THE CLASSIFICATION OF UNISERIAL $\mathfrak{sl}(2)\ltimes V(m)$-MODULES AND A NEW INTERPRETATION OF THE RACAH-WIGNER $6j$-SYMBOL

LEANDRO CAGLIERO AND FERNANDO SZECHTMAN

ABSTRACT. All Lie algebras and representations will be assumed to be finite dimensional over the complex numbers. Let $V(m)$ be the irreducible $\mathfrak{sl}(2)$-module with highest weight $m\geq 1$ and consider the perfect Lie algebra $\mathfrak{g}=\mathfrak{sl}(2)\ltimes V(m)$. Recall that a $\mathfrak{g}$-module is uniserial when its submodules form a chain. In this paper we classify all uniserial $\mathfrak{g}$-modules. The main family of uniserial $\mathfrak{g}$-modules is actually constructed in greater generality for the perfect Lie algebra $\mathfrak{g}=\mathfrak{s}\ltimes V(\mu)$, where $\mathfrak{s}$ is a semisimple Lie algebra and $V(\mu)$ is the irreducible $\mathfrak{s}$-module with highest weight $\mu\neq 0$. The fact that the members of this family are, but for a few exceptions of lengths 2, 3 and 4, the only uniserial $\mathfrak{sl}(2)\ltimes V(m)$-modules depends in an essential manner on the determination of certain non-trivial zeros of Racah-Wigner $6j$-symbol.

1. Introduction

All Lie algebras and representations considered in this paper are assumed to be finite dimensional over the complex numbers.

The problem of classifying all indecomposable modules of a given Lie algebra (or a family of Lie algebras) is usually hard. Very serious difficulties are encountered even for Lie algebras of very low dimensionality. Two notoriously difficult examples are furnished by the 2-dimensional abelian Lie algebra (see [GP], Corollary 1) and the 3-dimensional Euclidean Lie algebra $\mathfrak{e}(2)=\mathfrak{so}(2)\ltimes\mathbb{C}^2$ (see [Sa], Theorem 4.3).

As is well-known, the classification problem has a satisfactory answer for the class of all semisimple Lie algebras, and perfect Lie algebras possess favorable properties that make them suitable for consideration in this problem. Indeed, it is precisely the class of Lie algebras that enjoy an abstract Jordan decomposition [CS1], a crucial tool needed for the classification of the irreducible representations of semisimple Lie algebras. Additionally, a Lie algebra $\mathfrak{g}$ is perfect if and only if its solvable radical $\mathfrak{r}$ coincides with its nilpotent radical $[\mathfrak{g},\mathfrak{r}]$. Thus, a perfect Lie algebra $\mathfrak{g}$ has a Levi decomposition $\mathfrak{g}=\mathfrak{s}\ltimes\mathfrak{r}$, where $\mathfrak{s}$ is semisimple and the solvable radical $\mathfrak{r}$ acts trivially on every irreducible $\mathfrak{g}$-module, and hence nilpotently on every $\mathfrak{g}$-module. Important families of Lie algebras fall into this class, for instance, the truncated current algebras $\mathfrak{s}\otimes\mathbb{C}[x]/(x^r)$ (or generalizations), known as Takiff algebras. In §2 we present some other positive features of perfect Lie algebras from the representation theory point of view.

2000 *Mathematics Subject Classification.* 17B10, 22E70.

Key words and phrases. Uniserial or indecomposable module; perfect Lie algebra; Racah-Wigner $6j$-symbol.

The first author was supported in part by CONICET and SECYT-UNC grants.

The second author was supported in part by an NSERC discovery grant.

But even for the easiest perfect Lie algebra (other than semisimple), namely $\mathfrak{g} = \mathfrak{sl}(2) \ltimes \mathbb{C}^2$, the classification of the indecomposable representations is far from being achieved (see [DR] and [Pi]). Therefore a natural approach to this problem is to identify a distinguished class of indecomposable representations for which one could expect to obtain a reasonable classification.

This line of research has connections to other problemas in representation theory and it has been followed in a number of papers. For instance, some authors have considered embeddings of a given $\mathfrak{g}$ into a semisimple Lie algebra $\tilde{\mathfrak{g}}$ and considered the highest weight modules of $\tilde{\mathfrak{g}}$ to construct or classify indecomposable $\mathfrak{g}$-modules obtained by restriction (see, for instance, [Ca], [CMS], [Dd], [DP], [DR], [Pr]). More recently, in [CKR], the authors classified the simple objects in the category of $\mathbb{Z}$-graded finite-dimensional representations of $\mathfrak{s} \ltimes \mathfrak{r}$ (with $\mathfrak{s}$ semisimple and $\mathfrak{r}$ abelian). In [Pi], A. Piard classifies all indecomposable $\mathfrak{g}$-modules V, where $\mathfrak{g} = \mathfrak{sl}(2) \ltimes \mathfrak{r}$, $\mathfrak{r} = \mathbb{C}^2$ and $V/\mathfrak{r}V$ is irreducible. On the other hand, A. Khare [K] introduces a family of deformations H_f of the universal enveloping algebra of $\mathfrak{sl}(2) \ltimes \mathbb{C}^2$ and classifies all finite-dimensional simple modules of H_f which, in turn, are uniserial representations of $\mathfrak{sl}(2) \ltimes \mathbb{C}^2$. (Recall that a module V is uniserial if its submodules form a chain.)

In this paper we focus our attention on the class of uniserial representations. We think that, within this class, a classification can be achieved for certain families of Lie algebras and, moreover, that the members of this class might be viewed as building blocks to understand more general classes of indecomposable representations. Indeed, the following two facts support this belief.

First, one of the main results of this paper gives a complete classification of all uniserial $\mathfrak{g}$-modules for $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m)$, where $V(m)$ is the irreducible $\mathfrak{sl}(2)$-module with highest weight $m \geq 1$. As far as we know, this is the first time that a structurally defined class of indecomposable modules, other than the irreducible ones, has been simultaneously classified for all members of an infinite family of Lie algebras. The modules in our classification for $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(1)$ can be obtained from the simple modules of the algebra H_f (mentioned above) introduced by A. Khare [K], letting the polynomial parameter f vary. In contrast to the case of the indecomposable modules of the abelian and Euclidean Lie algebras, a classification of all uniserial modules for these and other solvable Lie algebras is attained in [CS2].

Secondly, uniserial modules are also considered (see [BH], [HZ], [HZ2]) as a starting point in terms of classification and as building blocks of other indecomposable modules in the case of certain finite dimensional associative algebras (notice that the Jordan Normal Form Theorem states that any $\mathbb{C}[x]$-module is a direct sum of uniserial modules). In the context of Lie algebras, all of Piard's indecomposable $\mathfrak{sl}(2) \ltimes V(1)$-modules mentioned above, as well as further indecomposable modules for more general perfect Lie algebras, can be constructed as a series of extensions of uniserial modules (see [CS3]).

A crucial step in the proof of our classification requires the determination of nontrivial zeros of the (classical) *Racah-Wigner* $6j$*-symbol* within certain parameters. The $6j$-symbol is a real number $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$ associated to six non-negative half-integer numbers j_1, j_2, j_3, j_4, j_5 and j_6, originally studied because it plays a central role in angular momentum theory (see for instance [CFS], [Ed], [RBMW]).

Our results provide a bridge between two related but different lines of research and we think that very interesting connections of this kind will appear by considering other families of Lie algebras.

1.1. **Main results.** Let $\mathfrak{g}$ be a Lie algebra with solvable radical $\mathfrak{r}$ and Levi decomposition $\mathfrak{g}=\mathfrak{s}\ltimes\mathfrak{r}$.

Let V be a $\mathfrak{g}$-module and let $0=V_0\subset V_1\subset\cdots\subset V_n=V$ be a composition series of V. By Lie's theorem $\mathfrak{r}$ acts via scalar operators on each composition factor $W_i=V_i/V_{i-1}$, $1\le i\le n$. In particular every W_i is an irreducible $\mathfrak{s}$-module.

The socle series $0=\mathrm{soc}^0(V)\subset\mathrm{soc}^1(V)\subset\cdots\subset\mathrm{soc}^m(V)=V$ is inductively defined by declaring $\mathrm{soc}^i(V)/\mathrm{soc}^{i-1}(V)$ to be the socle of $V/\mathrm{soc}^{i-1}(V)$, that is, the sum of all irreducible submodules of $V/\mathrm{soc}^{i-1}(V)$, for $1\le i\le m$.

As indicated earlier, V is uniserial if it has only one composition series, i.e., if the socle series of V has irreducible factors. A uniserial module is clearly indecomposable. A sequence $W_1,...,W_n$ of irreducible $\mathfrak{s}$-modules will be said to be *admissible* if there is a uniserial $\mathfrak{g}$-module with socle factors $\mathfrak{s}$-isomorphic to $W_1,...,W_n$. By considering dual modules, it is clear that a sequence $W_1,...,W_n$ is admissible if and only if so is $W_n^*,...,W_1^*$.

In general the classification of uniserial $\mathfrak{g}$-modules breaks down into two steps:
STEP 1. Determine all admissible sequences.
STEP 2. Given an admissible sequence, find all uniserial modules giving rise to it.

As mentioned earlier, perfect Lie algebras are well suited for consideration in this problem. Indeed, from $\mathfrak{g}=\mathfrak{s}\ltimes\mathfrak{r}$ we obtain the Levi decomposition $[\mathfrak{g},\mathfrak{g}]=\mathfrak{s}\ltimes[\mathfrak{g},\mathfrak{r}]$. Thus $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]$ if and only if $\mathfrak{r}=[\mathfrak{g},\mathfrak{r}]$. Here $[\mathfrak{g},\mathfrak{r}]=\mathfrak{r}\cap[\mathfrak{g},\mathfrak{g}]$ is not only the solvable radical of $[\mathfrak{g},\mathfrak{g}]$ but also the nilpotent radical of $\mathfrak{g}$, i.e., the ideal of all $x\in\mathfrak{g}$ such that $xV=0$ for every irreducible $\mathfrak{g}$-module V. It follows that $\mathfrak{g}$ is perfect if and only if $\mathfrak{r}$ annihilates every irreducible $\mathfrak{g}$-module. Thus if $\mathfrak{g}$ is perfect an irreducible $\mathfrak{g}$-module is nothing but an irreducible $\mathfrak{s}$-module annihilated by $\mathfrak{r}$. More generally, if $\mathfrak{g}$ is perfect then the terms of the socle series of V can be intrinsically obtained from $\mathfrak{r}$ as follows: $\mathrm{soc}^i(V)/\mathrm{soc}^{i-1}(V)$, for $1\le i\le n$, is simply the 0-weight space for the action of $\mathfrak{r}$ on $V/\mathrm{soc}^{i-1}(V)$.

We begin our paper in §2 and §3 by furnishing general criteria to recognize, construct and classify uniserial modules for perfect Lie algebras with abelian radical. These results turn out to be fundamental for the rest of the paper. As a first application we prove in §4 the following theorem.

Theorem 1.1. *Let $\mathfrak{s}$ be a non-zero semisimple Lie algebra. Let $b\in\mathbb{Z}_{\ge0}$ and let λ and μ be dominant integral weights of $\mathfrak{s}$, where $\mu\neq0$ and μ^* is the highest weight of $V(\mu)^*$. Consider the perfect Lie algebra $\mathfrak{g}=\mathfrak{s}\ltimes V(\mu)$. Then, up to isomorphism, there exists one and only one uniserial $\mathfrak{g}$-module, say $Z(\lambda,b)$, with socle factors $V(\lambda),V(\lambda+\mu^*),...,V(\lambda+b\mu^*)$.*

It is then clear that the dual $\mathfrak{g}$-module $Z(\lambda,b)^*$ is, up to isomorphism, the only uniserial $\mathfrak{g}$-module with socle factors $V(\lambda^*+b\mu),...,V(\lambda^*+\mu),V(\lambda^*)$.

We point out that, in the special case when μ is the highest weight of the adjoint representation of $\mathfrak{s}$, the $\mathfrak{g}$-module $Z(\lambda,b)$ is an example of a Kirillov-Reshetikhin module of the Takiff algebra $\mathfrak{s}\ltimes\mathfrak{s}\cong\mathfrak{s}\otimes\mathbb{C}[x]/(x^2)$, where the second factor $\mathfrak{s}$ is abelian (see [CM]; further information on Kirillov-Reshetikhin modules can be found in [BCFM], [CG], [G], [M]).

In §5 we exhibit explicit matrix realizations of these modules for $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m)$ and we spend considerable effort in §6 to achieve an axiomatic characterization of $Z(\lambda, b)$ and $Z(\lambda, b)^*$, which presents them as a particular subclass of the class of cyclic indecomposable $\mathfrak{g}$-modules.

Other uniserial $\mathfrak{g}$-modules are possible for $\mathfrak{g} = \mathfrak{s} \ltimes V(\mu)$, as indicated in §8. These exceptional modules, together with the modules $Z(\lambda, b)$ and $Z(\lambda, b)^*$, comprise all the uniserial modules of $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m)$. In fact, our main result reads as follows.

Theorem 1.2. *Let $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m)$, where $m \geq 1$. Then, up to a reversing of the order, the following are the only admissible sequences for $\mathfrak{g}$:*

Length 1. $V(a)$.

Length 2. $V(a), V(b)$, *where* $a + b \equiv m \mod 2$ *and* $0 \leq b - a \leq m \leq a + b$.

Length 3. $V(a), V(a+m), V(a+2m)$; *or*
$V(0), V(m), V(c)$, *where* $c \equiv 2m \mod 4$ *and* $c \leq 2m$.

Length 4. $V(a), V(a+m), V(a+2m), V(a+3m)$; *or*
$V(0), V(m), V(m), V(0)$, *where* $m \equiv 0 \mod 4$.

