PARTIALLY HYPERBOLIC DIFFEOMORPHISMS WITH A TRAPPING PROPERTY

RAFAEL POTRIE

ABSTRACT. We study partially hyperbolic diffeomorphisms satisfying a trapping property which makes them look as if they were Anosov at large scale. We show that, as expected, they share several properties with Anosov diffeomorphisms. We construct an expansive quotient of the dynamics and study some dynamical consequences related to this quotient.

1. INTRODUCTION

The purpose of this paper is twofold. On the one hand, we provide some mild contributions to the classification problem of partially hyperbolic diffeomorphisms in higher dimensions. On the other hand we study the dynamics of certain partially hyperbolic diffeomorphisms as well as give evidence of certain pathological phenomena which must be dealt with in order to understand better this kind of dynamics.

We shall consider partially hyperbolic diffeomorphisms $f : M \to M$ admitting a splitting of the form $TM = E^{cs} \oplus E^u$ in the pointwise sense (see below for precise definitions).

It is well known ([HPS]) that the unstable bundle E^u is uniquely integrable into a foliation called *unstable foliation* and denoted as $\mathcal{W}^u$. On the other hand, the integrability of the bundle E^{cs} is a subtler issue (see [BuW]). As in other results aiming at the classification of partially hyperbolic diffeomorphisms in higher dimensions (for example [Bo, Carr, Go]) we shall ignore this issue for the moment by assuming that f is *dynamically coherent*. We say that a partially hyperbolic diffeomorphism is *dynamically coherent* if there exists an f-invariant foliation $\mathcal{W}^{cs}$ tangent to E^{cs}. In dimension 3, there are several classification type results which do not assume integrability to start with (see for example [BoW, BBI, HP, HP$_2$, P$_3$]). See also section 6 of this paper where we remove this hypothesis under a different assumption.

This work will concern partially hyperbolic diffeomorphisms which verify a dynamical condition which makes them look, from far apart, as Anosov diffeomorphisms: We will say that a dynamically coherent partially hyperbolic diffeomorphism $f : M \to M$ with splitting $TM = E^{cs} \oplus E^u$ has a *trapping property* if there exists a continuous map $\mathcal{D}^{cs} : M \to \mathrm{Emb}^1(\overline{\mathbb{D}}^{cs}, M)$ such that $\mathcal{D}^{cs}(x)(0) = x$, the image of $\overline{\mathbb{D}}^{cs}$ (the closed unit ball of dimension $\dim E^{cs}$) by $\mathcal{D}^{cs}(x)$ is always contained in $\mathcal{W}^{cs}(x)$ and they verify the

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following trapping property:

$$f(\mathcal{D}^{cs}(x)(\overline{\mathbb{D}}^{cs})) \subset \mathcal{D}^{cs}(f(x))(\mathbb{D}^{cs}) \qquad \forall x \in M.$$

The main point of this paper is to recover the same type of results that are valid for Anosov diffeomorphisms ([Fr, Man, N]) in this setting. Several examples enjoy this trapping property and it is important in order to obtain dynamical consequences ([M, Carv, BV, P, BF, FPS, Rol]), however, the point here is to avoid the usual assumption that the trapping property is seen at "small scale" (a similar approach is pursued in [P$_4$] in the 3-dimensional case and with one dimensional center). See also [CP] for a related notion of *chain-hyperbolicity*.

A relevant point is to obtain results concerning partially hyperbolic diffeomorphisms without knowledge *a priori* on the global structure of the invariant foliations. To support this point of view we give in section 6 a weaker assumption which implies dynamical coherence and our trapping property.

1.1. **Statement of results.** In this paper, we will consider partially hyperbolic diffeomorphisms in one of the weakest forms (see [BDV, Appendix B] for a survey of possible definitions). We explicit the definition we shall use to avoid confusions with other references.

A C^1-diffeomorphism $f : M \to M$ is *partially hyperbolic* if there exists a Df-invariant continuous splitting $TM = E^{cs} \oplus E^u$ and $N \geq 1$ such that for every $x \in M$ and for every pair of unit vectors $v^{cs} \in E^{cs}(x)$ and $v^u \in E^u(x)$ one has that:

$$\|Df^N v^u\| > \max\{1, \|Df^N v^{cs}\|\}$$

Notice that with this definition f might be partially hyperbolic and f^{-1} not. This will not be a problem here, the results can be easily adapted to other settings. The definitions of *dynamical coherence* and the *trapping property* are the ones given in the introduction.

Theorem 1.1. *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism with splitting $TM = E^{cs} \oplus E^u$ and satisfying the trapping property. Assume moreover that one of the following holds:*

- $M = \mathbb{T}^d$ *or,*
- $\dim E^u = 1$

Then, $M = \mathbb{T}^d$ and f is isotopic to a linear Anosov automorphism of $\mathbb{T}^d$.

An expansive quotient of the dynamics will be constructed under the hypothesis of the theorem. This will be enough to obtain that results of non-existence of Anosov diffeomorphisms depending on the Lefschetz formula hold for partially hyperbolic diffeomorphisms in our setting (see for example [Sh, GH]). This is done in section 3 and the end of section 4. It might be that this quotient is of independent interest.

Under certain assumptions (resembling those of the theory of Franks-Newhouse-Manning [Fr, N, Man]) we will see in section 4 that the quotient map is in fact transitive by translating the proofs in the Anosov setting to ours.

In section 5 some mild dynamical consequences are derived and some questions which might will help to understand better the panorama are posed. Finally, in section 6 we give weaker conditions which ensure dynamical coherence.

Appendix A presents an example which improves a decomposition constructed in [Rob] showing that the quotients might be quite wild while respecting (some of) the dynamical conditions.

2. Notation

Along this paper M will denote a closed d-dimensional manifold and $f : M \to M$ a partially hyperbolic diffeomorphism. Except in section 6 we shall assume that f is dynamically coherent and verifies the trapping property. Along the paper we shall assume that $d \geq 3$ (for the case of $d = 2$ stronger results can be obtained with easier proofs, see for example [P_2, Section 4.A]).

Given any foliation $\mathcal{F}$ on M we shall denote as $\mathcal{F}(x)$ to the leaf through x, $\mathcal{F}_\varepsilon(x)$ to the ε-disk around x in the induced metric of the leaf and $\tilde{\mathcal{F}}$ will always denote the lift of $\mathcal{F}$ to the universal cover $\tilde{M}$ of M. Here, foliation means a continuous foliation with C^1-leafs tangent to a continuous distribution (foliations of class $C^{1,0+}$ according to [CC]).

3. An expansive quotient of the dynamics

Denote as $\overline{\mathcal{D}}_x^{cs}$ to $\mathcal{D}^{cs}(x)(\overline{\mathbb{D}}^{cs})$ and $\mathcal{D}_x^{cs}$ to $\mathcal{D}^{cs}(x)(\mathbb{D}^{cs})$.

