FULL DEMANDS OPTIMAL FRP-MORNDP SOLUTIONS OVER A CYCLE.

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Abstract. The FRP-MORNDP problem consists in finding a resilient and cost-optimal logical network, to be deployed over an existing optical infrastructure. This problem was first introduced in Risso (2010, Master's Thesis) for a specific application, and it was generalized later in his Doctoral Thesis, where it was proved that it is NP-Hard. In the middle, Almear et al (2011, degree project in Computer Engineering) found optimal numerical solutions for a subset within a particular family of instances. In this work we prove the correctness of those computer-aided constructions, although using purely analytical arguments. Furthermore, we present solutions over a wider set of instances for a higher number of nodes. These proofs are based in a quite tight new theoretical lower bound for the cost of the optimum. Besides, we give new examples to the converse of a necessary condition proved in Risso (2014 *PhD's Thesis*).

1. Introduction

In [4], a new optimization problem concerning with networks in modern communication system is introduced. It was called *Free Routing Protection Multi-Overlay Resilient Network Design Problem*, or FRP-MORNDP for short. For having a historical and technical perspective of this problem we invite the reader to overview [3]. Besides of thoroughly presenting the problem, this work proves that FRP-MORNDP is in the class of NP-Hard problems, which means that no efficient algorithm to solve it is known, and the prospective to find one in the near future is unlikely.

Due to the complexity of the problem, heuristic approaches like those developed in [3] are in general necessary for finding solutions. Conversely, finding optimal solutions analytically is achievable for some particular families of topologies, as is presented in [2]. There are also some computer-aided solutions like those found in [1], for networks with $n = 4, \dots, 8$ nodes and maximum capacity $b_{\bar{B}} = (3 - n + 2\lfloor n/2 \rfloor), \dots, \lceil n/2 \rceil \lfloor n/2 \rfloor$. These particular solutions are:

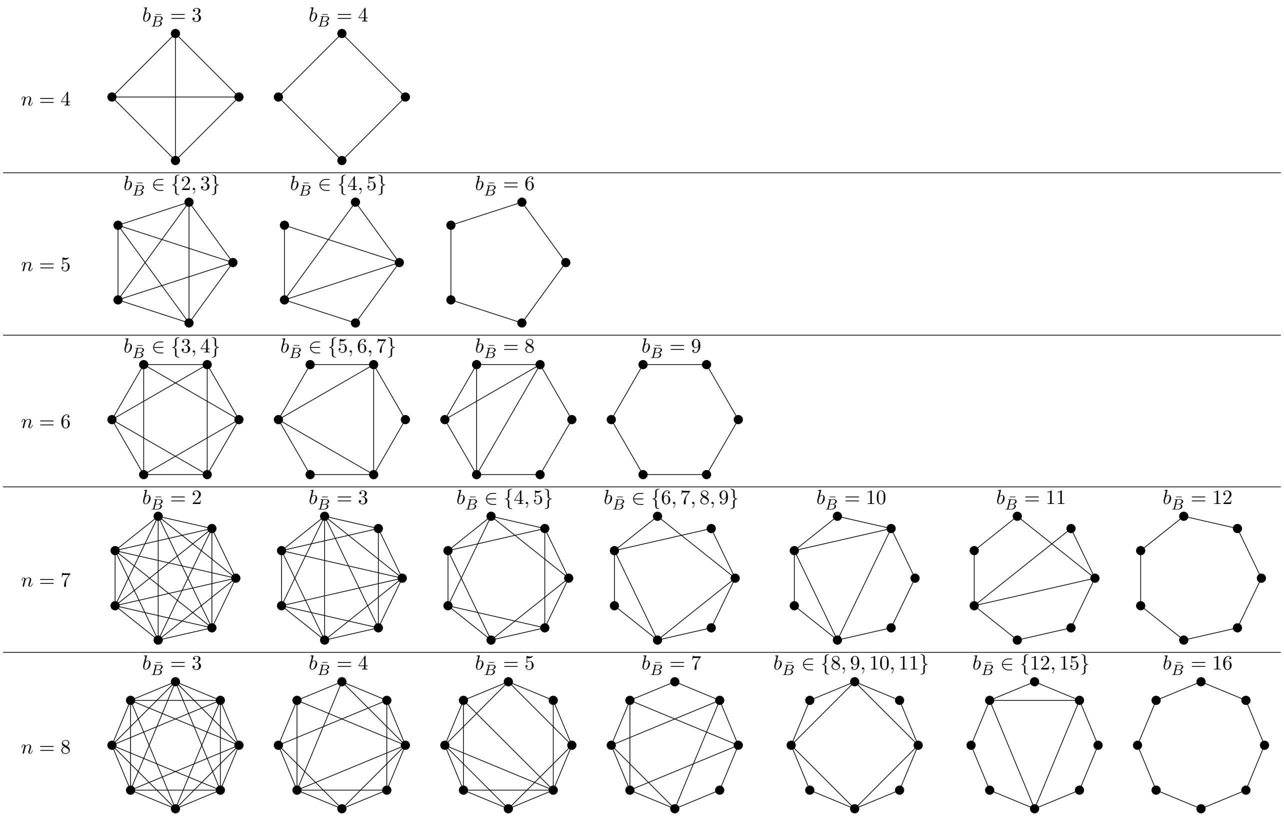


FIGURE 1. Numerical solutions found in [1] for different n's and $b_{\bar{B}}$'s.

The correctness of such constructions relies on the software used, which is prone to contain bugs, while in many cases the computer spends days making its calculations, turning then unfeasible a manual verification.

