}$$

From $0 \leq J(w_i)/J'(w_i) \leq f$ and the inequality on the left hand-side of (17), we obtain

$$|J'(w_1)| \geq \frac{1}{f}|J(w_1)| \geq \frac{\lambda}{\zeta f}|J'(w_0)|,$$

which together with the inequality on the right-hand sides of (17) gives

$$|J'(w_1)| \geq \frac{1}{2}\left(\zeta + \frac{\lambda}{\zeta f}\right)|J'(w_0)|.$$

But $x + c^2/x \geq 2c$ for every $x > 0$ and every $c \in \mathbb{R}$, and so

$$|J'(w_1)| \geq \left(\frac{\lambda}{f}\right)^{1/2}|J'(w_0)|.$$

The wanted inequality holds with $\beta_4'' = (\lambda/f)^{1/2}$. □

Lemma 7.23. *Let $\beta_4'' > 0$ be the constant in Proposition 7.22. Then*

$$\|D_zT^{j_{N-1}}v\| \geq \beta_4''\|D_zT^{j_0}v\| \qquad \text{for } v \in C_z(z).$$

Proof. By Lemmas 7.2, 7.3 and 7.5, it suffices to prove the inequality with $\|\cdot\|$ replaced by $|\cdot|_{J'}$. From the form of the matrices F_k, we see at once that $|v_{j_0}|_{J'} = |w_0|_{J'}$ and $|v_{j_{N-1}}|_{J'} = |w_1|_{J'}$. The notation here is as in the proof of Proposition 7.22. The wanted conclusion now follows from Proposition 7.22. □

Proposition 7.24. *There exists $\beta_4' > 0$ such that every block of type 4 satisfies* (9) *with $\beta' = \beta_4'$.*

Proof. The notation is an in the proof of Proposition 7.21 and the paragraph before it. Accordingly, a block φ of type 4 is given by $\varphi = \{z_{i_0}, \ldots, z_{i_N}\}$. We write $\varphi = \varphi_1 \cup \varphi_2$, where $\varphi_1 = \{z_{i_0}, \ldots, z_{j_{N-1}}\}$ and $\varphi_2 = \{z_{j_{N-1}}, \ldots, z_{i_N}\}$. It is easy to see that there are 2 integers $j_{N-1} < k_1 \leq k_2 \leq i_N$ such that $\varphi_2 = \psi_1 \cup \psi_2 \cup \psi_3$ with $\psi_1 = \{z_{j_{N-1}}, \ldots, z_{k_1}\}$ being an orbit starting at $z_{j_{N-1}} \in -E^+$ and ending at M^- after a sequence of collisions with M^0, $\psi_2 = \{z_{k_1}, \ldots, z_{k_2}\}$ being a block of type 2, and $\psi_3 = \{z_{k_2}, \ldots, z_{i_N}\}$ being a block of type 3.

Proposition 7.14 and Lemma 7.23 give $\|v_{j_0}\| \geq \gamma_1\|v_{i_0}\|$ and $\|v_{j_{N-1}}\| \geq \beta_4''\|v_{j_0}\|$, respectively. By Lemma 7.17, we obtain $\|v_{k_1}\| \geq \delta_1\|v_{j_{N-1}}\|$ with δ_1 independent of ψ_1 and v_0. Since ψ_2 and ψ_3 are blocks of type 2 and 3, we have $\|v_{k_2}\| \geq \beta_2'\|v_{k_1}\|$ and $\|v_{i_N}\| \geq \beta_3'\|v_{k_2}\|$ by Lemmas 7.19 and 7.20. Putting all together, we obtain the wanted conclusion with $\beta_4' = \beta_2'\beta_3'\beta_4''\gamma_1\delta_1$. □

Corollary 7.25. *If $T^mx \in E$, then* (9) *is satisfied.*

Proof. If $T^mx \in E$, (9) follows from Lemmas 7.18-7.20 and Proposition 7.24. □