We can define for each $x \in M$

$$A_x = \bigcap_{n \geq 0} f^n(\overline{\mathcal{D}}_{f^{-n}(x)}^{cs})$$

Some obvious properties satisfied by the sets A_x are:

- $f(A_x) = A_{f(x)}$ for every $x \in M$.
- The set A_x is a decreasing union of topological balls (it is a cellular set). In particular, A_x is compact and connected.

We would like to prove that the sets A_x constitute a partition of M, so that we can quotient the dynamics. For this, the following lemma is of great use:

Lemma 3.1. *For every $y \in \mathcal{W}^{cs}(x)$, there exists n_y such that $f^{n_y}(\overline{\mathcal{D}_y^{cs}}) \subset \mathcal{D}_{f^{n_y}(x)}^{cs}$. The number n_y varies semicontinuously on the point, that is, there exists U a small neighborhood of y such that for every $z \in U$ we have that $n_z \leq n_y$.*

PROOF. Consider in $\mathcal{W}^{cs}(x)$ the sets

$$E_n = \{y \in \mathcal{W}^{cs}(x) \ : \ f^n(\overline{\mathcal{D}_y^{cs}}) \subset \mathcal{D}_{f^n(x)}^{cs}\}$$

The sets E_n are clearly open (by continuity of f and $\mathcal{D}^{cs}$) and verify that $E_n \subset E_{n+1}$ because of the trapping property. Of course, $x \in E_n$ for every $n \geq 1$.

Using the trapping property and continuity of $\mathcal{D}^{cs}$ again, it follows that $\bigcup_{n \geq 0} E_n$ is closed. Indeed, let y belong to the closure of $\bigcup_{n \geq 0} E_n$, so, for $z \in \bigcup_{n \geq 0} E_n$ close enough to y one has that $f(\overline{\mathcal{D}_y^{cs}}) \subset \mathcal{D}_{f(z)}^{cs}$. Since $z \in E_k$ for some k one deduces that $y \in E_{k+1}$ showing that $\bigcup_n E_n$ is closed. Being non-empty, one deduces that $\bigcup_n E_n = \mathcal{W}^{cs}(x)$ as desired.

The fact that the numbers n_y varies semicontinuosly is a consequence of the fact that E_n is open (n_y is the first integer such that $y \in E_n$).

$\square$

One can show now that the sets A_x constitute a partition of M.

Proposition 3.2. *For $x, y \in M$ we have that $A_x = A_y$ or $A_x \cap A_y = \emptyset$.*

PROOF. There exists a uniform $n_0 > 0$ such that for every $x \in M$ and $y \in f(\mathcal{D}_{f^{-1}(x)}^{cs})$ we have that $f^{n_0}(\mathcal{D}_y^{cs}) \subset \mathcal{D}_{f^{n_0}(x)}^{cs}$.

To see this, notice that for each $x \in M$ there exists k_x with this property thanks to the previous Lemma and the fact that $f(\overline{\mathcal{D}_{f^{-1}(x)}^{cs}})$ is compact.

To prove the claim it is enough to show that the numbers k_x vary semicontinuously with the point x. This follows with a similar argument as the one in Lemma 3.1 by using continuity of the disks, the map f and the trapping property.

For $z \in A_x$ and $n \geq 0$ we have that $A_z \subset f^n(\mathcal{D}_{f^{-n}(x)}^{cs})$: Indeed, the existence of n_0 gives that $f^{n_0}(\mathcal{D}_{f^{-n_0}(f^{-n}(z))}^{cs}) \subset \mathcal{D}_{f^{-n}(x)}^{cs}$ which gives the desired inclusion. This gives $A_z \subset A_x$. By symmetry we deduce that given $x, y \in M$ the sets A_x and A_y are either disjoint or coincide.

$\square$

Consider two points x, y such that $y \in W^{uu}(x)$. We denote $\Pi_{x,y}^{uu} : \mathcal{D} \subset \mathcal{D}_x^{cs} \to \mathcal{D}_y^{cs}$ as the unstable holonomy from a subset of $\mathcal{D}_x^{cs}$ into a subset of $\mathcal{D}_y^{cs}$. An important useful property is the following:

Lemma 3.3. *The unstable holonomy preserves fibers, that is $\Pi_{x,y}^{uu}(A_x) = A_y$.*

PROOF. It is enough to show (by the symmetry of the problem) that $\Pi^{uu}(A_x) \subset A_y$. For n large enough we have that $f^{-n}(\Pi^{uu}(A_x))$ is very close to a compact subset of $\mathcal{D}_{f^{-n}(x)}^{cs}$ and thus $f^{-n}(\Pi^{uu}(A_x)) \subset \mathcal{D}_{f^{-n}(y)}^{cs}$ which concludes.

$\square$

Lemma 3.4. *The equivalence classes vary semicontinuously, i.e. if $x_n \to x$ then:*

$$\limsup A_{x_n} = \bigcap_{k > 0} \overline{\bigcup_{n > k} A_{x_n}} \subset A_x$$

PROOF. Using the invariance under unstable holonomy, it is enough to show that the classes vary semicontinuously inside center-stable manifolds. The proof of this fact is quite similar to the previous ones, particularly, the proof of Proposition 3.2.

$\square$

We get thus a continuous projection by considering the relation $x \sim y \Leftrightarrow y \in A_x$.

$$\pi : M \to M/_\sim$$

We denote as $g : M/_\sim \to M/_\sim$ the map given by $g([x]) = [f(x)]$ (that is $g \circ \pi = \pi \circ f$). Since π is continuous and surjective, it is a semiconjugacy.

Notice that a priori, the only knowledge of the topology of $M/_\sim$ one has is that it is the image by a cellular map of a manifold (some information on these maps can be found in the book [D] and references therein). For instance, we do not know a priori if the dimension of $M/_\sim$ is finite. This will follow from dynamical arguments after we prove Theorem 3.5 (combined with [M_2]).

Given a homeomorphism $h : X \to X$ of a compact metric space X, we denote the ε-stable (ε-unstable) set as

$$S_\varepsilon(x) = \{y \in X \ : \ d(h^n(x), h^n(y)) \leq \varepsilon \ for \ n \geq 0\}$$

$$U_\varepsilon(x) = \{y \in X \ : \ d(h^n(x), h^n(y)) \leq \varepsilon \ for \ n \leq 0\}$$

We say that a homeomorphism has *local product structure* if there exists $\delta > 0$ such that $d(x, y) < \delta$ implies that $S_\varepsilon(x) \cap U_\varepsilon(y) \neq \emptyset$.

A homeomorphism h is *expansive* (with expansivity constant α) if for every $x \in X$ we have that $S_\alpha(x) \cap U_\alpha(x) = \{x\}$.

Expansive homeomorphisms verify that $\operatorname{diam}(h^n(S_\varepsilon(x))) \to 0$ uniformly on x for $\varepsilon < \alpha$.

We shall also denote $W^{s(u)}(x) = \{y \in X \ : \ d(h^n(x), h^n(y)) \to 0 \ as \ n \to +\infty(n \to -\infty)\}$, the above remark implies that $S_\varepsilon(x) \subset W^s(x)$ for an expansive homeomorphism ($\varepsilon < \alpha$).