In this work we analytically prove the optimality of the solutions shown in Figure 1. Additionally, we present an infinite family that includes new optimal networks for the previous set of parameters. We verified their optimality twice, by hand and by computer search. Conversely to what happens for the computer-aided constructions presented in the Figure 1, those constructions presented along this article are verifiable by hand.

As we shall detail later on this article, there are many parameters to set up before constituting an instance to the FRP-MORNDP problem. They are: the potential topology for the logical network, the topology of an existing optical network (including the lengths of its links), the set of available capacities to dimension the logical links, the per-distance cost associated to each capacity, and the traffic demands to satisfy between each pair of logical nodes. This work tackles down the problem for the family of instances where physical topologies have a cycle structure, which is not only theoretically important but practical, since cycles (aka rings) are the basic building blocks of optical networks formerly designed to fulfill TDM transport requirements.

Under these premisses and following the scheme of [2, 3, 1], our aim is on parameters n (number of nodes of the graphs) and $b_{\bar{B}}$ (maximum capacity to dimension logical links) that are here supposed to be unique, leaving: demands, lengths and per-distance cost, as parameters of a second order of importance.

By using bounds for traffic and capacity we prove the optimality for such constructions. In general, the previous technique allows us to prove the optimality for most cases, but for a handful of singularities where ad-hoc arguments are used. Besides, for some instances alternate constructions are found, while under some premisses, families of solutions with an infinite number of member are introduced.

This article is organized as follows. In Section 2 we present basic definitions of graphs theory concerning to the FRP-MORNDP problem. In Section 3 we introduce a general lower bound for the cost of the optimal solution. We show three ad–hoc lower bound for cases $n = 4, 5, 7$ and $b_{\bar{B}} = 3$, and we present a stronger version of a necessary condition proved in [3]. In Section 4, we define an infinite family of graphs which contains new optimal networks. Besides, we present two new optimal networks for $n = 10, b_{\bar{B}} = 5$ and for $n = 25, b_{\bar{B}} = 6$, which do not belong to the infinite family previously defined, but arise from them by small modifications. In the same section, we describe the algorithms used to check the feasibility of the solutions. Finally, in Section 5, we present a new counterexample to the converse of a necessary condition proved in [3], which improves the former because it considers a full demand network rather than an artificial construction.

2. Basic Definitions

If A is a set, then $|A|$ denotes its cardinal and A^c its complement. The set of subset of A with cardinal k is denoted $\binom{A}{k}$, i.e. $\binom{A}{k} = \{S \subset A : |S| = k\}$.

A *(undirected simple) graph* $G = (V, E)$ consists in a non empty set of *vertex* or *nodes* $V = V(G)$ and a set of *edges* $E = E(G)$ such that $E \subset \binom{V}{2}$. If $\{v, w\} \in E$ then we say that v and w are *adjacent*. If $\emptyset \neq V_1 \subset V$, the graph *induced* by V_1 in G is defined by $G[V_1] = \left(V_1, \binom{V_1}{2} \cap \binom{V}{2}\right)$.

In this work we will extensively consider the *complete graph* $K_n = (\mathbb{Z}_n, \binom{\mathbb{Z}_n}{2})$ and the *cycle* $C_n = (\mathbb{Z}_n, \{\{i, i+1\} : i \in \mathbb{Z}_n\})$, where $\mathbb{Z}_n$ are the integer modulo n and the sums are take in $\mathbb{Z}_n$.

If the graph has *capacities* $b : E \to B \subset \mathbb{R}$ and *demands* given by a $|V| \times |V|$ matrix $D = ((d_{v\,w}))$ over the vertex set V, we say that the graph *satisfies* the demands if there is a set of paths $\mathcal{R}$ (aka routing map), such that for each $v, w \in V$ there exists a path $P_{v,w} \in \mathcal{R}$ joining v and w, and such that:

$$\forall e \in E \qquad \sum_{vw : e \in P_{v,w}} d_{v\,w} \leq b(e).$$

Given two non empty subsets of vertices $V_1, V_2 \subset V(G)$, we denote by $[V_1, V_2]$ to the set of edges of G with one end in V_1 and the other one in V_2. A *edge cut* is a subset of edges of the form $[V_1, \bar{V}_1]$. A minimal edge-cut is called *bond.* The *degree* $d_G(v)$ of a vertex v is the cardinal of $[\{v\}, \{v\}^c]$, i.e. $d_G(v) = |[\{v\}, \{v\}^c)]$. A graph is *disconnected* if it has an empty bond. A graph is *connected* if it is not disconnected. It is easy to verify that $[V_1, V_1^c]$ is a bond iff the graphs induced by V_1 and V_1^c are connected.

2.1. **Definitions concerned with the FRP-MORNDP problem.** The input of the FRP-MORNDP problem consists in a "physical" graph $G_P = (V, E)$ with lengths $\ell : E \to C \subset \mathbb{R}^+$ assigned to its edges, a matrix $D = ((d_{u\,v}))$ of positive demands over the vertex set V and a set of possible capacities or bit-rates $B \subset \mathbb{R}^+$, a per-distance cost $c : B \to \mathbb{R}^+$ such that $b' < b''$ implies $c(b')/b' > c(b'')/b''$ (economies of scale). A *feasible solution* of FRP-MORNDP consists in:

- a graph $G_L = (V, E_L)$, called *logical network* together with an assignment of capacities $b : E_L \to B$ and,
- a "routing" map $\rho : E_L \to G_P^*$ that maps each edge $e = vw \in E_L$ to a path $\rho(e)$ of G_P, called *lightpath*, with ends at v and w, such that for each "fault" of a physical edge $f \in E_P$, the "remaining" graph $G_L \setminus \{e : f \in \rho(e)\}$ can satisfy the demands given by D.

