

# MINIMIZING CONFIGURATIONS AND HAMILTON-JACOBI EQUATIONS OF HOMOGENEOUS N-BODY PROBLEMS

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*Dedicated to Professor Alain Chenciner on the occasion of his 70th anniversary*

ABSTRACT. For  $N$ -body problems with homogeneous potentials we define a special class of central configurations related with the reduction of homotheties in the study of homogeneous weak KAM solutions. For potentials in  $1/r^\alpha$  with  $\alpha \in (0, 2)$  we prove the existence of homogeneous weak KAM solutions. We show that such solutions are related to viscosity solutions of another Hamilton-Jacobi equation in the sphere of normal configurations. As an application we prove for the Newtonian three body problem that there are no smooth homogeneous solutions to the critical Hamilton-Jacobi equation.

## 1. INTRODUCTION.

We consider  $N$ -body problems with homogeneous potentials

$$U(x) = \sum_{i < j} \frac{m_i m_j}{r_{ij}^{2\kappa}}$$

where  $x = (r_1, \dots, r_N) \in E^N$  is a configuration of the  $N$  massive punctual bodies in some Euclidean space  $E$ , the positive constants  $m_i$  are their respective masses, and  $r_{ij} = \|r_i - r_j\|$ . The case in which  $E = \mathbb{R}^3$  and  $\kappa = 1/2$  is the classical Newtonian  $N$ -body problem. An old and natural question related to the study of the dynamics of such systems is the complete integrability. When the problem is completely integrable, the phase space can be foliated by invariant Lagrangian manifolds, and each leaf must then be contained in a level set of the energy function. In particular, if one of these Lagrangian manifolds is a graph over the space of configurations, it must correspond via the Legendre transformation with the graph of a closed 1-form  $\omega$  which satisfies  $H(x, \omega_x) = c$  for some constant  $c \in \mathbb{R}$ . Since the Hamiltonian of the system is the function on the cotangent bundle of the configuration space given by

$$H(x, p) = \frac{1}{2} \|p\|^2 - U(x)$$

and  $\inf U(x) = 0$ , we have that the last equation can not be solved if  $c < 0$ . Of course, the Hamiltonian is finite only in the open and dense set of configurations without collisions

$$\Omega = \{x = (r_1, \dots, r_N) \in E^N \mid r_i = r_j \iff i = j\}$$

which is simply connected when  $\dim E \geq 3$ . Therefore, these considerations lead us to investigate the existence of global solutions of the Hamilton-Jacobi equation  $H(x, d_x u) = c$  for  $c \geq 0$ . In this paper we study the critical case,

$$(1) \quad \|d_x u\|^2 = 2U(x)$$

and especially, the existence of global homogeneous solutions for this equation.

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In all what follows, the norm of a configuration in  $E^N$  will be the norm associated to the mass inner product, and  $(E^N)^*$  will be endowed with the corresponding dual norm.

The author has showed in [10], for values of  $\kappa \in (0, 1)$ , the existence of global viscosity solutions to the Hamilton-Jacobi equation (1) using weak KAM theory. Moreover, there are invariant solutions with respect to the obvious action of the compact Lie group  $O(E)$  in  $E^N$ . In [11] it was proved that, in the Newtonian case, every weak solution of (1) is invariant with respect to the action by translations of  $E$  in  $E^N$ . That is to say, each solution of (1) is uniquely determined by his restriction to the subspace  $V$  of  $E^N$  of configurations with center of mass at  $0 \in E$ . Thus, a configuration  $x = (r_1, \dots, r_N) \in E^N$  is in  $V$  if and only if  $\sum_{i=1}^N m_i r_i = 0$ . We also have that

$$V = \Delta^\perp = \{ (r, \dots, r) \mid r \in E \}^\perp$$

where the orthogonal complement is taken with respect to the mass inner product in  $E^N$ . Moreover, the translation invariance of a given function  $f : E^N \rightarrow \mathbb{R}$  implies that at each point of differentiability  $x \in E^N$  we have

$$d_{p(x)}(f|_V) = (d_x f)|_V$$

where  $p : E^N \rightarrow V$  is the orthogonal projection on  $V$  (in other words,  $p(x)$  is the unique translation of  $x$  with center of mass at  $0 \in E$ ). Therefore we have:

*A function  $u : E^N \rightarrow \mathbb{R}$  is a solution (in any possible way) of Hamilton-Jacobi equation (1) if and only if his restriction  $u|_V : V \rightarrow \mathbb{R}$  is a solution of the same Hamilton-Jacobi equation in  $V$ .*

For this reason, in all what follows we will only consider configurations in  $V$ , and functions  $u : V \rightarrow \mathbb{R}$ . As a subspace of  $E^N$ ,  $V$  is also an Euclidean space with the mass inner product, and his dual space will be considered with the corresponding dual norm.

On the other hand, non rotation invariant weak solutions can exist, and a simple example for the planar Kepler problem was suggested by Alain Chenciner and the author, and later it was found explicitly by Andrea Venturelli (We will explain more about these examples in [14]). More or less at the same time, Alain Chenciner asked if all these weak solutions are necessarily homogeneous functions modulo a constant. Again using the non rotation invariant examples, as well as some characteristic properties of weak KAM solutions, non homogeneous weak solutions can be constructed. Here we will prove the existence of homogeneous weak KAM solutions (theorem 3.5 bellow).

Until now the most fruitful application of the existence results of weak KAM solutions is that each one of them gives rise to a lamination of the configuration space by completely parabolic motions, showing the abundance of such motions (see [7, 13]). Moreover, the associated lamination defines the solution up to a constant, thus it is natural to expect that invariance properties of the solutions can be expressed in terms of properties of the lamination.

Let  $I(x)$  be the moment of inertia of a configuration  $x = (r_1, \dots, r_N)$  with respect to the origin of  $E$ , that is to say,

$$I(x) = \sum_{i=1}^N m_i r_i^2$$

and let  $S = \{ x \in V \mid I(x) = 1 \}$  be the sphere of normal configurations. In fact,  $I$  is the quadratic form associated to the mass inner product in  $V$  and  $S$  is the unit sphere in  $V$  of the corresponding norm. Every configuration  $x \neq 0$  has a unique polar decomposition, namely  $x = \lambda s$ , with  $\lambda > 0$  and  $s \in S$ . Therefore an homogeneous function  $u : V \rightarrow \mathbb{R}$  of degree  $\alpha$  is uniquely determined by his

restriction  $v$  to the unit sphere  $S$ , since we must have  $u(\lambda s) = \lambda^\alpha v(s)$ . We will show that, the equation must satisfy the function  $v$  in order to be an homogeneous solution of (1) the function  $u$ , is the Hamilton-Jacobi equation

$$(2) \quad (1 - \kappa)^2 v(s)^2 + \|d_s v\|^2 = 2U(s).$$

Note that this equation is not the Hamilton-Jacobi equation arising from a Tonelli Hamiltonian. However we will deduce for  $\kappa \in (0, 1)$  the existence of global viscosity solutions of Hamilton-Jacobi equation (2), see theorem 3.7 bellow.

Following the analogy with the Aubry-Mather theory we define a special type of central configurations, and we will prove that they are intimately related with the solutions of (2). Recall that a free time minimizer is a curve whose restriction to any compact interval minimizes the Lagrangian action in the set of all curves with the same extremities (see [7] for a detailed description of this concept in the Newtonian case). They correspond to the semistatic curves in the Aubry-Mather theory and must have critical energy (zero energy in our case). We also recall that a central configuration is a configuration  $x \in V$  for which there are homothetic motions passing through it. If that is the case, then only two (modulo translation of time) of these homothetic motions have zero energy, namely the parabolic ejection and the parabolic collision by  $x$ .

**Definition 1.1.** *A minimizing configuration is a central configuration such that the corresponding parabolic ejection is a free time minimizer. We will denote  $\mathcal{M}$  the set of normal minimizing configurations (i.e. in the sphere  $S$ ).*

The set of minimizing configurations is not empty. Note that the potential  $U$  has a minimum on the sphere  $S$ . If we denote

$$U_0 = \min \{ U(x) \mid x \in S \}$$

and

$$\mathcal{M}_0 = \{ x \in S \mid U(x) = U_0 \}$$

then it is clear that  $\mathcal{M}_0$  is not empty. We will call  $\mathcal{M}_0$  the set of normal *minimal configurations*. It is not difficult to prove that we have

$$\mathcal{M}_0 \subset \mathcal{M}$$

or in other words, that every minimal configuration is minimizing. An easy proof of this fact in the Newtonian case can be found in [7] (proposition 3.4), and the same proof works for any homogeneous potential with minor changes of the constants.