Length ≥ 5. $V(a), V(a+m), \dots, V(a+sm)$, *where* $s \geq 4$.

Moreover, each of these sequences arises from only one isomorphism class of uniserial $\mathfrak{g}$-modules, except for the sequence $V(0), V(m), V(m), V(0)$, $m \equiv 0 \mod 4$. The isomorphism classes of uniserial $\mathfrak{g}$-modules associated to this sequence are parametrized by the complex numbers.

Explicit matrix realizations illustrating this theorem are given in §5 and §8.

One major step towards the proof of the above theorem is the determination of all admissible sequences of length 3 for $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m)$ and this is done in §9. From the results of §2 and §3, it follows that $V(a), V(b), V(c)$ is an admissible sequence of length 3 for $\mathfrak{g}$ if and only if $V(m)$ occurs in $V(a) \otimes V(b)$ and $V(b) \otimes V(c)$, and $\mathcal{L}$ is abelian, where $\mathcal{L}$ is the Lie subalgebra of $\mathfrak{gl}(V)$, with $V = V(a) \oplus V(b) \oplus V(c)$, generated by $f(r) + g(r)$, $r \in V(m)$, and $f : V(m) \to \mathrm{Hom}(V(b), V(a))$ as well as $g : V(m) \to \mathrm{Hom}(V(c), V(b))$ are $\mathfrak{sl}(2)$-embeddings. In terms of matrices, $\mathcal{L}$ is the Lie subalgebra of $\mathfrak{gl}(a+b+c+3, \mathbb{C})$ generated by $\left\{ \begin{pmatrix} 0 & f(r) & 0 \\ 0 & 0 & g(r) \\ 0 & 0 & 0 \end{pmatrix} : r \in V(m) \right\}$. The determination of all a, b, c and m for which $\mathcal{L}$ is abelian requires the following theorem from §11.

Theorem 1.3. *Let a, b, c, p, q, k be non-negative integers for which there exist $\mathfrak{sl}(2)$-embeddings*

$$V(k) \xrightarrow{f_0} \mathrm{Hom}(V(c), V(a)), \quad V(k) \xrightarrow{f_1} V(p) \otimes V(q),$$

$$V(p) \xrightarrow{f_2} \mathrm{Hom}(V(b), V(a)), \quad V(q) \xrightarrow{f_3} \mathrm{Hom}(V(c), V(b));$$

and let $\begin{Bmatrix} \frac{q}{2} & \frac{k}{2} & \frac{p}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{Bmatrix}$ *be the Racah-Wigner* $6j$*-symbol associated to them. If*

$$f_4 = \begin{Bmatrix} \frac{q}{2} & \frac{k}{2} & \frac{p}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{Bmatrix} f_0$$

then, after a suitable normalization of f_i, $i=0,1,2,3,4$, (see §11 as well as Theorem 9.1 and its proof for precise details) the following diagram of $\mathfrak{sl}(2)$-morphisms is commutative, where g sends $\alpha\otimes\beta\to\alpha\beta$:

$$\begin{array}{ccc} V(k) & \xrightarrow{\quad f_4\quad} & \mathrm{Hom}(V(c),V(a)) \\ f_1\big\downarrow & & \big\uparrow g \\ V(p)\otimes V(q) & \xrightarrow{\ f_2\otimes f_3\ } & \mathrm{Hom}(V(b),V(a))\otimes\mathrm{Hom}(V(c),V(b)). \end{array}$$

In particular, $V(k)$ appears in the image of $g(f_2\otimes f_3)$ if and only if $\begin{Bmatrix} \frac{q}{2} & \frac{k}{2} & \frac{p}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{Bmatrix}\neq 0.$

Recall that the (classical) *Racah-Wigner 6j-symbol* is a real number $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$ associated to six non-negative half-integer numbers j_1, j_2, j_3, j_4, j_5, j_6 and, originally, it is implicitly defined (see [CFS]) in terms of the transition matrix between the following two basis of $\mathrm{Hom}_{\mathfrak{sl}(2)}\big(V(k),V(a)\otimes V(b)\otimes V(c)\big)$

$$\big\{V(k)\to V(p)\otimes V(c)\to V(a)\otimes V(b)\otimes V(c)\big\}_{p\geq 0}$$

and

$$\big\{V(k)\to V(a)\otimes V(q)\to V(a)\otimes V(b)\otimes V(c)\big\}_{q\geq 0}.$$

In fact, the 6j-symbol can be defined as above in a more general context, in particular for any semisimple multitensor category, see for instance [EFK].

Theorem 1.3 gives an explicit definition of the Racah-Wigner 6j-symbol, in contrast to the original implicit definition. Even though there are several formulas expressing the 6j-symbol as a sum of rational numbers we did not find in the literature any explicit definition of it in terms of the representation theory of $\mathfrak{sl}(2)$, as this theorem does. Our proof is based on a technical computation (performed in §11). P. Etingof has told us that he was aware of the result stated in this theorem and that it should be possible to obtain a proof of it based only on the original definition of the 6j-symbol and the representation theory of $\mathfrak{sl}(2)$.

In this paper, we use Theorem 1.3 to determine when the Lie algebra $\mathcal{L}$ mentioned above is abelian and we obtain the following result.

Theorem 1.4. *Let a, b, c and m be non-negative integers such that $V(m)$ is an $\mathfrak{sl}(2)$-submodule of both $V(a)\otimes V(b)$ and $V(b)\otimes V(c)$. Let $\mathcal{J}$ be the image of $\Lambda^2(V(m))$ under the map*

$$V(m)\otimes V(m)\to\mathrm{Hom}(V(b),V(a))\otimes\mathrm{Hom}(V(c),V(b))\to\mathrm{Hom}(V(c),V(a)).$$

Then the following conditions are equivalent:

(1) *$\mathcal{L}$ is abelian.*
(2) *$\mathcal{J}=0$.*
(3) $\begin{Bmatrix} \frac{m}{2} & \frac{k}{2} & \frac{m}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{Bmatrix}=0$ *for all non-negative integers k satisfying $k\equiv 2m-2 \mod 4$.*
(4) *Up to a swap of a and c we have: $c=0$, $b=m$, $a\equiv 2m \mod 4$ and $a\leq 2m$; or $b=c+m$ and $a=c+2m$.*
(5) $\mathrm{Hom}_{\mathfrak{sl}(2)}\big(\Lambda^2V(m),\mathrm{Hom}(V(c),V(a))\big)=0$.
(6) *There is a uniserial $\mathfrak{sl}(2)\ltimes V(m)$-module with socle factors $V(a),V(b),V(c)$.*

The proof of this theorem requires the determination of non-trivial zeros of the $6j$-symbol within certain parameters. Finding non-trivial zeros of the $6j$-symbol, is in general, a very difficult problem (see for instance [L], [R] or [ZR]) and we think that the above theorem might have applications to it.

2. Matrix recognition of uniserial modules

Let $\mathfrak{g}$ be a Lie algebra with solvable radical $\mathfrak{r}$ and Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$, and fix a representation $T : \mathfrak{g} \to \mathfrak{gl}(V)$. Given a basis B of V we let $M_B : \mathfrak{g} \to \mathfrak{gl}(d)$, $d = \dim(V)$, stand for the corresponding matrix representation.

By an $\mathfrak{s}$-basis of V of type 1 we understand a basis of the form $B = B_1 \cup \cdots \cup B_n$, where each B_i is a basis of an $\mathfrak{s}$-submodule W_i of V, and

$$0 \subset W_1 \subset W_1 \oplus W_2 \subset W_1 \oplus W_2 \oplus W_3 \subset \cdots \subset W_1 \oplus \cdots \oplus W_n = V \tag{2.1}$$

is the socle series of V. We likewise define an $\mathfrak{s}$-basis of type 2 by requiring that (2.1) be a composition series of V. Since V is completely reducible as an $\mathfrak{s}$-module it is clear that bases of both types exist. In either case B gives rise to a sequence $V_0, V_1, ..., V_n$ of $\mathfrak{g}$-modules defined by $V_0 = 0$ and $V_i = W_1 \oplus \cdots \oplus W_i$ for $1 \le i \le n$.

Lemma 2.1. *The ideal $[\mathfrak{g}, \mathfrak{r}]$ annihilates every irreducible $\mathfrak{g}$-module.*

Proof. An elementary proof can be found, for instance, in Lemma 2.4 of [CS1]. □

Corollary 2.2. *If B is any $\mathfrak{s}$-basis of V then $M_B(s)$ is block diagonal and $M_B(r)$ is strictly block upper triangular for all $s \in \mathfrak{s}$ and $r \in [\mathfrak{g}, \mathfrak{r}]$.*

Lemma 2.3. *If B is an $\mathfrak{s}$-basis of type 1 then none of the blocks in the first superdiagonal of $M_B(\mathfrak{r})$ is identically 0.*

Proof. Let B and $W_1, ..., W_n$, $V_0, V_1, ..., V_n$ be as in (2.1). Let $2 \le i \le n$ and suppose, if possible, that the block $(i-1, i)$ of $M_B(\mathfrak{r})$ is identically 0. It follows easily that $W_i \oplus V_{i-2}$ is a submodule of V. Clearly $(W_i \oplus V_{i-2})/V_{i-2}$ and V_{i-1}/V_{i-2} are non-zero submodules of V/V_{i-2} having trivial intersection. This contradicts the fact that the socle of V/V_{i-2} is V_{i-1}/V_{i-2}. □

Theorem 2.4. *The $\mathfrak{g}$-module V is uniserial if and only if given any $\mathfrak{s}$-basis B of type 2 none of the blocks in the first superdiagonal of $M_B(\mathfrak{r})$ is identically 0. If $\mathfrak{g}$ is perfect and there exists one $\mathfrak{s}$-basis B of type 2 such that none of the blocks in the first superdiagonal of $M_B(\mathfrak{r})$ is identically 0 then V is uniserial.*

Proof. If V is uniserial an $\mathfrak{s}$-basis of type 2 is also of type 1, so Lemma 2.3 applies. If V is not uniserial then some factor of its socle series is not irreducible. This factor is a completely reducible $\mathfrak{g}$-module, which easily yields an $\mathfrak{s}$-basis B of type 2 with at least one block in the first superdiagonal of $M_B(\mathfrak{r})$ identically 0.

Suppose next $\mathfrak{g}$ is perfect and let B be an $\mathfrak{s}$-basis of type 2 such that none of the blocks in the first superdiagonal of $M_B(\mathfrak{r})$ are identically 0. As indicated above, B gives rise to a series of $\mathfrak{s}$-modules $W_1, ..., W_n$ and $\mathfrak{g}$-modules $V_0, V_1, ..., V_n$ in such a way that (2.1) composition series of V. We will show that (2.1) is in fact the socle series of V.

Arguing by induction, it suffices to show that $\mathrm{soc}(V) = W_1$. Let U be a non-zero submodule of V. We wish to show that $W_1 \subseteq U$. Since $U \cap V \neq 0$ there exists a smallest index $1 \le j \le n$ such that $U \cap V_j \neq 0$. If $j = 1$ we are done by the irreducibility of W_1. Suppose, if possible, that $1 < j \le n$. The definition of j ensures the existence of $u\in U$ such that $u=w_1+\cdots+w_j$, where $w_i\in W_i$ for $1\le i\le j$ and $w_j\neq 0$. Let $r\in\mathfrak{r}$. Then $r\in[\mathfrak{g},\mathfrak{r}]$, since $\mathfrak{g}$ is perfect, so $ru=rw_2+\cdots+rw_j$, where $rw_i\in V_{i-1}$ for all $2\le i\le j$ by Lemma 2.1. In particular $\mathfrak{r}u\in U\cap V_{j-1}$. The choice of j forces forces $\mathfrak{r}u=0$, so $\mathfrak{r}w_j\subseteq V_{j-2}$. Let T be 0-weight space for the action of $\mathfrak{r}$ on V_j/V_{j-2}. As $\mathfrak{r}$ is an ideal of $\mathfrak{g}$, the subspace T is $\mathfrak{s}$-invariant. Since $w_j+V_{j-2}\in T$ the $\mathfrak{s}$-submodule of V_j/V_{j-2} generated by w_j+V_{j-2} is contained in T, i.e., $(W_j+V_{j-2})/V_{j-2}\subseteq T$, which means $\mathfrak{r}W_j\subseteq V_{j-2}$, a contradiction. □

Note 2.5. Let $\mathfrak{g}$ be any imperfect Lie algebra. Then there is a non-uniserial $\mathfrak{g}$-module with an $\mathfrak{s}$-basis B of type 2 such that none of the blocks in the first superdiagonal of $M_B(\mathfrak{r})$ is identically 0. It suffices to find a counterexample when $\mathfrak{g}=\mathbb{C}x$ is one dimensional, in which case we can take $x\mapsto\begin{pmatrix}1&1\\0&0\end{pmatrix}$.