Theorem 3.5. *The homeomorphism g is expansive with local product structure. Moreover, $\pi(\mathcal{W}^{cs}(x)) = W^s(\pi(x))$ and π is injective when restricted to the unstable manifold of any point.*

PROOF. The last two claims are direct from Lemma 3.1 and the definition of the equivalence classes respectively.

We must show the existence of a local product structure and that will establish expansivity also. First choose $\varepsilon > 0$ such that an unstable manifold of size 2ε cannot intersect the same center stable disk in more than one point. This is given by the continuity of the bundles E^{cs} and E^u.

Consider $x \in M$ and a neighborhood U of A_x. Using Lemma 3.4 one knows that there is a neighborhood V of A_x such that for every $y \in V$ one has that A_y is contained in U. One can choose U small enough so that for every $y \in V$ one has that $\mathcal{W}_\varepsilon^u(y) \cap \mathcal{D}_x^{cs}$ is exactly one point.

Moreover, by the continuous variation of the $\mathcal{D}^{cs}$-disks, one has that, maybe by choosing V smaller, it holds that for every $y, z \in V$ one has that $\mathcal{W}_{2\varepsilon}^u(y) \cap \mathcal{D}_z^{cs}$ is exactly one point.

Since the image of V by π is open, one gets a covering of $M/_\sim$ by open sets where there is product structure. By compactness one deduces that

there exists a local product structure for g and since the intersection point is unique one also obtains expansivity of g.

$\square$

3.1. **Some remarks on the topology of the quotient.** This section shall not be used in the remaining of the paper.

We shall cite some results from [D] which help understand the topology of $M/_\sim$. Before, we remark that Mañe proved that a compact metric space admitting an expansive homeomorphism must have finite topological dimension ([M$_2$]).

Corollary IV.20.3A of [D] implies that, since $M/_\sim$ is finite dimensional, we have that it is a locally compact ANR (i.e. absolute neighborhood retract). In particular, we get that $\dim(M/_\sim) \leq \dim M$ (see Theorem III.17.7). Also, using Proposition VI.26.1 (or Corollary VI.26.1A) we get that $M/_\sim$ is a $d-$dimensional homology manifold (since it is an ANR, it is a *generalized manifold*). More properties of these spaces can be found in section VI.26 of [D].

Also, in the cited book, one can find a statement of Moore's theorem (see section IV.25 of [D]) which states that a cellular decomposition of a surface is approximated by homeomorphisms. In particular, in our case, if $\dim E^{cs} = 2$, we get that $M/_\sim$ is a manifold (see also Theorem VI.31.5 and its Corolaries).

Lateer, we shall see that if $M = \mathbb{T}^d$ then the quotient $M/_\sim$ is also a manifold (indeed $M/_\sim$ is homeomorphic to $\mathbb{T}^d$). The same should hold for infranilmanifolds but we have not checked this.

4. TRANSITIVITY OF THE EXPANSIVE HOMEOMORPHISM

In this section $g : M/_\sim \to M/_\sim$ will denote the expansive quotient map we have constructed in the previous section. The quotient map will be as before denoted by $\pi : M \to M/_\sim$.

In general, it is not yet known if an Anosov diffeomorphism must be transitive. Since Anosov diffeomorphisms enter in our hypothesis, there is no hope of knowing if f or g will be transitive without solving this long-standing conjecture. We shall then work with similar hypothesis to the well known facts for Anosov diffeomorphisms, showing that those hypothesis that we know guaranty that Anosov diffeomorphisms are transitive imply transitivity of g as defined above.

Remark 4.1. It is well known that transitivity of g amounts to showing some form of uniqueness of basic pieces. This is quite direct if one assumes some knowledge on the structure of the foliations of f, for example, if for every $x, y \in M$ one has that $\mathcal{D}_x^{cs} \cap \mathcal{W}^u(y) \neq \emptyset$ then it follows that g is transitive. In this paper we rather concentrate on information which does not rely *a priori* on knowledge of the structure of the foliations.

In particular, we shall prove in this section the following two results:

Theorem 4.2. *Assume $f : M \to M$ is a dynamically coherent partially hyperbolic diffeomorphism with the trapping property and $\dim E^u = 1$. Then M is covered by $\mathbb{R}^d$ and homotopically equivalent to $\mathbb{T}^d$.*

Theorem 4.3. *Assume $f : M \to M$ is a dynamically coherent partially hyperbolic diffeomorphism with the trapping property and M is covered by $\mathbb{R}^d$ and homotopically equivalent to $\mathbb{T}^d$. Then f is isotopic to a linear Anosov automorphism L, the manifold M and the quotient $M/_\sim$ are homeomorphic to $\mathbb{T}^d$ and g is topologically conjugate to L.*

Putting together Theorems 4.2 and 4.3 can be compared to Franks-Newhouse theory ([Fr, N]) on codimension one Anosov diffeomorphisms. It is possible to prove directly (with an argument similar to the one of Newhouse but taking care on the quotients) that g is transitive when $\dim E^u = 1$ without showing that $M = \mathbb{T}^d$ (see [P$_2$, Section 5.4] for this approach).

Theorem 4.3 is reminiscent of Franks-Manning theory ([Fr, Man] see also [KH, Chapter 18.6]). It is natural to expect that property this result should hold if we consider M an infranilmanifold, but we have not checked this in detail.

It is reasonable to extend the conjecture about transitivity of Anosov diffeomorphisms to expansive homeomorphisms in manifolds with local product structure. See the results in [V, ABP].

4.1. **Proof of Theorem 4.2.** This proof is an adaptation of quite classical ideas (see for example the Appendix in [ABP]) with some arguments of [N].

The main point is to show that $\mathcal{W}^{cs}$ is a foliation by leafs homeomorphic to $\mathbb{R}^{d-1}$. This follows from the trapping property (and Lemma 3.1). Having this, one can lift the foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^u$ to the universal cover and show that a leaf of $\tilde{\mathcal{W}}^u$ cannot intersect a leaf of $\tilde{\mathcal{W}}^{cs}$ more than once using Haefliger's argument ([CC]) and the fact that all leafs are simply connected.

To prove that the universal cover of M is $\mathbb{R}^d$ one must show that given a leaf of $\tilde{\mathcal{W}}^u$ it intersects every leaf of $\tilde{\mathcal{W}}^{cs}$. This follows with exactly the same proof of Lemma (5.2) of [Fr] once one knows that every leaf of $\mathcal{W}^{cs}$ is dense.

Lemma 4.4. *Every leaf of $\mathcal{W}^{cs}$ is dense in M.*

PROOF. We use here some of the ideas of [N].