In the context of the Aubry-Mather theory, there is a well known conjecture due to Ricardo Mañé which says that, for a generic Tonelli Lagrangian on a given closed manifold  $M$ , the Mather set is reduced to an hyperbolic periodic orbit of the Lagrangian flow on the tangent bundle  $TM$ . When this happens all the theory becomes simple, for instance the semistatic curves are exactly the projection on  $M$  of orbits in the stable manifold of the Mather set, and there is only one weak KAM solution modulo a constant. Several years ago Renato Iturriaga asked to the author if there is an analogous conjecture in our context. He proposed that the correct conjecture must be: *For generic values of the masses, there is only one normal minimal configuration (modulo isometries) and it is non degenerate.* Of course, here the non degeneracy refers to the transversal directions to the action of the orthogonal group  $O(E)$  on the sphere  $S$ . Now we can see that the interesting question is to determine if the same happens with the minimizing configurations instead of the minimal ones. Note that the uniqueness of minimizing configuration implies  $\mathcal{M} = \mathcal{M}_0$ , but the only known results which proves this equality was obtained for some homogeneous  $N$ -body problems by Barutello and Secchi in [2]. Using a variational Morse-like index they prove that several colliding trajectories

are not minimizing. In particular they prove for the three body problem with equal masses that the collinear central configurations are not minimizing whenever  $\kappa > 3 - 2\sqrt{2}$  (note that the Newtonian case is included). Thus in these case the only minimizing configurations are the Lagrange equilateral configurations. In order to show the interest of this analysis, we will prove in section 5 the following theorem.

**Theorem 1.2.** *The critical Hamilton-Jacobi equation of the Newtonian three body problem has no smooth homogeneous solutions.*

## 2. THE LAX-OLEINIK SEMIGROUP AND WEAK KAM SOLUTIONS.

We need to recall briefly some facts about the Lagrangian action, and variational properties related to the Hamilton-Jacobi equation. Also we recall the definition of the Lax-Oleinik semigroup whose fixed points are precisely the weak KAM solutions. The proofs of the statements below can be found in [10].

If  $\gamma : [a, b] \rightarrow V$  is an absolutely continuous curve, then the Lagrangian action of  $\gamma$  is the value in  $(0, +\infty]$  defined by

$$A(\gamma) = \int_a^b \frac{1}{2} \dot{\gamma}(t)^2 + U(\gamma(t)) dt$$

where the square of the vector  $\dot{\gamma}(t)$ , which is defined for almost every  $t \in [a, b]$ , is taken with respect to the mass inner product in  $V$ . It is well known that if such a curve has a finite action, then it must be in the Sobolev space  $H^1([a, b], V)$ . Of course, since our system is autonomous, each curve can be parameterized in an interval of the form  $[0, t]$  by translation in time, and preserving his action. We will denote  $\phi(x, y, t)$  the infimum of the Lagrangian action in the set of all curves going from  $x$  to  $y$  in time  $t > 0$ . The infimum without restriction of time will be denoted  $\phi(x, y)$ . We know that for each  $\kappa \in (0, 1)$ , there is a positive constant  $\eta > 0$  such that the inequality

$$(3) \quad \phi(x, y) = \inf_{t>0} \phi(x, y, t) \leq \eta \|x - y\|^{1-\kappa}$$

holds for any pair of configurations  $x, y \in E^N$ .

The set of weak subsolutions of Hamilton-Jacobi equation (1) is

$$\mathcal{H} = \{ u : V \rightarrow \mathbb{R} \mid u(x) - u(y) \leq \phi(x, y) \text{ for all } x, y \in V \}$$

and will be endowed with the topology of uniform convergence on compact subsets. Since there is a trivial action of  $\mathbb{R}$  in  $\mathcal{H}$  given by addition of constants, we can deduce that  $\mathcal{H}$  is homeomorphic to  $\mathbb{R} \times \mathcal{H}_0$ , where

$$\mathcal{H}_0 = \{ u \in \mathcal{H} \mid u(0) = 0 \}$$

is a compact set of functions because of the Hölder estimate (3). The set of weak subsolutions  $\mathcal{H}$  is clearly convex. Another interesting property of  $\mathcal{H}$  is that it contains the infimum of any family of his elements whenever the infimum is finite.

**Lemma 2.1.** *If  $F \subset \mathcal{H}$  is such that  $u_F(x_0) = \inf \{ u(x_0) \mid u \in F \} > -\infty$  for some  $x_0 \in E^N$ , then  $u_F(x) = \inf \{ u(x) \mid u \in F \}$  is finite at every configuration  $x \in E^N$  and defines a weak subsolution  $u_F \in \mathcal{H}$ .*

*Proof.* Let  $x \in V$  be any configuration. Since  $F \subset \mathcal{H}$ , for each  $u \in F$  we have  $u(x) \geq u(x_0) - \phi(x_0, x)$ , thus

$$u(x) \geq u_F(x_0) - \phi(x_0, x)$$

which implies that  $u_F(x) \geq -\infty$  and that  $u_F(x_0) - u_F(x) \leq \phi(x_0, x)$ . Replacing now  $x$  and  $x_0$  by any two configurations  $x$  and  $y$  in the previous argument we conclude that  $u_F \in \mathcal{H}$ .  $\square$

The action in  $\mathcal{H}$  of the Lax-Oleinik semigroup  $(T_t)_{t \geq 0}$  is given by

$$T_t u(x) = \inf \{ u(y) + \phi(y, x, t) \mid y \in V \}$$

for  $t > 0$ , and  $T_0 u = u$  for all  $u \in \mathcal{H}$ . Note that we have

$$\mathcal{H} = \{ u : V \rightarrow \mathbb{R} \mid u \leq T_t u \text{ for all } t \geq 0 \} .$$

Note also that if  $u_1, u_2 \in \mathcal{H}$ , and  $u_1 \leq u_2$ , then  $T_t u_1 \leq T_t u_2$ . The action of the Lax-Oleinik semigroup is continuous, and the weak KAM theorem says that the set of fixed points is not empty.

**Definition 2.2.** *A function  $u : V \rightarrow \mathbb{R}$  is called a weak KAM solution if it is a fixed point of the Lax-Oleinik semigroup ( $u = T_t u$  for all  $t \geq 0$ ).*

Weak KAM solutions are viscosity solutions of Hamilton-Jacobi equation (1), a notion of weak solution that we will recall in section 3.3. They can be characterized between weak subsolutions as follows:

**Proposition 2.3.** *A function  $u$  is a weak KAM solution if and only if*

1.  $u \in \mathcal{H}$
2. *Given  $x \in V$  there is a curve  $\gamma$  defined for  $t \leq 0$  such that*
  - (a)  $\gamma(0) = x$
  - (b)  $u(x) - u(\gamma(t)) = A(\gamma|_{[t,0]})$  for all  $t \leq 0$ .

Note that the curves  $\gamma$  in the above proposition are free time minimizers of the Lagrangian action because we have

$$\phi(x, \gamma(t)) \geq u(x) - u(\gamma(t)) = A(\gamma|_{[t,0]}) \geq \phi(x, \gamma(t))$$

for all  $t \leq 0$ . In the Newtonian case it was proved in [7] that they are motions of zero energy and completely parabolic (for  $t \rightarrow -\infty$ ).

**Definition 2.4.** *We will say that a function  $u : V \rightarrow \mathbb{R}$  is a smooth solution of Hamilton-Jacobi equation (1) if it is differentiable and satisfy the equation at every configuration  $x$  such that  $U(x) < +\infty$  (at configurations  $x$  without collisions).*

In the collinear case, that is when  $\dim E = 1$ , we can have discontinuous smooth solutions. The reason for this is that the set of configuration without collisions has  $n!$  connected components and we can add to a given solution a different constant on each component, which results in a new solution. When minimizing curves avoid collisions (as happens in the Newtonian case, see [15]) we can deduce that a smooth solution  $u$  must be a weak subsolution, and must satisfy the Hölder estimate (3). Therefore, in this case smooth solutions must be Hölder continuous at collision configurations. On the other hand, the differentiability of a given weak KAM solution at some configuration without collisions  $x \in V$  is equivalent to the uniqueness of the calibrating curve given by proposition 2.3. This fact is of local nature and the proof can be found in [8]. The notion of calibrating curve appears several times in what follows, and for this reason we will now give a more general definition.

**Definition 2.5.** *Given a weak subsolution  $u \in \mathcal{H}$  and a curve  $\gamma : I \rightarrow V$ , we say that  $\gamma$  calibrates  $u$  if we have  $u(\gamma(b)) - u(\gamma(a)) = A(\gamma|_{[a,b]})$  whenever  $[a, b] \subset I$ .*

### 3. HOMOGENEOUS SOLUTIONS

**3.1. Preliminaries and existence of weak homogeneous solutions.** Suppose that  $u \in \mathcal{H}$  is an homogeneous function of degree  $\alpha$  (for example every constant function is in  $\mathcal{H}$  and homogeneous of degree 0). It is clear that if  $u$  is differentiable at some configuration  $x$ , and  $\lambda > 0$ , then  $u$  is also differentiable at the configuration

$\lambda x$  and  $d_{\lambda x}u = \lambda^{\alpha-1}d_xu$ . Therefore if  $u$  is a solution of Hamilton-Jacobi equation (1) we must have  $\|d_xu\|^2 = 2U(x)$ , and also

$$\lambda^{2(\alpha-1)} \|d_xu\|^2 = \|d_{\lambda x}u\|^2 = 2U(\lambda x) = 2\lambda^{-2\kappa}U(x)$$

from which we get that the degree of homogeneity must be  $\alpha = 1 - \kappa$ .

