On transitive expansive homeomorphisms of the plane.

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Abstract

We study the existence of transitive expansive homeomorphisms of $\mathbb{R}^2$. We prove that there are no (orientation preserving) transitive and uniformly expansive homeomorphisms of the plane.

1 Introduction

Expansiveness and transitivity are well known sources of chaotic behavior. Indeed any of these properties alone generates a complicated dynamical behavior when the ambient space is a compact manifold, see for instance [Le2, Ma, DPU, BDP, Hi, Vi]. For the non-compact case this is not necessarily true. For instance, a homothety H of center the origin O and ratio $r > 1$ defined on $\mathbb{R}^2$ is expansive but its dynamics is trivial, all points except the origin diverge to ∞ by forward iteration by H and converge to O by backward iteration. For the case of transitiveness the first thing is that it is not evident if it is possible to exhibit a transitive homeomorphism defined on the plane. The first to construct such examples seems to have been L.G. Shnirelman and later A.S. Besicovitch. In his article [Be1] Besicovitch built an example of a transitive homeomorphism of the plane. He proved that all forward orbits departing from a straight line are dense in $\mathbb{R}^2$ and conjectured that all points of the plane had this property. Later in [Be2] Besicovitch exhibited orbits of the same example that are not transitive disproving the conjecture. In fact there is a dense subset of $\mathbb{R}^2$ such that their forward orbits are not dense in $\mathbb{R}^2$ (see Lemma 2.6). It is shown in the book by Alpern and Prasad [AP] that examples like that of [Be1] are maximally chaotic in the sense of Alpern and Prasad (see [AP, Chapter 17]).

Mendes in [Me] studied Anosov diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ and proved that in that case $\Omega(f)$ is empty or just a single fixed point. Therefore such a diffeomorphism is expansive but not transitive.

Groisman has studied in [Gr1, Gr2] expansive homeomorphisms h of the plane giving conditions under which such a homeomorphism is conjugate to a linear hyperbolic transformation and in that case $\Omega(h)$ is a single fixed point or to a translation of the plane. In both cases the non wandering set is trivial.

The question arises whether it is possible for a homeomorphism to share both, expansiveness and transitivity. But if a homeomorphism has both properties at the same time we should expect rich dynamical properties even in non-compact spaces. We investigate if there exists a homeomorphism of $I\!R^2$ with both properties at the same time. The answer is negative if non trivial compact connected stable and unstable sets can be built in a neighborhood of the fixed point that necessarily exists due to Brouwer theory of homeomorphisms of the plane. This is the case when we ask for $f : I\!R^2 \to I\!R^2$ to be *uniformly* expansive and transitive (see definitions below).
Questions:

1. Are there transitive expansive homeomorphism defined on the plane (dropping the hypothesis of uniformity of expansiveness).

2. What can be said with respect to transitive expansive homeomorphisms defined on $I\!R^n$ for $n \geq 3$?

2 Lyapunov stable points.

Definition 2.1. *We say that the point $x \in I\!R^2$ is f-transitive if*

$$\overline{Orb(x)} = closure\big(\{f^n(x) \,:\, n \in Z\!\!Z\}\big) = I\!R^2 \,.$$

We say that x is positive (negative) f-transitive if the forward (resp.: backward) orbit by f is dense in $I\!R^2$, i.e., $\overline{Orb^+(x)} = I\!R^2$ (resp.: $\overline{Orb^-(x)} = I\!R^2$).

Remark 2.1. *Let $f : I\!R^2 \to I\!R^2$ be a homeomorphism that preserves orientation. Hence, since f is assumed to be transitive, from Brouwer theory of homeomorphisms preserving orientation of the plane it follows that there exists a fixed point for f. It is immediate that if there is a f-transitive point then there is a dense set of f-transitive points in $I\!R^2$.*

In the sequel we will omit to mention f when x is an f-transitive point. We will just say that it is a transitive point. Also we will say that f is transitive if it has a transitive point. The properties of transitive homeomorphisms $f : X \to X$ with X a compact manifold are well known. Moreover almost the same proofs are valid when X is a second countable complete metric space.

Lemma 2.2. *If x is transitive then either it is positive transitive or it is negative transitive. Moreover, if $\overline{Orb^-(x)} = I\!R^2$ then there is a residual subset $\mathcal{R}$ of $I\!R^2$ of positive transitive points, i.e., if $y \in \mathcal{R}$ then $\overline{Orb^+(y)} = I\!R^2$*

Proof. Let $x \in I\!R^2$ be such that $\overline{\mathrm{Orb}(x)} = I\!R^2$. Then there is a sequence $\{n_j\}_{j \in I\!N} \subset Z\!\!Z$ with $|n_j| \to +\infty$ such that $f^{n_j}(x) \to x$. Hence for every $k \in Z\!\!Z$ it holds that $f^{n_j+k}(x) \to f^k(x)$

when $j \to +\infty$. Either there is an infinite subsequence of $\{n_j\}$ of positive numbers or one of negative ones.

