Anomalous partially hyperbolic diffeomorphisms I: dynamically coherent examples

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Abstract

We build an example of a non-transitive, dynamically coherent partially hyperbolic diffeomorphism f on a closed 3-manifold with exponential growth in its fundamental group such that f^n is not isotopic to the identity for all $n \neq 0$. This example contradicts a conjecture in [HHU$_1$]. The main idea is to consider a well-understood time-t map of a non-transitive Anosov flow and then carefully compose with a Dehn twist.

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1   Introduction

In recent years, partially hyperbolic diffeomorphisms have been the focus of considerable study. Informally, partially hyperbolic diffeomorphisms are generalization of hyperbolic maps. The simplest partially hyperbolic diffeomorphisms admit an invariant splitting into three bundles: one of which is uniformly contracted by the derivative, another which is uniformly expanded, and a center direction whose behavior is intermediate. A more formal definition will soon follow. In this paper, we shall restrict to the case of these diffeomorphisms on closed 3-dimensional manifolds.

The study of partially hyperbolic diffeomorphisms has followed two main directions. One consist of studying conditions under which a volume preserving partially hyperbolic diffeomorphism is stably ergodic. This is not the focus of this article. See [HHU$_2$, W] for recent surveys on this subject.

The second direction, initiated in [BW, BBI], has as a long term goal of classifying these partially hyperbolic systems, at least topologically. Even in dimension 3, this goal seems quite ambitious but some partial progress has been made, which we briefly review below. This paper is intended to further this classification effort by providing new examples of partially hyperbolic diffeomorphisms. In a forthcoming paper ([BP]) we shall provide new transitive examples (and even stably ergodic); their construction uses some of the ideas of this paper as well as some new ones. Another viewpoint might be that these new examples throw a monkey wrench into the classification program. In light of these new partially hyperbolic diffeomorphisms, is there any hope to achieve any reasonable sense of a classification?

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1.1 Preliminaries

Before diving into a detailed exposition of these examples, we provide the necessary definitions and background. Let M be a closed 3-manifold, we say that a diffeomorphism $f : M \to M$ is *partially hyperbolic* if the tangent bundle splits into three one-dimensional[1] Df-invariant continuous bundles $TM = E^{ss} \oplus E^c \oplus E^{uu}$ such that there exists $\ell > 0$ such that for every $x \in M$:

$$\|Df^\ell|_{E^{ss}(x)}\| < \min\{1, \|Df^\ell|_{E^c(x)}\|\} \leq \max\{1, \|Df^\ell|_{E^c(x)}\} < \|Df^\ell|_{E^{uu}(x)}\|$$

Sometimes, the more restrictive notion of *absolute partial hyperbolicity* is used. This means that f is partially hyperbolic and there exists $\lambda < 1 < \mu$ such that:

$$\|Df^\ell|_{E^{ss}(x)}\| < \lambda < \|Df^\ell|_{E^c(x)}\| < \mu < \|Df^\ell|_{E^{uu}(x)}\|$$

For the classification of such systems, one of the main obstacles is understanding the existence of invariant foliations tangent to the center direction E^c. In general, the bundles appearing in the invariant splitting are not regular enough to guaranty unique integrability. In the case of the strong stable E^{ss} and strong unstable E^{uu} bundles, dynamical arguments insure the existence of unique foliations tangent to the strong stable and unstable bundle (see for example [HPS]). However, the other distributions need not be integrable.

The diffeomorphism f is *dynamically coherent* if there are 2-dimensional f-invariant foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent to the distributions $E^{ss} \oplus E^c$ and $E^c \oplus E^{uu}$, respectively. These foliations, when they exist, intersect along a 1-dimensional foliation $\mathcal{W}^c$ tangent to E^c. The diffeomorphism f is *robustly dynamically coherent* if there exists a C^1-neighborhood of f comprised only of dynamically coherent partially hyperbolic diffeomorphisms. There is an example of non dynamically coherent partially hyperbolic diffeomorphisms[2] on $\mathbb{T}^3$ (see [HHU$_3$]). This example is not transitive and it is not known whether every transitive partially hyperbolic diffeomorphisms on a compact 3 manifold is dynamically coherent. See [HP] and references therein for the known results on dynamical coherence of partially hyperbolic diffeomorphisms in dimension 3.

Two dynamically coherent partially hyperbolic diffeomorphisms $f\colon M \to M$ and $g\colon N \to N$ are *leaf conjugate* if there is a homeomorphism $h\colon M \to N$ so that h maps the center foliation of f on the center foliation of g and for any $x \in M$ the points $h(f(x))$ and $g(h(x))$ belong to the same center leaf of g.

Up to now the only known example of dynamically coherent partially hyperbolic diffeomorphisms were, up to finite lift and finite iterates, leaf conjugate to one of the following models:

1. linear Anosov automorphism of $\mathbb{T}^3$

2. skew products over a linear Anosov map of the torus $\mathbb{T}^2$

3. time one map of an Anosov flow.

It has been conjectured, first in the transitive case (informally by Pujals in a talk and then written in [BW]), and later in the dynamically coherent case (in many talks and minicourses [HHU$_1$]) that every partially hyperbolic diffeomorphism should be, up to finite cover and iterate, leaf conjugate to one of these three models. Positive results have been obtained in [BW, BBI, HP] and some families of 3-manifolds are now known only to admit partially hyperbolic diffeomorphisms that are leaf conjugate to the models.

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[1]One of the advantages of working with one-dimensional bundles is that the norm of Df along such bundles controls the contraction/expantion of every vector in the bundle. Compare with definitions of partial hyperbolicity when the bundles are not one-dimensional [W].