Consider a leaf $\mathcal{L}$ of $\mathcal{W}^{cs}$ and let $\Lambda = \overline{\mathcal{L}}$ which is a closed $\mathcal{W}^{cs}$-saturated set. We must show that $\Lambda = M$, for this, it is enough to show that every point $x \in \Lambda$ verifies that $\mathcal{W}^u(x) \cap \Lambda \neq \emptyset$ in both connected components of $\mathcal{W}^u(x) \setminus \{x\}$ (recall that since $\mathcal{W}^u(x)$ is one-dimensional $\mathcal{W}^u(x) \setminus \{x\}$ has two different connected components). Under this assumption, and using the fact that f^{-1} contracts uniformly leafs of $\mathcal{W}^u$ one deduces that Λ is also saturated by $\mathcal{W}^u$ showing that Λ is open and closed. Being non-empty, one deduces that $\Lambda = M$ as desired.

So, to prove the lemma it is enough to show the following:

Claim. *Every point $x \in \Lambda$ verifies that both connected components of $\mathcal{W}^u(x) \setminus \{x\}$ intersect Λ.*

PROOF. First, notice that without loss of generality[1], one can assume that $f(\Lambda) = \Lambda$.

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[1]A way to see this is to consider a spectral decomposition for g. If there is a leaf of $\mathcal{W}^{cs}$ which is not dense, one can consider Λ to be the preimage by π of an attractor of g. Considering an iterate, one has the desired property. See [N].

We claim that if $x \in \Lambda$ is a point such that one connected component of $\mathcal{W}^u(x) \setminus \{x\}$ does not intersect Λ then A_x is periodic (or equivalently, $\pi(x)$ is a periodic point for g). Moreover, there are finitely many such periodic points. To see this, notice that there exists $\varepsilon > 0$ such that if three points of the past orbit of $\pi(x)$ by g are at distance smaller than ε one has that the unstable manifold of one of the points in the orbit intersects Λ in both connected components. Since such points are invariant one deduces that $\pi(x)$ must be periodic for g.

Now assume there is a point x such that its unstable manifold does not intersect Λ on one side. Let Σ be the boundary of $\mathcal{D}_x^{cs}$ which is a topological sphere (of dimension ≥ 1 since $d \geq 3$). As we mentioned, every point in Σ verifies that the unstable manifold in both sides intersect Λ, and by continuity and the intersection point, one obtains a continuous map from $\varphi : \Sigma \times [0,1] \to M$ which verifies that $\varphi(z,0) = z$, $\varphi(z,1)$ is in Λ and maps $\{z\} \times [0,1]$ to a compact part of $\mathcal{W}^u(z)$. We can moreover assume that $\varphi(\Sigma \times \{t_0\})$ is contained in a leaf of $\mathcal{W}^{cs}$ for every t_0 using continuity (see [N]).

Since $\varphi(\Sigma \times \{t_0\})$ separates $\mathcal{W}^{cs}(\varphi(z,t_0))$ giving a compact region one can prove that the unstable manifold of x intersects $\mathcal{W}^{cs}(\varphi(z,t_0))$ for every $t_0 \in [0,1]$ and therefore that it intersects Λ giving a contradiction. See [N] for more details.

$\square$

Now, we have a global product structure in the universal cover which implies that $\tilde{M} = \mathbb{R}^d$ and moreover, we get that the space of leafs of the foliation $\tilde{\mathcal{W}}^{cs}$ is homeomorphic to the real line $\mathbb{R}$ (and can be identified with a single leaf of $\tilde{\mathcal{W}}^u$). The action by deck transformations induces an action on the space of leafs of $\tilde{\mathcal{W}}^{cs}$ which does not have fixed points since all leafs of $\mathcal{W}^{cs}$ are simply connected. By Holder's theorem this implies that $\pi_1(M)$ is free abelian and thus isomorphic to $\mathbb{Z}^k$. Since the universal cover is contractible, one deduces that M is homotopy equivalent to $\mathbb{T}^d$ as desired.

$\square$

4.2. **Proof of Theorem 4.3.** We shall follow the proof given in [KH] chapter 18.6.

Before we start with the proof, we shall recall Theorem 18.5.5 of [KH] (the statement is modified in order to fit our needs, notice that for an expansive homeomorphism with local product structure the specification property is verified in each basic piece):

Proposition 4.5 (Theorem 18.5.5 of [KH]). *Let X a compact metric space and $g : X \to X$ an expansive homeomorphism with local product structure. Then, there exists $h, c_1, c_2 > 0$ such that for $n \in \mathbb{N}$ we have:*

$$c_1 e^{nh} \leq P_n(g) \leq c_2 e^{nh}$$

where $P_n(g)$ is the number of fixed points of g^n.

We shall use several time the very well know Lefschetz formula which relates the homotopy type of a continuous function, with the index of its fixed points (see [Fr$_2$] Chapter 5).

Definition. Let $V \subset \mathbb{R}^k$ be an open set, and $F : V \subset \mathbb{R}^k \to \mathbb{R}^k$ a continuous map such that $\Gamma \subset V$ the set of fixed points of F is a compact set, then, $I_\Gamma(F) \in \mathbb{Z}$ (the index of F) is defined to be the image by $(id - F)_* :$ $H_k(V, V - \Gamma) \to H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\})$ of u_Γ where u_Γ is the image of 1 under the composite $H_k(\mathbb{R}^k, \mathbb{R}^k - D) \to H_k(\mathbb{R}^k, \mathbb{R}^k - \Gamma) \cong H_k(V, V - \Gamma)$ where D is a disk containing Γ.

Remark 4.6. In general, if we have a map from a manifold, we can include the manifold in $\mathbb{R}^k$ and extend the map in order to be in the hypothesis of the definition. The value of $I_\Gamma(F)$ does not depend on how we embed the manifold in $\mathbb{R}^k$.

For hyperbolic fixed points, it is very easy how to compute the index, it is exactly $sgn(det(Id - D_p f))$. Since the definition is topological, any time we have a set which behaves locally as a hyperbolic fixed point, it is not hard to see that the index is the same.

Lefshetz fixed point formula for the torus can be stated as follows:

Theorem 4.7 (Lefshetz fixed point formula ([Fr₂] p.34-38)). *Let $h : \mathbb{T}^d \to \mathbb{T}^d$ be an homeomorphism, so, the sum of the Lefshetz index along a covering of $Fix(h)$ by sets homeomorphic to balls equals $det(Id - h_*)$ where $h_* : H_1(\mathbb{T}^d, \mathbb{Z}) \to H_1(\mathbb{T}^d, \mathbb{Z})$ is the action of h in homology.*

The first thing we must show, is that the *linear* part of f, that is, the action $L = f_* : H_1(\mathbb{T}^d, \mathbb{Z}) \to H_1(\mathbb{T}^d, \mathbb{Z}) \in SL(d, \mathbb{Z})$ is a hyperbolic matrix.

Lemma 4.8. *The matrix L is hyperbolic.*

PROOF.We can assume (maybe after considering a double covering and f^2) that E^{cs} and E^u are orientable and its orientations preserved by Df. So, it is not hard to show that for every fixed point p of g^n, the index of $\pi^{-1}(p)$ for f is of modulus one and always of the same sign.