In the first case, assuming that $\{n_j\}$ itself is of positive numbers in order not to complicate notation, we obtain on account of the density of the x orbit, that given $z \in I\!R^2$ and $\epsilon > 0$ there is $k \in Z\!\!Z$ such that $\mathrm{dist}(f^k(x), z) < \epsilon/2$. Since $f^{n_j+k}(x) \to f^k(x)$ there is n_j big enough such that $\mathrm{dist}(f^{n_j+k}(x), f^k(x)) < \epsilon/2$ and moreover $n_j + k > 0$. It follows that $\mathrm{dist}(f^{n_j+k}(x), z) < \epsilon$ and hence $\overline{\mathrm{Orb}^+(x)} = I\!R^2$.

In the second case we have that there are infinitely many n_j which are negative and we may conclude that $\overline{\mathrm{Orb}^-(x)} = I\!R^2$ and so the same holds for $f^h(x)$ for any $h \in Z\!\!Z$. It follows that given any U and V non empty open subsets of $I\!R^2$ there is $n \in I\!N$, depending on U and V, such that $f^{-n}(V) \cap U \neq \emptyset$ and moreover $\bigcup_{n=1}^{\infty} f^{-n}(V)$ is open dense in $I\!R^2$. Taking a denumerable basis $\{V_k\}_{k \in I\!N}$ of open subsets of $I\!R^2$ we have by Baire Theorem that

$$\mathcal{R} = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} f^{-n}(V_k)$$

is a residual subset of $I\!R^2$ such that $y \in \mathcal{R}$ implies that $\overline{\mathrm{Orb}^+(y)} = I\!R^2$.

□

Corollary 2.3. *There are residual subsets of $I\!R^2$, $\mathcal{R}$, $\mathcal{R}_+$ and $\mathcal{R}_-$ such that points in $\mathcal{R}_+$ are positive transitive, points in $\mathcal{R}_-$ are negative transitive and points in $\mathcal{R} = \mathcal{R}_+ \bigcap \mathcal{R}_-$ have their positive and negative orbits dense.*

Proof. Follows from Lemma 2.2 and the fact that the intersection of residual sets is itself residual. Indeed, given a transitive orbit we find by Lemma 2.2 that it is positive transitive or negative transitive. In the second case there is a residual subset of positive transitive orbits of $I\!R^2$. Having a positive transitive orbit we have that given any U and V non empty open subsets of $I\!R^2$ there is $n \in I\!N$, depending on U and V, such that $f^n(V) \cap U \neq \emptyset$ and moreover $\bigcup_{n=1}^{\infty} f^n(V)$ is open dense in $I\!R^2$. From which we may find a residual subset of negative transitive orbits. □

Lemma 2.4. *Let V be an open subset of $I\!R^2$ such that $\overline{V}$ is compact and $f : I\!R^2 \to I\!R^2$ a transitive expansive homeomorphism. Then it holds that for all $n > 0$: $f^n(\overline{V}) \setminus \bigcup_{j=0}^{n-1} f^j(\overline{V}) \neq \emptyset$.*

Proof. If not there is $n_0 \in I\!N^+$ such that $f^{n_0}(\overline{V}) \subset \bigcup_{j=0}^{n_0-1} f^j(\overline{V})$. Since f is a homeomorphism we have

$$f^{n_0+1}(\overline{V}) \subset f\Big(\bigcup_{j=0}^{n_0-1} f^j(\overline{V})\Big) = \bigcup_{j=1}^{n_0} f^j(\overline{V}) \subset \bigcup_{j=0}^{n_0-1} f^j(\overline{V}) \, .$$

It follows that for all $n \in I\!N$: $f^n(\overline{V}) \subset \bigcup_{j=0}^{n_0-1} f^j(\overline{V})$ and therefore

$$\bigcup_{n\in I\!N} f^n(\overline{V}) \subset \bigcup_{j=0}^{n-1} f^j(\overline{V}).$$

But since $\overline{V}$ is compact we have that $\bigcup_{j=0}^{n-1} f^j(\overline{V})$ is compact. On the other hand since there is a residual subset of points in $\overline{V}$ which are forward transitive it holds that $\overline{\bigcup_{n\in I\!N} f^n(\overline{V})} = I\!R^2$ which is absurd. □

Remark 2.5. *Observe that Lemma 2.4 is valid in any* non compact *complete metric space provided that the hypothesis of the lemma about transitivity holds.*

Lemma 2.6. *Let $N(f) = I\!R^2 \backslash \mathcal{R}_+$. Then $N(f)$ is everywhere dense in $I\!R^2$.*

Proof. Assume that the result is false. Then there exists a point $q \in I\!R^2$ and $U(q)$ an open neighborhood of q such that $\forall\, x \in U(q) : x \in \mathcal{R}_+$. Since $I\!R^2$ is locally compact there exists V open neighborhood of q such that $\overline{V} \subset U(q)$ and $\overline{V}$ is a compact subset of $I\!R^2$.