Homogeneous functions can also be viewed as fixed points of an action of the multiplicative group  $\mathbb{R}^+$ . More precisely, for  $\lambda > 0$  and  $u : V \rightarrow \mathbb{R}$  is a given function, we can define the function  $S_\lambda u$  by

$$S_\lambda u(x) = \lambda^{\kappa-1}u(\lambda x)$$

which defines the group action. Therefore, a function  $u$  is homogeneous of degree  $1 - \kappa$  if and only if  $S_\lambda u = u$  for every  $\lambda > 0$ .

When we reduce the rotational symmetries, one of the main tools involved is the commutation of the  $O(E)$  action with the Lax-Oleinik semigroup. For every pair of configurations  $x, y \in V$ , for every  $g \in O(E)$ , and for every  $t > 0$ , we have that  $\phi(gx, gy, t) = \phi(x, y, t)$ . Therefore, using the notation  $gu$  for  $u \circ g$ , we can write  $T_t(gu) = gT_tu$  for every function  $u : V \rightarrow \mathbb{R}$  and every  $t \geq 0$ . In particular, if  $u \in \mathcal{H}$ , we have  $u \leq T_tu$  for all  $t \geq 0$ , hence  $gu \leq gT_tu = T_t(gu)$  which says that  $gu \in \mathcal{H}$ . This also implies that the set of invariant functions is preserved by the Lax-Oleinik semigroup: if  $u = gu$  then  $T_tu = T_t(gu) = gT_tu$ . By this way we get that the set of invariant weak KAM solutions is not empty.

We return now our attention to the reduction of homotheties. It is not difficult to see that the group  $(S_\lambda)_{\lambda>0}$  preserves the set of weak subsolutions  $\mathcal{H}$  as well as the set of weak KAM solutions. We will need the following lemma.

**Lemma 3.1.** *Given  $x, y \in V$ ,  $t > 0$ , and  $\lambda > 0$ , we have*

$$\phi(\lambda x, \lambda y, \lambda^{1+\kappa}t) = \lambda^{1-\kappa} \phi(x, y, t).$$

*Proof.* Let  $\gamma : [0, t] \rightarrow V$  be an absolutely continuous curve such that  $\gamma(0) = x$  and  $\gamma(t) = y$ . Define the curve  $\gamma_\lambda$  on the interval  $[0, \lambda^{1+\kappa}t]$  by

$$\gamma_\lambda(s) = \lambda \gamma(\lambda^{-(1+\kappa)}s).$$

A simple computation shows that  $A(\gamma_\lambda) = \lambda^{1-\kappa}A(\gamma)$ . Note that the curve  $\gamma_\lambda$  goes from  $\lambda x$  to  $\lambda y$ . Taking a minimizing sequence for the Lagrangian action in the set of curves going from  $x$  to  $y$  in time  $t$  we deduce that the inequality

$$\phi(\lambda x, \lambda y, \lambda^{1+\kappa}t) \leq \lambda^{1-\kappa} \phi(x, y, t)$$

is always verified. Therefore the reverse inequality is also verified, since we have

$$\phi(x, y, t) = \phi(\lambda^{-1}\lambda x, \lambda^{-1}\lambda y, \lambda^{-(1+\kappa)}\lambda^{(1+\kappa)}t) \leq \lambda^{-(1-\kappa)}\phi(\lambda x, \lambda y, \lambda^{1+\kappa}t).$$

□

We will see now that, although the two actions do not commute, there is a natural relation between them.

**Proposition 3.2.** *For any  $u : V \rightarrow \mathbb{R}$ ,  $\lambda > 0$  and  $t \geq 0$  we have*

$$T_t S_\lambda u = S_\lambda T_{\lambda^{(1+\kappa)}t} u.$$

*Proof.* For each  $x \in V$  we have

$$\begin{aligned} T_t S_\lambda u(x) &= \inf \{ \lambda^{\kappa-1}u(\lambda y) + \phi(y, x, t) \mid y \in V \} \\ &= \lambda^{\kappa-1} \inf \{ u(\lambda y) + \lambda^{1-\kappa}\phi(y, x, t) \mid y \in V \} \\ &= \lambda^{\kappa-1} \inf \{ u(\lambda y) + \phi(\lambda y, \lambda x, \lambda^{1+\kappa}t) \mid y \in V \} \\ &= \lambda^{\kappa-1} T_{\lambda^{(1+\kappa)}t} u(\lambda x) = S_\lambda T_{\lambda^{(1+\kappa)}t} u(x) \end{aligned}$$

□

From this relation we can deduce that the group  $(S_\lambda)_{\lambda>0}$  preserves both the set of weak subsolutions and the set of weak KAM solutions.

**Corollary 3.3.** *If  $u \in \mathcal{H}$  then  $S_\lambda u \in \mathcal{H}$  for all  $\lambda > 0$ .*

*Proof.* Fix  $\lambda > 0$  and suppose that  $u \leq T_t u$  for any  $t \geq 0$ . Therefore we have  $S_\lambda u \leq S_\lambda T_t u$ . Using proposition 3.2 we get that

$$S_\lambda u \leq T_{\lambda^{-(1+\kappa)}t} S_\lambda u$$

which is equivalent to say that  $S_\lambda u \leq T_t S_\lambda u$  for any  $t \geq 0$ .  $\square$

**Corollary 3.4.** *If  $u \in \mathcal{H}$  is a weak KAM solution, then  $S_\lambda u$  is also a weak KAM solution for any  $\lambda > 0$ .*

*Proof.* Fix  $\lambda > 0$  and  $u \in \mathcal{H}$  such that  $u = T_t u$  for any  $t \geq 0$ . Therefore we have

$$T_t S_\lambda u = S_\lambda T_{\lambda^{(1+\kappa)}t} u = S_\lambda u$$

for any  $t \geq 0$ , which says that  $S_\lambda u$  is a weak KAM solution.  $\square$

Now we are able to prove our first existence result.

**Theorem 3.5.** *If  $\kappa \in (0, 1)$  the set of homogeneous weak KAM solutions of the  $N$ -body problem with homogeneous potential of degree  $-2\kappa$  is not empty.*

*Proof.* From the weak KAM theorem proved in [10] we know that for  $\kappa \in (0, 1)$  there exists a weak KAM solution  $u \in \mathcal{H}$ . Moreover, adding a constant to  $u$  we can assume that  $u \in \mathcal{H}_0$ , that is to say, that  $u(0) = 0$ . Since  $S_\lambda u(0) = 0$  for every  $\lambda > 0$ , we can apply lemma 2.1 to define  $u_0 \in \mathcal{H}$  as

$$u_0 = \inf_{\lambda>0} S_\lambda u.$$

Thus we have  $u_0 \leq T_t u_0$  for all  $t \geq 0$ . On the other hand, since for each  $\lambda > 0$  we have  $u_0 \leq S_\lambda u$ , we also have  $T_t u_0 \leq T_t S_\lambda u$ , and we deduce that

$$T_t u_0 \leq \inf_{\lambda>0} T_t S_\lambda u.$$

Therefore lemma 3.4 implies that  $T_t u_0 \leq u_0$  for all  $t \geq 0$ . We have proved that  $u_0$  is a weak KAM solution. It remains to prove that  $u_0$  is homogeneous. For each  $\eta > 0$  we have

$$\begin{aligned} S_\eta u_0 &= \eta^{\kappa-1} \inf_{\lambda>0} S_\lambda u(\eta x) \\ &= \eta^{\kappa-1} \inf_{\lambda>0} \lambda^{\kappa-1} u(\lambda \eta x) \\ &= \inf_{\lambda>0} (\lambda \eta)^{\kappa-1} u(\lambda \eta x) \\ &= \inf_{\lambda>0} S_{\lambda \eta} u(x) = u_0(x) \end{aligned}$$

which prove that  $u_0$  is homogeneous.  $\square$

**3.2. Hamilton-Jacobi equation on the sphere.** Let  $u : V \rightarrow \mathbb{R}$  be a smooth solution of Hamilton-Jacobi equation (1). Let  $v : S \rightarrow \mathbb{R}$  be the restriction of  $u$  to the unit sphere  $S$ . If  $u$  is homogeneous we have

$$u(\lambda s) = \lambda^{1-\kappa} v(s)$$

for all  $s \in S$  and all  $\lambda > 0$ . Note that the Riemannian metric given by the mass inner product in  $V$  splits in polar coordinates  $(\lambda, s)$  as

$$dx^2 = d\lambda^2 + \lambda^2 ds^2$$

therefore

$$\|d_{(\lambda s)} u\|^2 = \left\| \frac{\partial u}{\partial \lambda}(\lambda s) \right\|^2 + \frac{1}{\lambda^2} \left\| \frac{\partial u}{\partial s}(\lambda s) \right\|^2$$

since we are taking the dual norm in the cotangent bundle. The partial derivatives are

$$\frac{\partial u}{\partial \lambda}(\lambda s) = (1 - \kappa)\lambda^{-\kappa}v(s) \quad \text{and} \quad \frac{\partial u}{\partial s}(\lambda s) = \lambda^{1-\kappa}d_s v$$

thus the Hamilton-Jacobi equation (1) can be written

$$(1 - \kappa)^2 \lambda^{-2\kappa} v(s)^2 + \lambda^{-2\kappa} \|d_s v\|^2 = 2U(\lambda s).$$

Since  $U$  is homogeneous of degree  $-2\kappa$ , the equation in  $v$  is the Hamilton-Jacobi equation (2)

$$(1 - \kappa)^2 v(s)^2 + \|d_s v\|^2 = 2U(s).$$

**3.3. Viscosity solutions.** We start this section recalling briefly the well known notion of viscosity solution of a first order Hamilton-Jacobi equation of the form

$$(4) \quad \mathbb{H}(x, d_x u, u) = 0$$

introduced by M. Crandall, L. Evans and P.-L. Lions (see for instance [5], [6]). We will assume that  $\mathbb{H}$ , the Hamiltonian, is a continuous function defined on  $T^*M \times \mathbb{R}$  where  $M$  is a compact smooth manifold, and moreover, that  $\mathbb{H}$  is smooth outside a singular set of the form  $T^*\Delta \times \mathbb{R}$  where  $\mathbb{H} = +\infty$  ( $\Delta \subset M$ ).