So, we know from the Lefshetz formula that

$$|det(Id - L^n)| = \sum_{g^n(p)=p} |I_{\pi^{-1}(p)}(f)| = \#Fix(g^n).$$

This implies that L^n is hyperbolic using Proposition 4.5 since the only way to have that estimate on the periodic orbits is that L is hyperbolic (see the argument in Lemma 18.6.2 of [KH]).

$\square$

It is standard to show the existence of a semiconjugacy $h : \mathbb{T}^d \to \mathbb{T}^d$ isotopic to the identity such that $h \circ f = L \circ h$. Its lift $\overline{h} : \mathbb{R}^d \to \mathbb{R}^d$ is given by shadowing, in particular, the iterations of the set $\overline{h}^{-1}(x)$ remain of bounded diameter.

Lemma 4.9. *The homeomorphism g factors as an intermediate semiconjugacy. More precisely, there exists $\tilde{h} : \mathbb{T}^d/_\sim \to \mathbb{T}^d$ continuous such that $\tilde{h} \circ \pi = h$.*

PROOF.It is enough to show that for every $x \in \mathbb{T}^d/_\sim$ there exists $y \in \mathbb{T}^d$ such that $\pi^{-1}(x) \subset h^{-1}(y)$.

For this, notice that any lifting of $\pi^{-1}(x)$ (that is, a connected component of the preimage under the covering map) to the universal covering $\mathbb{R}^d$ verifies that it's iterates remain of bounded size. This concludes by the remark above on $\overline{h}$.

$\square$

Now, we shall prove that if $\overline{f} : \mathbb{R}^d \to \mathbb{R}^d$ is any lift of f, then there is exactly one fixed fiber of π for $\overline{f}$.

Lemma 4.10. *Let $\overline{f}^n$ be any lift of f^n to $\mathbb{R}^d$. So, there is exactly one fixed fiber of π.*

PROOF. Since $\overline{f}^n$ is homotopic to L^n which has exactly one fixed point and each fixed fiber of π contributes the same amount to the index of $\overline{f}^n$ it must have exactly one fixed fiber.

$\square$

This allows us to show that g is transitive:

Proposition 4.11. *The homeomorphism g is transitive.*

PROOF. First, we show that there exists a basic piece of g which projects by $\tilde{h}$ to the whole $\mathbb{T}^d$. This is easy since otherwise, there would be a periodic point q in $\mathbb{T}^d \backslash \tilde{h}(\Omega(g))$ but clearly, the $g-$orbit of $\tilde{h}^{-1}(q)$ must contain non-wandering points (it is compact and invariant). This concludes, since considering a transitive point y of L and a point in $\Omega(g) \cap \tilde{h}^{-1}(y)$ we get the desired basic piece.

Now, let Λ be the basic piece of g such that $\tilde{h}(\Lambda) = \mathbb{T}^d$. Assume that there exists $\tilde{\Lambda} \neq \Lambda$ a different basic piece and z a periodic point of $\tilde{\Lambda}$, naturally, we get that $\tilde{h}^{-1}(\tilde{h}(z))$ contains also a periodic point z' in Λ. By considering an iterate, we can assume that z and z' are fixed by g.

So, we get that lifting $h^{-1}(\tilde{h}(z))$ to a lift which fixes $\pi^{-1}(z)$ and $\pi^{-1}(z')$ which contradicts the previous lemma.

$\square$

With this in hand, we will continue to prove that the fibers of h coincide with those of π proving that g is conjugated to L (in particular, $\mathbb{T}^d/_\sim \cong \mathbb{T}^d$).

First, we show a global product structure for the lift of f. Notice that when we lift f to $\mathbb{R}^d$, we can also lift its center-stable and unstable foliation. It is clear that both foliations in $\mathbb{R}^d$ are composed by leaves homeomorphic to $\mathbb{R}^{cs}$ and $\mathbb{R}^u$ respectively (the unstable one is direct, the other is an increasing union of balls, so the same holds).

Lemma 4.12. *Given $x, y \in \mathbb{R}^d$, the center stable leaf of x intersects the unstable leaf of y in exactly one point.*

PROOF. The fact that they intersect in at most one point is given by the fact that otherwise, we could find a horseshoe for the lift, and thus many periodic points contradicting Lemma 4.10 (for more details, see Lemma 18.6.7 in [KH]).

The proof that any two points have intersecting manifolds, is quite classical, and almost topological once we know that both foliations project into minimal foliations (see also Lemma 18.6.7 of [KH]).

$\square$

Now, we can conclude with the proof of Theorem 4.3.

To do this, notice that the map $\overline{h}$ conjugating $\overline{f}$ with L is proper, so the preimage of compact sets is compact. Now, assume that A_x, A_y are lifts of fibers of π such that $\overline{h}(A_x) = \overline{h}(A_y)$ we shall show they coincide.

Consider K such that if two points have an iterate at distance bigger than K then their image by $\overline{h}$ is distinct.

We fix $x_0 \in A_x$ and consider a box D_K^n of $\overline{f}^n(x_0)$ consisting of the points z of $\mathbb{R}^d$ such that $\mathcal{W}^u(z) \cap \mathcal{W}_K^{cs}(x_0) \neq \emptyset$ and $\mathcal{W}^{cs}(z) \cap \mathcal{W}_K^u(x_0) \neq \emptyset$.

It is not hard to show using Lemma 4.12 that there exists $\tilde{K}$ independent of n such that every pair of points in D_K^n in the same unstable leaf of $\mathcal{W}^u$ have distance along $\mathcal{W}^u$ smaller than $\tilde{K}$ (this is a compactness argument). An analogous property holds for $\mathcal{W}^{cs}$.

This implies that if $\overline{f}^n(A_y) \subset D_K^n$ for every $n \in \mathbb{Z}$ then A_y and A_x must be contained in the same leaf of $\mathcal{W}^{cs}$. In fact we get that $\overline{f}^{-n}(A_y) \subset \mathcal{W}_K^{cs}(\overline{f}^{-n}(x_0))$ for every $n \geq 0$ and so we conclude that $A_x = A_y$ using Lemma 3.1.

$\square$

4.3. Some manifolds which do not admit this kind of diffeomorphisms.

The arguments used in the previous section also allow to show that certain manifolds (and even some isotopy classes in some manifolds) do not admit dynamically coherent partially hyperbolic diffeomorphisms satisfying the trapping property.

A similar argument to the one used in the previous section yields the following result (see [GH] for sharper results in the same lines):

Theorem 4.13. *Let f be a partially hyperbolic diffeomorphism of M with the coherent trapping property, then, the action $f_* : H_*(M, \mathbb{Q}) \to H_*(M, \mathbb{Q})$ is strongly partially hyperbolic (it has both eigenvalues of modulus > 1 and < 1).*

This leads to a natural question: Is every dynamically coherent partially hyperbolic diffeomorphism with the trapping property homotopic to an Anosov diffeomorphism?. One should notice that expansive homeomorphisms admitting transverse stable and unstable foliations share many properties with Anosov diffeomorphisms (see for example [V, ABP]) but it is not known if every such homeomorphism is topologically conjugate to an Anosov diffeomorphism.