Proposition 7.26. *There exists a constant $\gamma_2 > 0$ such that if $z \in E \setminus R_m^+$ with $m > 1$, $\{T^{n(z)+1}z, \ldots, T^{m-1}z\} \subset M^0$, $T^jz \in S_1^-(\epsilon_0)$ for some $n(z) < j < m$, and $T^mz \in E$, then*

$$\|D_zT^mv\| \geq \gamma_2\|D_zT^jv\| \quad \text{for } v \in C_z(z).$$

Proof. Let $0 \neq v \in C_z(z)$, and define z_k and v_k for $0 \leq k \leq m$ as in the proof of Lemma 7.13. We study separately the two cases: $J(v_j)J'(v_j) \geq 0$ and $J(v_j)J'(v_j) < 0$. By Lemmas 7.3 and 7.7, it is enough to prove the desired inequality with $\|\cdot\|$ replaced by $\|\cdot\|_J$.

Suppose that $J(v_j)J'(v_j) \geq 0$. By Lemma 7.15, we have $|J'(v_m)| > \delta|J'(v_j)|$, and by arguing as in the proof of Lemma 7.16, one can easily prove that $|J(v_m)| \geq |J(v_j)|$. It follows at once that there exists $c > 0$ independent of the sequence $\{z, \ldots, T^mz\}$ as in the hypotheses of the proposition and $v \in C_z(z)$ such that $\|v_m\|_J \geq c\|v_j\|_J$.

Suppose now that $J(v_j)J'(v_j) < 0$. This means that v_j is focusing ($\tau^+(z_j, v_j) > 0$), and so $z \in M^+$. By Theorem 4.2, we have

$\tau^+(z_{n(z)}, v_{n(z)}) \leq f$. Recall that $f = \max_i f_i$ (see the paragraph before Lemma 7.15). From $z_k \in M^0$ for $n(z) < k \leq j$, it follows that $|J'(v_j)| = |J'(v_{n(z)})|$, and it is not difficult to see that $|J(v_j)| \leq |J(v_{n(z)})| \leq f|J'(v_j)|$. Thus $\|v_j\|_J \leq \delta_1 |v_j|_{J'}$, where $\delta_1 = (1+f^2)^{1/2}$. On the other hand, by applying Lemma 7.15 to $\{z, \ldots, T^m z\}$, we obtain $|v_m|_{J'} \geq \delta |v_j|_{J'}$. Therefore, using the trivial fact $\|v_m\|_J \geq |v_m|_{J'}$, we obtain $\|v_m\|_J \geq \delta\delta_1^{-1}\|v_j\|_J$. This completes the proof, because δ and δ_1 do not depend on the sequence $\{z, \ldots, T^m z\}$ and $v \in C_z(z)$. □

Proposition 7.27. *The non-contraction property is satisfied.*

Proof. Let $\varphi = \{z, \ldots, T^m z\}$ with $z \in E$ and $T^m z \in -E$. Since $M^- \subset E$ and $-M^- = M^-$, it is enough to prove the lemma for the case $T^m z \in -E^+$. If $T^m z \in -E^+ \cap E^+$, then there is nothing to prove. So, suppose that $T^m z \notin E^+$. There are two cases: $\varphi \subset M_i \subset M^+$ for some i, and $\varphi \not\subset M_i$ for every i such that $M_i \subset M^+$. In the first case, we have $z \in E_i$ and $m = n(z)$. By Proposition 7.14, it follows that (9) is satisfied with $\beta' = \gamma_1$. In the second case, we can write $\varphi = \varphi_1 \cup \varphi_2$, where $\varphi_1 = \{z, \ldots, T^k z\}$ with $z, T^k z \in E$, and $\varphi_2 = \{T^k z, \ldots, T^m z\}$ with $T^k z \in E^+$ and $T^m z \in -E^+$. Since φ_2 is an orbit of the type considered in the first case, we have $\|D_{T^k z} T^{m-k} v\| \geq \gamma_1 \|v\|$ for $v \in C_{T^k z}(T^k z)$. Note that φ_1 is a piece of orbit as in (9) so that by Corollary 7.25, there exists $\beta'_5 > 0$ independent of φ_1 such that $\|D_z T^k v\| \geq \beta'_5 \|v\|$ for $v \in C_z(z)$. Hence, we see again that (9) is satisfied with constant $\beta'_5\gamma_1$. This together with the fact that β'_5 and γ_1 do not depend on φ completes the proof. □