Consider the sequence of subsets of $I\!R^2$, $\{X_n\}_{n\in I\!N^+}$ given by

$$X_n = f^{-n}\big(f^n(\overline{V}) \backslash \bigcup_{j=0}^{n-1} f^j(\overline{V})\big).$$

By Lemma 2.4, $X_n \neq \emptyset$ and moreover we have

$$X_1 \supset X_2 \supset \cdots \supset X_n \supset X_{n+1} \supset \cdots.$$

If it were not true there exists $z \in X_{n+1} \backslash X_n$ that is

$$z \notin f^{-n}\big(f^n(\overline{V}) \backslash \bigcup_{j=0}^{n-1} f^j(\overline{V})\big) \quad \text{but} \quad z \in f^{-n-1}\big(f^{n+1}(\overline{V}) \backslash \bigcup_{j=0}^{n} f^j(\overline{V})\big).$$

Therefore, since $z \in \overline{V} \Longrightarrow f^n(z) \in f^n(\overline{V})$, from $f^n(z) \notin f^n(\overline{V}) \backslash \bigcup_{j=0}^{n-1} f^j(\overline{V})$ we conclude that $f^n(z) \in \bigcup_{j=0}^{n-1} f^j(\overline{V})$ and so there is $0 \leq k < n$ such that $f^n(z) \in f^k(\overline{V})$ from which $f^{n+1}(z) \in f^{k+1}(\overline{V})$ from which $f^{n+1}(z) \notin f^{n+1}(\overline{V}) \backslash \bigcup_{j=0}^{n} f^j(\overline{V})$ which is absurd.

Let us now consider a sequence $\{x_n\}_{n\in I\!N}$ such that $x_n \in X_n$. Since for all $n \in I\!N$: $X_n \subset \overline{V}$ which is compact, we have that there is a convergent subsequence of $\{x_n\}_{n\in I\!N}$. Without loss of generality let us assume that $x_n \to p$. By continuity of f^m we obtain that $f^m(x_n) \to f^m(p)$. Hence, since $X_n \subset X_m$ if $n \geq m$: $f^m(x_n) \in f^m(X_n) \subset f^m(X_m)$.

Observe that since

$$f^m(X_m)\bigcap\overline{V} = \left(f^m(\overline{V})\setminus\bigcup_{j=0}^{m-1} f^j(\overline{V})\right)\bigcap\overline{V}$$

we have that $f^m(X_m)\bigcap\overline{V} = \emptyset$ and so $f^m(x_n) \notin \overline{V}$ for every $1 \leq m \leq n$. Taking limits with $n \to \infty$ we obtain that $f^m(p) \notin \text{int}(\overline{V}) = V$ for all $m \in I\!N^+$. It follows that $\overline{\text{Orb}^+(p)} \neq I\!R^2$ which is absurd since $\overline{V} \subset \mathcal{R}_+$.

Thus for every non empty open set $U \subset I\!R^2$ there is a point $w \in N(f)$ and so we can conclude that $N(f)$ is dense in $I\!R^2$ finishing the proof of Lemma 2.6. □

Definition 2.2. *We say that x is a Lyapunov stable point for f if given $\epsilon > 0$ there is $\delta > 0$ such that if* $\text{dist}(x, y) < \delta$ *then* $\forall\, n \geq 0$ *it holds that* $\text{dist}(f^n(x), f^n(y)) < \epsilon$.

In a similar way we define Lyapunov stable points for f^{-1}, sometimes called totally unstable points.

In the sequel we prove that if $f : I\!R^2 \to I\!R^2$ is transitive and expansive then there are no Lyapunov stable points neither for f nor for f^{-1}.

Definition 2.3. *We define the distance between two continuous functions f, g defined in $I\!R^2$ as $\mathcal{D} : C(I\!R^2, I\!R^2) \times C(I\!R^2, I\!R^2) \to I\!R$*

$$\mathcal{D}(f, g) = \sup\{\min\{\text{dist}(f(x), g(x)), 1\}, x \in I\!R^2\}$$

It is easy to see that $\mathcal{D}$ is well defined.

Definition 2.4. *Let X be a complete metric space. We say that a homeomorphism $g : X \to X$ is recurrent if*

$$\liminf_{n\to+\infty} \mathcal{D}(g^n, id) = 0 \quad \textit{where } id : X \to X \textit{ is the identity map}.$$

A homeomorphisms $g : X \to X$ is periodic if there exist $p \geq 1$ such that $g^p = id$. Clearly a periodic homeomorphism is recurrent.

In general recurrent homeomorphisms need not to be periodic; consider for instance an irrational rotation in S^1. But in the case of $I\!R^2$ it is true that a recurrent homeomorphism is periodic as has been proved by Oversteegen and Tymchatyn, [OT].

Theorem 2.7. *Let $f : I\!R^2 \to I\!R^2$ be a recurrent homeomorphisms of the plane. Then f is periodic.*

Proof. See [OT, Theorem 3]. □

Using the previous result we can prove:

Theorem 2.8. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a transitive expansive homeomorphism. Then there are no Lyapunov stable points for f nor for f^{-1}.*

Proof. Arguing by contradiction assume that $x \in \mathbb{R}^2$ is a Lyapunov stable point for the homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$. Let also $y \in \mathbb{R}^2$ be a point with a transitive forward orbit (i.e., y is positive transitive). Let $\epsilon > 0$ and $\delta > 0$ be the corresponding number such that if $\text{dist}(x, x') < \delta$ then $\forall\, n \geq 0$ it holds that $\text{dist}(f^n(x), f^n(x')) < \epsilon/2$. Since y is positive transitive there is a first iterate $k > 0$ such that $f^k(y) \in B(x, \delta/2)$. Moreover, there is $\delta_1 > 0$ such that the diameter of $f^j(B(y, \delta_1))$ is less than $\delta/2$ for every $j = 0, 1, \ldots, k$. It follows, since x is Lyapunov stable, that y is Lyapunov stable. Indeed, if $\text{dist}(y, y') < \delta_1$ then $\text{dist}(f^n(y), f^n(y')) < \epsilon$ for all $n \geq 0$.