Also, let us remark that there exist examples of dynamically coherent partially hyperbolic diffeomorphisms which are isotopic to Anosov and robustly transitve while not satisfying the trapping property: See [P_2, Section 3.3.4]

5. Some dynamical consequences

In this section we shall look at what type of dynamical properties can be recovered in the spirit of [P] (see also [Carv, BV]).

We recall that a *quasi-attractor* Λ is a chain-recurrence class satisfying that it admits a decreasing basis of neighborhoods U_n satisfying that $f(\overline{U_n}) \subset U_n$ (see [P$_2$, Chapter 1] and references therein).

Since a quasi-attractor is saturated by unstable manifolds and the quotient we have defined which conjugates f to an expansive homeomorphism is injective on unstable manifolds, one expects that whenever the quotient map g is transitive (as it is ensured in some cases by Theorem A.1) there is a unique quasi-attractor. Unfortunately, showing this would involve showing that there are fibers of the semiconjugacy which are trivial and this is a subtle issue as the example presented in Appendix A shows.

We are however able to show uniqueness of the quasi-attractors under a mild assumption resembling *chain-hyperbolicity* as defined in [CP].

Proposition 5.1. *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism satisfying the trapping property. Assume moreover that the quotient map g defined above is transitive and that there exists a point $x \in M$ such that $A_x = \{x\}$. Then, f has a unique quasi-attractor.*

PROOF. Consider Λ a quasi-attractor for f and let $\pi : M \to M/\sim$ be the semiconjugacy to $g : M/\sim \to M/\sim$ constructed in section 3.

Since π is injective along unstable manifolds, one obtains that $\pi(\Lambda)$ contains the unstable set of any point $z \in M/\sim$ such that $z = \pi(y)$ with $y \in \Lambda$. Since g is expansive with local product structure and transitive, one know that the orbit of $W^u(z)$ is dense in $M/\sim$ (see for example [KH, Chapter 18]).

This implies that $\pi(\Lambda)$ is dense in $M/\sim$. Since π is continuous and Λ compact one deduces that $\pi(\Lambda) = M/\sim$. In particular, $\pi(\Lambda)$ contains $\pi(x)$.

As a consequence, we get that every quasi-attractor must intersect $A_x = \{x\}$. Since different quasi-attractors must be disjoint, this implies uniqueness of the quasi attractor under the assumptions of the proposition. $\qquad\square$

Remark 5.2. In the case where $E^{cs} = E^s \oplus E^c$ with $\dim E^c = 1$ and a trapping property is verified by leafs tangent to E^c one can show that f satisfies a trapping property. From the construction of π one sees that A_x is either a point or a closed interval. Using [J] one sees immediately that the conditions of the previous proposition are satisfied.

Another property, related to [P] is the following:

Proposition 5.3. *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism satisfying the trapping property. Assume moreover that the quotient map g defined above is transitive and the image of every open set of a center-stable leaf by π is either a point or has non-empty interior in the stable manifold of g. Then f has a unique quasi-attractor and every other chain-recurrence class of f is contained in a periodic disk of $\mathcal{W}^{cs}$.*

PROOF. Consider a quasi-attractor Λ. As in the previous proposition, one has that $\pi(\Lambda) = M/\sim$ and it is saturated by $\mathcal{W}^u$.

One can easily show that for every $x \in M$ the boundary of A_x is contained in Λ: indeed, consider any $y \in \partial A_x$ and a neighborhood U of y in $\mathcal{W}^{cs}(y)$.

From our hypothesis one has that $\pi(U)$ has non-empty interior in the stable manifold of $\pi(y)$.

Iterating backwards and using the semiconjugacy and using the density of unstable sets for g one obtains that $f^{-n}(U)$ intersects Λ for some n. Invariance of Λ and the fact that the choice of U was arbitrary gives that $y \in \overline{\Lambda} = \Lambda$.

The rest of the proposition follows by applying Proposition 2.1 of [P].

$\square$

As we have explained, the hypothesis we demand in this section might follow directly from the fact that f has the trapping property but the example presented in Appendix A strongly suggests that counterexamples might exist.

Question 1. *Does there exists a dynamically coherent partially hyperbolic diffeomorphism of $\mathbb{T}^3$ with splitting $T\mathbb{T}^3 = E^{cs} \oplus E^u$ and with the trapping property such that it admits more than one quasi-attractor? Such that it has chain-recurrence classes (different than the quasi-attractor) which are not contained in periodic center-stable discs?*

See [P$_4$] for related discussions.

6. A weaker trapping property and coherence

In this section we shall present a weaker trapping property without requiring dynamical coherence *a priori* and show that it is enough to recover the initial proposition. One would hope that this property is shared by certain partially hyperbolic diffeomorphisms isotopic to Anosov though it is not so clear that it holds (see [P$_3$] for results in this direction). The proof is completely analogous to the one presented in section 3 of [BF] but in a slightly different context. One important point is the fact that we do not assume that the trapping property occurs in a small region and hope this might find applications.

Let $f : M \to M$ be a partially hyperbolic diffeomorphism with splitting $TM = E^{cs} \oplus E^u$.

As before, we denote as $cs = \dim E^{cs}$ and $u = \dim E^u$ and $\mathbb{D}^\sigma$ is the σ-dimensional open disk and $\overline{\mathbb{D}}^\sigma$ its closure.

We will assume that f verifies the following property:

($*$) There exists a continuous map $B : M \to \mathrm{Emb}^1(\overline{\mathbb{D}}^{cs} \times \overline{\mathbb{D}}^u)$ such that $B(x)(0,0) = x$ for every $x \in M$, one has that $B(x)(\{a\} \times \overline{\mathbb{D}}^u) \subset \mathcal{W}^u(B(x)(a,0))$ and the following trapping property is verified:

$$\forall a \in \overline{\mathbb{D}}^{cs} \ \ \exists b \in \mathbb{D}^{cs} \ ; \ \ B(f(x))(\{b\} \times \overline{\mathbb{D}}^u) \subset f(B(x)(\{a\} \times \mathbb{D}^u))$$

The main result of this section is the following:

Theorem 6.1. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism verifying property ($*$), then, f is dynamically coherent with a trapping property.*

PROOF. First, we will denote as $\mathcal{D}_x^{cs}$ to the set of points $y \in B(x)(\mathbb{D}^{cs} \times \mathbb{D}^u)$ such that

$$f^n(y) \in B(f^n(x))(\mathbb{D}^{cs} \times \mathbb{D}^u)$$

We claim that $\mathcal{D}_x^{cs}$ is a manifold everywhere tangent to E^{cs} and moreover one has that a trapping property $f(\overline{\mathcal{D}_x^{cs}}) \subset \mathcal{D}_{f(x)}^{cs}$ is verified. Also, $\mathcal{D}_x^{cs}$ intersects every local unstable manifold of $\text{Image } B_x = B(x)(\mathbb{D}^{cs} \times \mathbb{D}^u)$ in a unique point.