[2]We should remark that this example is not absolutely partially hyperbolic. Moreover, it is isotopic to one of the known models of partially hyperbolic diffeomorphisms.

1.2 Statements of results

The aim of this paper is to provide a counter example to the conjecture stated above. Our examples are not isotopic to any of the models.

In order to present the ideas in the simplest way, we have chosen to detail the construction of a specific example on a (possibly the simplest) manifold admitting a non-transitive Anosov flow transverse to a non-homologically trivial incompressible two-torus; the interested reader should consult [Br] for more on 3-manifolds admitting such non-transitive Anosov flows. Our arguments go through directly in some other manifolds, but for treating the general case of 3-manifolds admitting non-transitive Anosov flows further work must be done.

Theorem 1.1. *There is a closed orientable 3-manifold M endowed with a non-transitive Anosov flow X and a diffeomorphism $f\colon M \to M$ such that:*

- *f is absolutely partially hyperbolic,*

- *f is robustly dynamically coherent,*

- *the restriction of f to its chain recurrent set coincides with the time-one map of the Anosov flow X, and*

- *for any $n \neq 0$, f^n is not isotopic to the identity.*

The manifold M on which our example is constructed also admits a transitive Anosov flow (see [BBY]).

As a corollary of our main theorem, we show that f is a counter example to the conjecture stated above in the non-transitive case (see [HHU$_1$, HP]):

Corollary 1.2. *Let f be the diffeomorphism announced in Theorem 1.1. Then for all n the diffeomorphism f^n does not admit a finite lift that is leaf conjugate to any of the following:*

- *linear Anosov diffeomorphisms on $\mathbb{T}^3$*

- *partially hyperbolic skew product with circle fiber over an Anosov diffeomorphism on the torus $\mathbb{T}^2$*

- *the time-one map of an Anosov flow.*

1.3 Organization of the paper

The paper is organized in the following manner. In Section 3 we describe a modified *DA* diffeomorphism of $\mathbb{T}^2$. This particular *DA* diffeomorphism of the torus may not seem the simplest but it has the necessary properties that make our example easy to present using only elementary methods. In section 4, we detail the construction of a non-transitive Anosov flow, following the construction of Franks and Williams in [FW]. In Section 5 we establish coordinates in a model space in order to prepare for the appropriate perturbation diffeomorphism—a Dehn twist along a separating torus T_1. Then, in Section 6, we choose the length of the neighborhood of the separating torus T_1. We present the example in Section 8 after providing criteria for establishing partial hyperbolicity in Section 7. Then, in Sections 9 and 10 we show that the example is dynamically coherent and not leaf conjugate to previously known examples; it is also not isotopic to the identity. Next, in Section 2, we informally outline the construction of our specific example.

2 Informal presentation of the example

The example is constructed in the following manner. We begin with a DA map with two sources instead of one. This choice makes it possible to easily show that our partially hyperbolic diffeomorphism has no non-trivial iterate isotopic to the identity. Next, we build a non-transitive Anosov flow (après [FW]) transverse to a torus T_1 (this is always the case for non-transitive Anosov flows [Br]). Then our example is obtained by composing the time N-map of this Anosov flow with a Dehn twist along a neighborhood of the torus T_1 of the form

$$\bigcup_{t \in [0,N]} X_t(T_1)$$

which is diffeomorphic to $[0,1] \times \mathbb{T}^2$.

The neighborhood and the time N is chosen in order to preserve partial hyperbolicity. In a nutshell, the idea is that a small C^1-perturbation always preserves partial hyperbolicity, and so, if we make the perturbation in a long enough neighborhood of T_1, by insuring that the time interval $[0,N]$ is sufficiently large, the effect of the Dehn twist can be made to appear negligible at the level of the derivative, even though the C^0-distance cannot be made arbitrarily small[3]. More precisely, we obtain conditions under which transversality between certain bundles are preserved under this kind of composition which allows to show partial hyperbolicity.

Since the perturbation is made in the wandering region of the time-N of the Anosov flow, the properties of the chain-recurrent set are preserved and it is possible to study the integrability of the center bundle by simply defining it in the obvious way and showing that it plays well with attracting and repelling regions as it approaches them. The fact that center leaves cannot be fixed when they pass through the fundamental region where the perturbation is made becomes a matter of checking that the new intersections cannot be preserved by the altered dynamics.

There are some reasons for which we present a specific example:

1. Even though most of our arguments are quite general and our Dehn twist perturbation can be applied to infinite family of Franks-Williams type Anosov flows, it is easier to first see these ideas presented for a single example. The construction of the perturbation and the fact that it preserves partial hyperbolicity is much easier to check in a specific case and we believe that this narrative makes the global argument more transparent.

2. It is possible to apply these techniques to other types of examples at the expense of having to check a few minor details. However, to perform the examples in any manifold admitting a non-transitive Anosov flow, some more work is required to guarantee the transversality of the foliations after perturbation. We believe this is beyond the scope of this paper and relegated to a more detailed study in a forthcoming article.

3. Mainly, for the specific example it is quite easy to give a direct and intuitive argument to show that the resulting dynamics has no iterate isotopic to the identity. This is carried out in Section 10 where we use the fact that the torus T_1 is homologically non-trivial and just utilize elementary algebraic topology to show that the action on homology is non-trivial. For the general case, showing that the perturbation is not isotopic to the identity requires more involved arguments. We remark that the paper [McC] solves this problem in many situations, but in a less elementary manner. Again, these details are best left to be expounded in another article.

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[3]Note that these statements are not precise and the remarks in this section are intended to impart our intuition to the reader.