7.4. **Conclusion of the proof of Condition L4.** Since the billiard map T is time-reversible, it is easy to check that the stable part of L4 for x is equivalent to the unstable part of L4 but with O replaced by $-O$. Also, note that by Proposition 5.3, the invariant set $\Lambda \subset M$ where T admits a local stable and an unstable manifold has full measure. For the definition of these manifolds, see Proposition A.3. The local stable manifold (resp. local unstable manifold) of $y \in \Lambda$ is denoted by $V^s(y)$ (resp. $v^u(y)$).

In view of the time-reversibility of T, we have $-V^s(y) = V^u(-y)$ and $-V^u(y) = V^s(-y)$ for every $y \in \Lambda$. Hence $\Lambda = -\Lambda$. From these considerations, it follows that the unstable part of L4 with O replaced by $E \cup -E$, and Λ_x replaced by Λ implies the full L4 (stable and unstable parts).

We will also need the following technical lemma. Its proof is exactly as the one of [14, Lemma 3.6]. The sets $W^s(y)$ and $W^u(y)$ are defined in Definition A.5.

Lemma 7.28. *The set Λ can be chosen so that it satisfies the following property: if $y \in \Lambda \cap E$ and $z \in W^u(y)$ (resp. $z \in W^s(y)$), then*

$T_zW^u(y) \subset C_y(y)$ (resp. $T_zW^u(y) \subset C'_y(y)$), where $\{(U_y, C_y)\}_{y\in E}$ is the family of cone fields defined in Subsection 4.3.

Proposition 7.29. *Every point $x \in H$ satisfies Condition L4.*

Proof. Suppose that $x \in H$, and let (l, N, O, K) be the quadruple of x as specified at the beginning of Subsection 6.2. Also, let ϵ_0 as in Lemma 7.6. We will prove the unstable part of L4 with O replaced by $E \cup -E$, Λ_x replaced by Λ and $\epsilon = \epsilon_0$. As explained at the beginning of this subsection, this implies L4.

Suppose that $y \in \Lambda \cap (E \cup -E)$, $z \in (E \cup -E) \cap W^u(y) \cap T^k S_1^-(\epsilon_0)$ for some $k > 0$. We study separately the cases $T^{-k}z \in E$ and $T^{-k}z \notin E$. For simplifying the notation, we will write $D_{i,j}$ for the restriction of $D_{T^iz}T^j$ to the tangent subspace of $W^u(T^iy)$ at T^iz with $i, j \in \mathbb{Z}$.

Before continuing with the proof, we need to make a remark. First, that the cone field (O, K) is equal to (U_y, C_y) for a proper $y \in E$. Then, note that by Lemma 7.28 and the invariance of the cone fields $\{(U_y, C_y)\}_{y\in E}$, the tangent space of any unstable manifold considered in this proof at a point $z \in E$ is always contained in the cone $C_z(z)$. This property is essential for applying Propositions 7.26 and 7.27, and will be used implicitly in this proof every time that one of the two propositions is applied.

We can now resume the proof. First, suppose that $T^{-k}z \in E$. Using Proposition 7.27, we immediately obtain that $\|D_0^{-k}\| \leq 1/\beta'$, where β' is the constant appearing in (9).

Now, suppose that $T^{-k}z \notin E$. Then, there are 2 possibilities, either $T^{-k}z \in M^0 \cap S_1^-(\epsilon_0)$ or $T^{-k}z \in M^+ \setminus E$. We analyze each possibility individually. First, define $m_\mp = \inf\{i > 0 \colon T^{-k\mp i}z \in E\}$.

Assume that $T^{-k}z \in M^0 \cap S_1^-(\epsilon_0)$. Applying Proposition 7.26 to $T^{-k-m_-}z$ with $m = m_- + m_+$ and $j = m_-$, we obtain $\|D_{-k+m_+,-m_+}\| \leq 1/\gamma_2$. Also, note that $\|D_{0,-k-m_+}\|$ is trivially equal to 1 if $m_+ = k$, and is not larger than $1/\beta'$ if $m_+ < k$ by Proposition 7.27. Combining the previous observations, we conclude that $\|D_{0,-k}\| \leq 1/(\gamma_2 \min\{1, \beta'\})$.