If the *cost of a ligthpath* $\rho(e)$ is its per-distance cost times its length, i.e.

$$c(\rho(e)) = \sum_{f \in \rho(e)} c(b(e))\ell(f),$$

then the cost of a solution G_L is given by $c(G_L) = \sum_{e \in E_L} c(\rho(e))$.

The FRP-MORNDP problem consists in **minimizing** $c(G_L)$ **over all feasible solutions** G_L. The FRP-MORNDP problem is NP-Hard as is proved in [3]. Even the subproblem of verifying whether a logical network satisfies the demands is NP-Hard, thus only heuristics algorithms are designed in [3] to find good quality solutions for a general case. However, for particular families of graphs, the exact solution is known. For instance, if $G_P = C_n$, $d_{v\,w} = 1$ for all $v \neq w$, $B = \{2\}$ and n odd, we proved that the optimal G_L is the complete graph with ρ being the path with the fewer number of edges (see [2]), no matter what c and ℓ are. Also in [2], if $n > 4$ is even, we have proved that there is no feasible solution for $B = \{2\}$ but there are for $B = \{3\}$. In [1], ILOG CPLEX v12.1 was used to find exact solutions for instances where $G_P = C_n$, $d_{v\,w} = 1$ for all $v \neq w$, $|B| = 1$ and $n = 4, 5, 6, 7, 8$. However, except for those cases falling back in the theorem proved in [2], the optimality of the other solutions depends on the correctness of the software used. In this work we prove that those constructions of Figure 1 are optimal, provided their feasibility, which can be verified by hand, by giving a routing map for each fault scenario. To prove their optimality, we will check out how these constructions fit to theoretical limits.

3. Lower bounds

In this section we are introducing some lower bounds for the cost of the optimal solution, which will allow us to prove the optimality of the solutions shown in Figure 1. In order to obtain a lower bound, we recall a result proved in [3][Lemma 5]. This result is a necessary condition for the feasibility of a solution to FRP-MORNDP, which states:

Lemma 1. *Given any solution G_L to* FRP-MORNDP. *In order for this solution to be feasible, it must be hold that for every edge-cut $b_P = [S, S^c]$ of G_P, then the condition:*

$$\sum_{v \in S, w \in S^c} d_{vw} \leq b_{\bar{B}} \left\lfloor \frac{|b_L|(|b_P| - 1)}{|b_P|} \right\rfloor \tag{1}$$

must be satisfied, where $b_{\bar{B}} = \max\{b \in B\}$, b_L is the edge-cut in G_L defined up from $[S, S^c]$ and d_{vw} is the traffic demand between nodes v and w. □

The original version of this lemma assumes that both b_P and b_L are bonds, however along its proof only their edge-cut character is used, so we shall used the result as it is enunciated above, which has the advantage of providing a bound for b_L despite of its bond character. In order to find a lower bound for the cost of the optimal solution, let $\mathcal{P} = \{[S_1, S_1^c], \ldots, [S_k, S_k^c]\}$ be a family of mutually disjoint edge-cuts, for instance, a partition of E_P in bonds. The ligthpaths have each of their edges in at most one element of the partition, so

$$\forall e \in G_L \qquad |E(\rho(e))| \geq \sum_{F \in \mathcal{P}} |\rho(e) \cap F|. \tag{2}$$

For each edge-cut $F = [S, S^c] \in \mathcal{P}$ let $\hat{b}_F$ be the minimum integer value for $|b_L|$ that verifies the inequality (1) of the previous Lemma, i.e., the minimum necessary number of logical links that traverse some edge-cut, which is:

$$\hat{b}_F = \hat{b}_{(S,S^c)} = \min \left\{ b \in \mathbb{Z} : \sum_{v \in S, w \in S^c} d_{vw} \leq b_{\bar{B}} \left\lfloor \frac{b(|b_P| - 1)}{|b_P|} \right\rfloor \right\}.$$

From the previous definition it is directly inferred that:

$$\hat{b}_F = \left\lceil \frac{|b_P|}{|b_P| - 1} \left\lceil \frac{1}{b_{\bar{B}}} \sum_{v \in S, w \in S^c} d_{vw} \right\rceil \right\rceil. \tag{3}$$

As a limit case, we assume during these proofs the following hypothesis: *there is only one capacity $b_{\bar{B}}$ for dimensioning links.*

Thus, whenever a feasible instance of FRP-MORNDP is given, we find a lower bound for the cost of any of its solutions (G_L, ρ) as follows:

$$c(G_L, \rho) = \sum_{e \in E_L} \sum_{f \in \rho(e)} c(b_{\bar{B}}) \ell(f) = c(b_{\bar{B}}) \sum_{f \in E_P} \sum_{e: f \in \rho(e)} \ell(f) \geq c(b_{\bar{B}}) \sum_{F \in \mathcal{P}} \sum_{f \in F} \sum_{e: f \in \rho(e)} \ell(f) =$$

$$c(b_{\bar{B}}) \sum_{F \in \mathcal{P}} \sum_{f \in F} \ell(f) |\{e \in E_L : f \in \rho(e)\}| \geq c(b_{\bar{B}}) \sum_{F \in \mathcal{P}} \sum_{f \in F} \min_{f \in F} \ell(f) |\{e \in E_L : f \in \rho(e)\}| =$$

$$c(b_{\bar{B}}) \sum_{F \in \mathcal{P}} (\min_{f \in F} \ell(f) \cdot \sum_{f \in F} |\{e \in E_L : f \in \rho(e)\}|) \geq c(b_{\bar{B}}) \sum_{F \in \mathcal{P}} \hat{b}_F \cdot \min_{f \in F} \ell(f).$$

where the third inequality comes from the fact that $\hat{b}_F$ corresponds to the minimum number of elements within a edge-cut of a feasible construction.