Since y is both positive transitive and Lyapunov stable we have that given $\varepsilon > 0$ there are $\delta_2 > 0$ and $N_0 > 0$ such that $\text{dist}(f^{N_0}(y), y) < \delta_2$ and consequently $\text{dist}(f^j(f^{N_0}(y)), f^j(y)) < \varepsilon$ for all $j \geq 0$. Since $\text{Orb}^+(y) = \mathbb{R}^2$ for any $z \in \mathbb{R}^2$ there is $n_i \to \infty$ such that $f^{n_i}(y) \to z$ and hence $f^{n_i+N_0}(y) \to f^{N_0}(z)$. Since for any $j \geq 0$, $\text{dist}(f^j(f^{N_0}(y)), f^j(y)) < \varepsilon$ we conclude that $\text{dist}(f^{N_0}(z), z) \leq \varepsilon$. Since z is arbitrarily chosen in $\mathbb{R}^2$ we have that $\mathcal{D}(f^{N_0}, id) \leq \varepsilon$. Choosing a sequence of positive numbers $\varepsilon_i \to 0$ when $i \to +\infty$ we conclude that f is recurrent. By Theorem 2.7 we have that there is $p > 1$ such that $f^p = id$ from which f cannot be expansive since given any $\epsilon > 0$ we may find two points v, w such that $\text{dist}(f^j(v), f^j(w)) < \epsilon$ for all $j = 0, 1, \ldots, p-1$ from which, since $f^p = id$ we obtain that $\text{dist}(f^j(v), f^j(w)) < \epsilon$ for every $j \in \mathbb{Z}$. This concludes the proof of Theorem 2.8. □

3 Main section.

Definition 3.1. *Let $\mathbb{R}^2$ be equipped with the usual Euclidean metric. We say that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an expansive homeomorphism, i.e., there is $\alpha > 0$ such that given $x, y \in \mathbb{R}^2$, $x \neq y$, there is $n \in \mathbb{Z}$ such that $\text{dist}(f^n(x), f^n(y)) > \alpha$. We say that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is infinite expansive if any $\alpha > 0$ is a constant of expansiveness for f.*

We have defined expansiveness for a homeomorphism of $\mathbb{R}^2$ but it is clear that we can define it for a homeomorphism defined in any metric space. Moreover although defined by a metric the concept of expansiveness depends only on the topology of the ambient space.

Lemma 3.1. *Let (X, d) be a compact metric space. If $h : X \to X$ is expansive then given any pair of points $x, y \in X$ with $d(x, y) \geq \alpha/2$ there is $N > 0$ such that $d(h^j(x), h^j(y)) \geq \alpha$ for some $j \in \mathbb{Z}$ with $|j| \leq N$.*

Proof. Otherwise there is a pair of sequences $\{x_k\}$, $\{y_k\}$ such that $d(x_k, y_k) \geq \alpha/2$ and $d(h^j(x_k), h^j(y_k)) < \alpha$ for all $j \in [-k, k]$. By compactness of X we may take converging subsequences $\{x_{k_l}\}$ and $\{y_{k_l}\}$ to different points x_∞ and y_∞. Hence $d(h^n(x_\infty), h^n(y_\infty)) < \alpha$ for every $n \in \mathbb{Z}$ contradicting expansiveness (because $x_\infty \neq y_\infty$). $\square$

This result is the base for the construction of a hyperbolic metric due to Fathi (see [Ft]) adapted to expansive homeomorphisms defined on compact metric spaces. Unfortunately Lemma 3.1 is not valid when the space X is not compact.

Counterexample: Consider a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(x,y) = r\big(\sqrt{x^2+y^2}\big) \cdot (x,y)$ where $r : \mathbb{R}^+ \to \mathbb{R}^+$ is given by the following continuous function

$$
\begin{aligned}
&r(t) = 2t \quad \text{for} \quad t \in [0,1] \\
&r(t) = 2 + (e^{1/2}-2)(t-1) \quad \text{for} \quad t \in [1,2] \\
&r(t) = e^{1/2} + \tfrac{e^{1/3}-e^{1/2}}{2e^{1/2}-2}(t-2) \quad \text{for} \quad t \in [2, 2e^{1/2}] \\
&r(t) = e^{1/3} + \tfrac{e^{1/4}-e^{1/3}}{2e^{1/2}(e^{1/3}-1)}(t-2e^{1/2}) \quad \text{for} \quad t \in [2e^{1/2}, 2e^{1/2}e^{1/3}] \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

For $t \geq 1$ we divide $\mathbb{R}^+$ in intervals $[2e^{1/2}e^{1/3}\cdots e^{1/(n-1)}, 2e^{1/2}e^{1/3}\cdots e^{1/(n-1)}e^{1/n}]$ where we define

$$
r(t) = e^{1/n} + \frac{e^{1/(n+1)} - e^{1/n}}{2e^{1/2}e^{1/3}\cdots e^{1/(n-1)}(e^{1/n}-1)}\big(t - 2e^{1/2}e^{1/3}\cdots e^{1/(n-1)}\big)\,.
$$

If now we have two points p and q which are in different rays from the origin then the distance between $h^n(p)$ and $h^n(q)$ tends to infinity when $n \to +\infty$. This follows from the fact that for any $n_0 > 0$, $\prod_{j=n_0}^{n} e^{1/j} \to +\infty$ when $n \to +\infty$. On the other hand, if p and q belong to the same ray from the origin we may assume that they are in Ox^+ and (iterating by h if it were necessary) that they belong to the interval $[1,2]$ and are at a distance $\Delta x > 0$ (if $h^{n_0}(p)$ belongs to $[1,2]$ but $h^{n_0}(q) \notin [1,2]$ the argument is similar). Hence $\operatorname{dist}(h^n(p), h^n(q)) \geq 2\prod_{j=2}^{n} e^{1/j}\Delta x \to +\infty$ when $n \to +\infty$. Thus h is (positively) infinite expansive.