Finally, assume that $T^{-k}z \in M^+ \setminus E$. Let $n = n(T^{-k}z)$. By breaking the sequence of consecutive numbers $\{-k, \ldots, 0\}$ into the sequences of consecutive numbers $\{-k, \ldots, -k+n\}$, $\{-k+n, \ldots, -k+m_+\}$ and $\{-k+m_+, \ldots, 0\}$, we see that $D_{0,-k} = D_{-k+n,-n} \circ D_{-k+m_+,-m_++n} \circ D_{0,-k+m_+}$. From Proposition 7.27, Lemma 7.17 and Proposition 7.14, we obtain $\|D_0^{-k+m_+}\| \leq 1/\beta'$, $\|D_{-k+m_+,-m_++n}\| \leq 1/\delta_1$ and $\|D_{-k+n,-n}\| \leq 1/\gamma_1$, respectively. Combining these inequalities, we conclude that $\|D_{0,-k}\| \leq 1/(\beta'\gamma_1\delta_1)$.

To complete the proof, we observe that the upper bounds of $\|D_{0,-k}\|$ for the different cases studied above do not depend on the data y, z, k of Condition L4. □

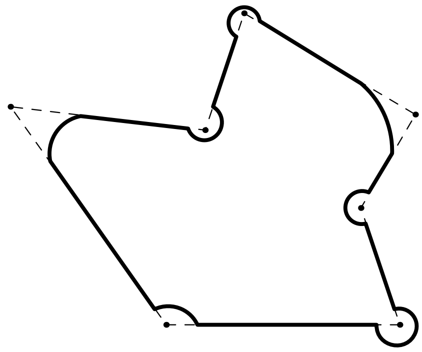

FIGURE 2. Polygon with pockets and bumps (solid curve) and original polygon (dashed curve).

8. DONNAY BILLIARDS AND THEIR GENERALIZATIONS

We now apply Theorem 5.6 to a large class of hyperbolic billiards. The purpose of this section is to provide concrete examples of new billiards with the Bernoulli property. For the definition of Bernoulli component, see Appendix A.

Definition 8.1. *A polygon with pockets and bumps is a planar domain Ω obtained from a polygon P by replacing a sufficiently small neighborhood of each vertex with internal angle less than π with a focusing arc (pocket) or a dispersing arc (bump), and of each vertex with internal angle greater than π by a dispersing arc so that the vertex lies outside the closure of Ω (see Fig. 2).*

Stadium-like domains with either parallel or non-parallel segments can be considered degenerate polygons with pockets. These domains correspond to quadrilaterals with a pocket or a bump replacing two vertexes instead of one. The results proved in this subsection apply also to billiards in stadium-like domains.

Note that polygons with pockets and bumps satisfy B1 by definition. In the rest if this section, we will assume that billiards in polygons with pockets and bumps satisfies Condition B2. This can achieved, for example, by using sufficiently short focusing arcs as pockets.

Proposition 8.2. *If a billiard in a polygon with pockets and bumps satisfies Condition B2, then it satisfies Conditions B3 and B4 as well. In particular, T has non-zero Lyapunov exponents a.e. on M.*

Proof. A general result for polygonal billiards states that every semi-orbit of a polygon billiard is either periodic or accumulates at least at one vertex of the polygon [18]. From this result and the assumption on the geometry of Ω, it follows that if $x \in N^- \cup N^+$, then x has to be periodic. Hence, $N^- = N^+ = N$ consists of periodic orbits. In a polygonal billiard, every periodic orbit is contained in a family of parallel orbits having the same period. This family consists of finitely many segments contained in horizontal segments ($\theta = const$) with endpoint belonging to R. Since the number of distinct strips in a polygonal billiard is countable, we immediately obtain $\mu(N) = 0$. Hence, B3 is satisfied. The fact that N consists of periodic orbits implies also that $N' = \emptyset$. Thus, $S_1^- \cap N^+ \subset N' = \emptyset$, and so Condition B4 is satisfied as well.