Lemma 2.6. *If U is a subspace of V let $U^0=\{f\in V^*\mid f(U)=0\}$. Then $U\mapsto U^0$ is an inclusion reversing bijective correspondence from the $\mathfrak{g}$-submodules of V to those of V^*. Moreover, if $U\subseteq W$ are $\mathfrak{g}$-submodules of V then $U^0/W^0\cong(W/U)^*$ via $f+W^0\mapsto\widetilde{f}$, where $\widetilde{f}(w+U)=f(w)$. In particular, if $0=V_0\subset V_1\subset\cdots\subset V_n=V$ is a (resp. the only) composition series of V, with composition factors $X_1,\dots,X_n$, then $0=V_n^0\subset\cdots\subset V_1^0\subset V_0^0=V^*$ is a (resp. the only) composition series of V^*, with composition factors $X_n^*,\dots,X_1^*$.*

Proof. Use the natural isomorphism of $\mathfrak{g}$-modules $V\to V^{**}$. □

Note 2.7. *Lemma 2.1 through Theorem 2.4 are valid, mutatis mutandis, for an arbitrary finite dimensional complex associative algebra $\mathfrak{A}$. Both $\mathfrak{r}$ and $[\mathfrak{g},\mathfrak{r}]$ are to be replaced by the Jacobson radical $\mathfrak{J}$ of $\mathfrak{A}$, and $\mathfrak{s}$ by a semisimple subalgebra of $\mathfrak{A}$ complementing $\mathfrak{J}$, whose existence is ensured by the Wedderburn-Malcev theorem.*

Lemma 2.6 is also valid if $\mathfrak{A}$ has an involution $a\mapsto a^*$ and we make V^* into an $\mathfrak{A}$-module via $(af)(v)=f(a^*v)$.

3. Admissible sequences

In this section $\mathfrak{g}=\mathfrak{s}\ltimes\mathfrak{r}$, where $\mathfrak{s}$ and $\mathfrak{r}$ are arbitrary Lie algebras.

Let $W_1,...,W_n$ be $\mathfrak{s}$-modules and set $V=W_1\oplus\cdots\oplus W_n$. Let $T_i:\mathfrak{s}\to\mathfrak{gl}(W_i)$ and $T:\mathfrak{s}\to\mathfrak{gl}(V)$ stand for the associated representations. As is well known, $\mathfrak{gl}(V)$ becomes an $\mathfrak{s}$-module via $s\cdot f=[T(s),f]$, for $s\in\mathfrak{s}$ and $f\in\mathfrak{gl}(V)$.

Suppose first that $X:\mathfrak{g}\to\mathfrak{gl}(V)$ is a representation extending T and denote by $Y:\mathfrak{r}\to\mathfrak{gl}(V)$ the restriction of X to $\mathfrak{r}$. Then

$$Y(s\cdot r)=Y([s,r])=X([s,r])=[X(s),X(r)]=[T(s),Y(r)]=s\cdot Y(r),$$

i.e., Y is a homomorphism of $\mathfrak{s}$-modules.

Suppose conversely that $Y:\mathfrak{r}\to\mathfrak{gl}(V)$ is a homomorphism of $\mathfrak{s}$-modules and let $X:\mathfrak{g}\to\mathfrak{gl}(V)$ be the linear extension of T and Y to $\mathfrak{g}$. Then

$$X([s,r])=Y([s,r])=Y(s\cdot r)=s\cdot Y(r)=[T(s),Y(r)]=[X(s),X(r)],$$

i.e., X preserves all brackets $[s,r]$. As T is a Lie homomorphism, X also preserves all brackets $[s_1,s_2]$. But Y, and hence X, need not preserve the brackets $[r_1,r_2]$.

In any case, we will identify $\mathfrak{gl}(V)$ with $\bigoplus_{1\le i,j\le n}\mathrm{Hom}(W_i,W_j)$ as $\mathfrak{s}$-modules by interpreting each linear map $W_i\to W_j$ as a linear map $V\to V$ that is 0 on all summands W_k of V with $k\neq i$. Suppose we are given $n(n-1)/2$ $\mathfrak{s}$-homomorphisms

$f_{i,j} : \mathfrak{r} \to \mathrm{Hom}(W_i, W_j)$, where $i > j$, and let $Y : \mathfrak{r} \to \mathfrak{gl}(V)$ be the $\mathfrak{s}$-homomorphism corresponding to them. At this point we make the simplifying assumption that $\mathfrak{r}$ be abelian. Then $Y([r,t]) = 0$ for all $r, t \in \mathfrak{r}$. On the other hand, in order to have $[Y(r), Y(t)] = 0$ it is necessary that the following maps vanish:

$$f_{i+1,i}(r)f_{i+2,i+1}(t) - f_{i+1,i}(t)f_{i+2,i+1}(r) \in \mathrm{Hom}(W_{i+2}, W_i), \quad 1 \le i \le n-2. \tag{3.1}$$

Equivalently, the Lie subalgebra of $\mathfrak{gl}(V)$ generated by all elements of $\mathfrak{gl}(V)$ of the form $f_{2,1}(r) + \cdots + f_{n,n-1}(r)$, $r \in \mathfrak{r}$, must be abelian. Obviously this condition is also sufficient if restrict our list of starting maps to $f_{2,1}, \ldots, f_{n,n-1}$.

Now each of the maps $\mathfrak{r} \times \mathfrak{r} \to \mathrm{Hom}(W_{i+2}, W_i)$ defined by (3.1) is alternating, thereby giving rise to a linear map $\Lambda^2\mathfrak{r} \to \mathrm{Hom}(W_{i+2}, W_i)$. A simple calculation shows that this is a homomorphism of $\mathfrak{s}$-modules. Thus a sufficient condition for the vanishing of (3.1) is $\mathrm{Hom}_{\mathfrak{s}}\big(\Lambda^2\mathfrak{r}, W_{i+2}^* \otimes W_i\big) = 0$ for all $i = 1, \ldots, n-2$. The necessity of this condition is examined in §9. Taking into account the preceding discussion and Theorem 2.4 the following criterion is established.

Proposition 3.1. *Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ be a Levi decomposition and let $W_1, ..., W_n$ be a sequence of irreducible $\mathfrak{s}$-modules.*

(a) Suppose that $W_1, ..., W_n$ is an admissible sequence (as defined in §1). Then $\mathrm{Hom}_{\mathfrak{s}}\big(\mathfrak{r}, W_{i+1}^* \otimes W_i\big) \neq 0$ for all $i = 1, \ldots, n-1$. In particular, if $\mathfrak{r}$ is irreducible, it must be a constituent of $W_2^* \otimes W_1$,...,$W_n^* \otimes W_{n-1}$.

(b) Assume $\mathfrak{g}$ is perfect and $\mathfrak{r}$ is abelian. Then $W_1, ..., W_n$ is admissible if and only if $\mathrm{Hom}_{\mathfrak{s}}\big(\mathfrak{r}, W_{i+1}^* \otimes W_i\big) \neq 0$ for all $i = 1, \ldots, n-1$ and for some choice of non-zero $\mathfrak{s}$-homomorphisms $f_{i+1,i} : \mathfrak{r} \to \mathrm{Hom}(W_{i+1}, W_i)$ the Lie subalgebra of $\mathfrak{gl}(V)$, $V = W_1 \oplus \cdots \oplus W_n$, generated by $f_{2,1}(r) + \cdots + f_{n,n-1}(r)$, $r \in \mathfrak{r}$, is abelian. In particular, if $\mathrm{Hom}_{\mathfrak{s}}\big(\Lambda^2\mathfrak{r}, W_{i+2}^* \otimes W_i\big) = 0$ for all $i = 1, \ldots, n-2$ and $\mathrm{Hom}_{\mathfrak{s}}\big(\mathfrak{r}, W_{i+1}^* \otimes W_i\big) \neq 0$ for all $i = 1, \ldots, n-1$ then $W_1, ..., W_n$ is admissible.

In regards to uniqueness, we have the following criterion.

Proposition 3.2. *Suppose $\mathfrak{g}$ is perfect, with Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$. Let $W_1, ..., W_n$ be irreducible $\mathfrak{s}$-modules satisfying:*

(a) $\dim(\mathrm{Hom}_{\mathfrak{s}}(\mathfrak{r}, W_{i+1}^* \otimes W_i)) = 1$ if $1 \le i < n$.

(b) $\dim(\mathrm{Hom}_{\mathfrak{s}}(\mathfrak{r}, W_j^* \otimes W_i)) = 0$ if $j - i \ge 2$.

Then there exists at most one $\mathfrak{g}$-module V, up to isomorphism, with socle factors $W_1, ..., W_n$.

Proof. Let V be one such a module. Since its socle factors are irreducible, V is uniserial. Let B be an $\mathfrak{s}$-basis of V. Since $\mathfrak{g}$ is perfect, Corollary 2.2 ensures that $M_B(\mathfrak{r})$ is strictly block upper triangular. By Lemma 2.3 none of the blocks in the first superdiagonal of $M_B(\mathfrak{r})$ are identically 0, while (b) guarantees that all other strictly upper triangular blocks of $M_B(\mathfrak{r})$ are identically 0. By (a) the blocks in the first superdiagonal of $M_B(\mathfrak{r})$ are uniquely determined up to a non-zero scalar (which depends only on the position of the block). Conjugating all $M_B(x)$, $x \in \mathfrak{g}$, by a suitable block diagonal matrix, with each block a scalar matrix, we can arbitrarily scale all blocks in the first superdiagonal. This yields the desired result. □

4. Existence and uniqueness of the uniserial module $Z(\lambda, b)$

The notation introduced here will be kept for the remainder of the paper. Let $\mathfrak{s}$ be a non-zero semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$, associated root system Φ, and fixed system of simple roots Π. The coroots $h_\alpha \in \mathfrak{h}$ associated to the simple roots $\alpha \in \Pi$ form a basis of $\mathfrak{h}$. The basis of $\mathfrak{h}^*$ dual to $\{h_\alpha \,|\, \alpha \in \Pi\}$ consists of the fundamental weights $\{\lambda_\alpha \,|\, \alpha \in \Pi\}$. Let Λ^+ stand for the dominant integral weights of $\mathfrak{h}$ associated to Π, i.e., the non-negative integral linear combinations of the fundamental weights λ_α. Given $\lambda, \mu \in \mathfrak{h}^*$ we declare $\lambda \leq \mu$ if $\mu - \lambda$ is a non-negative rational linear combination of simple roots. It is well-known that the inverse of the Cartan matrix has non-negative rational coefficients. It follows that all fundamental weights are strictly positive. Therefore, all non-zero dominant integral weights are strictly positive. This fact will be repeatedly and implicitly used below.

Let W stand for the Weyl group of Φ and write w_0 for the longest element of W, i.e., the one sending Π to $-\Pi$.

We fix $\mu \in \Lambda^+$ and let $V(\mu)$ stand for an irreducible $\mathfrak{s}$-module with highest weight μ. Define the dual weight $\mu^* = -w_0\mu \in \Lambda^+$, noting that $V(\mu)^* \cong V(\mu^*)$. We assume henceforth that that $\mu \neq 0$ and consider the perfect Lie algebra $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$, where $\mathfrak{r} = V(\mu)$. By the special case we mean the case $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m)$, $m \geq 1$.

Theorem 4.1. *Let $\lambda \in \Lambda^+$ and $b \geq 0$. Then, up to isomorphism, there exists one and only one uniserial $\mathfrak{g}$-module, say $Z(\lambda, b)$, that has length $b+1$ and socle factors $V(\lambda), V(\lambda+\mu^*), ..., V(\lambda+b\mu^*)$.*

Proof. In light of Proposition 3.1, existence follows provided

$$\mathrm{Hom}_{\mathfrak{s}}\big(V(\mu), V(\lambda^* + (i+1)\mu) \otimes V(\lambda + i\mu^*)\big) \neq 0, \quad i = 0, \dots, b-1$$

and

$$\mathrm{Hom}_{\mathfrak{s}}\big(\Lambda^2 V(\mu), V(\lambda^* + (i+2)\mu) \otimes V(\lambda + i\mu^*)\big) = 0, \quad i = 0, \dots, b-2.$$

Now

$$\mathrm{Hom}(V(\mu), V(\lambda^*+(i+1)\mu) \otimes V(\lambda+i\mu^*)) \cong V(\mu)^* \otimes V(\lambda^*+(i+1)\mu) \otimes V(\lambda+i\mu^*),$$

as $\mathfrak{s}$-modules, so

$$\mathrm{Hom}_{\mathfrak{s}}(V(\mu), V(\lambda^*+(i+1)\mu) \otimes V(\lambda+i\mu^*)) \cong \mathrm{Hom}_{\mathfrak{s}}(V(\lambda+(i+1)\mu^*), V(\mu^*) \otimes V(\lambda+i\mu^*))$$

as vector spaces. It is clear that the latter space is not only non-zero but in fact one dimensional.

Reasoning as above and using that fact that $(\Lambda^2 V(\mu))^* \cong \Lambda^2 V(\mu^*)$ we see that the vector space $\mathrm{Hom}_{\mathfrak{s}}(\Lambda^2 V(\mu), V(\lambda^* + (i+2)\mu) \otimes V(\lambda + i\mu^*))$ is isomorphic to $\mathrm{Hom}_{\mathfrak{s}}(V(\lambda + (i+2)\mu^*), \Lambda^2 V(\mu^*) \otimes V(\lambda + i\mu^*))$. But the latter is 0 since all weights of $\Lambda^2 V(\mu^*) \otimes V(\lambda + i\mu^*)$ are strictly less than $\lambda + (i+2)\mu^*$.