**Definition 3.6.** *Let  $u : M \rightarrow \mathbb{R}$  be a continuous function, and  $x_0 \in M$ .*

- *$u$  is a viscosity subsolution of (4) at  $x_0$  if for every  $\varphi \in C^1(M)$  such that  $\varphi(x_0) = u(x_0)$  and  $\varphi \geq u$  in a neighborhood of  $x_0$  we have*

$$\mathbb{H}(x, d_{x_0} \varphi, \varphi(x_0)) \leq 0$$

- *$u$  is a viscosity supersolution of (4) at  $x_0$  if for every  $\varphi \in C^1(M)$  such that  $\varphi(x_0) = u(x_0)$  and  $\varphi \leq u$  in a neighborhood of  $x_0$  we have*

$$\mathbb{H}(x, d_{x_0} \varphi, \varphi(x_0)) \geq 0$$

- *$u$  is a viscosity solution of (4) at  $x_0$  if it is both viscosity subsolution and viscosity supersolution at  $x_0$ .*
- *$u$  is a viscosity solution of (4) if it is a viscosity solution at each point  $x_0 \in M$ .*

**Theorem 3.7.** *Let  $u \in \mathcal{H}$  be an homogeneous weak KAM solution of the  $N$ -body problem, and  $v$  the restriction of  $u$  to the unit sphere  $S$ . Then  $v$  is a viscosity solution of the Hamilton-Jacobi equation (2).*

*Proof.* We know that  $u$  is a viscosity solution of Hamilton-Jacobi equation (1). Suppose that  $\varphi \in C^1(S)$  is such that  $\varphi \geq v$  and that  $\varphi(s) = v(s)$ . If  $\psi$  is the homogeneous extension of  $\varphi$  to  $V$  of degree  $1 - \kappa$ , then we have that  $\psi \geq u$  in a neighborhood of  $s$  and  $\psi(\lambda s) = u(\lambda s) = \lambda^{1-\kappa}\varphi(s)$  for all  $\lambda > 0$ . We also have that

$$\frac{\partial \psi}{\partial s}(s) = d_s \varphi.$$

Thus, since  $u$  is a viscosity subsolution at  $s$  we have

$$\|d_s \psi\|^2 = (1 - \kappa)^2 \varphi(s)^2 + \|d_s \varphi\|^2 \leq 2U(s)$$

and we conclude that  $v$  is a viscosity subsolution of (2) at  $s$ . A similar argument proves that  $v$  is also a viscosity supersolution.  $\square$



**3.4. Calibrating curves of homogeneous solutions.** Weak KAM solutions come with a lamination of calibrating curves, as it was explained in proposition 2.3 above. We start showing that the homogeneity of a weak KAM solution implies an invariance property of such calibrating curves.

**Lemma 3.8.** *If a weak KAM solution  $u$  is homogeneous then the set of calibrating curves is invariant under the action of  $\mathbb{R}^+$  given by*

$$\gamma \mapsto \gamma_\lambda \quad \gamma_\lambda(t) = \lambda \gamma(\lambda^{-(1+\kappa)}t)$$

for any  $\lambda > 0$ .

*Proof.* Suppose that  $u$  is homogeneous and that  $\gamma : [0, +\infty) \rightarrow V$  calibrates  $u$ . Fix  $\lambda > 0$ , and note that the curve  $\gamma_\lambda$  is also defined in  $[0, +\infty)$ . If we write  $x = \gamma(0)$  and  $y = \gamma(t)$  for some value of  $t > 0$ , we have that

$$u(x) - u(y) = A(\gamma|_{[0,t]}).$$

On the other hand, if we write  $t^* = \lambda^{(1+\kappa)}t$  we have that

$$\gamma_\lambda(0) = \lambda x \quad , \quad \gamma_\lambda(t^*) = \lambda y$$

and

$$A(\gamma_\lambda|_{[0,t^*]}) = \lambda^{1-\kappa} A(\gamma|_{[0,t]}).$$

Therefore, since  $u$  is homogeneous of degree  $1 - \kappa$ , we conclude that

$$u(\lambda x) - u(\lambda y) = \lambda^{1-\kappa}(u(x) - u(y)) = \lambda^{1-\kappa} A(\gamma|_{[0,t]}) = A(\gamma_\lambda|_{[0,t^*]})$$

which proves that the curve  $\gamma_\lambda$  is also calibrating.  $\square$

We will denote  $\pi : V \setminus \{0\} \rightarrow S$  the projection  $\pi(x) = I(x)^{-1/2}x$ , and  $\Omega$  will denote the open and dense set of configurations without collisions.

**Theorem 3.9.** *Let  $u \in \mathcal{H}$  be an homogeneous weak KAM solution of the  $N$ -body problem, and  $v$  the restriction of  $u$  to the unit sphere  $S$ . If  $\gamma : (a, b) \rightarrow \Omega$  is a calibrating curve for  $u$ ,  $\rho = I(\gamma)^{1/2}$  and  $\sigma = \pi \circ \gamma$ , then for all  $t \in (a, b)$  we have that  $v$  is differentiable on  $\sigma(t)$  and*

- (1)  $\dot{\rho} = (1 - \kappa)\rho^{-\kappa}v(\sigma)$
- (2)  $d_{\sigma(t)}v(\nu) = \langle \nu, \rho(t)^\kappa \dot{\sigma}(t) \rangle$  for all  $\nu \in T_{\sigma(t)}S$

**Remark.** The condition  $\gamma(t) \in \Omega$  is needless when Marchal's theorem applies (for instance in the Newtonian case) because calibrating curves are always minimizers and must avoid collisions.

*Proof.* Suppose now that  $\gamma : (a, b) \rightarrow \Omega$  calibrates the homogeneous function  $u$ . At each  $t \in (a, b)$  we have that  $U(\gamma(t)) < +\infty$ . Thus  $\gamma$  is an extremal without collisions of the Lagrangian action, hence  $\gamma$  is smooth. Since  $t \in (a, b)$  is an interior point,  $u$  is differentiable at  $\gamma(t)$  and the calibrating condition implies that  $d_{\gamma(t)}u$  is the Legendre transform of  $\dot{\gamma}(t)$  (see [8]). In other words, using the mass inner product we have

$$d_{\gamma(t)}u(\xi) = \langle \xi, \dot{\gamma}(t) \rangle$$

for any  $\xi \in V$ . By homogeneity,  $u$  is differentiable at  $\lambda\gamma(t)$  for all  $t \in (a, b)$  and any  $\lambda > 0$ . In polar coordinates we can write  $u(\lambda s) = \lambda^{1-\kappa}v(s)$ , thus  $v$  is differentiable at  $\sigma(t)$  for all  $t \in (a, b)$ . Also using polar coordinates we can write  $\gamma(t) = \rho(t)\sigma(t)$  where  $\rho(t) = I(\gamma(t))^{1/2}$  and  $\sigma(t) = \pi(\gamma(t))$ . At each time  $t \in (a, b)$  a vector  $\xi \in V$  can be written as  $\xi = r\sigma(t) + \nu$  with  $\langle \nu, \sigma(t) \rangle = 0$ . Thus we have

$$\begin{aligned} d_{\gamma(t)}u(\xi) &= \langle r\sigma(t) + \nu, \dot{\rho}(t)\sigma(t) + \rho(t)\dot{\sigma}(t) \rangle \\ &= r\dot{\rho}(t) + \rho(t)\langle \nu, \dot{\sigma}(t) \rangle \end{aligned}$$

and also

$$d_{\gamma(t)}u(\xi) = (1 - \kappa)\rho(t)^{-\kappa}v(\sigma(t))r + \rho(t)^{1-\kappa}d_{\sigma(t)}v(\nu).$$

Since  $\xi \in V$  is arbitrary, we have that

$$\dot{\rho} = (1 - \kappa)\rho^{-\kappa}v(\sigma)$$

and that for all  $t \in (a, b)$

$$d_{\sigma(t)}v(\nu) = \langle \nu, \rho(t)^\kappa \dot{\sigma}(t) \rangle \quad \text{for all } \nu \in T_{\sigma(t)}S$$