The second part of the proposition follows from Proposition 5.3. □

Chernov and Troubetzkoy proved that a billiard in convex polygons with pockets Ω is ergodic if the pockets are arcs of circles with the full circles contained in Ω [11]. Here, we consider the more general situation of polygons with pockets and bumps. The condition in Definition 8.1 that each vertex with internal angle greater than π is replaced by a dispersing arc so that the vertex lies outside the closure of the polygon plays a crucial role in the proof of the next theorem.

Theorem 8.3. *The map T of a billiard in a polygon with pockets and bumps that satisfies Condition B2 has the Bernoulli property.*

Proof. Conditions B3 and B4 are satisfied by Proposition 8.2. By Theorem 5.6, every point of H has neighborhood contained (mod 0) in one Bernoulli component. Now, suppose that $M_i \subset M^- \cup M^+$. Since R is countable (see [15, Propositions 6.17-6.19]), $N' = \emptyset$ and $N \cap M_i = \emptyset$, the set $H \cap M_i$ is connected. Hence, every set $M_i \subset M^- \cup M^+$ is contained (mod 0) in one Bernoulli component. We cannot claim the same for sets $M_i \subset M^0$, because in this case[2], N may disconnect $H \cap M_i$. Thus, to prove that T is Bernoulli, we cannot use Corollary 5.8.

The alternative approach that we take is quite simple: we prove that the entire set $M^- \cup M^+$ is contained (mod 0) in one Bernoulli component. This together with $\mu(N) = 0$ implies that T is Bernoulli. Indeed, by Theorem A.6, the number of Bernoulli components of T is equal to the minimum integer $n > 0$ such that $T^n B = B$ for some Bernoulli component B. By the geometry of Ω, it is always possible to find two distinct curved components Γ_i and Γ_j such that $\mu(M_j \cap T M_i) > 0$. Thus, if B is the Bernoulli component containing $M^- \cup M^+$, then $TB = B$, and so $n = 1$.

To prove that $M^- \cup M^+$ is contained (mod 0) in a single Bernoulli component, we start by observing that the polygon P decomposes into finitely many triangles with pairwise disjoint interiors. The fact that this decomposition is not unique is irrelevant for our purposes. Let Δ be one of triangles of the decomposition. By construction of Ω, every vertex with internal angle greater than π lies outside Ω. Then, we claim that the union of the sets M_i corresponding to the pockets and bumps attached to the vertexes of Δ are contained (mod 0) in the

[2]This is indeed the case whenever the polygon P has two parallel sides facing each other.

same Bernoulli component. In fact, let Γ_i, Γ_j, Γ_k be the pockets or bumps attached at the vertexes of Δ. If B is the Bernoulli component containing M_i, then by Theorem A.6 states that there exists a Bernoulli component B' such that $TB = B'$. It is easy to see that $\mu(M_j \cap TM_i) > 0$ and $\mu(M_k \cap TM_i) > 0$. Thus, M_j and M_k must belong to B'. By symmetry of the configuration, also M_i and M_j belong the same Bernoulli component. We can then conclude that M_i, M_j, M_k belong to the same Bernoulli component. In a stadium-like domain, the previous argument does not work, because there are only 2 curved components Γ_i and Γ_j attached to the 3 vertexes of Δ. However, in this case, we have i) $\mu(M_j \cap TM_i) > 0$ and $\mu(M_j \cap TM_i) > 0$, and ii) $\mu(M_j \cap T^2 M_i) > 0$. Claim i) is obvious. Claim ii) follows from the fact that there is a positive measure set of orbits starting at M_i and reaching M_j after one collision with flat components of $\partial\Omega$. Since each M_i and M_j is contained in a Bernoulli component, and all the Bernoulli components are cyclically permuted with the same period $n > 0$ by Theorem A.6, claim i) implies that $n \leq 2$, whereas claim ii) implies that n is either 3 or one of its divisors. Hence $n = 1$. The same conclusion is clearly true for all the sets M_i corresponding to pockets and bumps attached to vertexes of two adjacent triangles Δ_1 and Δ_2. Hence, the entire set $M^- \cup M^+$ belongs (mod 0) to a Bernoulli component. □

Donnay billiards (see Theorem 3.8) are billiard in convex polygons with pockets. The next result then follows directly from Theorem 8.3.