As for uniqueness, note that $\mathrm{Hom}_{\mathfrak{s}}\big(V(\mu), V(\lambda^*+j\mu) \otimes V(\lambda+i\mu^*)\big) = 0$ provided $j - i \geq 2$. This follows as above by observing that all weights of $V(\mu^*) \otimes V(\lambda+i\mu^*)$ are strictly less than $\lambda + j\mu^*$. Now apply Proposition 3.2. □

Corollary 4.2. *In the notation of Theorem 4.1, there exists one and only one uniserial $\mathfrak{g}$-module, namely $Z(\lambda, b)^*$, that has length $b+1$ and whose socle factors are $V(\lambda^* + b\mu), ..., V(\lambda^* + \mu), V(\lambda^*)$.*

Proof. Immediate consequence of Theorem 4.1 and Lemma 2.6. □

Note 4.3. *It is clear that all modules constructed in Theorem 4.1 and Corollary 4.2 are non-isomorphic from each other, except in the obvious case $b = 0$ and $\lambda = \lambda^*$, when $Z(\lambda, b) = V(\lambda) \cong V(\lambda)^* = Z(\lambda, b)^*$.*

Note 4.4. *In the special case* $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m)$ *these results read as follows. Given integers* $\ell, b \geq 0$ *there exists one and only one uniserial* $\mathfrak{g}$*-module having socle factors either* $V(\ell), V(\ell+m), ..., V(\ell+bm)$ *or* $V(\ell+bm), ..., V(\ell+m), V(\ell)$*. These modules will be respectively denoted by* $Z(\ell, b)$ *and* $Z(\ell, b)^*$*.*

Note 4.5. *Suppose* $\mathfrak{g}$ *is perfect with Levi decomposition* $\mathfrak{s} \ltimes \mathfrak{r}$ *such that* $[\mathfrak{r}, \mathfrak{r}] = 0$*. Let* W_1, W_2 *be irreducible* $\mathfrak{s}$*-modules. By Proposition 3.1 there exists a uniserial* $\mathfrak{g}$*-module* V *with socle factors* W_1, W_2 *if and only if* $\operatorname{Hom}_{\mathfrak{s}}\big(\mathfrak{r}, W_2^* \otimes W_1\big) \neq 0$*.*

Let $H = \operatorname{Hom}_{\mathfrak{s}}(\mathfrak{r}, W_2^* \otimes W_1)$ *and set* $P = P(H)$*, the associated projective space (i.e., the points of* P *are the the lines of* H *through the origin). It is not difficult to see that the isomorphism classes of such modules* V *are parametrized by the points of* P*. In particular, if* $\dim(H) > 1$ *there are infinitely many such classes.*

In the special case $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m)$ *the possibility* $\dim(H) > 1$ *never arises. Indeed, we have* $W_1 = V(a)$*,* $W_2 = V(b)$*, with* $0 \leq a \leq b$ *(otherwise consider* V^**). Our previous comments and the Clebsch-Gordan formula ensure that such* V *exists if and only if* m *is in the list of numbers* $b-a, b-a+2, ..., b+a-2, b+a$*. But then* V *will be unique since* $V(b) \otimes V(a)$ *is multiplicity free and Proposition 3.2 applies.*

5. An explicit matrix realization of the $\mathfrak{sl}(2) \ltimes V(m)$-module $Z(\ell, b)$

In this section we consider the special case $\mathfrak{g} = \mathfrak{sl}(2) \times V(m)$ and construct a matrix realization of $Z(\ell, b)$, where $\ell \geq 0$ and $b \geq 0$. Taking the opposite transpose of our representation yields a matrix version of $Z(\ell, b)^*$.

The Lie algebra $\mathfrak{g}$ has basis $e, h, f, v_0, v_1, ..., v_m$, subject to the following relations:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h, \tag{5.1}$$

$$[v_i, v_j] = 0, \quad 0 \leq i, j \leq m, \tag{5.2}$$

$$[h, v_i] = (m-2i)v_i, \quad [e, v_i] = (m-(i-1))v_{i-1}, \quad [f, v_i] = (i+1)v_{i+1}, \tag{5.3}$$

where $0 \leq i \leq m$ and $v_{-1} = 0 = v_{m+1}$.

For $a \geq 0$ let $H(a), E(a), F(a)$ be the matrices in $\mathfrak{gl}(a+1)$ all of whose entries are 0, except that the diagonal of $H(a)$ is $(a, a-2, \dots, -a+2, -a)$ and, if $a \geq 1$, the first superdiagonal of $E(a)$ is $(a, \dots, 2, 1)$ and the first subdiagonal of $F(a)$ is $(1, 2, \dots, a)$. Set $H(\ell, b) = H(\ell) \oplus \cdots \oplus H(\ell+bm)$, $E(\ell, b) = E(\ell) \oplus \cdots \oplus E(\ell+bm)$ and $F(\ell, b) = F(\ell) \oplus \cdots \oplus F(\ell+bm)$.

For $a \geq 0$ we define the $m+1$ rectangular matrices $W_0(a), ..., W_m(a)$, all of size $(a+1) \times (a+m+1)$, as follows:

$$W_0(a) = (0_{(a+1)\times m} I_{a+1}), W_1(a) = (0_{(a+1)\times(m-1)} I_{a+1} 0_{(a+1)\times 1}),$$
$$W_2(a) = (0_{(a+1)\times(m-2)} I_{a+1} 0_{(a+1)\times 2}), ..., W_m(a) = (I_{a+1} 0_{(a+1)\times m}).$$

We next scale these matrices to produce

$$V_i(a) = (-1)^i \binom{m}{i} W_i(a), \quad 0 \leq i \leq m.$$

For $0 \leq i \leq m$ let $V_i(\ell, b)$ be the block partitioned matrix all of whose blocks are equal to 0, except that if $b \geq 1$ the blocks along the first superdiagonal are $V_i(\ell), V_i(\ell+m), ..., V_i(\ell+(b-1)m)$.

Lemma 5.1. *The map* $h \mapsto H(\ell, b)$*,* $e \mapsto E(\ell, b)$*,* $f \mapsto F(\ell, b)$*,* $v_i \mapsto V_i(\ell, b)$*, where* $0 \leq i \leq m$*, defines a matrix representation of* $\mathfrak{g}$ *with associated module* $Z(\ell, b)$*.*

Proof. Since $H(\ell,b), E(\ell,b), F(\ell,b)$ satisfy (5.1) we see that the map $h \mapsto H(\ell,b)$, $e \mapsto E(\ell,b)$, $f \mapsto F(\ell,b)$ defines a matrix representation of $\mathfrak{sl}(2)$ whose associated module decomposes as $V(\ell) \oplus V(\ell+m) \oplus \cdots \oplus V(\ell+bm)$.

Given $a \geq 0$ let $U_i(a)$ be the matrix partitioned into 4 blocks, whose (1,2) block is $W_i(a)$ and all other blocks are 0. Direct calculation shows that

$$[H(a) \oplus H(a+m), U_0(a)] = mU_0(a), \quad [E(a) \oplus E(a+m), U_0(a)] = 0$$

and

$$[F(a) \oplus F(a+m), U_i(a)] = -(m-i)U_{i+1}(a),\ 0 \leq i \leq m. \tag{5.4}$$

Thus $U_0(a), \ldots, U_m(a)$ is a basis for an $\mathfrak{sl}(2)$-module, say S_a, isomorphic to $V(m)$. From (5.4) we get

$$f^i U_0(a) = (-1)^i m(m-1)\cdots(m-(i-1))U_i(a) = (-1)^i m!/(m-i)! U_i(a),\ \ 0 \leq i \leq m,$$

It follows that

$$f^i U_0(a)/i! = (-1)^i \binom{m}{i} U_i(a), \quad 0 \leq i \leq m$$

is a basis for S_a upon which h, e, f act as in (5.3). Hence $V_0(\ell,b), \ldots, V_m(\ell,b)$ is a basis of an $\mathfrak{sl}(2)$-module upon which h, e, f act via $H(\ell,b), E(\ell,b), F(\ell,b)$ as in (5.3).

Next we verify that the relations (5.2) are preserved. By means of the actions of e and f on $V_0(\ell,b), \ldots, V_m(\ell,b)$, it suffices to verify $[V_0(\ell,b), V_m(\ell,b)] = 0$. This easily reduces to the case $b = 2$, which is confirmed through a simple calculation.

We thus have a matrix representation of $\mathfrak{g}$. By Theorem 2.4 the associated module, say V, is uniserial with socle factors $V(\ell), V(\ell+m), ..., V(\ell+bm)$, so $V \cong Z(\ell,b)$ by Proposition 3.2. □

Here we present a matrix realization for $m = 2$ and $V \cong Z(1,2)$.

$$\left(\begin{array}{cc|cccc|cccccc}
h & e & v_2 & -2v_1 & v_0 & 0 & & & & & & \\
f & -h & 0 & v_2 & -2v_1 & v_0 & & & & & & \\
\hline
 & & 3h & 3e & 0 & 0 & v_2 & -2v_1 & v_0 & 0 & 0 & 0 \\
 & & f & h & 2e & 0 & 0 & v_2 & -2v_1 & v_0 & 0 & 0 \\
 & & 0 & 2f & -h & e & 0 & 0 & v_2 & -2v_1 & v_0 & 0 \\
 & & 0 & 0 & 3f & -3h & 0 & 0 & 0 & v_2 & -2v_1 & v_0 \\
\hline
 & & & & & & 5h & 5e & 0 & 0 & 0 & 0 \\
 & & & & & & f & 3h & 4e & 0 & 0 & 0 \\
 & & & & & & 0 & 2f & h & 3e & 0 & 0 \\
 & & & & & & 0 & 0 & 3f & -h & 2e & 0 \\
 & & & & & & 0 & 0 & 0 & 4f & -3h & e \\
 & & & & & & 0 & 0 & 0 & 0 & 5f & -5h
\end{array}\right)$$

A matrix realization of $Z(1,2)^*$ is obtained by taking the opposite transpose of the above matrix. A suitable change of basis presents a realization of $Z(1,2)^*$ as the following block upper triangular matrices:

$$\left(\begin{array}{cccccc|cccc|cc}
5h & 5e & 0 & 0 & 0 & 0 & 10v_0 & 0 & 0 & 0 & & \\
f & 3h & 4e & 0 & 0 & 0 & 4v_1 & 6v_0 & 0 & 0 & & \\
0 & 2f & h & 3e & 0 & 0 & v_2 & 6v_1 & 3v_0 & 0 & & \\
0 & 0 & 3f & -h & 2e & 0 & 0 & 3v_2 & 6v_1 & v_0 & & \\
0 & 0 & 0 & 4f & -3h & e & 0 & 0 & 6v_2 & 4v_1 & & \\
0 & 0 & 0 & 0 & 5f & -5h & 0 & 0 & 0 & 10v_2 & & \\
\hline
 & & & & & & 3h & 3e & 0 & 0 & 3v_0 & 0 \\
 & & & & & & f & h & 2e & 0 & 2v_1 & v_0 \\
 & & & & & & 0 & 2f & -h & e & v_2 & 2v_1 \\
 & & & & & & 0 & 0 & 3f & -3h & 0 & 3v_2 \\
\hline
 & & & & & & & & & & h & e \\
 & & & & & & & & & & f & -h
\end{array}\right)$$

6. Characterization of the uniserial modules $Z(\lambda,b)$ and $Z(\lambda,b)^*$

We adhere to the notation introduced in §4. Let V be a $\mathfrak{g}$-module. By a weight vector we mean a non-zero common eigenvector for the action of $\mathfrak{h}$ on V. A highest weight vector, or just a maximal vector, is a weight vector that is annihilated by all e_α, $\alpha\in\Pi$. The weight spaces $\mathfrak{r}_\mu$ and $\mathfrak{r}_{w_0\mu}$ as well as the root spaces $\mathfrak{s}_\alpha$, $\alpha\in\Pi$, are all one dimensional and we fix a spanning vector for each of them, say $u_\mu\in\mathfrak{r}_\mu$, $u_{w_0\mu}\in\mathfrak{r}_{w_0\mu}$ and $e_\alpha\in\mathfrak{s}_\alpha$.

Lemma 6.1. *Let V be a $\mathfrak{g}$-module and let $v\in V$.*

(a) Let $\alpha\in\Pi$. Then $e_\alpha u_{w_0\mu}^i v=0$ for all $i\geq 0$ if and only if $e_\alpha v=0$ and $[e_\alpha,u_{w_0\mu}]v=0$.

(b) Let $i\geq 0$. If $e_\alpha u_{w_0\mu}^i v=0$ for all $\alpha\in\Pi$ then $\mathfrak{r}u_{w_0\mu}^i v$ is included in the $\mathfrak{s}$-submodule of V generated by $u_{w_0\mu}^{i+1}v$.

(c) Let $\alpha\in\Pi$. If $e_\alpha v=0$ then $e_\alpha u_\mu^i v=0$ for all $i\geq 0$.

Proof. (a) Necessity is obvious. As for sufficiency, we argue by induction. The base case $i=0$ is given. Suppose the result is true for some $i\geq 0$. Since $\mathfrak{r}$ is abelian

$$e_\alpha u_{w_0\mu}^{i+1}v=u_{w_0\mu}e_\alpha u_{w_0\mu}^i v+[e_\alpha,u_{w_0\mu}]u_{w_0\mu}^i v=0+u_{w_0\mu}^i[e_\alpha,u_{w_0\mu}]v=0.$$

(b) It suffices to prove this for elements of $\mathfrak{r}$ of the form $[e_{\alpha_1},\dots,[e_{\alpha_s},u_{w_0\mu}]\dots]$ as these span $\mathfrak{r}$. If $s=0$ the multibracket reduces to $u_{w_0\mu}$ and the result is true by definition. This case and the stated hypothesis yield the result by induction.

(c) Since $[e_\alpha,u_\mu]=0$ we obtain $e_\alpha u_\mu^i v=u_\mu^i e_\alpha v=0$. ☐

Theorem 6.2. *Let V be a $\mathfrak{g}$-module, $\lambda\in\Lambda^+$, $b\geq 0$ and let $u=u_{w_0\mu}$ (resp. $u=u_\mu$). Then $V\cong Z(\lambda,b)$ (resp. $V\cong Z(\lambda,b)^*$) if and only if there is a vector $v\in V$ satisfying conditions (C1),(C2),(C3) (resp. (C1),(C2),(C3)*) stated below.*

- *(C1) v is a maximal vector of weight $\lambda+b\mu^*$ (resp. λ^*) that generates V as a $\mathfrak{g}$-module.*
- *(C2) $u^b v\neq 0$, $u^{b+1}v=0$.*
- *(C3) $[e_\alpha,u_{w_0\mu}]v=0$ for all $\alpha\in\Pi$.*
- *(C3)* $\mathfrak{r}u_\mu^i v$ is included in the $\mathfrak{s}$-module generated by $u_\mu^{i+1}v$ for all $0\leq i\leq b$.*

Moreover, in such case v is unique up to scaling.