3   Modified DA map on $\mathbb{T}^2$

Our construction begins with a diffeomorphism of the torus. We use a modified DA map, with two sources instead of one.

We simply state the properties of the required map below. The classical construction of the Derived from Anosov (DA) diffeomorphism is well known—see [Ro] for instance. Our modified DA map is obtained by lifting (some iterate of) the classical DA map to some 2-folded cover of $\mathbb{T}^2$. Alternatively, one may begin with a linear Anosov diffeomorphism with 2 fixed points and then create two sources by "blowing up" these fixed points and their unstable manifolds.

Proposition 3.1. *There exists a diffeomorphism $\varphi\colon \mathbb{T}^2 \to \mathbb{T}^2$ with the following properties:*

- *the non-wandering set of φ consists in one non-trivial hyperbolic attractor A and two fixed sources σ_1, σ_2*

- *the stable foliation of the hyperbolic attractor coincides with a linear (for the affine structure on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$) irrational foliation on $\mathbb{T}^2 \setminus \{\sigma_1, \sigma_2\}$*

4   Building non-transitive Anosov flows

In this section, we briskly run through the relevant details of the classical Franks-Williams construction in ([FW]) of a non-transitive Anosov flow. Note that the map ϕ in [FW] is the standard DA map with a single source, and so, the presentation below has been adapted to our context.

Let (M_0, Z) be the suspension of the DA-diffeomorphism φ given in Theorem 3.1. We denote by γ_i the periodic orbit of the flow of Z corresponding to the sources σ_i, and by A_Z the hyperbolic attractor of the flow of Z.

Lemma 4.1. *There is a convex map $\alpha\colon (0, \frac{1}{2}) \to \mathbb{R}$ tending to $+\infty$ at 0 and $\frac{1}{2}$, whose derivative vanishes exactly at $\frac{1}{4}$, and so that, for any $i \in 1, 2$ there is a tubular neighborhood Γ_i of γ_i whose boundary is an embedded torus $T_i \simeq \mathbb{T}^2$ so that*

- *T_i is transverse to Z, and therefore to the weak stable foliation W_Z^{cs} of the attractor A_Z; We denote by F_i^s the 1-dimensional foliation induced by W_Z^{cs} on T_i:*

- *there are coordinates $\theta_i\colon \mathbb{T}^2 \to T_i$ so that, in these coordinates:*

  - *F_i^s has exactly 2 compact leaves $\{0\} \times S^1$ and $\{\frac{1}{2}\} \times S^1$*
  - *given any leaf L of F_i^s in $(0, \frac{1}{2}) \times S^1$ there is $t \in \mathbb{R}$ so that L is the projection on $\mathbb{T}^2$ of the graph of $\alpha(x) + t$.*
  - *given any leaf L of F_i^s in $(\frac{1}{2}, 1) \times S^1$ there is $t \in \mathbb{R}$ so that L is the projection on T^2 of the graph of $\alpha(x - \frac{1}{2}) + t$.*

Proof. See page 165 of [FW]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Notice that the expression of F_1^s and F_2^s are the same in the chosen coordinates. We denote by F_i^u the image of F_i^s by the translation $(x, y) \mapsto (x + \frac{1}{4}, y)$.

Corollary 4.2. *F_i^s and F_j^u are transverse. Moreover F_i^s and F_i^u are invariant under any translation in the second coordinates (that is $(x, y) \mapsto (x, y + t)$, $t \in \mathbb{R}$).*

We are now ready to define our manifold M and the vector-field X on M.

Let M^+ be the manifold with boundary obtained by removing to M_0 the interior of $\Gamma_1 \cup \Gamma_2$, and we denote by Z^+ he restriction of Z to M^+. We denote by T_i^+ the
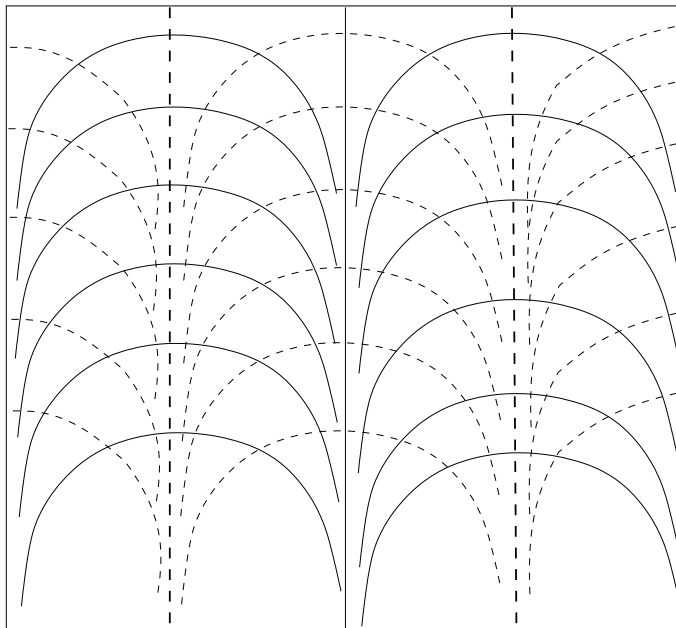
Figure 1: Gluing of the two foliations making them transverse. Notice that both foliations are invariant under vertical translations.

boundary component of M^+ corresponding to the boundary of Γ_i. Notice that T_i^+ are tori transverse to Z^+ and Z^+ points inwards to M^+.

Let M^- be another copy of M^+ and we denote by Z^- the restriction of $-Z$ to M^-. We denote by T_i^- the boundary component of M^-, there are transverse to Z^- and Z^- is points outwards of M^-.