Corollary 8.4. *Donnay billiards have the Bernoulli property.*

The conclusion of Theorem 8.3 is robust for sufficiently small perturbations of the pockets and bumps.

Definition 8.5. *Two polygons with pockets and bumps Ω_1 and Ω_2 are ϵ-close if Ω_2 is obtained from Ω_1 by replacing each pocket (resp. bump) with another pocket (resp. bump) of the same length, ϵ-close in the C^6-topology (resp. C^3-topology) and ϵ-close in the Hausdorff distance.*

Proposition 8.6. *Suppose that the billiard in a polygon with pockets and bumps Ω_1 satisfies B2. Then there exists $\epsilon > 0$ such that if Ω_2 is ϵ-close to Ω_1, then the billiard in Ω_2 is Bernoulli.*

Proof. Condition B2 is an open condition. Thus, if ϵ is sufficiently small, then the billiard in Ω_2 satisfies B2, and so it is Bernoulli by Theorem 8.3. □

We conclude this section with a result concerning hyperbolic billiards with domains for which $\Gamma^0 = \emptyset$, i.e., without straight boundary components. For these billiards, the Bernoulli property follows quite directly from Corollary 5.8. We observe that Sinai billiards [28] – those with domains bounded only by strictly convex outwards arcs – belong to this class of billiards. For them, the Bernoulli property was first proved in [17].

Theorem 8.7. *Let Ω be a billiard domain without straight boundary components, and suppose that its map T satisfies Conditions B1-B4. Then T is Bernoulli.*

Proof. For billiards satisfying the hypotheses of the theorem, the set R is countable (see [15, Propositions 6.17-6.19]), and we trivially have $N = NR = \emptyset$. Thus, each set $H \cap M_i$ is connected, and B5 is satisfied. The wanted conclusion now follows from Corollary 5.8. □

Appendix A. Local Ergodic Theorem

We state the Local Ergodic Theorem proved in [14], and recall the relevant definitions. The formulation of this theorem in its general form requires a series of technical definitions, which are not needed for 2-dimensional billiards. Thus, to avoid unnecessary technicalities, we specialize the presentation of the Local ergodic Theorem to 2-dimensional billiards. Accordingly, T will denote the billiard map for some planar domain Ω throughout this appendix.

The definition of a cone field and related notions are given in Section 4.

Definition A.1. *A point $x \in M \setminus \partial M$ is called* sufficient *if there exist*

(i) an invariant continuous cone field K on $O \cup T^{-N}O$ such that $\sigma_K(D_yT^N) > 3$ for every $y \in T^{-N}O$.
(ii) a neighborhood O of $T^l x$ and an integer $N > 0$ such that O and R_N^- are disjoint,
(iii) an invariant continuous cone field K on $O \cup T^{-N}O$ such that $\sigma_K(D_yT^N) > 3$ for every $y \in T^{-N}O$.

Every time we need to emphasize the role of the data l, N, O, K in this definition, we will write that x is a sufficient point with quadruple (l, N, O, K).

Definition A.2. *A point $x \in M \setminus \partial M$ is called u-*essential *if for every $\alpha > 0$, there exist $n_{x,\alpha} \in \mathbb{N}$, a neighborhood $O_{x,\alpha}$ of x with $O_{x,\alpha} \cap R^+_{n_{x,\alpha}} = \emptyset$ and a continuous invariant cone field $(O_{x,\alpha} \cup T^{n_{x,\alpha}} O_{x,\alpha}, K_{x,\alpha})$ such that $\sigma^*_{K_{x,\alpha}}(D_y T^{n_{x,\alpha}}) > \alpha$ for every $y \in O_{x,\alpha}$. Analogously, a point $x \in M \setminus \partial M$ is called s-*essential *if, in the previous definition, T and $R^+_{n_{x,\alpha}}$ are replaced by T^{-1} and $R^-_{n_{x,\alpha}}$, respectively.*