Proof. This naturally breaks into two parts.

Sufficiency. Let W_i be the $\mathfrak{s}$-submodule of V generated by $u^i v$ for $i=0,\dots,b$. Then (C1)-(C3) and Lemma 6.1 ensure that $u^i v$ is a maximal vector of weight $\lambda+(b-i)\mu^*$ (resp. $\lambda^*+i\mu$), so $W_i\cong V(\lambda+(b-i)\mu^*)$ (resp. $W_i\cong V(\lambda^*+i\mu)$). Let $V_{b+1}=0$ and set

$$V_{b-i}=W_b\oplus\cdots\oplus W_{b-i},\quad 0\leq i\leq b.$$

We claim that V_{b-i} is a $\mathfrak{g}$-submodule of V for all $i=0,\dots,b$. It suffices to show that $\mathfrak{r}V_{b-i}\subset V_{b-(i-1)}$. Since the 0-weight space of $\mathfrak{r}$ acting on any $\mathfrak{g}$-module is $\mathfrak{s}$-invariant and $V_{b-i}/V_{b-(i-1)}$ is $\mathfrak{s}$-irreducible, it suffices to prove that $\mathfrak{r}$ has a common 0-eigenvector in $V_{b-i}/V_{b-(i-1)}$. We contend that $u^{b-i}v+V_{b-(i-1)}\in V_{b-i}/V_{b-(i-1)}$ is non-zero and annihilated by $\mathfrak{r}$. That $u^{b-i}v\notin V_{b-(i-1)}$ follows from the fact that $u^{b-i}v$ is a maximal vector and none of the maximal vectors of $V_{b-(i-1)}$ has the same weight as $u^{b-i}v$. It follows from (C2) and Lemma 6.1 (resp. (C2) and (C3)*) that $\mathfrak{r}u^{b-i}v\in V_{b-(i-1)}$. This proves our contention and hence the claim.

The $\mathfrak{g}$-invariance of V_0 and (C1) now yield $V = V_0$. We have shown that

$$0 = V_{b+1} \subset V_b \subset \cdots \subset V_0 = V$$

is a composition series of the $\mathfrak{g}$-module V, with composition factors

$$V_{b-i}/V_{b-(i-1)} \cong V(\lambda + i\mu^*) \quad (\text{resp. } V_{b-i}/V_{b-(i-1)} \cong V(\lambda^* + (b-i)\mu)).$$

Since $uu^{b-i}v \in W_{b-(i-1)}$, $0 < i \le b$, Theorem 2.4 ensures that V is uniserial with socle factors $V(\lambda), V(\lambda+\mu^*), ..., V(\lambda+b\mu^*)$ (resp. $V(\lambda^*+b\mu), ..., V(\lambda^*+\mu), V(\lambda^*)$). From the uniqueness part of Theorem 4.1 we conclude that $V \cong Z(\lambda, b)$ (resp. $V \cong Z(\lambda, b)^*$)

Necessity. By assumption the socle series of V, say

$$0 = V_{b+1} \subset V_b \subset \cdots \subset V_0 = V,$$

has irreducible factors

$$V_{b-i}/V_{b-(i-1)} \cong V(\lambda + i\mu^*) \quad (\text{resp. } V_{b-i}/V_{b-(i-1)} \cong V(\lambda^* + (b-i)\mu).$$

Up to scaling V has a unique maximal vector, say v_i, of weight $\lambda + i\mu^*$ (resp. $\lambda^* + (b-i)\mu)$). In any uniserial module, a vector belonging only to the last term of the socle series generates the entire module. Hence V is generated by $v = v_b \notin V_1$.

We know from Lemma 2.1 that $\mathfrak{r}V_{b-i} \subseteq V_{b-(i-1)}$. In particular $u^{b+1}v = 0$.

Suppose next $V \cong Z(\lambda, b)$. If $0 \le i \le b$ and $u^i v \ne 0$ then $u^i v$ is a maximal vector since its weight, namely $\lambda + (b-i)\mu^*$, is the highest in V_i. Since $v \ne 0$ and $u^{b+1}v = 0$ there is an index i satisfying $0 \le i \le b$, $u^i v \ne 0$ and $u^{i+1}v = 0$. Our preceding comment implies $e_\alpha u^i v = 0$ for all $\alpha \in \Pi$. This and $uu^i v = 0$ yield $\mathfrak{r}u^i v = 0$, that is, $u^i v \in \mathrm{soc}(V) \cong V(\lambda)$. But the highest weight in $V(\lambda)$ is λ and $u^i v$ has weight $\lambda + (b-i)\mu^*$, so $i = b$. Thus $u^i v$ is a maximal vector for all $0 \le i \le b$.

Suppose finally $V \cong Z(\lambda, b)^*$. Thus V has socle factors $V(\lambda^* + b\mu), \dots, V(\lambda^*)$ and we view these as $\mathfrak{s}$-submodules of V. We claim that u sends a maximal vector of $V(\lambda^* + i\mu)$ into one of $V(\lambda^* + (i+1)\mu)$ for all $0 \le i < b$. By assumption $\mathfrak{r}V(\lambda^* + i\mu)$ is non-zero and included in $V(\lambda^* + (i+1)\mu)$. Since $\mathfrak{r}$ is an irreducible $\mathfrak{s}$-module, it follows that $u_\mu V(\lambda^* + i\mu) \ne 0$, so there is a weight vector w in $V(\lambda^* + i\mu)$ not annihilated by u_μ. Since $[e_\alpha, u_\mu] = 0$ for all $\alpha \in \Pi$ by repeatedly applying the e_α to $u_\mu w$ we may assume that $u_\mu w$ is a maximal vector of $V(\lambda^* + (i+1)\mu)$, in which case w must have weight $\lambda^* + i\mu$. This proves our claim. Since v is a maximal vector of $V(\lambda^*)$ we deduce that $u^i v$ is a maximal vector of $V(\lambda^* + i\mu)$ for all $0 \le i \le b$. Moreover, $\mathfrak{r}u^b v \subseteq \mathfrak{r}V(\lambda^* + b\mu) = 0$ and $\mathfrak{r}u^i v \subseteq \mathfrak{r}V(\lambda^* + i\mu) \subseteq V(\lambda^* + (i+1)\mu)$, which is $\mathfrak{s}$-generated by $u^{i+1}v$ for all $0 \le i < b$. □

7. A natural construction of the $\mathfrak{s} \ltimes V(\mu)$-module $Z(0, b)$

As mentioned in the Introduction, there have been recent constructions of indecomposable modules for a Lie algebra $\mathfrak{g}$ by embedding $\mathfrak{g}$ into a semisimple Lie algebra $\mathfrak{t}$ and restricting an irreducible $\mathfrak{t}$-module to $\mathfrak{g}$. In this section we use the characterization given in Theorem 6.2 to produce $Z(0, b)$ in the spirit just described.

We adopt the notation introduced at the beginning of §4 and §6. In particular, $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$, where $\mathfrak{r} = V(\mu)$. Consider the $\mathfrak{s}$-module $W = \mathfrak{r}^* \oplus \mathbb{C}w$, where $\mathfrak{s}$ acts trivially on $\mathbb{C}w$. We make W into a $\mathfrak{g}$-module as follows:

$$(s + r)(f + aw) = sf + f(r)w, \quad s \in \mathfrak{s}, r \in \mathfrak{r}, f \in \mathfrak{r}^*, a \in \mathbb{C}. \tag{7.1}$$

This gives a representation $\mathfrak{g} \to \mathfrak{sl}(W)$ (which is an embedding if $\mathfrak{r}$ is faithful and, in particular, if $\mathfrak{s}$ is simple).

Let $f \in \mathfrak{r}^*$ be the only linear functional such that $f(u_{w_0\mu}) = 1$ and $f(z) = 0$ for any z belonging to a weight space in $\mathfrak{r}$ of weight different from $w_0\mu$. It is clear from this definition that

$$f \in (\mathfrak{r}^*)_{-w_0\mu} = (\mathfrak{r}^*)_{\mu^*}.$$

Fix $b \geq 0$ and let $X = S^b(W)$, the bth symmetric power of W. This is an irreducible $\mathfrak{sl}(W)$-module. We view X as a $\mathfrak{g}$-module via the Lie homomorphism $\mathfrak{g} \to \mathfrak{sl}(W)$. Let V be the $\mathfrak{g}$-submodule of X generated by f^b.

Theorem 7.1. *The $\mathfrak{g}$-module V is isomorphic to $Z(0, b)$. Moreover, the $\mathfrak{g}$-module X is indecomposable, with trivial socle, full socle series*

$$0 = \mathrm{soc}^0(X) \subset \mathrm{soc}^1(X) \subset \mathrm{soc}^2(X) \subset \cdots \subset \mathrm{soc}^{b+1}(X) = X, \tag{7.2}$$

and socle factors

$$\mathrm{soc}^{i+1}(X)/\mathrm{soc}^i(X) \cong S^i(\mathfrak{r}^*) \cong S^i(V(\mu^*)), \quad 0 \leq i \leq b.$$

Thus the socle factors of V, namely $V(0), V(\mu^*), ..., V(b\mu^*)$, are precisely the top summands of the socle factors of X. In particular, X itself need not be uniserial, but it is so in the very special case $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(1)$, when $X = V$.

Proof. The first assertion follows at once from Theorem 6.2 applied to $v = f^b$. Indeed, v is clearly a maximal vector of V of weight $b\mu^*$ that generates V as a $\mathfrak{g}$-module. Moreover,

$$u^b f^b = b! w^b \neq 0 \text{ and } u^{b+1} f^b = 0, \quad u = u_{w_0\mu}$$

and the very definition of f gives

$$[e_\alpha, u]f = f([e_\alpha, u])w = 0, \text{ so } [e_\alpha, u]f^b = 0, \quad \alpha \in \Pi, \ u = u_{w_0\mu}.$$

As remarked in the Introduction, $\mathrm{soc}^{i+1}(X)/\mathrm{soc}^i(X)$ is the 0-weight space for the action of $\mathfrak{r}$ on $X/\mathrm{soc}^i(X)$. The formula (7.1) makes it clear that

$$\mathrm{soc}^{i+1}(X) = w^b S^0(\mathfrak{r}^*) \oplus w^{b-1} S^1(\mathfrak{r}^*) \oplus \cdots \oplus w^{b-i} S^i(\mathfrak{r}^*), \quad 0 \leq i \leq b$$

which gives the isomorphisms of $\mathfrak{s}$-modules

$$\mathrm{soc}^{i+1}(X)/\mathrm{soc}^i(X) \cong w^{b-i} S^i(\mathfrak{r}^*) \cong S^i(\mathfrak{r}^*), \quad 0 \leq i \leq b.$$

The remaining assertions now follow immediately. □

The special case $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m)$ can be translated as follows. Let $\mathfrak{g} \to \mathfrak{gl}(m+2)$ be the matrix representation defined in §5 for the uniserial $\mathfrak{g}$-module with socle factors $V(0), V(m)$.

Let S be the algebra of polynomials in $m + 2$ variables $X_1, ..., X_{m+2}$. This is a module for $\mathfrak{gl}(m + 2)$, where each basic matrix E_{ij} acts via derivations on S by means of $M_{X_i} \circ \partial/\partial X_j$, i.e., partial differentiation with respect to X_j followed by multiplication by X_i.

Given $b \geq 0$, the subspace X of S of all homogeneous polynomials of degree b is $\mathfrak{gl}(m + 2)$-stable. We may thus view X as a $\mathfrak{g}$-module via $\mathfrak{g} \to \mathfrak{gl}(m + 2)$, and consider the $\mathfrak{g}$-submodule V of X generated by X_2^b. It follows immediately from Theorem 6.2 that $V \cong Z(0, b)$.

Note finally that in the very special case $m = 1$ we have $X = V$. In this case, by factoring the terms of the socle series of V we obtain all $\mathfrak{g}$-modules $Z(\ell, b)$, $\ell \geq 0$.

8. Other uniserial modules

The uniserial modules $Z(\lambda, b)$ and their duals are not the only possible ones, even in the special case $\mathfrak{g} = \mathfrak{sl}(2) \times V(m)$. We already noted this when dealing with uniserial modules of length in Note 4.5. In this section we produce further exceptions, in this case of lengths 3 and 4. It is be shown in §10 that no other exceptions exist.

We maintain throughout the notation introduced in §4. In particular, $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$, where $\mathfrak{r} = V(\mu)$.

Lemma 8.1. *Let $\lambda \in \Lambda^+$. There is a unique uniserial $\mathfrak{g}$-module with socle factors $V(0), V(\mu^*), V(\lambda)$ provided $V(\lambda)$ occurs once in $V(\mu^*) \otimes V(\mu^*)$ but not in $\Lambda^2(V(\mu^*))$, or equivalently, when $V(\lambda)$ occurs once in $S^2(V(\mu^*))$ and $V(\mu^*) \otimes V(\mu^*)$.*

Proof. This follows easily from Propositions 3.1 and 3.2, except for uniqueness when $\lambda = \mu^*$. In this case there is a uniserial $\mathfrak{g}$-module V with socle factors $V(0), V(\mu^*), V(\mu^*)$ and we need to establish the uniqueness of V up to isomorphism.