Since this bound does not depend on the routing function ρ, it is also valid for the optimal solution and we have the result:

Theorem 1. *The cost of an optimal solution to* FRP-MORNDP *is at least*

$$c(b_{\bar{B}}) \sum_{F\in\mathcal{P}} \hat{b}_F \cdot \min_{f\in F} \ell(f),$$

for every family $\mathcal{P}$ of mutually disjoint edge-cuts. □

Analogously, if $\mathcal{P}$ is a "double partition", i.e., each edge belongs to exactly two sets of $\mathcal{P}$, then we have the following bound:

$$\min c(G_L,\rho) \geq \frac{1}{2} c(b_{\bar{B}}) \sum_{F\in\mathcal{P}} \hat{b}_F \cdot \min_{f\in F} \ell(f), \qquad \forall \mathcal{P} \text{ double partition.}$$

In what follows we will extensively consider double partition in physical bonds of cardinal 2, with $\ell(f) = 1$ for all f, so we will have

$$\min c(G_L,\rho) \geq \frac{1}{2} c(b_{\bar{B}}) \sum_{F\in\mathcal{P}} \hat{b}_F, \quad \forall \mathcal{P} \text{ double partition: } |F| = 2\ \forall F \in \mathcal{P}. \tag{4}$$

3.1. **Double partition of the first kind.** Consider the double partition generated by each vertex against its complement, i.e. $\mathcal{P} = \{[\{v\},\{v\}^c]\}_{v\in V}$, then

$$\hat{b}_{(\{v\},\{v\}^c)} = \left\lceil \frac{d_{G_P}(v)}{d_{G_P}(v)-1} \left\lceil \frac{\sum_{w\neq v} d_{v\,w}}{b_{\bar{B}}} \right\rceil \right\rceil.$$

Replicating the parameters sets used in Figure 1, let $B = \{b_{\bar{B}}\}$, $\ell(f) = c(b_{\bar{B}}) = d_{v\,w} = 1$, for all $f \in E_P$ and $v,w \in V$, while $G_P = C_n$. Thus $d_{G_P}(v) = 2$ and $\hat{b}_{(\{v\},\{v\}^c)} = 2\lceil (n-1)/b_{\bar{B}} \rceil$. Besides we can apply bound (4) to obtain

$$c(G_L,\rho) \geq \frac{1}{2} \sum_{v\in V} 2 \left\lceil \frac{n-1}{b_{\bar{B}}} \right\rceil = n \left\lceil \frac{n-1}{b_{\bar{B}}} \right\rceil.$$

Unfortunately, this bound is not very good. In particular, it is not good enough to prove the optimality of the solutions shown in Figure 1, but we mention it here because it is quite general since it can be applied to many families of instances.

3.2. **Diameter Partition of C_n.** To deal with the graphs in Figure 1, let us consider the double-partition of the set of edges of C_n with bonds determined by a "half and a half" partition. More precisely, consider the subsets of the form $H_i = \{i, (i+1), (i+2), \ldots, (i+\lceil n/2 \rceil)\}$ -where the sums are taken in $\mathbb{Z}_n$-, and the partition $\mathcal{P}$ induced by them, i.e., $\mathcal{P} = \{[H_i, H_i^c]\}_{i=0}^{n-1}$. Because of (3) it holds that:

$$\hat{b}_{[H_i,H_i^c]} = 2\left\lceil \frac{\lceil n/2\rceil\lfloor n/2\rfloor}{b_{\bar{B}}}\right\rceil$$

$\mathcal{P}$ is a double-partition, so because of (4) we obtain a lower bound $\check{C}(n, b_{\bar{B}})$:

$$c(G_L,\rho) \geq \check{C}(n,b_{\bar{B}}) = n\left\lceil \frac{\lceil n/2\rceil\lfloor n/2\rfloor}{b_{\bar{B}}}\right\rceil .$$

In Table 1, we show the values of $\check{C}$ and the cost of the graphs found in Figure 1 which can be considered as upper bounds for the optimal cost. As we can see, there are only three non coincidence. They are for $(n, b_{\bar{B}}) = (4,3), (5,3), (7,3)$.

TABLE 1. Upper bound found in [1] vs theoretical lower bounds

$n\backslash b_{\bar{B}}$		2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
4	[1]		**10**	4												
	$\check{C}$		**8**	4												
5	[1]	15	**15**	10	10	5										
	$\check{C}$	15	**10**	10	10	5										
6	[1]		18	18	12	12	12	12	6							
	$\check{C}$		18	18	12	12	12	12	6							
7	[1]	42	**33**	21	21	14	14	14	14	14	14	7				
	$\check{C}$	42	**28**	21	21	14	14	14	14	14	14	7				
8	[1]		48	32	32	–	24	16	16	16	16	16	16	16	16	8
	$\check{C}$		48	32	32	**24**	24	16	16	16	16	16	16	16	16	8

Complementarily, each matching closes the optimality of the corresponding solution found in Figure 1.

Let us remark that the case $n = 8$, $b_{\bar{B}} = 6$ was badly reported in [1]. Besides, there were some cases (e.g. $n = 7, b_{\bar{B}} = 10$) for which the computer aborted without the certainty of having found an optimal solution, certainty that now, under the light of the present work, we do have.