To show this, notice first that expansivity of the unstable manifolds implies that $\mathcal{D}_x^{cs}$ cannot intersect an unstable manifold more than once. Also, the trapping property verified by the maps B gives that every point $a \in \mathbb{D}^{cs}$ verifies that $B(x)(\{a\} \times \mathbb{D}^u)$ intersects $\mathcal{D}_x^{cs}$. The fact that it is a C^1 manifold everywhere tangent to E^{cs} follows by classical graph transform arguments (see [HPS] or [KH, Chapter 6]).

An important fact of the above is that one can view $\mathcal{D}_x^{cs}$ as a limit of disks D_x^n where D_x^n is any disk inside $\text{Image } B_x$ with the following property:

- There exists a disk D in $\text{Image } B_{f^n(x)}$ which is the image by $B(f^n(x))$ of a 1-Lipchitz graph over $\mathbb{D}^{cs} \times \{0\}$ which intersects $\mathcal{W}_{loc}^u(f^n(x))$ in a point z. The disk D_x^n is the connected component containing $f^{-n}(z)$ of $f^{-n}(D) \cap \text{Image } B_x$.

Any family of such disks will converge to $\mathcal{D}_x^{cs}$ by the arguments sketched above (see the proof of Theorem 3.1 of [BF] for more details in a similar context).

To finish the proof is then enough to show that the plaques $\mathcal{D}_x^{cs}$ are coherent in the sense that if $y \in \mathcal{D}_x^{cs}$ then $\mathcal{D}_x^{cs} \cap \mathcal{D}_y^{cs}$ is relatively open in $\mathcal{D}_x^{cs}$. To see that this holds in general we shall argue in a similar way as in Lemma 3.1 to take advantage of the trapping property as well as the continuity of the map B.

For each $x \in M$ we consider the set $\mathcal{W}_x^{cs}$ defined as $\bigcup_n f^{-n}(\mathcal{D}_{f^n(x)}^{cs})$. Notice that $\mathcal{W}_x^{cs}$ is an immersed copy (in principle not injective) of $\mathbb{R}^{\dim E^{cs}}$ in M. We shall use in $\mathcal{W}_x^{cs}$ the topology induced by this immersion (i.e. the intrinsic topology and not the one given as a subset of M).

To show that E^{cs} is integrable, it is enough to show that $\mathcal{W}_x^{cs}$ is a partition of M and that each leaf is injectively immersed. Assume then that $y \in \mathcal{W}_x^{cs}$, we must show that $\mathcal{D}_y^{cs} \subset \mathcal{W}_x^{cs}$. This will conclude since by local uniqueness this gives that $\mathcal{W}_x^{cs}$ is injectively immersed and that the sets $\mathcal{W}_x^{cs}$ are disjoint or coincide.

Consider, for $x \in M$ the sets $E_n = \{y \in \mathcal{W}_x^{cs} : f^n(\overline{\mathcal{D}_y^{cs}}) \subset \mathcal{D}_{f^n(x)}^{cs}\}$. If one shows that $\mathcal{W}_x^{cs} = \bigcup_n E_n$ one completes the proof of the Theorem since $f^n(\overline{\mathcal{D}_y^{cs}}) \subset \mathcal{D}_{f^n(x)}^{cs}$ implies that $\mathcal{D}_y^{cs} \subset \mathcal{W}_x^{cs}$ which as argued above will imply that $\{\mathcal{W}_x^{cs}\}_{x \in M}$ is an f-invariant foliation tangent to E^{cs}.

The proof that the union $\bigcup_n E_n$ is closed in $\mathcal{W}_x^{cs}$ is the same as in Lemma 3.1. The proof of openness is slightly more delicate that in that case since we do not know coherence in principle. However, coherence is easy to establish for points which are nearby and that this is exactly what we need to show to show openness.

To see that E_n is open consider z is close enough to y verifying that $f^n(\overline{\mathcal{D}_y^{cs}}) \subset \mathcal{D}_{f^n(x)}^{cs}$. We must show that $z \in E_n$. To see this, it is enough to show that given $y \in M$ one has that for z in a small neighborhood of y

in $\mathcal{D}_y^{cs}$ it holds that $f(\mathcal{D}_z^{cs}) \subset \mathcal{D}_{f(y)}^{cs}$. By continuity of B and the trapping property, it follows that for z in a neighborhood of y in $\mathcal{D}_y^{cs}$ the image by f of Image B_z traverses the image of $B_{f(y)}$. The characterization of $\mathcal{D}_z^{cs}$ as limits of disks as explained above implies that $f(\mathcal{D}_z^{cs}) \subset \mathcal{D}_y^{cs}$ as desired and concludes the proof.

$\square$

It is natural to expect that for a partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ with splitting $T\mathbb{T}^3 = E^{cs} \oplus E^u$ (with $\dim E^{cs} = 2$) isotopic to a linear Anosov automorphism with two-dimensional stable bundle, property $(*)$ will be satisfied. To show this, one possibility would be to show injectivity of the semiconjugacy to the linear model along unstable manifolds but we have not succeed in doing so. A positive answer would improve the results of $[\text{P}_3]$ in this context.

APPENDIX A. A NON-TRIVIAL DECOMPOSITION OF THE PLANE ADMITTING HOMOTHETIES

We shall denote as $d_2 : \mathbb{R}^2 \to \mathbb{R}^2$ to the map

$$d_2(x) = \frac{x}{2}.$$

The goal of this appendix is to prove the following Theorem:

Theorem A.1. *There exists a C^∞-diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ and a constant $K > 0$ such that the following properties are verified:*

- *There exists a Holder continuous cellular map $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $d_{C^0}(h, id) < K$ and $d_2 \circ h = h \circ f$.*
- *There exist open sets V_1 and V_2 such that*
  - $\overline{V_1} \cap \overline{V_2} = \emptyset$
  - $h(V_i) = \mathbb{R}^2$ *for $i = 1, 2$.*
  - $f(\overline{V_i}) \subset V_i$ *for $i = 1, 2$.*
- *The C^∞ norm of f and f^{-1} is smaller than K.*

A direct consequence of this Theorem is the existence of $h : \mathbb{R}^2 \to \mathbb{R}^2$ whose fibers are all non trivial and cellular (decreasing intersection of topological disks), the existence of these decompositions of the plane had been shown by Roberts [Rob].

A.1. **Construction of f.** We start by considering a curve $\gamma = \{0\} \times [-\frac{1}{4}, \frac{1}{4}]$.

Clearly, $\gamma \subset B_0 = B_1(0)$ the ball of radius one on the origin. Consider also the sets $B_n = B_{2^n}(0)$ for every $n \geq 0$. It follows that

$$\mathbb{R}^2 = \bigcup_{n \geq 0} B_n$$

We shall define $f : \mathbb{R}^2 \to \mathbb{R}^2$ with the desired properties in an inductive manner, first in B_0 and then in the annulus $B_n \setminus B_{n-1}$ with arbitrary $n \geq 1$.