Let B be an $\mathfrak{s}$-basis of V which yields identical matrix representations of $\mathfrak{s}$ on W_2, W_3 in the notation of (2.1). Our hypotheses ensure that each of the blocks (1,2), (2,3), (1,3) of $M_B(\mathfrak{r})$ is completely determined, up to a scalar, once B is fixed. Moreover, this scalar must be non-zero in the first two cases. The (1,3) block of $M_B(\mathfrak{r})$ is a scalar multiple, say by $a \in \mathbb{C}$, of the (1,2) block. Conjugating by the block matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$$

we obtain a new matrix representation which is identical to the first except that the (1,3) block of $M_B(\mathfrak{r})$ is now surely 0. The result follows. □

Under the hypotheses of Lemma 8.1, there is a unique uniserial module with socle factors $V(\lambda^*), V(\mu), V(0)$, dual to the above. Clearly, there is at most one uniserial module of both types, namely the self-dual module with socle factors $V(0), V(\mu), V(0)$, where $\mu = \mu^*$. We next find explicit conditions for the existence of such a module. Let $\mu = \mu^*$. Then, up to scaling, there is one and only one non-zero $\mathfrak{s}$-invariant bilinear form $\phi : V(\mu) \times V(\mu) \to \mathbb{C}$, necessarily non-degenerate. By our discussion in §3 there is a uniserial $\mathfrak{g}$-module with socle factors $V(0), V(\mu), V(0)$ if and only if the $\mathfrak{s}$-homomorphism $F : \Lambda^2 V(\mu) \to \mathrm{Hom}(\mathbb{C}, \mathbb{C})$ associated to (3.1) is trivial. But $F_{v \wedge w}(a) = a(\phi(v, w) - \phi(w, v))$ for all $a \in \mathbb{C}$ and $v, w \in V(\mu)$. This is 0 if and only if ϕ is symmetric. We have proven

Lemma 8.2. *If $\mu = \mu^*$ there is a uniserial module with socle factors $V(0), V(\mu), V(0)$ if and only if the non-zero $\mathfrak{s}$-invariant bilinear form on $V(\mu)$ is symmetric.*

Note 8.3. *Suppose that $\mu = \mu^*$ and the hypotheses of Lemma 8.1 are met for $\lambda = \mu$, i.e., $V(\mu)$ occurs once in $S^2(V(\mu))$ and $V(\mu) \otimes V(\mu)$. Then, clearly, there is a uniserial module with socle factors $V(0), V(\mu), V(\mu), V(0)$. However, in this case the isomorphism classes of such modules are parametrized by the complex numbers. Indeed, once all diagonal blocks of $M_B(\mathfrak{s})$ as well as the first superdiagonal blocks $M_B(\mathfrak{r})$ have been fixed and the block (1,3) of $M_B(\mathfrak{r})$ has been cleared, there is no way to modify the block (2,4) of $M_B(\mathfrak{r})$.*

We next adapt the above observations to the special case $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m)$. Recall that $V(m) \otimes V(m) = V(2m) \oplus V(2m-2) \oplus \cdots \oplus V(2) \oplus V(0)$, where

$S^2(V(m)) = V(2m) \oplus V(2m-4) \oplus \cdots$ and $\Lambda^2(V(m)) = V(2m-2) \oplus V(2m-6) \oplus \cdots$. Thus $V(\ell)$ appears in $S^2(V(m))$ if and only if $\ell \leq 2m$ and $2m \equiv \ell \mod 4$. In particular, $V(0)$ appears in $S^2(V(m))$ (that is, the non-zero $\mathfrak{sl}(2)$-invariant bilinear form on $V(m)$ is symmetric) if and only if m is even. We have shown

Lemma 8.4. *Let $\ell \geq 0$. Then there is a unique uniserial module with socle factors $V(0), V(m), V(\ell)$ if $\ell \leq 2m$ and $2m \equiv \ell \mod 4$, in which case the dual module is uniserial with socle factors $V(\ell), V(m), V(0)$. Moreover, there is a unique uniserial module with socle factors $V(0), V(m), V(0)$ if and only if m is even. Furthermore, if $m \equiv 0 \mod 4$ there is a parametrization by $\mathbb{C}$ of the isomorphism classes of uniserial modules with the same socle factors $V(0), V(m), V(m), V(0)$.*

Example 8.5. A matrix realization of a uniserial $\mathfrak{sl}(2) \ltimes V(3)$-module with socle factors $V(0)$, $V(3)$, $V(2)$ is:

$$\left(\begin{array}{c|cccc|ccc}
0 & -v_3 & 3v_2 & -3v_1 & v_0 & & & \\
\hline
 & 3h & 3e & 0 & 0 & -3v_1 & 3v_0 & 0 \\
 & f & h & 2e & 0 & -2v_2 & v_1 & v_0 \\
 & 0 & 2f & -h & e & -v_3 & -v_2 & 2v_1 \\
 & 0 & 0 & 3f & -3h & 0 & -3v_3 & 3v_2 \\
\hline
 & & & & & 2h & 2e & 0 \\
 & & & & & f & 0 & e \\
 & & & & & 0 & 2f & -2h
\end{array}\right).$$

The one parameter family, parametrized by $z \in \mathbb{C}$, of non-isomorphic uniserial $\mathfrak{sl}(2) \ltimes V(4)$-modules with socle factors $V(0)$, $V(4)$, $V(4)$, $V(0)$ is given by:

$$\left(\begin{array}{c|ccccc|ccccc|c}
0 & v_4 & -4v_3 & 6v_2 & -4v_1 & v_0 & & & & & & \\
\hline
 & 4h & 4e & 0 & 0 & 0 & 6v_2 & -12v_1 & 6v_0 & 0 & 0 & z\,v_0 \\
 & f & 2h & 3e & 0 & 0 & 3v_3 & -3v_2 & -3v_1 & 3v_0 & 0 & z\,v_1 \\
 & 0 & 2f & 0 & 2e & 0 & v_4 & 2v_3 & -6v_2 & 2v_1 & v_0 & z\,v_2 \\
 & 0 & 0 & 3f & -2h & e & 0 & 3v_4 & -3v_3 & -3v_2 & 3v_1 & z\,v_3 \\
 & 0 & 0 & 0 & 4f & -4h & 0 & 0 & 6v_4 & -12v_3 & 6v_2 & z\,v_4 \\
\hline
 & & & & & & 4h & 4e & 0 & 0 & 0 & v_0 \\
 & & & & & & f & 2h & 3e & 0 & 0 & v_1 \\
 & & & & & & 0 & 2f & 0 & 2e & 0 & v_2 \\
 & & & & & & 0 & 0 & 3f & -2h & e & v_3 \\
 & & & & & & 0 & 0 & 0 & 4f & -4h & v_4 \\
\hline
 & & & & & & & & & & & 0
\end{array}\right).$$

Let $\mathfrak{g} = \mathfrak{sl}(3) \ltimes \mathbb{C}^3$ and let λ_1 and λ_2 be the fundamental weights of $\mathfrak{sl}(3)$. We now show that there exists a unique uniserial $\mathfrak{g}$-module with socle factors

$$V(2\lambda_2),\ V(\lambda_1 + \lambda_2),\ V(2\lambda_1).$$

Notice that in contrast to the other examples considered so far, none of the differences between the highest weights of these three $\mathfrak{sl}(3)$-modules is a dominant weight.

Let $\mathfrak{r} = \mathbb{C}^3$. According to Propositions 3.1 and 3.2, it suffices to prove that

$$\mathrm{Hom}_{\mathfrak{sl}(3)}(\mathfrak{r}, V(\lambda_1 + \lambda_2)^* \otimes V(2\lambda_2)) \neq 0,\ \mathrm{Hom}_{\mathfrak{sl}(3)}(\mathfrak{r}, V(2\lambda_1)^* \otimes V(\lambda_1 + \lambda_2)) \neq 0,$$

and

$$\mathrm{Hom}_{\mathfrak{sl}(3)}(\mathfrak{r}, V(2\lambda_1)^* \otimes V(2\lambda_2)) = 0 = \mathrm{Hom}_{\mathfrak{sl}(3)}(\Lambda^2\mathfrak{r}, V(2\lambda_1)^* \otimes V(2\lambda_2)).$$

Since $\lambda_1^* = \lambda_2$ and

$$\mathfrak{r} \cong V(\lambda_1), \quad \Lambda^2\mathfrak{r} \cong V(\lambda_2),$$

the above conditions follow from the following tensor product decompositions:

$$\begin{aligned} V(\lambda_1+\lambda_2)^* \otimes V(2\lambda_2) &= V(\lambda_1+\lambda_2) \otimes V(2\lambda_2) \\ &= V(\lambda_1+3\lambda_2) \oplus V(2\lambda_1+\lambda_2) \oplus V(2\lambda_2) \oplus V(\lambda_1), \\ V(2\lambda_1)^* \otimes V(2\lambda_2) &= V(2\lambda_2) \otimes V(2\lambda_2) \\ &= V(4\lambda_2) \oplus V(\lambda_1+2\lambda_2) \oplus V(2\lambda_1). \end{aligned}$$

Here is an example with $\mathfrak{s} = \mathfrak{so}(m)$, $m \geq 3$, and $\mu = \lambda_1$, the first fundamental weight. A matrix representation of $\mathfrak{so}(m) \ltimes V(\lambda_1)$ that is uniserial with socle factors $V(0), V(\lambda_1), V(0)$ can be obtained as follows.

Let U be a vector space of dimension m, and let $f : U \times U \to \mathbb{C}$ be a non-degenerate symmetric bilinear form. The subalgebra of $\mathfrak{gl}(U)$ preserving f is $\mathfrak{so}(m)$, and U is the natural module for $\mathfrak{so}(m)$.

Set $n = m + 2$ and let J be the $n \times n$ matrix with 1's along the secondary diagonal and 0's elsewhere. The $n \times n$ matrices X satisfying $X'J + JX = 0$, where X' indicates the transpose of X, form $\mathfrak{so}(n)$. The appearance of such X is

$$X = \begin{pmatrix} a & x_1 & \cdots & x_m & 0 \\ y_1 & * & * & * & -x_m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_m & * & * & * & -x_1 \\ 0 & -y_m & \cdots & -y_1 & -a \end{pmatrix}$$

where the inner $*$ follow the same pattern as the outer entries, i.e., X is skew-symmetric relative to the secondary diagonal. The subalgebra of $\mathfrak{so}(n)$ formed by all X having 0's in the outer rows/columns is clearly isomorphic to $\mathfrak{so}(m)$. Moreover, the subspace of all X having 0's in the first column, last row, and all inner entries, is normalized by $\mathfrak{so}(m)$, in such a way that together they form a subalgebra isomorphic to $\mathfrak{so}(m) \ltimes U$. Theorem 2.4 ensures that this is a uniserial representation, whose socle factors are clearly $V(0), U, V(0)$.

In particular when $m = 3$ we obtain a uniserial module for $\mathfrak{sl}(2) \ltimes V(2)$ with socle factors $V(0), V(2), V(0)$. Explicitly, we have the embedding of $\mathfrak{sl}(2) \ltimes V(2)$ into $\mathfrak{so}(5)$

$$\begin{pmatrix} 0 & a & b & c & 0 \\ 0 & h & e & 0 & -c \\ 0 & f & 0 & -e & -b \\ 0 & 0 & -f & -h & -a \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which makes $\mathbb{C}^5$ into a uniserial module with socle factors $V(0), V(2), V(0)$.

Thus the isomorphism $\mathfrak{sl}(2) \cong \mathfrak{so}(3)$ yields a uniserial module for $\mathfrak{sl}(2) \ltimes V(2)$ with socle factors $V(0), V(2), V(0)$, where $V(2)$ is the natural module for $\mathfrak{so}(3)$. But when we identify $\mathfrak{sl}(2)$ with $\mathfrak{sp}(2)$ the invariant form on $V(1)$ is skew-symmetric and no uniserial module for $\mathfrak{sl}(2) \ltimes V(1)$ with socle factors $V(0), V(1), V(0)$ exists, as indicated above.

It is perhaps worth noting that when we pass to a perfect Lie algebra whose radical is nilpotent of class 2, in addition to all modules arising from the abelian case, we may obtain some new ones as well. As an illustration, let $m \geq 1$, let U be a vector space of dimension $2m$ and let $f : U \times U \to \mathbb{C}$ be a non-degenerate skew-symmetric bilinear form. The subalgebra of $\mathfrak{gl}(U)$ preserving f is $\mathfrak{sp}(2m)$ and

U is the natural module for $\mathfrak{sp}(2m)$. The Heisenberg algebra $\mathfrak{h}(2m+1)$ can be defined on the vector space $U \oplus \mathbb{C}$ by declaring $[u+a, v+b] = f(u,v)$. Then $\mathfrak{sp}(2m)$ acts via derivations on $\mathfrak{h}(2m+1)$ by $[x, u+a] = x(u)$, and we may then form the perfect Lie algebra $\mathfrak{sp}(2m) \ltimes \mathfrak{h}(2m+1)$. We have a natural Lie epimorphism $\mathfrak{sp}(2m) \ltimes \mathfrak{h}(2m+1) \to \mathfrak{sp}(2m) \ltimes U$, which allows us to view every uniserial module for $\mathfrak{sp}(2m) \ltimes U$ as one for $\mathfrak{sp}(2m) \ltimes \mathfrak{h}(2m+1)$ in which the center acts trivially. We wish to construct a uniserial module V for $\mathfrak{sp}(2m) \ltimes \mathfrak{h}(2m+1)$ with socle factors $V(0), U, V(0)$. Since the $\mathfrak{sp}(2m)$-invariant form on U, namely f, is skew-symmetric, our earlier comments ensure that no such module exists for $\mathfrak{sp}(2m) \ltimes U$. The reader will also note that the center of $\mathfrak{sp}(2m) \ltimes \mathfrak{h}(2m+1)$ does not act trivially on V.