□

**Corollary 3.10.** *Let  $u \in \mathcal{H}$  be an homogeneous weak KAM solution of the  $N$ -body problem, and  $v$  the restriction of  $u$  to the unit sphere  $S$ . If  $\gamma : (a, b) \rightarrow \Omega$  is a calibrating curve for  $u$ , then  $v(\pi(\gamma))$  is strictly increasing unless  $\gamma$  is homothetic.*

*Proof.* If  $\gamma = \rho\sigma$  as before, then theorem 3.9 implies

$$\frac{d}{dt}v(\sigma(t)) = \langle \dot{\sigma}(t), \rho(t)^\kappa \dot{\sigma}(t) \rangle = \rho(t)^\kappa \|\dot{\sigma}(t)\|^2 \geq 0.$$

Therefore  $v(\sigma)$  is not decreasing. On the other hand, if we have  $v(\sigma(c)) = v(\sigma(d))$  for some values of  $c < d$  then

$$0 = \int_c^d \rho(t)^\kappa \|\dot{\sigma}(t)\|^2 dt$$

which implies  $\dot{\sigma}(t) = 0$  for all  $t \in [c, d]$  because  $\rho > 0$ . We deduce that  $\gamma$  is homothetic on the interval  $[c, d]$ . It is clear that this implies that  $\gamma$  is homothetic over whole his domain (and that  $\sigma$  is a central configuration). □

#### 4. MINIMIZING CONFIGURATIONS

As we have say, a minimizing configuration is a central configuration such that his parabolic ejection is a free time minimizer. The parabolic ejection of a central configuration  $s$  is a curve  $\gamma_s : [0, +\infty) \rightarrow V$  of the form

$$(5) \quad \gamma_s(t) = \alpha_s t^{c_\kappa} s$$

where  $\alpha_s$  and  $c_\kappa$  are positive constants which depends on the subscripts. We need to compute explicitly these constants. For the sake of simplicity the configuration  $s$  will be supposed of unit norm, that is,  $s \in S$ . In order to be a motion the curve  $\gamma_s$ , it must satisfies the Newton's equation of motion  $\ddot{\gamma}_s = \nabla U(\gamma_s)$  (we recall that the gradient is taken with respect to the mass inner product). The explicit computation gives

$$\ddot{\gamma}_s(t) = c_\kappa(c_\kappa - 1)\alpha_s t^{c_\kappa-2} s$$

and

$$\nabla U(\gamma_s(t)) = \alpha_s^{-(2\kappa+1)} t^{-c_\kappa(2\kappa+1)} \nabla U(s).$$

Therefore, the equation of motion will be satisfied if and only if

- (a)  $\nabla U(s) = \lambda s$  for some constant  $\lambda$ ,
- (b)  $c_\kappa - 2 = -c_\kappa(2\kappa + 1)$ , and
- (c)  $\lambda \alpha_s^{-(2\kappa+1)} = c_\kappa(c_\kappa - 1)\alpha_s$ .

The first condition says that  $s$  is a central configuration, or in other words, that  $s$  is a critical point of the restriction of  $U$  to the sphere  $S$ . We will see that if condition (a) is satisfied, then the equations (b) and (c) have unique solution, namely, the ones that make the curve  $\gamma_s$  a zero energy motion.

Since  $U$  is homogeneous of degree  $-2\kappa$ , the Euler's theorem gives

$$\langle \nabla U(s), s \rangle = -2\kappa U(s).$$

Thus, if (a) is satisfied, we must have  $-2\kappa U(s) = \langle \lambda s, s \rangle = \lambda$ . From (b) we deduce that

$$c_\kappa = \frac{1}{1 + \kappa}$$

which takes the well known value  $2/3$  in the Newtonian case. Finally, replacing the found values of  $\lambda$  and  $c_\kappa$  in equation (c) we get that

$$\alpha_s = (2(1 + \kappa)^2 U(s))^{1/2(1+\kappa)}$$

which takes the well known value  $(9U(s)/2)^{1/3}$  in the Newtonian case. If now we compute the kinetic energy and the potential function

$$T(t) = \frac{1}{2} \|\dot{\gamma}_s(t)\|^2 \quad \text{and} \quad U(t) = U(\gamma_s(t))$$

we obtain

$$(6) \quad T(t) = U(t) = 2^{-\kappa/(1+\kappa)} (1 + \kappa)^{-2\kappa/(1+\kappa)} U(s)^{1/(1+\kappa)} t^{-2\kappa/(1+\kappa)}$$

which shows that  $\gamma_s$  is a zero energy motion.

Because of lemma 3.1, we can deduce that the critical potential action  $\phi(x, y)$  is homogeneous of degree  $1 - \kappa$ , that is, we have

$$(7) \quad \phi(\lambda x, \lambda y) = \lambda^{1-\kappa} \phi(x, y)$$

for any pair of configurations  $x, y \in V$  and any value of  $\lambda > 0$ . Since the curve  $\gamma_s$  is invariant under the blow-up transformations used in lemma 3.1 or in lemma 3.8, it is easy to see that if the equality

$$A(\gamma_s |_{[0,t]}) = \phi(0, \gamma_s(t))$$

holds for some  $t_0 > 0$ , then it must also hold for every  $t > 0$ . In particular, since the restriction of a free time minimizer to a subinterval is also a free time minimizer, if the previous equality holds for some  $t_0 > 0$  then we will have

$$A(\gamma_s |_{[a,b]}) = \phi(\gamma_s(a), \gamma_s(b))$$

for every compact interval  $[a, b] \subset [0, +\infty)$ . This is precisely the condition required to be the parabolic ejection  $\gamma_s$  a free time minimizer.

We now introduce an auxiliary function  $\psi : S \rightarrow \mathbb{R}$ . Given normal configuration  $s \in S$ , let  $t(s) > 0$  be the time in which the curve  $\gamma_s$  above defined (5) pass through the configuration  $s$ , that is such that  $\gamma_s(t(s)) = s$ , and set

$$(8) \quad \psi(s) = A(\gamma_s |_{[0,t(s)]}) .$$

Of course, we have  $\psi(s) \geq \phi(s, 0)$  for all  $s \in S$ . The above discussion shows that if  $\psi(s) = \phi(s, 0)$  then the parabolic ejection  $\gamma_s$  is a free time minimizer. Therefore we have proved the following proposition, which gives a characterization of the set of normal minimizing configurations.

**Proposition 4.1.** *A normal configuration  $s \in S$  is minimizing configuration if and only if satisfies  $\psi(s) = \phi(s, 0)$ .*

**Corollary 4.2.** *The set of normal minimizing configurations  $\mathcal{M} \subset S$  is compact.*

Let us compute the auxiliary function  $\psi$ . First we need to compute the time  $t(s)$  of a given configuration  $s \in S$ . Clearly we have  $\gamma_s(t(s)) = s$  if and only if  $\alpha_s t(s)^{c_\kappa} = 1$ , thus we deduce that

$$t(s) = \alpha_s^{-1/c_\kappa} = (2(1 + \kappa)^2 U(s))^{-1/2} .$$

Since  $T(t) = U(t)$  for any  $s \in S$ , even is the configuration  $s$  is not minimizing, we can write

$$\psi(s) = \int_0^{t(s)} T(t) + U(t) dt = 2 \int_0^{t(s)} U(t) dt$$

which gives

$$\psi(s) = \frac{1}{1 - \kappa} (2U(s))^{1/2}$$

and in the Newtonian case takes the value  $2\sqrt{2U(s)}$ . It is not surprising to note that the explicit computation of the auxiliary function  $\psi$  gives (modulo a square root) one of the terms of the Hamilton-Jacobi equation (2). Using the function  $\psi$  we can reformulate equation (2) in a more suggestive way as

$$(9) \quad v(s)^2 + \frac{1}{(1 - \kappa)^2} \|d_s v\|^2 = \psi(s)^2$$

Another important role of the minimizing configurations is that they allow to define the critical Busemann functions, since their associated parabolic ejections are minimizing geodesics (geodesic rays) for the Jacobi metric on the zero energy level. More precisely, if  $s \in S$  is a minimizing configuration, the corresponding Busemann critical function is defined by

$$b_s(x) = \lim_{t \rightarrow +\infty} (\phi(x, ts) - \phi(ts, 0)) .$$

These functions are homogeneous weak KAM solutions and are studied by Andrea Venturelli and the author in [14]. At the same time in which these papers were written, Héctor Sánchez-Morgado and Boris Percino announced similar results [16].

## 5. SMOOTH HOMOGENEOUS SOLUTIONS.

This section is devoted to the Newtonian case  $2\kappa = 1$  in a space of dimension at least two. The reason of this restriction is that we want to apply several results which are until now only proved for the Newtonian potential like [1, 7, 17] or which are not true in the collinear case, like [4, 9, 15].

The main application of the analysis developed in the previous sections is the following theorem. Recall that the unit sphere  $S \subset V$  is a Riemannian manifold as a submanifold of  $V$  endowed with the mass inner product. We will denote  $K \subset S$  the compact set of normal configurations with partial collisions.