But given any $\alpha > 0$ and $N > 0$ we may find points $p, q \in Ox^+$ at a distance $\alpha/2$ such that $\operatorname{dist}(h^n(p), h^n(q)) \leq \alpha$ for $1 \leq n \leq N$. This follows form the fact that

$$
\prod_{j=n_0}^{n_0+N} e^{1/j} = e^{\sum_{j=n_0}^{n_0+N} 1/j} < 2 \Longleftrightarrow \sum_{j=n_0}^{n_0+N} \frac{1}{j} < \ln(2)\,.
$$

The last inequality holds for arbitrarily large N provided that n_0 is great enough. Thus Lemma 3.1 is not valid in our setting.

For this reason we introduce a new concept named *uniform expansiveness* (see [Se]).

Definition 3.2. *Let (X, d) be a complete metric space and $h : X \to X$ be an expansive homeomorphism with $\alpha > 0$ a constant of expansiveness. We say that h is uniformly expansive if given $\epsilon > 0$ with $\epsilon \leq \alpha$ there is $N \in \mathbb{N}$ such that for any pair of points $x, y \in X$ with $d(x, y) \geq \epsilon$ there is $j \in \mathbb{Z}$ such that $d(h^j(x), h^j(y)) \geq \alpha$ and $|j| \leq N$.*

Clearly Lemma 3.1 implies that any expansive homeomorphism defined on a compact metric space X is automatically uniform expansive. On the other hand infinite expansiveness is clearly metric dependent. In fact it is well known that we can define a new metric in $\mathbb{R}^2$ such that it becomes bounded with respect to this new metric.

3.1 Lyapunov function

Now we wish to construct a new metric D for $\mathbb{R}^2$ defining the topology of the plane, such that f is hyperbolic with respect to this new metric. Since we are not in a compact metric space we need the extra hypothesis of uniform expansiveness to repeat arguments of Fathi in order to built this metric. Following Fathi, see [Ft, Section 5] let us define $n(x, y)$, x and y in $\mathbb{R}^2$ as follows. For every element $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ define

$$\begin{cases} n(x,y) = +\infty & \text{if } x = y \\ n(x,y) = \min\left\{n_0 \in \mathbb{N} \,:\, \max_{|j|\leq n_0} \operatorname{dist}(f^j(x), f^j(y)) > \alpha\right\} & \text{if } x \neq y\,. \end{cases}$$

Clearly $n(x, y) = n(y, x)$. Let $\beta > 1$ and define $\rho(x, y) := \beta^{-n(x,y)}$. Then it follows that $\rho(x, y) = \rho(y, x)$ and $\rho(x, y) = 0$ if and only if $x = y$. Since f is expansive $\rho(x, y)$ defines the same topology of the plane as the usual Riemannian metric, although ρ is not a metric. If we have that

$$\max_{|j|\leq n-1} \rho(f^j(x), f^j(y)) \leq \frac{1}{\beta}$$

then $\max\{\rho(f^n(x), f^n(y)), \rho(f^{-n}(x), f^{-n}(y))\} \geq \beta^n \rho(x, y)$. By uniform expansiveness the following property holds **There is $\alpha > 0$ an expansivity constant such that for α there is $m > 0$ such that if $x, y \in \mathbb{R}^2$ and $\operatorname{dist}(x, y) \geq \alpha/2$ then $n(x, y) \leq m$.** Let $x, y, z \in \mathbb{R}^2$. We have by definition of $n(x, y)$ that $\operatorname{dist}(f^{n(x,y)}(x), f^{n(x,y)}(y)) > \alpha$ or $\operatorname{dist}(f^{-n(x,y)}(x), f^{-n(x,y)}(y)) > \alpha$. To fix ideas suppose that the former case occurs. Then $\operatorname{dist}(f^{n(x,y)}(x), f^{n(x,y)}(z)) > \alpha/2$ or $\operatorname{dist}(f^{n(x,y)}(z), f^{n(x,y)}(y)) > \alpha/2$. In the first case we have that $n(x, z) \leq n(x, y) + m$ while in the second case we have $n(y, z) \leq n(x, y) + m$. Therefore we have that $\beta^{-n(x,y)} \leq \beta^m \cdot \beta^{-n(x,z)} \implies \rho(x, y) \leq \beta^m \cdot \rho(x, z)$ or $\beta^{-n(x,y)} \leq \beta^m \cdot \beta^{-n(y,z)} \implies \rho(x, y) \leq \beta^m \cdot \rho(y, z)$. Let us choose $\beta > 1$ such that $\beta^m \leq 2$. We obtain that

$$\forall\, x, y, z \in \mathbb{R}^2 \,:\, \rho(x, y) \leq 2 \cdot \max\{\rho(x, z),\, \rho(y, z)\}\,. \tag{1}$$

Hence we may apply Frinks metrization theorem to find a metric $\delta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ from ρ.