We denote by $\psi \colon \partial M^+ \to \partial M^-$ the diffeomorphisms sending T_i^+ on T_i^-, $i = 1, 2$, and whose expression in the $\mathbb{T}^2$ coordinates are $(x, y) \mapsto (x + \frac{1}{4}, y)$.

We denote by M the manifold obtained by gluing M^+ with M^- along the diffeomorphism ψ. We can now restate Franks-Williams theorem in our setting:

Theorem 4.3. *There is a smooth structure on M, coinciding with the smooth structures on M^+ and on M^- and there is a smooth vector-field X on M, whose restrictions to M^+ and M^- are Z^+ and Z^- respectively.*

Furthermore X is an Anosov vector-field whose non-wandering set consists exactly in a non-trivial hyperbolic attractor A_X contained in M^+ and a non-trivial hyperbolic repeller R_X contained in M^-.

A non-singular vector field X on a 3-manifold is an *Anosov vector field* if the time-t map X_t is partially hyperbolic for some $t > 0$ (see [Br, FW] for the classical definition). It is easy to show that the time t-map of an Anosov flow will always be absolutely partially hyperbolic.

Remark 4.4. Since the bundle generated by X must be necessarily the center bundle of X_t it follows that the center direction is always integrable for Anosov flows. Indeed, it is a classical result (see e.g. [HPS]) that Anosov flows are dynamically coherent[4]. We call W_X^{cs} and W_X^{cu} the center stable and center unstable foliations (sometimes called weak stable and weak unstable foliations in the context of Anosov flows) and W_X^{ss} and W_X^{uu} the strong stable and unstable foliations respectively. We denote as E_X^{cs}, E_X^{cu}, E_X^{ss} and E_X^{uu} to the tangent bundles of these foliations.

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[4]This is not the way it is usually stated. Dynamical coherence is modern terminology.

The above construction immediately generalizes to examples arising from DA maps with n sources instead of 2, where $n \geq 1$. We call these flows *Franks-Williams type Anosov flows.* The arguments in the next sections are very general and apply to all Franks-Williams type Anosov flows.

5 A perturbation on a model space

In this section we shall perform a perturbation in a certain model space.

Consider the torus T_1 and $X_1(T_1)$ where X_1 is the time one map of the flow of X. Then T_1 and $X_1(T_1)$ bound a manifold diffeomorphic to $[0,1] \times \mathbb{T}^2$, which is a fundamental domain of X_1. We shall use the convention that sets of the form $[a,b] \times \{p\}$ are *horizontal* and those of the form $\{t\} \times \mathbb{T}^2$ are *vertical.*

The projection of the vector-field X on this coordinates is $\frac{\partial}{\partial t}$.

We denote by $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$ the 1-dimensional foliations $(\{t\} \times F_1^s)_{t \in [0,1]}$ and $(\{t\} \times F_1^u)_{t \in [0,1]}$ and by $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ the 2-dimensional foliations $[0,1] \times F_1^s$ and $[0,1] \times F_1^u$.

Lemma 5.1. *Let $\mathcal{G} \colon [0,1] \times \mathbb{T}^2 \to [0,1] \times \mathbb{T}^2$ be the diffeomorphism defined as $(t,x,y) \mapsto (t, x, y + \rho(t))$. Where $\rho : [0,1] \to [0,1]$ is a monotone smooth function such that it is identically zero in a neighborhood of 0 and identically 1 in a neighborhood of 1.*

Then $\mathcal{G}(\mathcal{F}^{cu})$ is transverse to $\mathcal{F}^{ss}$ and $\mathcal{G}(\mathcal{F}^{uu})$ is transverse to $\mathcal{F}^{cs}$.

Proof. This is a direct consequence of Lemma 4.1, Corollary 6.3 and the fact that $\mathcal{G}$ makes translations only in the y-direction of the coordinates given by that lemma. $\square$

6 The strong stable and unstable foliations on a fundamental domain of X_N for large N

For any $N > 0$ we consider the fundamental domain $\mathcal{U}_N$ of the diffeomorphism X_N (time N of the flow of X) restricted to $M \setminus (A_X \cup R_X)$ bounded by T_1 and $X_N(T_1)$.

This fundamental domain $\mathcal{U}_N$ is canonically identified with $[0,N] \times T_1$: the projection of T_1 is the identity map on $\{0\} \times T_1$ and the projection of X is $\frac{\partial}{\partial t}$.

The intersection of the weak stable and weak unstable 2-foliations W_X^{cs}, W_X^{cu} as well as the (1-dimensional) strong stable and strong unstable foliations W_X^{ss} and W_X^{uu} with $\mathcal{U}_N$; will be denoted by W_N^{cs}, W_N^{cu} W_N^{ss}, W_N^{uu}.

Lemma 6.1. *The expression of the tangent space of W_N^{cs}, W_N^{cu} W_N^{ss}, W_N^{uu} at a point $(t,x,y) \in [0,N] \times T_1$ only depends on $(x,y) \in T_1$: it depends neither of $t \in [0,N]$ nor on N.*

Proof. This follows from the fact that the bundles are invariant under the flow and the vector-field X is $\frac{\partial}{\partial t}$ in this coordinates. $\square$

We denote by $H_N \colon \mathcal{U}_N \to [0,1] \times \mathbb{T}^2$ the diffeomorphisms defined by $H_N(t,p) = (\frac{t}{N}, \theta_1^{-1}(p))$.

We denote $\mathcal{F}_N^{cs} = H_N(W_N^{cs})$, $\mathcal{F}_N^{cu} = H_N(W_N^{cu})$, $\mathcal{F}_N^{ss} = H_N(W_N^{ss})$, $\mathcal{F}_N^{uu} = H_N(W_N^{uu})$.