3.2.1. *Remark.* Another important remark is the following. For some values of $b_{\bar{B}}$ the bound $\check{C}$ remains constant. For instance, for $n = 5$ with $b_{\bar{B}} = 3,4,5$ or $n = 6$ with $b_{\bar{B}} = 5,6,7,8$. Thus, whenever an optimal solution verifying the bound for one of these $b_{\bar{B}}$'s is found, it will automatically turn out to be an optimal solution for greater $b_{\bar{B}}$'s under the same $\check{C}$. It is worth pointing out that solutions for small $b_{\bar{B}}$'s are sometimes found much more quicker than those for greater capacities. For instance, the solution for $n = 7$ and $b_{\bar{B}} = 6$ was found by [1], in 2 minutes 20 seconds, while the solutions for $b_{\bar{B}} = 7,8,9,10,11$ were found in approximately 12, 43, 180, 5220 and 287 minutes respectively.

3.3. **Case** $n = 4$, $b_{\bar{B}} = 3$. This was found in [1], however it is easy to check. Indeed, since a network with cost 8 can only be attained by the shortest path routing, there is only one fundamental possible solution (plus symmetric variants), and it is unfeasible. Complementarily, the shortest alternative to route a lightpath which is not a shortest path is two units longer.

3.4. **Case** $n = 5$, $b_{\bar{B}} = 3$. Follows by applying bound (1) to a bond determined by one vertex and its complement. Indeed, this gives us $4 \leq 3\lfloor d_{G_L}(v)/2 \rfloor$, thus $d_{G_L}(v) \geq 4$ and the logical network must be complete.

3.5. **Case** $n = 7$, $b_{\bar{B}} = 3$. We consider the bonds with two adjacent vertices, i.e. b_i of the form $F_i = [\{i, (i+1)\}, \{i, (i+1)\}^c], i = 0, \ldots, 6$, where the sums are taken in $\mathbb{Z}_7$. In that case $\hat{b}_{F_i}$ is $2\lceil 2 \times 5/3 \rceil = 8$. So at least 8 edges must traverse such bonds. We are looking for constructions with the lowest lengths for their lightpaths. The minimum possible distribution of lengths is of two lightpaths of length 1, four of length 2 and two of length 3.

After adding up these costs for the seven bonds $b_0, \ldots, b_6$, we are counting the costs of lightpaths of length 1 twice, and four times those of lightpaths of length 2 and 3, thus, the total length is at least

$$7 \times \left(\frac{2 \times 1}{2} + \frac{4 \times 2 + 2 \times 3}{4} \right) = 31.5,$$

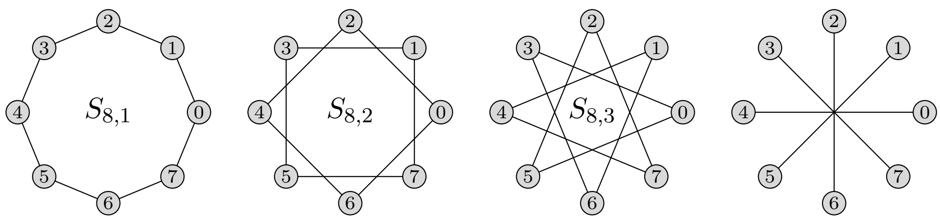
Hence, 32 would be a lower bound for the cost. Such configuration is unfeasible as we shall see immediately. First of all let us notice that it should consist of 7 lightpaths of lengths 1 and 2, which would add up a partial cost of 21. The complementary portion of the cost would be $11 = 32 - 21$, which is impossible to obtain with lightpaths of length 3. The next possible value is 33 and it is feasible indeed, because it matches the cost found in Figure 1. Therefore, the optimality of that construction is proven.

Up to this point we formally proved that all those solutions numerically found with CPLEX are actually optimal. These solutions are for $n = 4, \ldots, 8$, with $b_{\bar{B}}$ ranging within the theoretical boundaries found in [2] and [3].

4. Optimal networks for greater n

In this section we present a family of logical networks that seems to be optimal for many values of the parameters. We prove its optimality for some small values of n. Besides, in some cases they do not match with the graphs found in [1], so they constitute examples of the non unicity of the optimal solution.

4.1. **Definition of the family $G_{n,b}$.** We consider a family of logical networks $G_{n,b}$ defined as follows. First of all, we shall define an order relation between the edges of K_n. Let us limit to consider $V(K_n) = \mathbb{Z}_n$, i.e., the integers modulo n represented by $0, 1, 2, \ldots, n-1$. When an edge is of the form $\{i, i+j\}$ and $j \leq n/2$, we say that its length is j. Given $n \geq 3$ and $k < n/2$, let $S_{n,k}$ denote the set of 2-regular connected subgraphs of K_n with edges of length k. If n is even and $k = n/2$, then $S_{n,k}$ are copies of K_2. For illustrative purposes, the following figure sketches the sets of edges $S_{8,1}$, $S_{8,2}$, $S_{8,3}$ and $S_{8,4}$.



$S_{8,4}$

The degenerated case of n even and $k = n/2$ is not of our interest, so, from now we will suppose, $k < n/2$.

If $\gcd(n,k) = 1$ then $S_{n,k}$ consists of exactly one cycle of length n. However, when $\gcd(n,k) > 1$, $S_{n,k}$ consists of exactly $\gcd(n,k)$ cycles of length $c_{n,k} = n/\gcd(n,k)$. Indeed, $(n/\gcd(n,k))k = n(\gcd(n,k)/k) \equiv 0 \mod n$, so $c_{n,k} \leq n/\gcd(n,k)$. Conversely, if $c_{n,k}$ is the length of the cycle, then $c_{n,k}k \equiv 0 \mod n$, i.e. $n | c_{n,k}k$. Dividing by $\gcd(n,k)$ we have $n/\gcd(n,k) \mid c_{n,k}(k/\gcd(n,k))$. Thus $n/\gcd(n,k) \mid c_{n,k}$, because $\gcd(n/\gcd(n,k), k/\gcd(n,k)) = 1$, thus $n/\gcd(n,k) \leq c_{n,k}$. In the example above, $\gcd(8,2) = 2$ and we have 2 cycles of length $8/2 = 4$.