Since $\delta(x,y) \le \rho(x,y) \le 4\cdot\delta(x,y)$ if $\max_{j\le n-1}\delta(f^j(x),f^j(y)) \le \frac{1}{4\beta}$ then $\max\{\delta(f^n(x),f^n(y)),\delta(f^{-n}(x),f^{-n}(y))\} \ge \frac{\beta^n}{4}\delta(x,y)$. Let $n_0>0$ such that $\frac{\beta^{n_0}}{4}>1$ and let $\lambda = \left(\frac{\beta^n}{4}\right)^{1/n_0}$. We define a new metric $D : I\!\!R^2 \to I\!\!R$ by

$$D(x,y) = \max_{-n_0+1\le j\le n_0-1} \frac{\delta(f^j(x),f^j(y))}{\lambda^{|j|}}$$

As in [Ft, Section 5] we may conclude that if $D(x,y) \le \frac{1}{4\beta\lambda^{n_0-1}}$ then

$$\max\{D(f(x),f(y)),D(f^{-1}(x),f^{-1}(y))\} \ge \lambda D(x,y)\,.$$

Now we find a new difference with respect to the compact case studied in [Ft].
If $D(x,y) > \frac{1}{4\beta\lambda^{n_0-1}}$ in the compact case we may conclude that there is $\epsilon>0$ such that $\max\{D(f(x),f(y)),D(f^{-1}(x),f^{-1}(y))\} > \epsilon$. But in the non-compact case this not need to be true. On the other hand we do have that for a given $R>0$ if $\|x\|\le R$, $\|y\|\le R$, and $D(x,y) > \frac{1}{4\beta\lambda^{n_0-1}}$ then $\max\{D(f(x),f(y)),D(f^{-1}(x),f^{-1}(y))\} > \epsilon(R) > 0$. Observe that we are using both metrics at the same time, the standard Riemannian one, dist and the metric D. Clearly $\epsilon(R)$ is a non-increasing function depending on R. If it were the case that $\epsilon(R)$ is bounded away from zero then we can procceed as in [Ft]. In any case we can work in a neighborhood of radius R (with respect to the Riemannian metric of the plane) of p the f-fixed point that we will identify with the origin of coordinates O of the plane.

We will work with the distance D till new advise, the only reference to the Riemannian metric will be the norm $\|\cdot\|$. So when we speak of the ϵ-local stable set of a point x we are thinking in the set $W^s_\epsilon(x) = \{y \in I\!\!R^2 \,:\, D(f^n(x),f^n(y)) \le \epsilon\}$. Similarly with respect to the local unstable sets.

Let us find a Lyapunov function for f, in the sense of [Le1], from Fathi's metric. Since $\lambda>1$ there is $n_0>0$ such that $\lambda^{n_0}\ge 2$. We define $U(x,y)$ in a neighborhood N of the diagonal of $I\!\!R^2\times I\!\!R^2$ such that if $(x,y)\in N$ then

$$\max\{D(f^n(x),f^n(y)),D(f^{-n}(x),f^{-n}(y))\} \ge \lambda^n D(x,y) \quad \text{for } \ 0\le n\le 2n_0\,.$$

Then we define

$$U(x,y) = \sum_{i=0}^{n_0-1}\sum_{j=0}^{n_0-1} D(f^{i+j}(x),f^{i+j}(y))\,.$$

Observe that U is a distance defined on N such that $V(x,y) = \Delta U(x,y) = U(f(x),f(y)) - U(x,y)$ is

$$V(x,y) = \Delta U(x,y) = \sum_{i=0}^{n_0-1}\left[D(f^{i+n_0}(x),f^{i+n_0}(y)) - D(f^i(x),f^i(y))\right]$$

and the second difference of U, $W(x,y) = \Delta^2 U(x,y) = \Delta V(x,y) = U(f^2(x), f^2(y)) - 2U(f(x), f(y)) + U(x,y)$ has the property that

$$W(x,y) = D(f^{2n_0}(x), f^{2n_0}(y)) - 2D(f^{n_0}(x), f^{n_0}(y)) + D(x,y)$$

is positive.

Indeed, either $D(f^{2n_0}(x), f^{2n_0}(y)) \geq 2D(f^{n_0}(x), f^{n_0}(y))$ or $D(x,y) \geq 2D(f^{n_0}(x), f^{n_0}(y))$ from which the result follows. Thus $U(x,y)$ is a positive Lyapunov function for f defined in the neighborhood N of the diagonal of $I\!R^2 \times I\!R^2$ such that its second difference is positive too.