Set $n = m+1$ and let J be the $n \times n$ matrix with 1's along the secondary diagonal and 0's everywhere else. Let S be the $2n \times 2n$ skew-symmetric invertible matrix

$$S = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}.$$

The $2n \times 2n$ matrices X satisfying $X'S + SX = 0$ form $\mathfrak{sp}(2n)$. The appearance of such X is

$$X = \begin{pmatrix} a & x_1 & \cdots & x_m & y_1 & \cdots & y_m & z \\ b_1 & * & * & * & * & * & * & y_m \\ \vdots & * & * & * & * & * & * & \vdots \\ b_m & * & * & * & * & * & * & y_1 \\ c_1 & * & * & * & * & * & * & -x_m \\ \vdots & * & * & * & * & * & * & \vdots \\ c_m & * & * & * & * & * & * & -x_1 \\ d & c_m & \cdots & c_1 & -b_m & \cdots & -b_1 & -a \end{pmatrix},$$

where the inner $*$ follow the same pattern as the outer entries. More explicitly, if we partition X into 4 blocks of size $n \times n$, that is, $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then B, C are symmetric relative to the secondary diagonal and D is the opposite of the transpose of A relative to the secondary diagonal. In particular, the inner $*$ form a subalgebra isomorphic to $\mathfrak{sp}(2m)$, which together with the first row, and last column, with $a = 0$, form a subalgebra isomorphic to $\mathfrak{sp}(2m) \ltimes \mathfrak{h}(2m+1)$. This makes the column space $V = \mathbb{C}^{2n}$ into a uniserial module for $\mathfrak{sp}(2m) \ltimes \mathfrak{h}(2m+1)$ with socle factors $V(0), U, V(0)$.

In particular when $m = 1$ we get a uniserial module for $\mathfrak{sl}(2) \ltimes \mathfrak{h}(3)$ whose socle factors are $V(0), V(1), V(0)$. Explicitly, this is obtained through the following embedding of $\mathfrak{sl}(2) \ltimes \mathfrak{h}(3)$ into $\mathfrak{sp}(4)$:

$$\begin{pmatrix} 0 & x & y & z \\ 0 & a & b & y \\ 0 & c & -a & -x \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

9. Admissible sequences of length 3 for $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m)$

Let $\mathfrak{g}$ be an arbitrary Lie algebra and let X, Y, Z be $\mathfrak{g}$-modules. Then the map $\mathrm{Hom}(Y,X) \otimes \mathrm{Hom}(Z,Y) \to \mathrm{Hom}(Z,X)$, defined by $\alpha \otimes \beta \mapsto \alpha\beta$, is an epimorphism of $\mathfrak{g}$-modules.

Let M,N be $\mathfrak{g}$-modules and let $f:M\to \mathrm{Hom}(Y,X)$, $g:N\to\mathrm{Hom}(Z,Y)$ be homomorphisms of $\mathfrak{g}$-modules. They give rise to the homomorphism of $\mathfrak{g}$-modules

$$M\otimes N \xrightarrow{f\otimes g} \mathrm{Hom}(Y,X)\otimes \mathrm{Hom}(Z,Y)\to \mathrm{Hom}(Z,X). \tag{9.1}$$

We are interested in the image, say $\mathcal{I}$, of this map. Here is a matrix interpretation. Let d_X,d_Y,d_Z be the dimensions of X,Y,Z and fix bases B_X,B_Y,B_Z on them. Let $M_X:\mathfrak{g}\to\mathfrak{gl}(d_X)$, $M_Y:\mathfrak{g}\to\mathfrak{gl}(d_Y)$, $M_Z:\mathfrak{g}\to\mathfrak{gl}(d_Z)$ be the matrix representations associated to the modules X,Y,Z relative to the bases B_X,B_Y,B_Z. Consider the $\mathfrak{g}$-module $U=X\oplus Y\oplus Z$ of dimension $d=d_X+d_Y+d_Z$ and basis $B=B_X\cup B_Y\cup B_Z$. Then the matrix representation $M:\mathfrak{g}\mapsto\mathfrak{gl}(d)$ associated to U relative to B is

$$M(t)=\begin{pmatrix} M_X(t) & 0 & 0\\ 0 & M_Y(t) & 0\\ 0 & 0 & M_Z(t)\end{pmatrix},\quad t\in\mathfrak{g}.$$

We view $\mathfrak{gl}(d)$ as a $\mathfrak{g}$-module by means of $t\cdot A=[M(t),A]$ for $t\in\mathfrak{g}$ and $A\in\mathfrak{gl}(d)$. Let $M_{a,b}$ stand for the space of all complex matrices of size $a\times b$. Then

$$\left\{\begin{pmatrix} 0 & A & 0\\ 0&0&0\\ 0&0&0\end{pmatrix} : A\in M_{d_X,d_Y}\right\}\ \text{and}\ \left\{\begin{pmatrix} 0 & 0 & 0\\ 0&0&A\\ 0&0&0\end{pmatrix} : A\in M_{d_Y,d_Z}\right\}$$

are $\mathfrak{g}$-submodules of $\mathfrak{gl}(d)$, respectively isomorphic to $\mathrm{Hom}(Y,X)$ and $\mathrm{Hom}(Z,Y)$. Let

$$\mathcal{D}=\left\{\begin{pmatrix} 0 & f(m) & 0\\ 0&0&g(n)\\ 0&0&0\end{pmatrix} : m\in M, n\in N\right\},$$

where, by abuse of notation, $f(m)$ and $g(n)$ stand for their own matrices relative to the bases B_Y,B_X and B_Z,B_Y. Then $\mathcal{I}$ is, relative to the bases B_Z,B_X, the subspace generated by all matrices

$$\left\{\begin{pmatrix} 0 & 0 & f(m)g(n)\\ 0&0&0\\ 0&0&0\end{pmatrix} : m\in M, n\in N\right\}.$$

Note that $\mathcal{A}=\mathcal{D}\oplus\mathcal{I}$ is the associative algebra generated by $\mathcal{D}$.

We are also interested in the case $M=N$. In this case we have the $\mathfrak{g}$-module decomposition $M\otimes M=\Lambda^2(M)\oplus S^2(M)$. Let $\mathcal{J}$ be the the image of $\Lambda^2(M)$ under (9.1). Then $\mathcal{J}$ is, relative to the bases B_Z,B_X, the subspace generated by all matrices

$$\left\{\begin{pmatrix} 0 & 0 & f(m)g(n)-f(n)g(m)\\ 0&0&0\\ 0&0&0\end{pmatrix} : m,n\in M\right\}$$

and, in this case, $\mathcal{L}=\mathcal{D}_{\mathrm{diag}}\oplus\mathcal{J}$ is the Lie algebra generated by

$$\mathcal{D}_{\mathrm{diag}}=\left\{\begin{pmatrix} 0 & f(m) & 0\\ 0&0&g(m)\\ 0&0&0\end{pmatrix} : m\in M\right\}.$$

Let a,b,c,p,q be non-negative integers. We next focus attention on the case:

$$\mathfrak{g}=\mathfrak{sl}(2),\ M=V(p),\ N=V(q),\ X=V(a),\ Y=V(b),\ Z=V(c).$$

We wish to determine the $\mathfrak{g}$-module structure of $\mathcal{I}$ (resp. $\mathcal{J}$ when $p=q$). Since the tensor product of irreducible $\mathfrak{sl}(2)$-modules is multiplicity free, $\mathcal{I}$ is independent of the choice of f and g (as long as they are non-zero) and determining $\mathcal{I}$ is equivalent to finding all $k \geq 0$ such that $V(k)$ is a submodule of $\mathcal{I}$.

Theorem 9.1. *Let a, b, c, p, q, k be non-negative integers. Assume the existence of $\mathfrak{sl}(2)$-embeddings $V(p) \to \mathrm{Hom}(V(b), V(a))$ and $V(q) \to \mathrm{Hom}(V(c), V(b))$, and let $\mathcal{I}$ be the image of the corresponding $\mathfrak{sl}(2)$-homomorphism*

$$V(p) \otimes V(q) \to \mathrm{Hom}(V(b), V(a)) \otimes \mathrm{Hom}(V(c), V(b)) \to \mathrm{Hom}(V(c), V(a)). \tag{9.2}$$

Then $V(k)$ appears in $\mathcal{I}$ if and only if the Wigner-Racah $6j$-symbol (see §12)

$$\begin{Bmatrix} \frac{q}{2} & \frac{k}{2} & \frac{p}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{Bmatrix} \neq 0. \tag{9.3}$$

Proof. Let $V(p) \to V(a) \otimes V(b)$ and $V(q) \to V(b) \otimes V(c)$ be $\mathfrak{sl}(2)$-embeddings, and let $j : V(b) \to V(b)^*$ be an $\mathfrak{sl}(2)$-isomorphism. Let $\mathcal{K}$ be the image of the corresponding $\mathfrak{sl}(2)$-homomorphism

$$V(p) \otimes V(q) \to V(a) \otimes V(b) \otimes V(b) \otimes V(c) \xrightarrow{\mu} V(a) \otimes V(c),$$

where

$$\mu(x \otimes y_1 \otimes y_2 \otimes z) = (j(y_1))(y_2)x \otimes z, \quad x \in V(a),\ y_1, y_2 \in V(b),\ z \in V(c). \tag{9.4}$$

It is not difficult to see that $\mathcal{K}$ is independent of the choice of j and the given embeddings, and that $\mathcal{I} \cong \mathcal{K}$. Hence, it suffices to prove the result for $\mathcal{K}$.

If $V(k)$ does not occur in $V(p) \otimes V(q)$ or $V(a) \otimes V(c)$ then $V(k)$ does not occur in $\mathcal{K}$ and, moreover, the left hand side of (9.3) is 0. Thus, we may assume that $V(k)$ appears in $V(p) \otimes V(q)$ and $V(a) \otimes V(c)$. In §11 we furnish a concrete embedding $\iota_r^{s,t} : V(r) \to V(s) \otimes V(t)$ for any non-negative integers such that $|t - s| \leq r \leq t + s$ and $t + s \equiv r \mod 2$, as well as a fixed isomorphism $j_r : V(r) \to V(r)^*$ for any $r \geq 0$. These data yield a specific $\mathfrak{sl}(2)$-homomorphism $\phi : V(k) \to V(a) \otimes V(c)$

$$V(k) \xrightarrow{\iota_k^{p,q}} V(p) \otimes V(q) \xrightarrow{\iota_p^{a,b} \otimes \iota_q^{b,c}} V(a) \otimes V(b) \otimes V(b) \otimes V(c) \xrightarrow{\mu} V(a) \otimes V(c). \tag{9.5}$$

But, up to scaling, the only $\mathfrak{sl}(2)$-homomorphism $V(k) \to V(a) \otimes V(c)$ is $\iota_k^{a,c}$. Hence

$$\phi = \lambda \iota_k^{a,c}$$

for a unique scalar λ. A long and technical calculation (performed independently in Theorem 11.2) shows that

$$\lambda = C \begin{Bmatrix} \frac{q}{2} & \frac{k}{2} & \frac{p}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{Bmatrix},$$

where C is a non-zero scalar explicitly defined in §11. The result now follows. □

Corollary 9.2. *Keep the hypothesis of Theorem 9.1 and suppose that $p = q$. Let $\mathcal{J}$ be the image of $\Lambda^2(V(p))$ under (9.2). Then $V(k)$ appears in $\mathcal{J}$ if and only if $k \equiv 2p - 2 \mod 4$ and (9.3) holds.*

Proof. Clearly $V(k)$ appears in $\mathcal{J}$ if and only if $V(k)$ appears in $\mathcal{I}$ and $\Lambda^2(V(p))$. Recalling that $V(k)$ appears in $\Lambda^2(V(p))$ if and only if $0 \leq k \leq 2p$ and $2p - 2 \equiv k \mod 4$, the result follows from Theorem 9.1. □

Example 9.3. Let us consider the $\mathfrak{sl}(2)$-homomorphism

$$V(4)\otimes V(4)\to \mathrm{Hom}(V(6),V(4))\otimes \mathrm{Hom}(V(4),V(6))\to \mathrm{Hom}(V(4),V(4)),$$

i.e., $p=q=a=c=4$ and $b=6$. It turns out that

$$\left\{\begin{matrix}\frac{4}{2} & \frac{0}{2} & \frac{4}{2}\\ \frac{4}{2} & \frac{6}{2} & \frac{4}{2}\end{matrix}\right\}=-\frac{1}{5},\quad \left\{\begin{matrix}\frac{4}{2} & \frac{2}{2} & \frac{4}{2}\\ \frac{4}{2} & \frac{6}{2} & \frac{4}{2}\end{matrix}\right\}=0,\quad \left\{\begin{matrix}\frac{4}{2} & \frac{4}{2} & \frac{4}{2}\\ \frac{4}{2} & \frac{6}{2} & \frac{4}{2}\end{matrix}\right\}=\frac{4}{35},\quad \left\{\begin{matrix}\frac{4}{2} & \frac{6}{2} & \frac{4}{2}\\ \frac{4}{2} & \frac{6}{2} & \frac{4}{2}\end{matrix}\right\}=\frac{1}{14},\quad \left\{\begin{matrix}\frac{4}{2} & \frac{8}{2} & \frac{4}{2}\\ \frac{4}{2} & \frac{6}{2} & \frac{4}{2}\end{matrix}\right\}=\frac{1}{70}.$$

This shows that $\mathcal{I}=V(0)\oplus V(4)\oplus V(6)\oplus V(8)$ and thus $\mathcal{J}=V(6)$. Similarly, it can be shown that for $\mathfrak{sl}(2)$-homomorphism

$$V(4)\otimes V(4)\to \mathrm{Hom}(V(2),V(4))\otimes \mathrm{Hom}(V(4),V(2))\to \mathrm{Hom}(V(4),V(4)),$$

we have $\mathcal{I}=V(0)\oplus V(2)\oplus V(4)\oplus V(8)$ and thus $\mathcal{J}=V(2)$.