Another way to see the length of the cycles in $S_{n,k}$ is by observing that the vertices of each cycle are the right (and left) cosets of the (cycle) subgroup $\langle k \rangle$ of $\mathbb{Z}_n$ generated by k.

I.e., $\langle k\rangle = \{k, 2k, 3k, \ldots\}$ and the vertices of the cycles of $S_{n,k}$ are $\{\langle k\rangle, 1+\langle k\rangle, 2+\langle k\rangle, \ldots\}$. The order (i.e. length) of each cycle is $c_{n,k} = |\langle k\rangle| = n/\gcd(n,k)$, since $\gcd(n,k)$ is the smallest integer such that $\gcd(n,k)k \equiv 0 \mod n$.

Given any cycle C in $S_{n,k}$, let i_C be the smallest index of those vertices in C. In the previous example, $S_{8,2}$ consists of two cycles C and C'. For one of these cycles, e.g. C, it holds that $i_C = 0$, while $i_{C'} = 1$ for the other.

Thus, given any cycle C in $S_{n,k}$ and any node x in C, there exists a unique $h_x \in \{0, \ldots, c_{n,k}\}$ such that $x \equiv i_C + h_x k \pmod n$. For instance, in the unique cycle of $S_{8,3}$, $i_C = 0$ and $1 \equiv 0 + 3\cdot 3 \pmod 8$, while $4 \equiv 0 + 4 \cdot 3 \pmod 8$. Therefore $h_1 = 3$ and $h_4 = 4$.

Now let us define a total order on $E(K_n)$ in the following way:

- The edges of $S_{n,k}$ always precede those of $S_{n,k+1}$.
- If C and C' are in $S_{n,k}$, then the edges of C precede those of C' iff $i_C < i_{C'}$.
- Finally, when $\{x, x+k\}$ and $\{x', x'+k\}$ are part of the same cycle $C \in S_{n,k}$, then $\{x, x+k\}$ precedes $\{x', x'+k\}$ iff $h_x < h_{x'}$.

For instance, if $n = 8$ we have

$$\begin{aligned}
&(0,1) < (1,1) < (2,1) < (3,1) < (4,1) < (5,1) < (6,1) < (7,1) < \\
&(0,2) < (2,2) < (4,2) < (6,2) < (1,2) < (3,2) < (5,2) < (7,2) < \\
&(0,3) < (3,3) < (6,3) < (1,3) < (4,3) < (7,3) < (2,3) < (5,3) < \\
&(0,4) < (4,4) < (1,4) < (5,4) < (2,4) < (6,4) < (3,4) < (7,4).
\end{aligned}$$

Suppose we index the edges of K_n in such a way that $e_1 < e_2 < \cdots < e_m$. consider the sum s_k of the length of the first k edges, i.e.,

$$s_k = |e_1| + \cdots + |e_k|.$$

Then, let k_b such that $s_{k_b} \leq nb$ but $s_{1+k_b} \geq (n+1)b$. If $s_{k_b} = nb$, then we define $E(G_{n,b}) = \{e_1, e_2, \ldots, e_{k_b}\}$. If $s_{k_b} < nb$ we define $E(G_{n,b}) = \{e_1, e_2, \ldots, e_{k_b+1}\} \setminus \{\{0, s_{k_b+1} - b\}\}$. We are giving some examples in Figure 2.

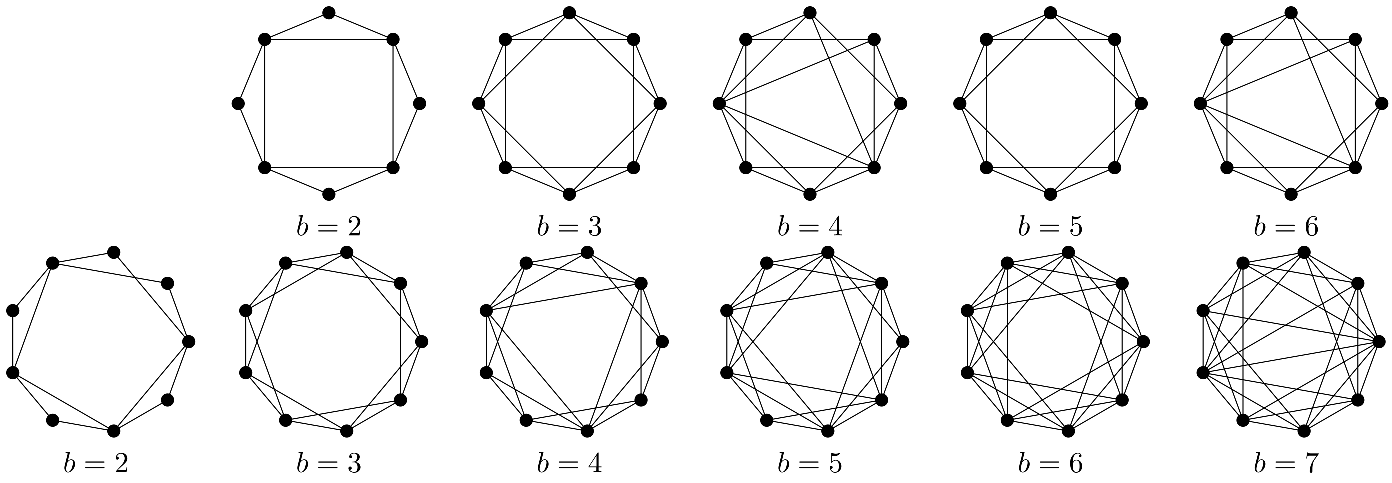


FIGURE 2. $G_{n,b}$ for $n = 8, 9$.