3.2 Construction of stable and unstable sets

Let x be a birrecurrent point or more generally any point x such that $\alpha(x) \neq \emptyset$ and $\omega(x) \neq \emptyset$. We assume that $\|x\| < R$. Let also $\{n_k\}_{k \in Z\!\!Z}$ be a sequence of integers such that $n_k \to +\infty$ when $k \to +\infty$, $n_k \to -\infty$ when $k \to -\infty$ and the limits of $f^{n_k}(x)$ when $k \to +\infty$ and $k \to -\infty$ exist, say $\lim_{k \to +\infty} f^{n_k}(x) = z$ and $\lim_{k \to -\infty} f^{n_k}(x) = w$. Let $2\sigma > 0$ be a constant of expansivity for f with respect to the distance U such that if $U(u,v) \leq \sigma$ then $\|u - v\| \leq \varepsilon$. We also denote by $B_U(u,r)$ the connected component containing u of $\{v \in I\!R^2 \,|\, U(u,v) < r\}$.

Theorem 3.2. *For such an x as above $W^s_\sigma(x)$ and $W^u_\sigma(x)$ contain nontrivial compact connected subsets $C(x) \subset W^s_\sigma(x)$ and $D(x) \subset W^u_\sigma(x)$ such that $C(x) \cap D(x) = \{x\}$. Here the metric involved is that given by U.*

Proof. For a given $\delta > 0$ such that $\delta < \sigma$ there is $n_k > 0$ and $y_k \in B_U(f^{n_k}(x), \delta)$ such that for some l_k with $0 \leq l_k < n_k$ it holds that $U(f^{l_k}(x), f^{-n_k+l_k}(y_k)) \geq \sigma$ and $U(f^n(x), f^{-n_k+n}(y_k)) < \sigma$ for every $n \in [l_k+1, n_k]$. For, if it were not true that such an l_k exists then for all $k > 0$ and $0 \leq n \leq n_k$ and for all $y \in B_U(f^{n_k}(x), \delta)$ it holds that $U(f^n(x), f^{-n_k+n}(y)) < \sigma$. Letting $k \to \infty$ we find that z is a Lyapunov stable point for f^{-1} contradicting that there are no Lyapunov stable points (Theorem 2.8). Thus there is l_k such that $U(f^{l_k}(x), f^{-n_k+l_k}(y_k)) \geq \sigma$. Choose an arc $\gamma = \gamma_k : [0,1] \to I\!R^2$ joining $f^{n_k}(x)$ with y_k contained in $B_U(f^{n_k}(x), \delta)$ and let

$$t_k = \sup\{t \in [0,1] \,|\, U(f^n(x), f^{-n_k+n}(\gamma(t))) \leq \sigma, \forall\, n \in [l_k, n_k]\}\,.$$

It follows by continuity that $U(f^{l_k}(x), f^{-n_k+l_k}(\gamma(t_k))) = \sigma$. Since $\Delta^2 U$ is positive and $U(f^{l_k+1}(x), f^{-n_k+l_k+1}(\gamma(t_k))) < \sigma$ we have that $U(f^{l_k-1}(x), f^{-n_k+l_k-1}(\gamma(t_k))) > \sigma$. So that for every $n \in [0, l_k]$ there is $t_{n,k} \in [0, t_k]$ such that $U(f^n(x), f^{-n_k+n}(\gamma(t_{n,k}))) = \sigma$ while $U(f^l(x), f^{-n_k+l}(\gamma(t_{n,k}))) < \sigma$ for all $l \in [n+1, n_k]$. Renaming the points we may assume that $y_k = \gamma(t_{0,k})$. With this change $U(f^n(x), f^{-n_k+n}(y_k)) \leq \sigma$ for all $0 \leq n \leq n_k$ and moreover for all $t \in [0, t_{0,k}]$, $U(f^n(x), f^{-n_k+n}(\gamma(t))) \leq \sigma$ for all $0 \leq n \leq n_k$.

Let $\delta_k > 0$ and $\delta_k \to 0$ when $k \to +\infty$. For such k we find k_1 depending on k such that for some $y_k \in B_U(f^{n_{k_1}}(x), \delta_k)$ there is l_k with $0 \leq l_k < n_{k_1}$ such that it holds that $U(f^{l_k}(x), f^{-n_{k_1}+l_k}(y_k)) \geq \sigma$ and $U(f^n(x), f^{-n_{k_1}+n}(y_k)) < \sigma$ for every $n \in [l_k + 1, n_{k_1}]$. Deleting some values of n_k if it were necessary and renaming the values of the subsequence $\{n_k\}$ we may assume that $n_{k_1} = n_k$ simplifying the notation. For every k there is defined $\gamma = \gamma_k$ such that

$$U(f^n(x), f^{-n_k+n}(y_k)) \leq \sigma \ \text{ for all } \ 0 \leq n \leq n_k \ \text{ and}$$

$$\forall\, t \in [0, t_{0,k}] : U(f^n(x), f^{-n_k+n}(\gamma(t))) \leq \sigma \forall\, 0 \leq n \leq n_k\,.$$

Letting $k \to +\infty$ and the Hausdorff limit of a converging subsequence of $\{\gamma_k\}$ we find a continuum (compact connected set) $C(x)$ joining x with $\partial B_U(x, \sigma) = \{z \in I\!\!R^2 : U(x, z) = \sigma\}$ such that for all $y \in C(x)$ and $n \geq 0$ we have $U(f^n(x), f^n(y)) \leq \sigma$ that is $C(x) \subset W^s_\sigma(x)$.

In a similar way, since x is a birrecurrent point, we may construct $D(x) \subset W^u_\sigma(x)$ a non trivial continuum joining $\partial B_U(x, \sigma)$ with x and such that for all $y \in D(x)$ and every $n \leq 0$ we have $U(f^n(x), f^n(y)) \leq \sigma$.

