

On the Complexity of the Diameter Constrained Reliability

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Abstract

Let $G = (V, E)$ be a simple graph with $|V| = n$ nodes and $|E| = m$ links, a subset $K \subseteq V$ of *terminals*, a vector $p = (p_1, \dots, p_m) \in [0, 1]^m$ and a positive integer d , called *diameter*. We assume nodes are perfect but links fail stochastically and independently, with probabilities $q_i = 1 - p_i$. The *diameter-constrained reliability* (DCR for short), is the probability that the terminals of the resulting subgraph remain connected by paths composed by d links, or less. This number is denoted by $R_{K,G}^d(p)$.

The general DCR computation is inside the class of \mathcal{NP} -Hard problems, since it subsumes the complexity that a random graph is connected. In this paper, the computational complexity of DCR-subproblems is discussed in terms of the number of terminal nodes $k = |K|$ and diameter d . Either when $d = 1$ or when $d = 2$ and k is fixed, the DCR is inside the class \mathcal{P} of polynomial-time problems. The DCR turns \mathcal{NP} -Hard when $k \geq 2$ is a fixed input parameter and $d \geq 3$.

The cases where $k = n$ or k is a free input parameter and $d \geq 2$ is fixed are not studied in prior literature. Here, the \mathcal{NP} -Hardness of the case $k = n$ (or k free) and $d \geq 3$ is established. The complexity of the case $k = n$ and $d = 2$ requires further research.

Contents

1. Introduction	2
2. Terminology	2
3. Complexity	3
4. Main theorem	5
5. Concluding Remarks	6
6. Acknowledgments	7
References	7

1. Introduction

The definition of DCR has been introduced by Héctor Cancela and Louis Petingi, inspired in delay-sensitive applications over the Internet infrastructure [PR01]. Nevertheless, its applications over other fields of knowledge enriches the motivation of this problem in the research community [Col99].

We wish to communicate special nodes in a network, called *terminals*, by d hops or less, in a scenario where nodes are perfect but links fail stochastically and independently. The all-terminal case with $d = n - 1$ is precisely the probability that a random graph is connected, or *classical reliability problem* (CLR for short). Arnon Rosenthal proved that the CLR is inside the class of \mathcal{NP} -Hard problems [Ros77]. As a corollary, the general DCR is \mathcal{NP} -Hard as well, hence intractable unless $\mathcal{P} = \mathcal{NP}$.

The focus of this paper is the computational complexity of DCR subproblems, in terms of the number of terminals k and diameter d . In Section 2 a formal definition of the DCR problem is provided as a particular instance of a coherent stochastic binary system. The computational complexity of the DCR is discussed in terms of the diameter and number of terminals in Section 3. The main contribution of this paper is included in Section 4. Specifically, the DCR is in the computational class of \mathcal{NP} -Hard problems in the all-terminal scenario ($k = n$) with a given diameter $d \geq 3$. Concluding remarks and open problems are summarized in Section 5

2. Terminology

We are given a system with m components. These components are either “up” or “down”, and the binary state is captured by a word $x = (x_1, \dots, x_m)$. Additionally we have a structure function $\phi : \{0, 1\}^m \rightarrow \{0, 1\}$ such that $\phi(x) = 1$ if and only if the system works under state x . When the components work independently and stochastically, with certain probabilities of operation $p = (p_1, \dots, p_m)$, the pair (ϕ, p) defines a *stochastic binary system*, or SBS for short, following the terminology of Michael Ball [Bal86]. An SBS is *coherent* whenever $x \leq y$ implies that $\phi(x) \leq \phi(y)$, where the partial order set $(\leq, \{0, 1\}^m)$ is bit-wise (i.e. $x \leq y$ if and only if $x_i \leq y_i$ for all $i \in \{1, \dots, m\}$). If $\{X_i\}_{i=1, \dots, m}$ is a set of independent binary random variables with $P(X_i = 1) = p_i$ and $X = (X_1, \dots, X_m)$, then $r = E(\phi(X)) = P(\phi(X) = 1)$ is the *reliability* of the SBS.