The cone field (O, K) in Definition A.1 is eventually strictly invariant. By a well-known result [23, 31, 32], it follows that all the Lyapunov exponents of T are non-zero a.e. on the set $\bigcup_{k\in\mathbb{Z}} T^k O$. This fact combined with the Katok-Strelcyn theory [20] gives Proposition A.3 below (Part (3) is proved in [14, Proposition 5.3]). For the definition of absolute continuity of a foliation, we refer the reader to [10, 20].

Proposition A.3. *Let $x \in M \setminus \partial M$ be a sufficient point with quadruple (l, N, K, O). Then there exist an invariant set $\Lambda_x \subset \bigcup_{k\in\mathbb{Z}} T^k O$ with $\mu(\Lambda_x) = \mu(\bigcup_{k\in\mathbb{Z}} T^k O) > 0$ and two families $V^s = \{V^s(y)\}_{y\in\Lambda_x}$ and $V^u = \{V^u(y)\}_{y\in\Lambda_x}$ consisting of C^2 submanifolds such that for every $y \in \Lambda_x$, we have*

(1) $V^s(y) \cap V^u(y) = \{y\}$,
(2) $V^s(y)$ and $V^u(y)$ are embedded open intervals,
(3) $T_y V^s(y) \subset K'(y)$ and $T_y V^u(y) \subset K(y)$ provided that $y \in O \cup T^{-N}O$,
(4) $TV^s(y) \subset V^s(Ty)$ and $T^{-1}V^u(y) \subset V^u(T^{-1}y)$,
(5) $d(T^n y, T^n z) \to 0$ exponentially as $n \to +\infty$ for every $z \in V^s(y)$, and the same is true as $n \to -\infty$ for every $z \in V^u(y)$,
(6) $V^s(y)$ and $V^u(y)$ vary measurably with $y \in \Lambda_x$,
(7) the families V^s and V^u have the absolute continuity property.

Definition A.4. *The submanifolds forming the families V^s and V^u are called* local stable manifolds *and* local unstable manifolds*, respectively.*

Definition A.5. *Let x be a sufficient point of $M \setminus \partial M$, and let Λ_x be the set in Proposition A.3. For every $y \in \Lambda_x$, we denote by $W^u(y)$ the connected component of*

$$\bigcup_{k\geq 0} T^k V^u(T^{-k}y)$$

containing y. Analogously, denote by $W^s(y)$ the set obtained by replacing T with T^{-1} and V^u with V^s in the definition of $W^u(y)$.

We now recall the Spectral Decomposition Theorem. It applies to a larger class of hyperbolic system with singularities than billiards, but here we formulate it only for billiards. For its proof, one has to combine two results: [20, Theorem 13.1, Part II], which extends Pesin'result for smooth systems [27] to systems with singularities, and [9, Theorem 3.1]. See also [26], for results similar to those of [9].

Theorem A.6. *Suppose that the billiard map T has non-vanishing Lyapunov exponents a.e. on M. Then there exist countably many pairwise disjoint measurable subsets $E_0, E_1, \dots$ of M such that*

(1) $M = \bigcup_{i=0}^{\infty} E_i$,
(2) $\mu(E_0) = 0$, and $\mu_i(E_i) > 0$ for $i \in \mathbb{N}$,
(3) $TE_i = E_i$, and $(T|_{E_i}, \mu|_{E_i})$ is ergodic for $i \in \mathbb{N}$,
(4) for each $i \in \mathbb{N}$, there exist $m_i \in \mathbb{N}$ pairwise disjoint measurable subsets $B_{i,1}, \dots, B_{i,m_i}, B_{i,m_i+1} = B_{i,1}$ of M such that $E_i = \bigcup_{j=1}^{m_i} B_{i,j}$, $TB_{i,j} = B_{i,j+1}$ and $(T^{m_i}|_{B_{i,j}}, \mu|_{B_{i,j}})$ is Bernoulli for every $j = 1, \dots, m_i$.