Lemma 6.2. *For every $N > 0$, $\mathcal{F}_N^{cs} = \mathcal{F}^{cs}$ and $\mathcal{F}_N^{cu} = \mathcal{F}^{cu}$, where $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ are the 2-dimensional foliation on $[0,1] \times \mathbb{T}^2$ defined in Section 5.*

The tangent bundle to $\mathcal{F}_N^{ss}$ and to $\mathcal{F}_N^{uu}$ converges in the C^0 topology to the tangent bundle to the foliations $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$, defined in Section 5, as N goes to $+\infty$.

Proof. The first assertion is a direct consequence of the fact that $\mathcal{F}_N^{cs}$ and $\mathcal{F}_N^{cu}$ are saturated by the orbits of the flow and Lemma 6.1.

The second assertion follows from the fact that the diffeomorphism is independent of N in the first coordinate and compresses the t coordinate by $\frac{1}{N}$. $\square$

As a direct consequence of Lemmas 5.1 and 6.2 we obtain

Corollary 6.3. *Let $\mathcal{G}$ be the diffeomorphism defined in Lemma 5.1. Then , there is $N_0 > 0$ so that for any $N \geq N_0$ one has:*
$\mathcal{G}(\mathcal{F}_N^{cu})$ *is transverse to* $\mathcal{F}_N^{ss}$ *and* $\mathcal{G}(\mathcal{F}_N^{uu})$ *is transverse to* $\mathcal{F}_N^{cs}$.

7 Establishing partial hyperbolicity

We just recall a classical criterion for partial hyperbolicity. As before, we remain in dimension 3 for simplicity.

Let g be a diffeomorphism on a compact 3-dimensional manifold M. Assume that there is a codimension 1 submanifold cutting M in two compact manifolds with boundary M^+ and M^- so that M^+ is an *attracting region* for f (i.e. $f(M^+) \subset \mathrm{Int}(M^+)$) and M^- is a *repelling region* (i.e. attracting region for g^{-1}).

Assume that the maximal invariant set A in M^+ and the maximal invariant set R in M^- admit a (absolute) partially hyperbolic splitting $E^{ss} \oplus E^c \oplus E^{uu}$.

Then E^{ss} and $E^{ss} \oplus E^c$ admit a unique invariant extension E^{ss} and E^{cs} on $M \setminus R$ and symmetrically E^{uu} and $E^c \oplus E^{uu}$ admit a unique invariant extension E^{uu} and E^{cu} on $M \setminus A$.

Remark 7.1. Assume that g coincides with an Anosov flow Z in a neighborhood U of A. Then, the bundles E^{cs} and E^{ss} in $M \setminus R$ coincide exactly with the tangent spaces of the foliation $f^{-n}(W_Z^{cs} \cap U)$ and $f^{-n}(W_Z^{ss} \cap U)$. A symmetric property holds for E^{cu} and E^{uu}. Notice that the center bundle cannot be a priori extended to these sets.

A compact set $\mathcal{U}$ will be called a *fundamental domain* of $M \setminus (A \cup R)$ if every orbit of g restricted to $M \setminus (A \cup R)$ intersects $\mathcal{U}$ in at least one point.

Theorem 7.2. *Let $g : M \to M$ be a C^1-diffeomorphism. Assume that:*

- *there exists a codimension one submanifold cutting M into an attracting and a repelling regions with maximal invariant sets A and R which are (absolutely) partially hyperbolic*

- *there exists a compact fundamental domain $\mathcal{U}$ of $M \setminus (A \cup R)$ such that if E_A^{ss}, E_A^{cs} denote the extensions of the bundles E^{ss} and $E^{ss} \oplus E^c$ of A to $M \setminus R$ and E_R^{uu}, E_R^{cu} denote the extensions of $E^{uu}, E^c \oplus E^{uu}$ of R to $M \setminus A$ then E_A^{cs} is transverse to E_R^{uu} and E_R^{cu} is transverse to E_A^{ss} at each point of $\mathcal{U}$.*

Then g is (absolutely) partially hyperbolic on M.

Proof. We have a well defined splitting $E^{ss} \oplus E^c \oplus E^{uu}$ above A and R and we can define the bundles E^{ss} and E^{uu} everywhere as $E^{ss} := E_A^{ss}$ and $E^{uu} := E_R^{uu}$ in $M \setminus (A \cup R)$.

The transversality conditions we have assumed on $\mathcal{U}$ allows us to define the E^c bundle in $M \setminus (A \cup R)$ as the intersection between E_A^{cs} and E_R^{cu}. These intersect in a one-dimensional subbundle thanks to our transversality assumptions.

Now, let us show that the splitting we have defined is (absolutely) partially hyperbolic. To see this, it is enough to show that the decomposition is continuous. Indeed, if this is the case, one can use the fact that given a neighborhood U of $A \cup R$ there exists $N > 0$ such that every point outside U verifies that every iterate larger than N belongs to U. Together with continuity of the bundles and the partially hyperbolicity along $A \cup R$ this allows to show that if $g|_{A \cup R}$ is (absolutely) partially hyperbolic, then it must be (absolutely) partially hyperbolic globally.

To show continuity of the bundles, notice first that since for a point $x \in M \setminus (A \cup R)$ the bundle E_R^{uu} is transverse to E_A^{cs} one has that as one iterates forward the point $g^n(x)$ approaches A while the bundle $D_x g^n(E_R^{uu})$ must approach E^{uu} by the transversality and domination. The same argument shows that E^{ss} is also continuous. The fact

that E^c glues well with E^c along A follows from the fact that E_A^{cs} is invariant and E^c is transverse to E_A^{ss} in E_A^{cs}. The symmetric argument gives continuity of E^c as one approaches R and this concludes the proof.