Since 2σ is a constant of expansivity for f with respect to U we also have that $C(x) \cap D(x) = \{x\}$. This finishes the proof of Theorem 3.2. □

Corollary 3.3. *For every point $z \in I\!\!R^2$ there are non trivial continua $C(z) \subset W^s_\sigma(z)$ and $D(z) \subset W^u_\sigma(z)$ of diameter bounded away from zero such that $C(z) \cap D(z) = \{z\}$.*

Proof. Since birrecurrent points are dense in $I\!\!R^2$ we can take the Hausdorff limit of convergent subsequences of $C(x_n)$ and $D(x_n)$ whit x_n birrecurrent and $x_n \to z$ when $n \to +\infty$. □

Now, as far as the point x has norm $\|x\| \leq R$ we have that if $C(x)$ and $D(x)$ have diameter σ with respect to Fathi's metric, then we also have that its diameter is greater than $\epsilon > 0$ with respect to the Euclidean metric. Thus from this point we may return to the usual metric of $I\!\!R^2$.

Remark 3.4.

- *Since we have stable and unstable sets, as in [Le2], we can prove that $C(x)$ and $D(x)$ are locally connected and in particular arc-wise connected. Moreover, the local stable and unstable sets built for the fixed point p are a finite union of arcs intersecting at p.*
- *In the same way as in [Le2] there is an open dense subset $\mathcal{A}$ of points of $I\!\!R^2$ with local product structure. That is, for any point $x \in \mathcal{A}$ there is a neighborhood $N(x)$ such that if $y, z \in N(x)$ then $C(y) \cap D(z) \cap N(x) \neq \emptyset$.*
- *If S (resp.: U) is a stable (resp.: unstable) prompt of the fixed point then $S \backslash \{p\} \subset \mathcal{A}$ (resp.: $U \backslash \{p\} \subset \mathcal{A}$). This follows from the fact that singular points cannot accumulate.*

3.3 Main Theorem

Lemma 3.5. *Let p be a fixed point of f and U be a prompt of $W^u(p)$. Then U intersects $W^s(p)$ in a point $y \neq p$.*

Proof. Since bi-recurrent points are dense, there is one of them contained in $\mathcal{A}$ and sufficiently near U such that $C(x) \cap U = \{z\}$. Let S be a prompt of $W^s(p)$ such that S and U are consecutive in the sense that there are no other prompts between them. Since the forward orbit of x is dense there is some positive iterate $f^N(x)$ arbitrarily close to S such that $D(f^N(x))$ cuts S. Since the distance between $f^N(z)$ and $f^N(x)$ tends to zero when $n \to \infty$ we also obtain that $D(f^N(z)) \cap S \neq \emptyset$. Thus U intersects S.

□

Theorem 3.6. *There are no preserving orientation uniform expansive and transitive homeomorphisms of the plane.*

Proof. Assume that such a homeomorphism f exists. Let p be a fixed point of f. Let U be a prompt of $W^u(p)$ and S be a prompt of $W^s(p)$ Consider that these prompts are consecutive in the sense that there are not other prompts between them. By Lemma 3.5 there exists a point $y \neq p$ of intersection between U and S. Let then consider the Jordan curve J determined by an arc S^* of S and an arc U^* of U such that $S^* \cap U^* = \{p, y\}$.

By Remark 3.4 there are not singular points belonging to this Jordan curve since singular points cannot accumulate (see [Le2]). Also, by the same reason there is only a finite number of singularities in the bounded connected component determined by J (see Fig. 1).

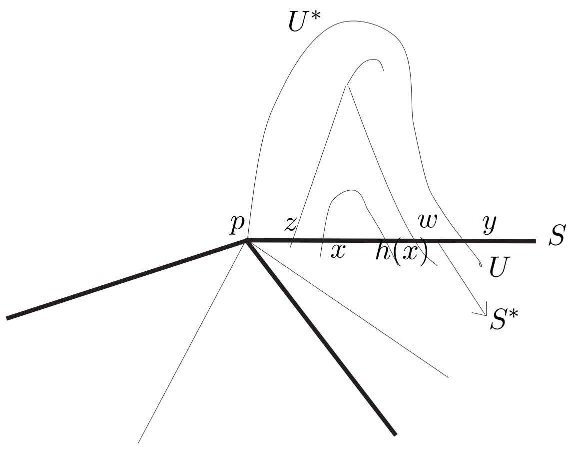

Figure 1:

Thus we can consider an arc $\widetilde{zw} \subset S^*$ such that for all $x \in \widetilde{zw}$, $W^u(x)$ does not belong to any prompt of a singularity. Let us consider the function h defined on $\widetilde{zw}$ sending any point $x \in \widetilde{zw}$ to the first cut of the unstable set of x with $\widetilde{zw}$. Repeating the arguments of Lemma 3.5 we see that this cut exists since $W^u(x)$ is not contained in the unstable set of a singularity. The fact that in the considered region there are not singularities implies also that $h : \widetilde{zw} \to \widetilde{zw}$ is a continuous function. Thus h has a fixed point q. Its existence implies either that q is a singularity contradicting that there are not singularities in S, or there is a closed unstable curve which implies the existence of a Lyapunov stable point for f^{-1} which contradicts Theorem 2.8. This finishes the proof. □

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