Now, consider a simple graph $G = (V, E)$, a subset $K \subseteq V$ and a positive integer d . Let us choose an arbitrary order of the edge-set $E = \{e_1, \dots, e_m\}$, $e_i \leq e_{i+1}$. For each subgraph $G_x = (V, E_x)$ with $E_x \subseteq E$, we identify a binary

word $x \in \{0, 1\}^m$, where $x_i = 1$ if and only if $e_i \in E_x$; this is clearly a bijection. A subgraph $G_x = (V, E_x)$ is d - K -connected if $d_x(u, v) \leq d, \forall \{u, v\} \subseteq K$, where $d_x(u, v)$ is the distance between nodes u and v in the graph G_x . Then, we define the structure $\phi : \{0, 1\}^m \rightarrow \{0, 1\}$ such that $\phi(x) = 1$ if and only if the graph G_x is d - K -connected. If we assume nodes are perfect but links fail stochastically and independently ruled by the vector $p = (p_1, \dots, p_m)$, the pair (ϕ, p) is a coherent SBS. Its reliability, denoted by $R_{K,G}^d(p)$, is called *diameter constrained reliability*, or DCR for short. A particular case is $R_{K,G}^{n-1}(p)$, called *classical reliability*, or CLR for short.

In all coherent SBS, a *pathset* is a state x such that $\phi(x) = 1$. A *minpath* is a state x such that $\phi(x) = 1$ but $\phi(y) = 0$ for all $y < x$ (i.e. a minimal pathset). A *cutset* is a state x such that $\phi(x) = 0$, while a *mincut* is a state x such that $\phi(x) = 0$ but $\phi(y) = 1$ if $y > x$ (i.e. a minimal cutset).

3. Complexity

The class \mathcal{NP} is the set of problems polynomially solvable by a non-deterministic Turing machine [GJ79]. A problem is \mathcal{NP} -Hard if it is at least as hard as every problem in the set \mathcal{NP} (formally, if every problem in \mathcal{NP} has a polynomial reduction to the former). It is widely believed that \mathcal{NP} -Hard problems are intractable (i.e. there is no polynomial-time algorithm to solve them). An \mathcal{NP} -Hard problem is \mathcal{NP} -Complete if it is inside the class \mathcal{NP} . Stephen Cook proved that the joint satisfiability of an input set of clauses in disjunctive form is an \mathcal{NP} -Complete decision problem; in fact, the first known problem of this class [Coo71]. In this way, he provided a systematic procedure to prove that a certain problem is \mathcal{NP} -Complete. Specifically, it suffices to prove that the problem is inside the class \mathcal{NP} , and that it is at least as hard as an \mathcal{NP} -Complete problem. Richard Karp followed this hint, and presented the first 21 combinatorial problems inside this class [Kar72]. Leslie Valiant defines the class $\#\mathcal{P}$ of counting problems, such that testing whether an element should be counted or not can be accomplished in polynomial time [Val79]. A problem is $\#\mathcal{P}$ -Complete if it is in the set $\#\mathcal{P}$ and it is at least as hard as any problem of that class.

Recognition and counting minimum cardinality mincuts/minpaths are at least as hard as computing the reliability of a coherent SBS [Bal86]. Arnon Rosenthal proved the CLR is \mathcal{NP} -Hard [Ros77], showing that the minimum cardinality mincut recognition is precisely Steiner-Tree problem, included in Richard Karp's list. The CLR for both two-terminal and all-terminal cases are still \mathcal{NP} -Hard, as Michael Ball and J. Scott Provan proved by reduction