Define $f_0 : \overline{B_0} \to B_0$ a C^∞ embedding and open sets $V_1^0 = \mathbb{R} \times (-1/5, -1/6)$ and $V_2^0 = \mathbb{R} \times (1/6, 1/5)$ such that:

(a) f_0 coincides with d_2 in a small neighborhood of ∂B_0.
(b) $\bigcap_{n \geq 0} f_0^n(B_0) = \gamma$.

(c) $f_0(\overline{V_i^0}) \subset V_i^0$ for $i = 1, 2$.

Assume now that for some sufficiently large constants K_1 and $K_2 > 0$ we have defined a C^∞-diffeomorphism $f_n : B_n \to B_{n-1}$ and disjoint open connected sets V_1^n and V_2^n (homeomorphic to a band $\mathbb{R} \times (0, 1)$) such that:

(I1) $f_n|_{B_{n-1}} = f_{n-1}$ and $V_i^{n-1} \subset V_i^n$ for $i = 1, 2$.
(I2) The C^∞-distance between f_n and d_2 in B_n is smaller than K_1.
(I3) $f_n(\overline{V_i^n}) \subset V_i^{n-1}$ for $i = 1, 2$ and $f_n^n(\overline{V_i^n})$ disconects B_0.
(I4) V_i^n contains balls of radius $\frac{1}{10}$ in every ball of radius $\frac{K_2}{2}$ in B_n.
(I5) f_n coincides with d_2 in a $\frac{K_2}{10}$-neighborhood of ∂B_n.

We must now construct f_{n+1} and the sets V_i^{n+1} assuming we had constructed f_n and V_i^n.

To construct f_{n+1} and V_i^{n+1} we notice that in order to verify (I1), it is enough to define f_{n+1} in $B_n \backslash B_{n-1}$ as well as to add to V_i^n an open set in $B_{n+1} \backslash B_n$ which verifies the desired hypothesis.

Consider $d_2^{-1}(V_i^n) \cap B_{n+1} \backslash B_n$. Since V_i^n satisfies property (I4) one has that $d_2^{-1}(V_i^n)$ contains a ball of radius $\frac{1}{5}$ in every ball of radius K_2 of $B_{n+1} \backslash B_n$ for $i = 1, 2$.

Now, we consider a diffeomorphism φ_n which is $K_1 - C^\infty$-close to the identity, coincides with the identity in the $\frac{K_2}{10}$-neighborhoods of ∂B_{n+1} and ∂B_n and such that $\varphi_n(V_i^n)$ contains a ball of radius $\frac{1}{10}$ in every ball of radius $\frac{K_2}{2}$ of B_{n+1} for $i = 1, 2$. The existence of such φ_n is assured provided the value of K_1 is large enough with respect to K_2.

We define then f_{n+1} in $B_{n+1} \backslash B_n$ as $d_2 \circ \varphi_n^{-1}$ which clearly glues together with f_n and satisfies properties (I2) and (I5).

To define V_i^{n+1} we consider a very small $\varepsilon > 0$ (in order that $\varphi_n(V_i^n)$ also verifies (I4)) and for each boundary component C of $\varphi_n(V_i^n)$ (which is a curve) we consider a curve C' which is at distance less than ε of C inside $\varphi_n(V_i^n)$ and such that each when it approaches $C \cap \partial B_n$ the distance goes to zero and when it approaches $C \cap \partial B_{n+1}$ the distance goes to ε. This allows to define new V_i^{n+1} as the open set delimited by these curves united with the initial V_i^n. It is not hard to see that it will satisfy (I3) and (I4).

We have then constructed a C^∞-diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is at C^∞ distance K_1 of d_2 and such that there are two disjoint open connected sets V_1 and V_2 such that $f(\overline{V_i}) \subset V_i$. and such that both of them are $\frac{K_2}{2}$-dense in $\mathbb{R}^2$.

A.2. **Proof of the Theorem.** We first show the existence of a continuous function $h : \mathbb{R}^2 \to \mathbb{R}^2$ conjugating f to d_2 which is close to the identity.

This argument is quite classical: consider a point $x \in \mathbb{R}^2$, so, since $d_{C^0}(f, d_2) < K_1$ we get that the orbit $\{f^n(x)\}$ is in fact a K_1-pseudo-orbit for d_2. Since d_2 is infinitely expansive, there exists only one orbit $\{d_2^n(y)\}$ which $\alpha(K_1)$-shadows $\{f^n(x)\}$ and we define $h(x) = y$ (in fact, in this case, it suffices with the past pseudo-orbit to find the shadowing).

We get that h is continuous since when $x_n \to x$ then the pseudo-orbit which shadows must rest near for more and more time, and then, again by expansivity, one concludes. This implies also that h is onto since it is at bounded distance of the identity.

Now, consider any ball B of radius $100\alpha(K)$ in $\mathbb{R}^2$, it is easy to see that $f(B)$ is contained in a ball of radius $50\alpha(K)$ and then, we get a way to identify the preimage of points by h. Consider a point $x \in \mathbb{R}^2$, we get that

$$h^{-1}(h(x)) = \bigcap_{n>0} f^n(B_{100\alpha(K)}(f^{-n}(x)))$$

which implies that h is cellular.

It only remains to show that the image under h of both V_1 and V_2 is the whole plane.

Lemma A.2. $h(V_i) = \mathbb{R}^2$ for $i = 1, 2$.

PROOF.We shall show that $h(\overline{V_i})$ is dense. Since it is closed, this will imply that it is in fact the whole plane, and using the semiconjugacy and the fact that $f(\overline{V_i}) \subset V_i$ we get the desired property.

To prove that $h(\overline{V_i})$ is dense, we consider an arbitrary open set $U \subset \mathbb{R}^2$. Now, choose n_0 such that $d_2^{-n_0}(U)$ contains a ball of radius $10\alpha(K)$. We get that $h^{-1}(d_2^{-n_0}(U))$ contains a ball of radius $9\alpha(K)$ and thus, since $\alpha(K) > K$, we know that since V_i is $K/2$-dense, we get that $V_i \cap h^{-1}(d_2^{-n_0}(U)) \neq \emptyset$. So, since $f(V_i) \subset V_i$ we get that $V_i \cap f^{n_0} \circ h^{-1}(d_2^{-n_0}(U)) \neq \emptyset$ which using the semiconjugacy gives us that $h(V_i) \cap U \neq \emptyset$.

This concludes.

$\square$

Holder continuity of h follows as in Theorem 19.2.1 of [KH] (see also [P_2]). Notice that the exponent of Holder continuity cannot be larger than $\frac{1}{2}$ since the boundary of V_i is sent as a space-filling curve.

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CMAT, Facultad de Ciencias, Universidad de la República, Uruguay
URL: www.cmat.edu.uy/∼rpotrie
E-mail address: rpotrie@cmat.edu.uy