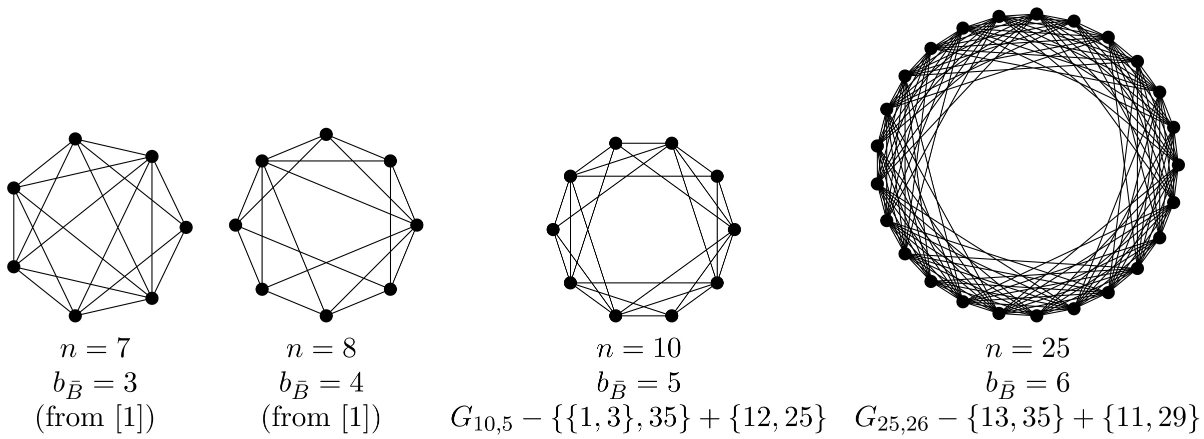


FIGURE 3. Exotic graphs for $n = 7, 8, 10, 25$.

It can be verified that *if we route each physical edge of* $G_{n,b}$ *by the shortest path, then for each edge* $e \in EC_n$*, there are exactly* b *lightpaths that contain* e*, i.e.* $|\{f \in EG_{n,b} : e \in \rho(f)\}| = b$.

4.2. **Exotic graphs.** The family $G_{n,b}$ is optimal for many values of the parameters, nevertheless, for some values it is not. In some of this cases we have found optimal graphs which are quite close to one in $G_{n,b}$. We show these ad hoc graph in Figure 3. We include those graph found in [1], for which the corresponding $G_{n,b}$ is not optimal.

4.3. **Optimality of $G_{n,b}$.** Let us remark that for each n, the bound $\check{C}(n, b_{\bar{B}})$ is attained by exactly one $G_{n,b}$. In fact, it is easy to compute b given $b_{\bar{B}}$: b will be $\check{C}(n, b_{\bar{B}})/n$, i.e.

$$b(b_{\bar{B}}) = \left\lceil \frac{\lceil n/2 \rceil \lfloor n/2 \rfloor}{b_{\bar{B}}} \right\rceil .$$

For instance, if $b_{\bar{B}} = \lceil n/2 \rceil \lfloor n/2 \rfloor$, i.e., the maximum demand between two set of nodes, then $b = 1$ and $G_{n,b}$ is the cycle C_n. In the other extreme, if n is odd and $b_{\bar{B}} = 2$, then $G_{n,b}$ is the complete graph K_n as predicted in [2].

Say this, the graphs $G_{n,b(b_{\bar{B}})}$ seems to be optimal for every n and $b_{\bar{B}}$ except for some exceptional values of the parameters. For instance,

$$(n, b_{\bar{B}}) \in \{(5,3), (7,3), (10,5), (12,3), (25,6), (29,2)\}.$$

For this exceptions, the graphs described in previous subsection are optimal for

$$(n, b_{\bar{B}}) \in \{(5,3), (7,3), (10,5), (25,6)\}.$$

The first two were found in [1] while the last two are modifications of the corresponding $G_{n,b}$ graphs. On the other hand for $(n, b_{\bar{B}}) \in \{(12,3), (29,2)\}$, we have not found graphs attained the bound $\check{C}$, but the graphs $G_{12,12} + 11$ and $G_{29,2} + 11$ are feasible and have cost $\check{C} + 1$, i.e., one unit more than the lower bound. It is worth to say that we have a proof, though not include in this work, that $G_{12,12}$ is not optimal, but we still have not the corresponding result for $G_{25,26}$.

Interestedly, if n is even and $b_{\bar{B}} = 3$, the feasible logical network introduced in [2] is the complete graph K_n "without diameters", i.e., $K_n^- = K_n - \{\{i, i+l\} : i \in \mathbb{Z}_n\}$, but $G_{n,\lceil \lceil n/2 \rceil \lfloor n/2 \rfloor \rceil /3}$ is that graph only for $n \leq 8$. Curiously, the graphs treated in [1] were for $n \leq 8$, so the authors and their advisors never noticed that difference. This induced them to think that the graphs K_n^- were optimal. This claim is not true, since $G_{n,b(3)}$ are optimal at least for $n = 10, 14, 16, 18, 20, 22, 24, 26, 28, 30$.