**Theorem 5.1.** *Let  $U$  be the Newtonian potential ( $\kappa = 1/2$ ) and suppose that  $\dim E \geq 2$ . Let  $u : V \rightarrow \mathbb{R}$  be an homogeneous smooth solution of Hamilton-Jacobi equation (1) (in the sense of 2.4) and  $v$  the restriction of  $u$  to the unit sphere  $S$ . Let  $\nabla v$  be his gradient vector field, which is a smooth vector field on  $S \setminus K$ . For  $s \in S \setminus K$ , let  $\theta_s : (a_s, b_s) \rightarrow S$  be the maximal solution of  $\dot{\theta} = \nabla v(\theta)$  with  $\theta_s(0) = s$ . Let  $Z_v = \{s \in S \mid \nabla v(s) = 0\}$ . Then we have:*

- (a)  $Z_v$  is a subset of  $\mathcal{M}$ , the set of minimizing configurations.
- (b) If  $s \in A = S \setminus (K \cup Z_v)$  then
  - (i)  $a_s = -\infty$  and the  $\alpha$ -limit set satisfies  $\alpha(s) \subset Z_v$ .
  - (ii)  $b_s < +\infty$  and there is  $r(s) \in K$  such that

$$\lim_{t \rightarrow b_s} \theta(s) = r(s) .$$

- (iii) The map  $r : A \rightarrow K$  continuous and surjective.

*Proof.* Let  $s \in Z_v$  and Since  $v$  is the restriction of  $u$  to the unit sphere  $S$ , then it is clear that we have

$$|v(s)| = |u(s) - u(0)| \leq \phi(s, 0) \leq \psi(s) .$$

Therefore, if  $v$  has a critical point at some configuration  $s \in S$ , then equation (9) implies that  $|v(s)| = \psi(s)$ . Thus we must have  $\psi(s) = \phi(s, 0)$ , which implies that

$s$  is a minimizing configuration as a consequence of proposition 4.1. Thus we have proved item (a).

Suppose now that  $s \in S \setminus K$  and that  $\nabla v(s) \neq 0$ . Let  $\gamma : (-\infty, 0]$  be the unique calibrating curve for  $u$  such that  $\gamma(0) = s$  (the uniqueness is ensured by the differentiability of  $u$  at  $s$ ). We know that  $\gamma$  is a free time minimizer, and it is proved in [7] that  $\gamma(t)$  is therefore completely parabolic for  $t \rightarrow -\infty$ . It is known that the normalized configuration of such motions tends to the set of central configurations. A simple proof of this fact for homogeneous potential was written by Alain Chenciner (see ‘‘Théorème fondamental’’ in [3]). On the other hand, a well known theorem of Shub [17] says that the set of normal central configurations is a compact subset of  $S \setminus K$ . Of course this is obvious when the conjecture of finiteness of the set of similarity classes of central configurations holds, like in the three body problem, or in many other cases (see for instance the work of Albouy and Kaloshin [1] and the references therein). We have thus that  $\sigma(t) = \pi(\gamma(t))$  tends to the set of central configurations when  $t \rightarrow -\infty$ .

By theorem 3.9 (2) we have, for each  $t \leq 0$ ,

$$\dot{\sigma}(t) = \rho(t)^{-1/2} \nabla v(\sigma(t))$$

where  $\rho(t) = I(\gamma(t))^{1/2}$ . Thus  $\sigma$  is a reparametrization of a segment of  $\theta$ . If  $\sigma(t) = \theta(\tau(t))$  then we have that  $\tau(0) = 0$  and that

$$\dot{\sigma}(t) = \dot{\tau}(t) \dot{\theta}(\tau(t)) = \dot{\tau}(t) \nabla v(\sigma(t))$$

for all  $t \leq 0$ . Therefore  $\tau$  satisfies  $\dot{\tau}(t) = \rho(t)^{-1/2}$ . By integration we get

$$\tau(t) = - \int_t^0 \rho(u)^{-1/2} du.$$

In a completely parabolic motion all mutual distances grow like  $t^{2/3}$ , thus we have  $\rho(u) \sim |u|^{2/3}$  for  $u \rightarrow -\infty$ . Therefore  $\tau(t) \sim -|t|^{2/3}$  for  $t \rightarrow -\infty$ , which proves that  $a_s = -\infty$  and that  $\theta(t)$  tends to the compact set of central configurations for  $t \rightarrow -\infty$ . In particular, the  $\alpha$ -limit set  $\alpha(s)$  is a well defined compact connected set of central configurations. Since each point in  $\alpha(s)$  is recurrent, and regular orbits of a gradient flow are never recurrent, we conclude that each point in  $\alpha(s)$  is an equilibrium point, meaning that  $\alpha(s) \subset Z_v$ . The statement b.(i) is thus proved.

We will prove now statements b.(ii) and b.(iii). As before, let  $\gamma : (-\infty, 0]$  be the unique calibrating curve for  $u$  such that  $\gamma(0) = s$ . By lemma 5.2 bellow,  $\gamma$  can be extended to a maximal motion over an interval  $(0, T_s)$  with  $T_s > 0$  which also be called  $\gamma$ . Moreover, this extension  $\gamma$  also calibrates  $u$ .

Since the extended curve  $\gamma$  is maximal, We conclude that either  $\gamma$  presents a pseudocollision at time  $t = T_s$  or there is a collision configuration  $c_s \in V$  such that  $\lim_{t \rightarrow T_s} \gamma(t) = c_s$ . Recall that Painlevé has proved that pseudocollisions can only occur when the number of bodies is at least  $N \geq 4$ . If  $\gamma$  has a collision at time  $t = T_s$  there are two possibilities: either  $c_s$  is a total collision, either  $c_s$  is a partial collision.

We discuss now these three cases.

*First case:  $c_s$  is a total collision.* As before, we write  $\gamma(t) = \rho(t)\sigma(t)$  for  $t \leq T_s$  where  $\sigma = \pi(\gamma)$  and  $\rho = I(\gamma)^{1/2}$ . Writing  $\sigma(t) = \theta(\tau(t))$  we have again that  $\dot{\tau}(t) = \rho(t)^{-1/2}$  and that  $\tau(0) = 0$ , but this time we know that

$$\rho(t) \sim (T_s - t)^{2/3}$$

for  $t \rightarrow T_s$  because  $\gamma$  presents a total collision at time  $T_s$  (see also [3]). Thus by integration we get again, for  $t \in [0, T_s)$

$$\tau(t) = \int_0^t \rho(u)^{-1/2} du.$$

If  $\alpha > 0$  is such that  $\rho(u) \geq \alpha(T_s - u)^{2/3}$  we deduce the upper bound

$$\tau(t) \leq \alpha \frac{3}{2} T_s^{2/3}$$

for all  $t \in (0, T_s)$ , and that the limit  $\tau_0$  of  $\tau(t)$  for  $t \rightarrow T_s$  exists. Now we use again the fact that  $\gamma$  presents a total collision at time  $T_s$  in order to guarantee that  $\sigma(t)$  tends to the set of central configurations when  $t \rightarrow T_s$ . Since  $\sigma(t) = \theta(\tau(t))$  and the set of central configurations is a compact subset of  $S \setminus K$  we have that

$$\lim_{t \rightarrow T_s} \theta(\tau(t)) = \lim_{\tau \rightarrow \tau_0} \theta(\tau) = \theta(\tau_0) = s_0$$

where  $s_0$  is some central configuration. Thus we have proved that  $\sigma(t)$  converges to a central configuration  $s_0$ . Of course we have that  $\nabla v(s_0) \neq 0$  since the vector field  $\nabla v$  is uniquely integrable in  $S \setminus K$ .

On the other hand, using a blow-up technique we can prove that the parabolic collision by  $s_0$  calibrates  $u$ , which implies that  $\nabla v(s_0) = 0$ . To see this, we start by translating the domain of the calibrating curve  $\gamma$  in order to have the total collision at time  $t = 0$ . Therefore we can suppose that  $\gamma : (-\infty, 0]$  is a calibrating curve of  $u$ , that  $\gamma$  has total collision at  $t = 0$ , and that  $\gamma$  is completely parabolic for  $t \rightarrow -\infty$ . Moreover, we have that his normalized shape has a limit, that is to say that  $\lim_{t \rightarrow 0} \sigma(t) = s_0$ . Now we apply lemma 3.8 to obtain a family of calibrating curves  $(\gamma_\lambda)_{\lambda > 0}$  with exactly the same properties. Recall that  $\gamma_\lambda$  is defined for  $t \leq 0$  by

$$\gamma_\lambda(t) = \lambda \gamma(\lambda^{-3/2} t).$$

It is not difficult to prove with well known arguments that  $\gamma_\lambda$  converges uniformly on compact subsets for  $\lambda \rightarrow +\infty$  to an homothetic curve  $\gamma_0$  by  $s_0$ . For instance, restricting the curves to  $[-1, 0]$ , we have that

$$A(\gamma_\lambda |_{[-1, 0]}) = u(\gamma_\lambda(-1)) - u(0) = \phi(\gamma_\lambda(-1), 0).$$

Since the Lagrangian action is lower semicontinuous,

$$\lim_{\lambda \rightarrow +\infty} \gamma_\lambda(t) = \gamma_0(t)$$

uniformly in  $t \in [-1, 0]$ , and the potential action as well as the function  $u$  are continuous, we also have that

$$A(\gamma_0 |_{[-1, 0]}) \leq \lim_{\lambda \rightarrow +\infty} A(\gamma_\lambda |_{[-1, 0]}) = u(\gamma_0(-1)) - u(0).$$

Therefore the curve  $\gamma_0$  is an homothetic calibrating curve of  $u$ , and as such it must be the parabolic collision by  $s_0$ . In particular we have  $\nabla v(s_0) = 0$ , and the possibility of  $c_s$  to be a total collision is excluded.