$\square$

8 The example: perturbation of the time N of the flow

In this section we construct the example announced in Theorem 1.1 and prove it is (absolute) partially hyperbolic.

We fix $N \geq N_0$ as in Corollary 6.3.

Consider $M \setminus (A_X \cup R_X)$. Let V_1 be the X-invariant open subset of M consisting in the point whose orbit crosses T_1.

Consider the diffeomorphism $\mathcal{G}$ on $[0,1] \times \mathbb{T}^2$ defined in Lemma 5.1. We consider $G : M \to M$ defined as the identity outside $\mathcal{U}_N$ and $G = H_N^{-1} \circ \mathcal{G} \circ H_N$ in $\mathcal{U}_N$. Then G is a smooth diffeomorphism from how $\mathcal{G}$ was defined.

Define the diffeomorphism $f : M \to M$:

$$f = G \circ X_N$$

We will prove that the diffeomorphism f defined above satisfies all the conclusion of Theorem 1.1. We begin by first demonstrating that f is partially hyperbolic.

Theorem 8.1. *The diffeomorphism f is absolutely partially hyperbolic.*

Proof. First notice that the chain recurrent set of f coincides with the non-wandering set of X_N and thus of X, that is $A_X \cup R_X$. Furthermore the dynamics of f coincides with the one of X_N in a neighborhood of $A_X \cup R_X$. In particular, f is absolute partially hyperbolic in restriction to $A_X \cup R_X$.

According to Theorem 7.2 we must check that the extension E_A^{ss} and E_A^{cs} of the bundles E_X^{ss}, E_X^{cs} in A_X and the extensions E_R^{cu}, E_R^{uu} of E_X^{cu} and E_X^{uu} on R_X satisfy the transversality conditions between E_A^{cs} and E_R^{uu} and E_A^{ss} and E_R^{cs} in a compact fundamental domain $\mathcal{U}$.

Recall that E_A^{ss}, E_A^{cs} in a neighborhood of A_X are the tangent bundles to the strong stable and center stable foliations of A_X and E_R^{cu}, E_R^{uu} coincide with the tangent bundles to the strong unstable and center unstable foliations of R_X (see Remark 7.1 and notice that f coincides with an Anosov flow in neighborhoods of A_X and R_X).

Notice that f admits a fundamental domain having two connected components, one (denoted as $\Delta_{1,N}$) bounded by T_1 and $X_N(T_1)$ and the other bounded by T_2 and $X_N(T_2)$ (denoted as $\Delta_{2,N}$). Therefore $M \setminus (A_X \cup R_X)$ has two connected components V_1 and V_2 which are the sets of points whose orbits pass through one or the other fundamental domains.

On V_2 the diffeomorphism f coincides with X_N and the bundles coincide with the invariant bundles of X, so that we get the transversality conditions for free.

Thus we just have to show the transversality in $\Delta_{1,N} = \mathcal{U}_N$.

Notice that f coincides with X_N on

$$\bigcup_{t \geq 0} X_t(T_1).$$

As $\bigcup_{t \geq 0} X_t(T_1)$ is positively invariant and the orbits tends to A_X the stable and center stable foliations of f coincide on that set with those of X.

In particular, E_A^{cs} and E_A^{ss} in $\mathcal{U}_N$ coincide with E_X^{cs} and E_X^{ss} respectively.

Notice also that f coincides with X_N on

$$\bigcup_{t \leq -N} X_t(T_1).$$

As $\bigcup_{t \leq -N} X_t(T_1)$ is negatively invariant and the negative orbits tends to R_X the unstable and center unstable foliations of f coincide on that set with those of X.

In particular, we obtain that $E_R^{cu} = G_*(E_X^{cu})$ and $E_R^{uu} = G_*(E_X^{uu})$ in $\mathcal{U}_N$ which satisfy the transversality conditions thanks to Corollary 6.3.

$\square$

The bundles of f will be denoted as E_f^{ss}, E_f^c and E_f^{uu}. As usual, we denote $E_f^{cs} = E_f^{ss} \oplus E_f^c$ and $E_f^{cu} = E_f^c \oplus E_f^{uu}$.

9 Dynamical coherence

Here we prove that the diffeomorphism f is robustly dynamically coherent and that it cannot be leaf conjugate to the time one map of an Anosov flow. Also, the same result holds for any iterate and any finite lift of f.

Lemma 9.1. *There exists $K > 1$ so that for any unit vector $v \in E_f^c$ and any $n \in \mathbb{Z}$ one has*

$$\frac{1}{K} < \|Df^n(v)\| < K.$$

Proof. Notice that given any neighborhoods U_R and U_A of R_X and A_X, there is $n_0 > 0$ so that $f^{n_0}(M \setminus U_R)$ is contained in U_A.

We choose U_A as being a positively X_t-invariant neighborhood of A_X on which f coincides with X_N. In U_A the bundles E_f^{ss} and E_f^{cs} coincide with E_X^{ss} and E_X^{cs}. As E_f^c is transverse to E_f^{ss} on the compact manifold M, we get that any unit vector of E_f^c in U_A has a component in $\mathbb{R}X$ uniformly bounded from below and above and a component in E_X^{ss} uniformly bounded (from above). Therefore, the positive iterates of v by Df are uniformly bounded from below and above.