The sets E_i and $B_{i,j}$ are called an *ergodic component* of T and a *Bernoulli component* of T, respectively. These sets are uniquely defined up to a set of zero measure.

We need one last definition before formulating the Local Ergodic Theorem.

Definition A.7. *Let (O_1, C_1) and (O_2, C_2) be two cone fields. We say that (O_1, C_1) and (O_2, C_2) are* jointly invariant *if $D_x T^k C_1(x) \subset C_2(T^k x)$ for every $x \in O_1$ and $k > 0$ such that $T^k x \in O_2$, and $D_x T^k C_2(x) \subset C_1(T^k x)$ for every $x \in O_2$ and $k > 0$ such that $T^k x \in O_1$.*

Note that in the previous definition, we neither require that the sets O_1 and O_2 are disjoint nor that the cone fields C_1 and C_2 are invariant. However, it is easy to see that C_1 and C_2 are invariant in the following sense: if $x \in O_1$ and $k_2 > k_1 > 0$ such that $T^{k_1} x \in O_2$ and $T^{k_2} x \in O_1$, then $D_x T^{k_2} C_1(x) \subset C_1(T^{k_2} x)$. The same is true for C_2, once O_1 has been replaced by O_2.

Definition A.8. *A subset $\Sigma \subset M$ is called* regular *if it is a union of finitely many arcs $\Sigma_1, \ldots, \Sigma_k$ of class C^2 that can only intersect at their boundaries. The arcs $\Sigma_1, \ldots, \Sigma_k$ are called the* components *of Σ.*

Definition A.9 (Regularity)**.** *We say that T satisfies Condition L1 if the singular sets R_k^+ and R_k^- are regular for every $k > 0$.*

In the rest of this subsection, we assume that $x \in M \setminus \partial M$ is a sufficient point with quadruple (l, N, O, K). Let Λ_x be the subset associated to x as in Proposition A.3.

Definition A.10 (Alignment)**.** *We say that x satisfies Condition L2 if the sets $O \cap R_k^+$ and $O \cap R_k^-$ are regular for every $k > 0$, and the tangent subspace*[3] *$T_y \Sigma$ is contained in $K(y)$ (resp. $K'(y)$) for every $k > 0$, every component Σ of $O \cap R_k^-$ (resp. $O \cap R_k^+$) and every $y \in \Sigma \cap T^{-N} O$ (resp. $\Sigma \cap O$).*

Definition A.11 (Ansatz)**.** *We say that x satisfies Condition L3 if the set of u-essential points of S_1^- (resp. s-essential points of S_1^+) has full m_--measure (resp. m_+-measure), and if y is any of such points, then the cone fields (O, K) and $(O_{y,\alpha}, K_{y,\alpha})$ are jointly invariant for every $\alpha > 0$.*

Given $A \subset M$ and $\epsilon > 0$, we call the set $A(\epsilon) = \{x \in M : d(x, A) < \epsilon\}$ the *ϵ-neighborhood* of A.

Definition A.12 (Contraction)**.** *We say that x satisfies Condition L4 if there exist $\beta > 0$ and $\epsilon > 0$ such that*

$$\left\| D_z T^{-k}|_{T_z W^u(y)} \right\| \leq \beta \quad \left(\textit{resp. } \left\| D_z T^{k}|_{T_z W^s(y)} \right\| \leq \beta \right)$$

for every $y \in O \cap \Lambda_x$ and every $z \in O \cap W^u(y) \cap T^k S_1^-(\epsilon)$ (resp. $O \cap W^s(y) \cap T^{-k} S_1^+(\epsilon)$) with $k > 0$.

[3]In the general version of the theorem, the tangent space $T_y \Sigma$ has to be replaced by its Lagrangian skew-orthogonal complement with respect to the symplectic form ω.

Theorem A.13 (Local Ergodic Theorem)**.** *Suppose that L1 is satisfied, and that $x \in M \setminus \partial M$ is a sufficient point satisfying L2-L4. Then there exists a neighborhood of x contained up to a set of zero μ-measure in a Bernoulli component of T.*

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