*Note that we have proved the following:* Any calibrating curve of an homogeneous weak KAM solution with a total collision is homothetic.

*Second case:  $c_s$  is a partial collision.* Once again we write  $\gamma(t) = \rho(t)\sigma(t)$  for  $t \leq T_s$  where  $\sigma = \pi(\gamma)$  and  $\rho = I(\gamma)^{1/2}$ . Writing  $\sigma(t) = \theta(\tau(t))$  we have again that  $\dot{\tau}(t) = \rho(t)^{-1/2}$ , that  $\tau(0) = 0$  hence that

$$\tau(t) = \int_0^t \rho(u)^{-1/2} du$$

for all  $t < T_s$ . Since  $\lim_{t \rightarrow T_s} \rho(t) = \rho_0 = I(c_s)^{-1/2} > 0$ , we can say that  $\tau(t)$  tends to the convergent integral

$$\tau_0 = \int_0^{T_s} \rho(u)^{-1/2} du$$

when  $t \rightarrow T_s$ . Moreover, if  $r(s) = \rho_0^{-1} c_s$  is the normalized configuration of  $c_s$ , then  $r(s) \in K$  and we have that

$$r(s) = \lim_{t \rightarrow T_s} \sigma(t) = \lim_{\tau \rightarrow \tau_0} \theta(\tau).$$

Note that this implies that the maximal solution  $\theta$  is defined until  $\tau_0 > 0$  so we deduce that  $b_s = \tau_0 < +\infty$ , and that in this second case statement b.(ii) holds.

*Third case:  $\gamma$  has a pseudocollision at time  $t = T_s$*  We will exclude this possibility. In that case  $\gamma(t)$  has no limit for  $t \rightarrow T_s$ . As before we write in polar coordinates  $\gamma = \rho\sigma$ . Our first step, will be to prove that  $\rho(t)$  is bounded in  $[0, T_s)$ . By theorem 3.9 we know that  $2\dot{\rho}(t) = \rho(t)^{-1/2}v(\sigma(t))$ . Since  $v$  is continuous in  $S$ , we deduce that there exists a positive constant  $M > 0$  for which

$$(10) \quad \frac{d}{dt} \rho(t)^{3/2} = \frac{2}{3} \rho(t)^{1/2} \dot{\rho}(t) \leq M$$

for all  $t \in [0, T_s)$ . Thus  $\rho(t)$  is bounded, and  $\gamma(t)$  must be contained in a compact subset of  $V$  for all  $t \in [0, T_s)$ . Since we are assuming that  $\gamma(t)$  has no limit, we must have at least two limits points for  $t \rightarrow T_s$ . This is impossible because  $\gamma$  is a free time minimizer.

It remains to prove statement b.(iii). Let  $s_0 \in K$  and  $\gamma : (-\infty, 0]$  be a calibrating curve of  $u$  such that  $\gamma(0) = s_0$ . It is clear that for every  $t < 0$  the configuration  $s = \pi(\gamma(t))$  is in  $A$  and that  $r(s) = s_0$ . Therefore the map  $r : A \rightarrow K$  is surjective.

In order to prove the continuity, note that it suffices to prove the continuity of the map  $s \mapsto c_s$  since  $r(s) = \pi(c_s)$ . Let  $(s(n))_{n>0}$  be a convergent sequence in  $A$  such that  $\lim s(n) = s \in A$ . For each  $n > 0$ , we know that there is a configuration with partial collisions  $c_n$  and a positive time  $T_n > 0$  with the following property: the maximal calibrating curve of  $u$  passing by  $s_n$  at time zero is a curve  $\gamma_n : (-\infty, T_n] \rightarrow V$  such that  $\gamma_n(T_n) = c_n$ . Now observe that for each  $n > 0$  we have that  $\gamma_n$  is differentiable at  $t = 0$  and that the Legendre transform of  $\dot{\gamma}_n(0)$  is precisely  $d_{s(n)}u$  which tends to  $d_s u$  since  $u$  is smooth. Thus we have that  $\dot{\gamma}_n(0) \rightarrow w$ , where  $w \in V$  is the unique vector such that his Legendre transform is precisely  $d_s u$ . If  $\gamma : (-\infty, T_s]$  is the maximal calibrating curve of  $u$  passing by  $s$  at time zero, we must have for the same reason  $\dot{\gamma}(0) = w$ , hence we have proved that  $\lim \dot{\gamma}_n(0) = \dot{\gamma}(0)$ .

*Claim 1 :  $\lim T_n = T_s$ .* This is the more delicate part of the proof, and uses both Tonelli's and Marchal's theorems (the reader can found the proof of these fundamental theorems for instance in [8, 12] for the first one, and [4, 9, 15] for the second). Let  $\epsilon > 0$  and  $0 < t < T_s$ . The continuity of the Lagrangian flow on the phase space implies that, for sufficiently large values of  $n > 0$  the curves  $\gamma_n$  are defined at least until time  $t$ , and that  $\|\gamma_n(t) - \gamma(t)\| \leq \epsilon$ . In particular we have that  $\liminf T_n \geq T_s$ . Suppose that  $\limsup T_n \geq T_s$ . If such is the case, it must exists  $\delta > 0$  and a subsequence  $(\gamma_{n_k})_{k>0}$  such that each curve  $\gamma_{n_k}$  is defined until time  $T_{n_k} > T_s + \delta$ . Let now  $T^* \in (T_s, T_s + \delta)$ . We know that the sequence  $\gamma_{n_k}(T^*)$  is bounded; in fact we have bounded uniformly these values when we have exclude the case of pseudocollisions, see the inequality (10) above. Taking again a subsequence, we can assume that there exists the limit

$$r = \lim_k \gamma_{n_k}(T^*).$$

For each  $k > 0$ , the curve  $\gamma_n$  is a free time minimizer. Thus in addition we have,

$$A(\gamma_{n_k} |_{[0, T^*]}) = \phi(\gamma_{n_k}(T^*), s_n)$$

from which we deduce that

$$A(\gamma_{n_k} |_{[0, T^*]}) \rightarrow \phi(r, s).$$

Therefore Tonelli's theorem applies, and we deduce the existence of a subsequence of these curves which converges uniformly to an absolutely continuous curve  $\tilde{\gamma}$  defined on  $[0, T^*]$ . The lower semicontinuity of the Lagrangian action gives

$$A(\tilde{\gamma}) \leq \lim A(\gamma_{n_k} |_{[0, T^*]}) = \phi(r, s)$$

which says that  $\tilde{\gamma}$  is also a free time minimizer. The proof of the claim ends as follows: for every  $T \in (0, T_s)$  we know that  $\gamma_{n_k} |_{[0, T]}$  also converges uniformly to  $\gamma_0$ . Thus we have  $\tilde{\gamma}(t) = \gamma(t)$  for every  $t \in (0, T_s)$ . We conclude that  $\tilde{\gamma}(T_s) = c_s$  is a configuration of partial collisions, which contradicts Marchal's theorem since a minimizer can not present a collision at any interior point of his domain. Therefore we have proved the claim that  $T_n \rightarrow T_s$ , but also

*Claim 2* : Given  $\epsilon > 0$  and  $t < T_s$  we have, for  $n > 0$  large enough, that  $t < T_n$  and that  $\|\gamma_n(t) - \gamma(t)\| \leq \epsilon$

To complete the proof of the theorem, we now prove that

$$\lim_n c_n = \lim_n \lim_{t \rightarrow T_n} \gamma_n(t) = \lim_n \lim_{t \rightarrow T_s} \gamma(t) = c_s.$$

In what follows, we will assume that  $c_n$  does not converges to  $c_s$ , and arrive at a contradiction. As before, we use the fact that  $\gamma_n(t)$  is uniformly bounded as consequence of inequality (10). Therefore, we can assume (taking a subsequence if necessary) that  $\lim c_n = c^* \neq c_s$ .