to counting minimum cardinality $s - t$ cuts [PB83]. As a consequence, the general DCR is \mathcal{NP} -Hard as well. Later effort has been focused to particular cases of the DCR, in terms of the number of terminals $k = |K|$ and diameter d . When $d = 1$ all terminals must have a direct link, $R_{K,G}^1 = \prod_{\{u,v\} \subseteq K} p_{u,v}$, where $p_{u,v}$ denotes the probability of operation of link $\{u,v\} \in E$, and $p_{u,v} = 0$ if $\{u,v\} \notin E$. The problem is still simple when $k = d = 2$. In fact, $R_{\{u,v\},G}^2 = 1 - (1 - p_{u,v}) \prod_{w \in V - \{u,v\}} (1 - p_{u,w} p_{w,v})$. Héctor Cancela and Louis Petingi rigorously proved that the DCR is \mathcal{NP} -Hard when $d \geq 3$ and $k \geq 2$ is a fixed input parameter, in strong contrast with the case $d = k = 2$ [CP04]. Its proof is the main source of inspiration of this paper, and will be revisited in Section 4. The literature offers at least two proofs that the DCR has a polynomial-time algorithm when $d = 2$ and k is a fixed input parameter [Sar13, CCR⁺13]. Pablo Sartor et. al. present a recursive proof [Sar13], while Eduardo Canale et. al. present an explicit expression for $R_{K,G}^2$ that is computed in a polynomial time of elementary operations [CCR⁺13]. Table 1 summarizes the known results for the computational complexity of the DCR in terms of d and k .

	k (fixed)		$k = n$ or free
	2	3...	
2	$O(n)$ [CP04]	$O(n)$ [CCR ⁺ 13]	Unknown
3	\mathcal{NP} -Hard [CP04]		Unknown
\vdots			
d	\mathcal{NP} -Hard [Ros77]		\mathcal{NP} -Hard [PB83]
\vdots			
$n - 2$	\mathcal{NP} -Hard [Ros77]		\mathcal{NP} -Hard [PB83]
$n - 1$			
\vdots	\mathcal{NP} -Hard [Ros77]		\mathcal{NP} -Hard [PB83]
\vdots			

Figure 1. DCR Complexity in terms of the diameter d and number of terminals $k = |K|$

4. Main theorem

The DCR is inside the class of \mathcal{NP} -Hard problems in the all-terminal case with diameter $d \geq 3$. The main source of inspiration is the article authored by Héctor Cancela and Louis Petingi [CP04], where they proved that the DCR is \mathcal{NP} -Hard when $d \geq 3$ and $k \geq 2$ is a fixed input parameter. There, the authors prove first that the result holds for $k = 2$, and they further generalize the result for fixed $k \geq 2$. For our purpose it will suffice to revisit the first part. Before, we state a technical result first proved by Michael Ball and Scott Provan [BP83]. Recall that a *vertex cover* in a graph $G = (V, E)$ is a subset $V' \subseteq V$ such that V' meets all links in E . The graph G is bipartite if there exists a bipartition $V = V_1 \cup V_2$ such that $E \subseteq V_1 \times V_2$.

LEMMA 4.1. [BP83] *Counting the number of vertex covers of a bipartite graph is $\#\mathcal{P}$ -Complete.*

PROPOSITION 4.2. [CP04] *The DCR is \mathcal{NP} -Hard when $k = 2$ and $d \geq 3$.*

Proof. Let $d' = d - 3 \geq 0$ and $P = (V(P), E(P))$ a simple path with node set $V(P) = \{s, s_1, \dots, s_{d'}\}$ and edge set $E(P) = \{\{s, s_1\}, \{s_1, s_2\}, \dots, \{s_{d'-1}, s_{d'}\}\}$. For each bipartite graph $G = (V, E)$ with $V = A \cup B$ and $E \subseteq A \times B$ we build the following auxiliary network:

$$(1) \ G' = \{(A \cup B \cup V(P) \cup \{t\}, E \cup E(P) \cup \{\{s_{d'}, a\}, a \in A\} \cup \{\{b, t\}, b \in B\}),$$

where all links of G' are perfect but links in $I = \{\{s_{d'}, a\}, a \in A\} \cup \{\{b, t\}, b \in B\}$, which fail independently with identical probabilities $p = 1/2$. Consider the terminal set $K = \{s, t\}$. The auxiliary graph G' is illustrated in Figure 2. The reduction from the bipartite graph to the two-terminal instance is polynomial.