The same happens in a neighborhood U_R of R_X when looking at backward iterates. Now the result is established by using the invariance of E_f^c and the fact that any orbit spends at most n_0 iterates outside $U_R \cup U_A$. $\square$

This property has a dynamical consequence on curves tangent to the center direction. Let us recall a definition:

Definition 9.2. Consider a diffeomorphism g with an invariant bundle $E \subset TM$, one says that g is *Lyapunov stable* in the direction of E if given any $\varepsilon > 0$ there is $\delta > 0$ so that any path γ tangent to E of length smaller than δ verifies that the forward iterates $g^n(\gamma)$ have length smaller than ε.

As a direct consequence of Lemma 9.1 one gets:

Corollary 9.3. *The diffeomorphism f is Lyapunov stable in the direction E_f^{cs}. Symmetrically, f^{-1} is Lyapunov stable in the direction E_f^{cu}.*

Proof. Just notice that for any unit vector v tangent to E_f^{cs} the forward iterates $Df^n(v)$ have uniformly bounded norm. $\square$

This permits us to prove

Theorem 9.4. *The diffeomorphism f is robustly dynamically coherent. Moreover, the bundle E_f^c is uniquely integrable.*

Proof. We refer the reader to [HPS, Section 7] or [HHU$_2$, Section 7] for precise definitions of some notions which will appear in this proof (which are classical in the theory of partially hyperbolic systems). Related arguments appear in [BBI].

It is proved in [HHU$_2$, Theorem 7.5] (see also [HPS, Theorem 7.5]) that E_f^{cs} is tangent to a unique foliation provided f is Lyapunov stable with respect to E_f^{cs}. Therefore, by Corollary 9.3 the bundle E_f^{cs} is tangent to a unique foliation tangent to E_f^{cs}. Moreover, it is shown that under this assumptions, the unique foliation W_f^{cs} must be *plaque-expansive* in the sense of [HPS, Section 7].

Applying the same result for f^{-1} and E_f^{cu} we deduce dynamical coherence and using [HPS, Theorem 7.1] we deduce that the center-stable and center-unstable foliations are *structurally stable* (in particular, they exist for small C^1 perturbations of f). This concludes the proof of robust dynamical coherence.

Finally, unique integrability of E_f^c follows by classical arguments using the fact that curves tangent to E_f^c are Lyapunov stable for f and f^{-1} because of Lemma 9.1 (see also [HHU$_2$, Corollary 7.6]). $\square$

Now we show that the example cannot be leaf conjugate to the time-one map of an Anosov flow and this establishes the result claimed in Corollary 1.2.

First we must show something about the leafs of the center-foliation W_f^c of f. Recall from the previous section that $\mathcal{U}_N = \bigcup_{0 \le t \le N} X_t(T_1)$.

Lemma 9.5. *The connected components of $W_f^c \cap \mathcal{U}_N$ are arcs which join T_1 with $X_N(T_1)$ with uniformly bounded length.*

Proof. This is direct from the fact that the perturbation preserves the $\mathbb{T}^2 \times \{t\}$ coordinates (in the coordinates given by H_N) and the original center direction was positively transverse to those fibers. Since $\mathbb{T}^2 \times [0,1]$ is compact, one obtains that these arcs are of bounded length and join both boundaries of $\mathcal{U}_N$. $\square$

We can now show:

Theorem 9.6. *There are center leaves which are not fixed for no iterate of f. Consequently, there is no finite lift or finite iterate of f which is leaf conjugate to the time-one map of an Anosov flow.*

Proof. Using the uniqueness, one knows that the f-invariant foliation W_f^c tangent to E_f^c is obtained by intersecting the preimages of W_X^{cs} with the forward images of W_X^{cu}, where W_X^{cs} and W_X^{cu} are defined in neighborhoods of A_X and R_X respectively.

In particular, we get that restricted to $\mathcal{U}_N$ which is a fundamental domain for f in the complement of $A_X \cup R_X$, we have that W_f^c consists of $W_X^{cs} \cap G(W_X^{cu})$ which is an arc joining T_1 and $X_N(T_1)$ as proved in Lemma 9.5.

Notice that f coincides with X_N in a neighborhood of T_1 (and $G = id$ in a neighborhood of T_1) so that W_f^c consists of horizontal lines (i.e. of the form $[0, \varepsilon) \times \{p\}$ or $(1 - \varepsilon, 1] \times \{p\}$ in the coordinates given by H_N) in neighborhoods of T_1 and $f(T_1)$. Moreover, one has that the image of the center line which is of the form $[0, \varepsilon) \times \{p_0\}$ in a neighborhood of T_1 is sent by f to the arc of W_f^c which is of the form $(1 - \varepsilon, 1] \times \{p_0\}$ (again because $G = id$ in a neighborhood of T_1 and $X_N(T_1)$).

On the other hand, those arcs cannot be joined inside $W_X^{cs} \cap G(W_X^{cu})$ if they are not the leaves corresponding to circles in T_1 so that we deduce that the center leaves cannot be fixed by f. The same argument implies that this is not possible for f^k for any $k \ge 0$ since f coincides with X_N once it leaves $\mathcal{U}_N$. $\square$

Remark 9.7. This in stark contrast with the results of [BW] where it is shown that when f is transitive, if certain center leaves are fixed, then all center leaves must be. It is also important to note that in this example the leaves of both the center-stable and center-unstable foliations are fixed by f but the connected components of their intersections—the center leaves—are not.