Let us call  $4d = \|c^* - c_s\| > 0$  and fix  $\delta > 0$ . Let  $t < T_s$  be such that  $\|\gamma(t) - c_s\| \leq d$ . Choose  $n_0 > 0$  such that  $\|c_n - c^*\| \leq d$  for every  $n > n_0$ . Using now the *claim 2* above with  $\epsilon = d$ , we see that we can choose  $n(t) > n_0$  such that  $\|\gamma_{n(t)}(t) - \gamma(t)\| \leq d$ . We can also assume that  $n(t) > n_0$  is large enough to have

$$|T_{n(t)} - T_s| \leq \delta/2.$$

Therefore we have

$$\|c_{n(t)} - \gamma_{n(t)}(t)\| \geq \|c^* - c_s\| - 3d = d$$

for all  $t < T_s$  such that  $\|\gamma(t) - c_s\| \leq d$ . Moreover, each  $\gamma_n$  is a free time minimizer and  $c_n = \gamma_n(T_n)$  for each  $n > 0$ , therefore we also have

$$A(\gamma_{n(t)} |_{[t, T_{n(t)}]}) = \phi(c_{n(t)}, \gamma_{n(t)}(t)) = \phi(c_{n(t)}, \gamma_{n(t)}(t), T_{n(t)} - t).$$

Let us write, to simplify the notation,  $x_t = c_{n(t)}$ ,  $y_t = \gamma_{n(t)}(t)$  and  $\tau_t = T_{n(t)} - t$ . Accordingly we can write  $\|x_t - y_t\| \geq d > 0$  and

$$\phi(x_t, y_t, \tau_t) = \phi(x_t, y_t).$$

Note that if  $|T_s - t| < \delta/2$  then we have  $\tau_t < \delta$ . This is impossible for  $\delta$  small enough as consequence of lemma 5.3 bellow. This ends the proof of the theorem.  $\square$

**Lemma 5.2.** *Suppose that  $U$  is the Newtonian potential and that  $\dim E \geq 2$ . Let  $u : V \rightarrow \mathbb{R}$  be a smooth solution of Hamilton-Jacobi equation (1) in the sense of 2.4. Let  $x \in E^N$  be a configuration without collisions and  $\gamma : (-\infty, 0] \rightarrow V$  be a calibrating curve of  $u$  such that  $\gamma(0) = x$ . Then the maximal solution of the motion equation of the  $N$  bodies which extends  $\gamma$  is defined until a positive finite time  $a \in (0, +\infty)$ .*



*Proof.* We start recalling that If  $\gamma : (a, b) \rightarrow V$  is calibrating of a function  $u \in \mathcal{H}$  then,  $\gamma$  is a free time minimizer, hence has no collisions as consequence of Marchal's theorem [15, 4]. In particular  $\gamma$  is a differentiable motion of zero energy. Moreover,  $u$  is differentiable at  $\gamma(t)$  and the Legendre transform of  $\dot{\gamma}(t)$  is precisely the derivative of  $u$  at  $\gamma(t)$  for all  $t \in (a, b)$ . On the other hand, if we only know that  $\gamma|_{(a, t]}$  calibrates  $u$  for some  $t \in (a, b)$  but we also know that  $u$  is differentiable at  $\gamma(t)$ , then  $\gamma$  must calibrate  $u$  on a bigger interval  $(a, t + \epsilon)$  for some  $\epsilon > 0$  (see [8]).

Let  $\gamma^* : (-\infty, a) \rightarrow V$ ,  $a \in \mathbb{R} \cup \{+\infty\}$  be the maximal motion extending  $\gamma$ . If  $\gamma^*$  is not a calibrating curve of  $u$ , then we can define

$$\tau = \max \{ t \in (0, a) \mid \gamma^*|_{(-\infty, t]} \text{ calibrates } u \}$$

and clearly we have  $0 < \tau < a$ . We deduce that  $u$  can not be differentiable at  $\gamma^*(\tau)$  which contradicts the above considerations. We conclude that  $\gamma^*$  is calibrating for  $u$  and as such, it is a free time minimizer. On the other hand, we know that there are no complete free time minimizers (see [7] theorem 1.2). Therefore we must have  $a < +\infty$ .  $\square$

**Lemma 5.3.** *For every pair of configurations  $x, y \in V$  and any  $t > 0$ , we have*

$$\phi(x, y, T) \geq \frac{1}{2} \|x - y\|^2 T^{-1}.$$

*Proof.* Let  $\gamma : [0, T] \rightarrow V$  be any absolutely continuous curve such that  $\gamma(0) = x$  and  $\gamma(T) = y$ . Neglecting the integral of the Newtonian potential in the definition of the Lagrangian action we deduce that

$$2A(\gamma) \geq \int_0^T \|\dot{\gamma}(t)\|^2 dt.$$

Applying the Bunyakovsky inequality we can write

$$\begin{aligned} \|x - y\| &\leq \int_0^T \|\dot{\gamma}(t)\| dt \\ &\leq \left( \int_0^T dt \right)^{1/2} \left( \int_0^T \|\dot{\gamma}(t)\|^2 \right)^{1/2} \\ &\leq T^{1/2} (2A(\gamma))^{1/2} \end{aligned}$$

from which we get

$$2A(\gamma) \geq \|x - y\|^2 T^{-1}$$

so the proof is obtained taking the infimum over all possible curve  $\gamma$ .  $\square$

**Proof of theorem 1.2.** Suppose that there exists a smooth homogeneous solution  $u : V \rightarrow \mathbb{R}$  of Hamilton-Jacobi equation (1). Let  $v$  the restriction of  $u$  to the unit sphere  $S$ . Theorem 5.1 says that the set  $Z_v$  of critical points of  $v$  is contained in  $\mathcal{M}$  which has at most five connected components, namely three corresponding to Euler configurations, and the corresponding to the Lagrange equilateral configurations (two components in the planar case and only one if  $k = \dim E \geq 3$ ). On the other hand, the compact set  $K$  of normal partial collisions on  $V$  has three connected components, which we will call  $K_{12}$ ,  $K_{23}$  and  $K_{31}$ . We note that the open set  $A = S \setminus (Z_v \cup K)$  is connected. This is clear for the planar or the spatial three body problem, since the compact sets  $\mathcal{M}$ , and  $K_{ij}$  are a finite number of orbits of the action of the orthogonal group  $O(E)$  which has dimension  $k(k-1)/2$ , where  $k = \dim E$ , and  $\dim S = 2k - 1$ . Thus we have codimension 2 for  $k \in \{2, 3\}$ . But in fact  $A$  is connected for every  $k \geq 2$ , see proposition 5.4 bellow. Finally, applying part b.(iii) of theorem 5.1 we conclude that  $K = r(A)$  is connected, which we know to be false.  $\square$

We are convinced that the following proposition may be useful to generalize the application of the here developed techniques to the case of more than three bodies. When the number of bodies  $N \geq 4$  the set  $K$  becomes connected, suggesting that other topological invariants should be considered. Of course, a constructive proof could be established but surely it would be more cumbersome than presented here.

**Proposition 5.4.** *For the Newtonian  $N$ -body problem we have that, if the set of similarity classes of central configurations is finite, then the open set of noncentral configurations without collisions is connected.*

*Proof.* Of course we are excluding the collinear case where the set has  $n!$  connected components. Let  $x, y \in V$  two given configurations which are not central nor collision configurations. It is proven in [7] (theorem 3.1) that there is a free time minimizer  $\gamma : [0, T] \rightarrow V$  such that  $\gamma(0) = x$  and  $\gamma(T) = y$ . By Marchal's theorem we know that  $\gamma(t)$  has no collisions for  $t \in (0, T)$ . Moreover, since the set of central configurations is closed, we have that  $\gamma(t)$  is noncentral if  $t$  is close to 0 or if  $t$  is close to  $T$ . Let us define

$$C = \{ t \in (0, T) \mid \gamma(t) \text{ is a central configuration} \}.$$

We claim that  $C$  is finite. Otherwise  $C$  must accumulate at some  $t^* \in (0, T)$ . Thus using the finiteness hypothesis we can choose a sequence  $t_n \rightarrow t^*$ , and fixed central configuration  $z$  such that  $\gamma(t_n)$  is similar to  $z$  for all  $n > 0$ . Writing  $\gamma(t) = (r_1(t), \dots, r_N(t))$  and  $z = (z_1, \dots, z_N)$  we have that the equalities

$$(11) \quad \alpha_{ijkl}(t) = \frac{\|r_i(t) - r_j(t)\|}{\|r_k(t) - r_l(t)\|} = \frac{\|z_i - z_j\|}{\|z_k - z_l\|}$$

hold for each  $t = t_n$  and every choice of  $i, j, k, l \in \{1, \dots, N\}$  such that  $i \neq j$  and  $k \neq l$ . Since  $\gamma$  has no collisions, the functions  $\alpha_{ijkl}$  are analytic functions of  $t$  on  $(0, T)$ . Therefore each one of the functions  $\alpha_{ijkl}$  is constant, which means that  $\gamma(t)$  is similar to  $z$  for all  $t \in [0, T]$  contradicting the fact that the configurations  $x$  and  $y$  are not central.

We have proved that there is at most a finite set of times  $0 < t_1 < \dots < t_k < T$  such that  $\gamma(t)$  is a central configuration. We will now perturb the curve  $\gamma$  slightly, thereby avoiding central configurations. This can be achieved perturbing slightly only one of the functions  $r_i(t)$  in small neighborhoods of times  $t = t_i$ . If the perturbation  $\gamma'$  is sufficiently small, we have for  $t - t_i$  small, that  $\gamma'(t)$  is neither similar to  $\gamma(t_i)$  nor any other central configuration.  $\square$

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