4.4. **Feasibility analysis.** We have check the feasibility of the solution by hand for $n = 5, \ldots, 10$. Let us remarks, that this verification is facilitated by the symmetry of the graphs, which implies that at most five edge faults need to be verified, and in many cases only one edge fault. Besides, although the problem of feasibility is NP-complete, we applies a polynomial heuristic algorithm which seems to works quite well. The algorithm is as follows. For each fault edge, e.g., $\{1, 2\}$, we try to find i paths from vertex $n - i$ to vertices $n - i + 1, n - i + 2, \ldots, n$. The algorithm solves a max flow problem using Ford-Fulkerson from $n-i$ to an auxiliary vertex v^* adjacent from $n-i+1, n-i+2, \ldots, n$. We assign capacity $b_{\bar{B}}$ to each edge of the graph, and capacity 1 to the new edges incidence to v^*. In practice we have used two different algorithm. The first one is the described previously, the second one tries to find a shortest path from vertex $n-i$ to vertices $n-i+1, n-i+2, \ldots, n$ "forward", i.e., without passing through vertices smaller than $n-i$. If that is not possible, then tries to find a shortest path using all vertices. Both algorithm works well for the majority of the cases, but the first does not work very well for large values of $b_{\bar{B}}$, while the second one does not work very well for small ones, so they are quite complementary. Nevertheless, it seems possible to add the characteristics of the second algorithm to the first one in order to have only one, but we haven't time to do it. It is worth to say that what we verify by hand was something similar to the second algorithm. Besides, the actual technology applied by real-word networks, use shortest path algorithms, so, the second algorithm seems to be more realistic. However, we have try the shortest path algorithm alone, but it doesn't work very well.

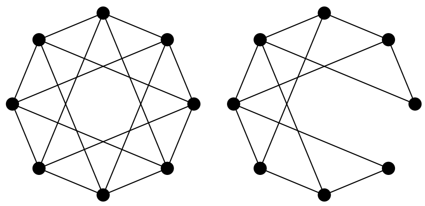
Table 2. Parameter for new optimal solutions for FRS-MORDP found in this work, and the missing ones

n	9	10	11	12	13	14	15	16	17	18	19
$b_{\bar{B}}$	2–20	3–25	2–30	4–36	2–42	3–49	2–56	3–64	2–72	3–81	2–90
% of $b_{\bar{B}}$	100%	100%	100%	97%	100%	100%	100%	100%	100%	100%	100%
n	20	21	22	23	24	25	26	27	28	29	30
$b_{\bar{B}}$	3–100	2–110	3–121	2–132	3–144	2–156	3–169	2–182	3–196	2–104,106–210	3–225
% of $b_{\bar{B}}$'s	100%	100%	100%	100%	100%	100%	100%	100%	100%	99.52%	100%

In Table 2, we present the values of the parameters for which we have found new optimal networks. For each n we give the maximum capacities for which an optimal solution was found. We compute the percent of the possible values of the maximum capacities. The reader can observe that for only two cases, $n = 12, b_{\bar{B}} = 3$ and $n = 29, b_{\bar{B}} = 105$, we have not found an optimum.

5. Counterexamples to Lemma 1's sufficiency

In [3], a counterexample for the sufficiency of Lemma 1's (considering bonds instead of edge-cuts) is given in an network with non full demands. During the thesis defense, it was mention that in practice the condition seems to be sufficient because no counterexample was found. In particular, for those feasible solutions found in [1]. Now we will describe a non feasible networks verifying inequality (1).



(a) (b)

Figure 4.

The counter example is for $n = 8$ and $b_{\bar{B}} = 4$. Let us consider as logical network the graph $G = Cayley(\mathbb{Z}_8, \{\pm 1, \pm 3\})$, i.e. $V(G) = \mathbb{Z}_8$ and $E(G) = \{\{i, i \pm j\} : i \in \mathbb{Z}_n, j \in \{1, 3\}\}$. It is shown in Figure 4 (a). We route the ligthpaths through the shortest path. There are four kind of bonds, those leaving 1, 2 3 or 4 vertices in one side and 7,6,5 and 4 in the other side respectively. Their corresponding demands are $1 \times 7 = 7$, $2 \times 6 = 12$, $3 \times 5 = 15$ and $4 \times 4 = 16$ which is the maximum. The size of the physical bonds is always 2, and the cardinal of b_L are, respectively, $4, 6, 8, 8$, thus, the inequality is verified, since $7 \leq 4 \times \lfloor 4/2 \rfloor = 8$, $12 \leq 4 \times \lfloor 6/2 \rfloor = 12$, $15 \leq 4 \times \lfloor 8/2 \rfloor = 16$, $16 \leq 4 \times \lfloor 8/2 \rfloor = 16$.

Now, let us prove that the solution is not feasible. Indeed, consider a physical link fault. It will leave the logical network as in Figure 4 (b). Let us prove that it is not possible to route the traffic in that case. First we prove that we need to route the traffic through the smallest paths. Indeed, the sum of the distance between pair of vertices is $(1+2+1+2+3+2+3)+(1+2+1+2+3+2)+(1+2+1+2+3)+(1+2+1+2)+(1+2+1)+(1+2)+1 = 48$, but we have 12 edges, thus, the capacity should be saturated in each edge when routing the demands. Finally, let v one of the vertices with degree two. It will use its incident edges 7 times, 4 one and 3 the other. This leaves 1 free capacity, which will not be used by any other vertex demand.

It is worth to say that there are many other examples. For instance, the graph $(\mathbb{Z}_8, \{\{i, i \pm j\} : i \in \mathbb{Z}_8, j \in \{2, 3\}\})$ is easy to verify it is a counter example. Other possible counterexamples are those $G_{n,b}$ corresponding to exceptional graph described in subsection 4.2, however we have not proved yet.

6. Acknowledgements

Some part of this work was done during the scientific visit of the first author to Facultad Politécnica, Universidad Nacional de Asunción, Paraguay.

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