Since the 3-manifold M admits an Anosov flow, $\pi_1(M)$ has exponential growth, and so, the manifold M does not support a partially hyperbolic diffeomorphisms isotopic to an Anosov diffeomorphism or a skew product. This is because Anosov diffeomorphisms only exist on $\mathbb{T}^3$ in dimension 3 (see for example [HP] and references therein) and skew products are only defined (in [BW]) on circle bundles over $\mathbb{T}^2$—both these types of 3-manifolds are associated with polynomial growth in their fundamental groups. These growth properties are immune to taking finite covers, and so, the same holds true for any finite cover of M. Therefore, if a finite lift or iterate of f were leaf conjugate to one of the three models, it would have to be the time-one map of an Anosov flow. In this case, there would exist an iterate of f on a finite cover that would fix every center leaf. We have shown that this is not the case, and thus, Theorem 9.6 implies Corollary 1.2.

10 Non-trivial isotopy class

In this section we will complete the proof of Theorem 1.1 by showing that no iterate of f is isotopic to the identity. It is at this point only that we shall use the fact that the original DA-diffeomorphism has at least two sources since this provides a curve intersecting the torus where the modification is made which is homologically non trivial. So we restrict ourselves to the case where there are exactly two sources, but it should be clear that all we have done until now works with any number of such sources.

Theorem 10.1. *For every $k \neq 0$, f^k is not isotopic to the identity. More precisely, the action of f^k on homology is non-trivial.*

Proof. It suffices to show that the action on homology is not trivial.

To establish this, first note that for the suspension manifold M_0 the periodic orbits where we did the DA-construction are homologically non-trivial. This implies that after removing the solid torus, the circles in the same direction in the boundary are still homologically non-trivial. The gluing we have performed preserves this homology class, so that it remains homologically non-trivial after the gluing too [5].

Notice moreover that there is a representative γ_1 of this homology class which does not intersect T_1 simply by making a small homotopy of this loop which makes it disjoint from T_1.

Consider a closed curve γ_2 intersecting T_1 only once (it enters by T_1 and then comes back by T_2) we know that γ_2 is not homologous to γ_1. This can be shown by considering a small tubular neighborhood U of T_1 and a closed 1-form which is strictly positive[6] in U and vanishes outside U. It is clear that such a 1-form has a non-vanishing integral along γ_2 and a vanishing one along γ_1 proving the desired claim.

From how G was chosen it is clear that $G_*^k([\gamma_2]) = [\gamma_2] + k[\gamma_1]$. Since X_N is isotopic to the identity, we also have that $(G \circ X_N)_* = G_*$.

This implies that for every $k \neq 0$, the map $(f^k)_* = (G \circ X_N)_*^k = G_*^k$ is different from the identity.

$\square$

Acknowledgements

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[5] A way to see this is by constructing a closed 1-form in each piece which integrates one in the desired circle. These 1-forms glue well to give a closed 1-form integrating one in the same curve.

[6] If $U \sim T_1 \times [-1, 1]$ and we call θ to the variable on $[-1, 1]$ it suffices to choose $f(\theta)d\theta$ with f smooth, positive on $(-1, 1)$ and vanishing at ± 1.

References

[BBY] C. Bonatti, F. Beguin, B. Yu; Building Anosov flows on 3-manifolds, *Preprint*, arXiv:1408.3951.

[BP] C. Bonatti, R. Potrie, Anomalous partially hyperbolic diffeomophisms II: stably ergodic examples, *In Preparation.*

[BW] C. Bonatti, A. Wilkinson; Transitive partially hyperbolic diffeomorphisms on 3-manifolds. *Topology* **44** (2005), no. 3, 475-508.

[BBI] M. Brin, D.Burago, S. Ivanov; On partially hyperbolic diffeomorphisms of 3-manifolds with commutative fundamental group. *Modern dynamical systems and applications* 307–312, Cambridge Univ. Press, Cambridge, 2004.

[Br] M.Brunella; Separating the basic sets of a nontransitive Anosov flow, *Bull. London Math. Soc.* **25** (1993), no. 5, 487-490.

[FW] J. Franks, R. Williams; Anomalous Anosov flows, *Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979)*, pp. 158-174, Lecture Notes in Math., 819, Springer, Berlin, 1980.

[HP] Andy Hammerlindl, Rafael Potrie Classification of partially hyperbolic diffeomorphisms in 3-manifolds with solvable fundamental group, *Preprint*, arXiv:1307.4631

[HHU$_1$] F. Rodriguez Hertz, M.A. Rodriguez Hertz, R. Ures, *Partial hyperbolicity in 3-manifolds* Talk by M.A. Rodriguez Hertz at Palis-Balzan conference, IHP 2013[7]

[HHU$_2$] F. Rodriguez Hertz, M.A. Rodriguez Hertz, R. Ures, A survey in partially hyperbolic dynamics, Fields Institute Communications, Partially Hyperbolic Dynamics , Laminations and Teichmuller Flow, Giovanni Forni, Mikhail Lyubich, Charles Pugh, Michael Shub editors, Vol. 51, 35–88, (2007). .

[HHU$_3$] F. Rodriguez Hertz, M.A. Rodriguez Hertz, R. Ures; A non-dynamically coherent example $\mathbb{T}^3$, arXiv:1409.0738.

[HPS] M. Hirsch, C. Pugh and M. Shub, Invariant Manifolds, *Springer Lecture Notes in Math.*, **583** (1977).

[McC] D. McCullough, Virtually geometrically finite mapping class groups of 3-manifolds, *Journal of Differential Geometry* **33** (1991) 1-65.

[Ro] R. C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos, 2nd edition*, CRC Press (1998).

[W] A. Wilkinson, *Conservative partially hyperbolic dynamics*, 2010 ICM Proceedings (2010).

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[7]Notes available on the web page
http://www.impa.br/opencms/pt/eventos/extra/2013_balzan/attach/apresentacao_maria_alejandra_hertz.pdf

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