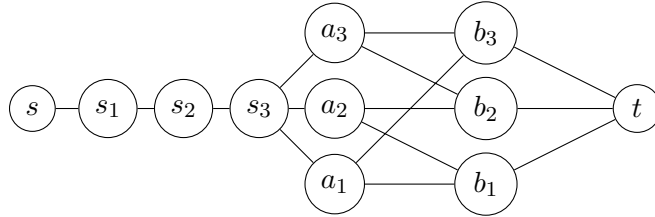


Figure 2. Example of auxiliary graph G'' with terminal set $\{s, t\}$ and $d = 6$, for the particular bipartite instance C_6 .

A set cover $A' \cup B' \subset A \cup B$ induces a cutset $I' = \{\{s_{d'}, a\}, a \in A'\} \cup \{\{b, t\}, b \in B'\}$ (i.e. if all links in I' fail, the nodes $\{s, t\}$ are not connected). Reciprocally, that cutset determines a set cover. Therefore, the number of

cutsets $|\mathcal{C}|$ is precisely the number of vertex covers of the bipartite graph $|\mathcal{B}|$. Moreover:

$$|\mathcal{B}| = 2^{|A|+|B|}(1 - R_{\{s,t\},G'}^d(1/2)).$$

Thus, the DCR for the two-terminal case is at least as hard as counting vertex covers of bipartite graphs. \square

The main result is perhaps a direct Corollary of Proposition 4.2:

THEOREM 4.3. *The DCR is \mathcal{NP} -Hard when $k = n$ and $d \geq 3$.*

Proof. Extend the auxiliary graph $G' = (V', E')$ to $G'' = (V'', E'')$, where $V'' = V'$ and $E'' = E' \cup \{\{a, a'\}, a \neq a', a, a' \in A\} \cup \{\{b, b'\}, b \neq b', b, b' \in B\}$. In words, just add links in order to connect all nodes from A , and all nodes from B . We keep the same probabilities of operation that in G' , and the new links are perfect.

Consider now the all-terminal case $K = V''$ for G'' , and given diameter $d \geq 3$. The key is to observe that the cutsets in the all-terminal scenario for G'' are precisely the $s - t$ cutsets in G' , and they have the same probability. Indeed, each pair of terminals from the set A are directly connected by perfect links; the same holds in B . The distance between s and $s_{d'}$ is $d' = d - 3 < d$, so these nodes (and all the intermediate ones) respect the diameter constraint. Finally, if there were an $s - t$ path (i.e. a path from s to t), the diameter of the resulting subgraph of G'' would be exactly d . Therefore, $R_{\{s,t\},G'}^d = R_{V'',G''}^d$, and again:

$$|\mathcal{B}| = 2^{|A|+|B|}(1 - R_{\{s,t\},G'}^d(1/2)) = 2^{|A|+|B|}(1 - R_{V'',G''}^d(1/2)).$$

Thus, the DCR for the all-terminal case is at least as hard as counting vertex covers of bipartite graphs. \square

5. Concluding Remarks

The reliability evaluation of a particular stochastic binary system has been discussed, called diameter constrained reliability (DCR). When the number of terminals k or diameter d are free, the DCR is \mathcal{NP} -Hard, since it subsumes the classical reliability problem. The cases $d = 1$ or $d = 2$ and k fixed belong to the set \mathcal{P} of polynomially solvable problems. In contrast, the DCR turns \mathcal{NP} -Hard when $k \geq 2$ is fixed and $d \geq 3$. In this paper we proved that the DCR is \mathcal{NP} -Hard when $k = n$ and $d \geq 3$. As a corollary, the result holds when $d \geq 3$ and k is an free parameter as well.

The computational complexity of the DCR is still unknown when $d = 2$ and $k = n$. It is worth to notice that when all links fail independently with

identical probability $p = 1/2$, all graphs occur with the same probability. Therefore, the DCR when $d = 2$ and $k = n$ is at least as hard as counting all subgraphs with diameter 2 of a given graph of order n .

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