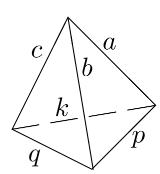
Definition 9.4. Given three non-negative integers a, b and c, we will say that the triple (a,b,c) satisfies the *triangle condition* if a, b and c are the side lengths of a (possibly degenerate) triangle and $a+b+c$ is even.

From the Clebsch-Gordan formula for the decomposition of the tensor product of two $\mathfrak{sl}(2)$-modules, we know that $V(k)$ is a submodule of $V(a)\otimes V(b)$ if and only if $|a-b|\le k\le a+b$ and $k\equiv a+b \mod 2$. It is clear that this is the same as saying that (a,b,k) satisfies the triangle condition.

In terms of Theorem 9.1, it is clear that a necessary condition for $V(k)$ to appear in the image $\mathcal{I}$ of (9.2) is that the four triples

$$(k,p,q),\quad (p,a,b),\quad (q,b,c),\quad (k,a,c) \tag{9.6}$$

satisfy the triangle condition. These four triangle conditions can be depicted by the following labeled tetrahedron:



We point out, however, that the above four triangle conditions do not imply the existence of an euclidean metric tetrahedron with side lengths a, b, c, p, q and k (as indicated in the above picture); it is known that an additional condition on the Cayley-Menger determinant is required for that (see, [Bl], [GV] or [WD]).

Note also that the four triangle conditions (9.6) are not sufficient for $V(k)$ to appear in the image of (9.2), as shown in Example 9.3. According to Theorem 9.1, $V(k)$ will not appear in the image of (9.2) if and only if $\left\{\begin{matrix}\frac{q}{2} & \frac{k}{2} & \frac{p}{2}\\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2}\end{matrix}\right\}=0$. We recall that finding the non-trivial zeros of the 6j-symbol is a well studied and very difficult problem (see, for instance, [L], [R] and the references therein, or more recently [ZR]). In particular it is known that

$$\left\{\begin{matrix}a & a-1 & a\\ a & a+1 & 2\end{matrix}\right\}=0 \quad\text{and}\quad \left\{\begin{matrix}\frac{j}{2} & \frac{2j-2}{2} & \frac{j}{2}\\ \frac{3j-8}{2} & \frac{2j-6}{2} & \frac{j}{2}\end{matrix}\right\}=0$$

for all integers $a\ge 2$ and $j\ge 4$ (see equations (4.14) and (4.15) in [L]).

We can now state the following theorem which is important for the classification of the uniserial modules of the Lie algebra $\mathfrak{sl}(2) \ltimes V(m)$.

Theorem 9.5. *Let a, b, c and m be non-negative integers such that (a,b,m) and (b,c,m) satisfy the triangle condition, and let $\mathcal{J}$ be the image of $\Lambda^2(V(m))$ under (9.2) when $p=m=q$.*

Let $f : V(m) \to \mathrm{Hom}(V(b),V(a))$ and $g : V(m) \to \mathrm{Hom}(V(c),V(b))$ be non-zero $\mathfrak{sl}(2)$-homomorphisms and consider the Lie subalgebra, say $\mathcal{L}$, of $\mathfrak{gl}(V)$, where $V = V(a)\oplus V(b)\oplus V(c)$, generated by $f(r)+g(r)$, $r \in V(m)$.

Then the following conditions are equivalent:

(1) $\mathcal{L}$ is abelian.
(2) $\mathcal{J}=0$.
(3) $\begin{Bmatrix} \frac{m}{2} & \frac{k}{2} & \frac{m}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{Bmatrix} = 0$ for all non-negative integers k satisfying $k \equiv 2m-2 \mod 4$.
(4) Up to a swap of a and c we have: $c=0$, $b=m$, $a\equiv 2m \mod 4$ and $a \le 2m$; or $b=c+m$ and $a=c+2m$.
(5) $\mathrm{Hom}_{\mathfrak{sl}(2)}\big(\Lambda^2 V(m), \mathrm{Hom}(V(c),V(a))\big)=0$.
(6) There is a uniserial $\mathfrak{sl}(2)\ltimes V(m)$-module with socle factors $V(a),V(b),V(c)$.

Proof. (1)⇒(2) We already noted that $\mathcal{L} = \mathcal{D}_{\mathrm{diag}} \oplus \mathcal{J}$ is generated by $\mathcal{D}_{\mathrm{diag}}$, so $\mathcal{J}=0$ if $\mathcal{L}$ is abelian.

(2)⇒(3) If $0\le k\le 2m$ and (a,c,k) satisfies the triangle condition then (2) and Corollary 9.2 imply that $\begin{Bmatrix} \frac{m}{2} & \frac{k}{2} & \frac{m}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{Bmatrix} = 0$. For all other $k \equiv 2m-2 \mod 4$, this $6j$-symbol is zero by definition.

(3)⇒(4) As the $6j$-symbol is invariant under the permutation of any two columns ([CFS]), we may assume that $a \ge c$.

Since (a,b,m) and (b,c,m) satisfy the triangle condition it follows that $m \ge a-b$, $m \ge b-c$ and thus $2m \ge a-c$.

We claim that

$$a-c \equiv 2m \mod 4. \tag{9.7}$$

Otherwise, $k=a-c\equiv 2m-2 \mod 4$ and, since (a,c,k) is a degenerate triangle, it follows from Property (iii) in the list of properties of the $6j$-symbols in §12, that $\begin{Bmatrix} \frac{m}{2} & \frac{k}{2} & \frac{m}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{Bmatrix} \neq 0$ which contradicts (3).

If $a-c=2m$, the triangle conditions on (a,b,m) and (b,c,m) imply that $b=m+c$ and $a=c+2m$. Otherwise, $a-c<2m$ and we will prove that $c=0$ and thus $b=m$, $a\equiv 2m \mod 4$ and $a\le 2m$. Since $a-c<2m$, it follows from (9.7) that $a-c\le 2m-4$. If $c\ge 1$ then

$$(h,m,m), \quad (h,a,c), \quad (m,a,b), \quad (m,b,c)$$

satisfy the triangle condition for $h=a-c$ and $h=a-c+2$ and we are in a position to apply Lemma 12.1 to

$$j_1=\frac{a-c}{2}, \quad j_2=j_3=\frac{m}{2}, \quad j_4=\frac{b}{2}, \quad j_5=\frac{c}{2}, \quad j_6=\frac{a}{2}.$$

We obtain that $\begin{Bmatrix} \frac{m}{2} & \frac{k}{2} & \frac{m}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{Bmatrix} = \begin{Bmatrix} \frac{k}{2} & \frac{m}{2} & \frac{m}{2} \\ \frac{b}{2} & \frac{c}{2} & \frac{a}{2} \end{Bmatrix} \neq 0$ for either $k = a-c+2$ or $k = a-c+6$. Since both k satisfy $k \equiv 2m-2 \mod 4$, this contradicts (3).

(4)⇒(5) Immediate from the decompositions of $\Lambda^2(V(m))$ and $V(a) \otimes V(c)$.

(5)⇒(1) This is obvious.

(6)⟺(1) If $m \geq 1$ this follows from Proposition 3.1. If $m = 0$ then $a = b = c$ and it is easy to see that conditions (1) and (6) are both true. □

10. Classification of uniserial $\mathfrak{sl}(2) \ltimes V(m)$-modules

We are finally in a position to show that, except for a few exceptions of lengths 2, 3 and 4, every uniserial $\mathfrak{sl}(2) \ltimes V(m)$-module is isomorphic to $Z(\ell, b)$ or its dual.

Theorem 10.1. *Let $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m)$, where $m \geq 1$. Then, up to a reversing of the order, the following are the only admissible sequences for $\mathfrak{g}$:*

Length 1. $V(a)$.

Length 2. $V(a), V(b)$, *where* $a + b \equiv m \mod 2$ *and* $0 \leq b - a \leq m \leq a + b$.

Length 3. $V(a), V(a+m), V(a+2m)$; *or*
$V(0), V(m), V(c)$, *where* $c \equiv 2m \mod 4$ *and* $c \leq 2m$.

Length 4. $V(a), V(a+m), V(a+2m), V(a+3m)$; *or*
$V(0), V(m), V(m), V(0)$, *where* $m \equiv 0 \mod 4$.

Length ≥ 5. $V(a), V(a+m), \dots, V(a+sm)$, *where* $s \geq 4$.

Moreover, each of these sequences arises from only one isomorphism class of uniserial $\mathfrak{g}$-modules, except for the sequence $V(0), V(m), V(m), V(0)$, $m \equiv 0 \mod 4$. The isomorphism classes of uniserial $\mathfrak{g}$-modules associated to this sequence are parametrized by the complex numbers, as described in Note 8.3.

Proof. That the stated sequences are admissible is proven in §4 and §8, while the uniqueness, up to isomorphism, of the uniserial modules arising from such sequences follows from Proposition 3.2, except for the sequence $V(0), V(m), V(m), V(0)$, where $m \equiv 0 \mod 4$, which is handled in Note 8.3.

It remains to prove that, up to a reversing of the order, the only admissible sequences are as indicated. Those of length 2 are considered in Note 4.5. Let $V(a), V(b), V(c)$ be an admissible sequence of length 3, which is condition (1) of Theorem 9.5. This is equivalent to condition (4) of Theorem 9.5, so all admissible sequences of length 3 are as stated. Next let $V(a_1), \dots, V(a_n)$ be an admissible sequence of length $n \geq 4$. By Lemma 2.6 we may assume that $a_1 \leq a_n$. Since any submodule or quotient of a uniserial module is also uniserial, we see that $V(a_{i-1}), V(a_i), V(a_{i+1})$ is also admissible for any $1 < i < n$. Applying this fact in combination with our determination of all admissible sequences of length 3, we deduce the following: $a_i \neq 0$ for $1 < i < n$; either $a_1, \dots, a_n$ is strictly increasing, in which case it does so by a fixed increment of m, or else $n = 4$ and $a_1 = 0, a_2 = m, a_3 = m, a_4 = 0$. □

11. A new interpretation of the Wigner-Racah $6j$-symbol and the calculation of the scalar λ

Let k be a non-negative integer and let e_k denote a highest weight vector of the irreducible $\mathfrak{sl}(2)$-module $V(k)$ of highest weight k. If $\{H, E, F\}$ is the standard basis of $\mathfrak{sl}(2)$ then $\mathcal{B}_k = \{F^r e_k : r = 0, \dots, k\}$ is a basis of $V(k)$ and

$$HF^r e_k = (k - 2r)\, F^r e_k,$$
$$EF^r e_k = r(k + 1 - r)\, F^{r-1} e_k.$$

We know that $V(k)$ is isomorphic to the dual $\mathfrak{sl}(2)$-module $V(k)^*$. In fact, if $\{(F^r e_k)^* : r = 0, \dots, k\}$ is the dual basis of $\mathcal{B}_k$, then $(F^k e_k)^*$ is the highest weight vector of $V(k)^*$ and the map $j_k : V(k) \to V(k)^*$, given by

$$F^r e_k \mapsto (-1)^r (F^{k-r} e_k)^* \tag{11.1}$$

is, up to a scalar, the unique $\mathfrak{sl}(2)$-module isomorphism between $V(k)$ and $V(k)^*$. Suppose (a, b, k) satisfies the triangle condition. In this case $V(k)$ occurs with multiplicity one in $V(a) \otimes V(b)$ and if

$$x_k^{a,b} = \frac{a + b - k}{2}$$

then

$$v_k^{a,b} = \sum_{r=0}^{x_k^{a,b}} (-1)^r \frac{\binom{x_k^{a,b}}{r}}{\binom{x_k^{a,b}+k}{a-r}}\; F^r e_a \otimes F^{x_k^{a,b} - r} e_b \tag{11.2}$$

is, up to a scalar, the unique highest weight vector of weight k in $V(a) \otimes V(b)$. We denote by

$$\iota_k^{a,b} \in \mathrm{Hom}_{\mathfrak{sl}(2)}\big(V(k), V(a) \otimes V(b)\big)$$

the unique $\mathfrak{sl}(2)$-module homomorphism sending e_k to $v_k^{a,b}$.

Let us assume that the four triples

$$(k, p, q), \quad (p, a, b), \quad (q, b, c), \quad (k, a, c)$$

satisfy the triangle condition. Let ϕ be the map defined by (9.5), that is

$$\phi = \mu \circ \big(\iota_p^{a,b} \otimes \iota_q^{b,c}\big) \circ \iota_k^{p,q},$$

where $\mu : V(a) \otimes V(b) \otimes V(b) \otimes V(c)$ is defined in (9.4) by means of the isomorphism $j = j_b : V(b) \to V(b)^*$ given in (11.1). Explicitly, we have

$$\mu(x \otimes F^r e_b \otimes F^{b-s} e_b \otimes z) = (-1)^r \delta_{r,s} x \otimes z, \quad x \in V(a), z \in V(c), 0 \le r, s \le b.$$

As noted at the end of the proof of Theorem 9.1, we have $\phi = \lambda \iota_k^{a,c}$. Thus

$$\phi(e_k) = \lambda\, v_k^{a,c},$$

where $v_k^{a,c}$ is defined in (11.2). We will now compute λ. First we have

$$\iota_k^{p,q} e_k = \sum_{r_1=0}^{x_k^{p,q}} (-1)^{r_1} \frac{\binom{x_k^{p,q}}{r_1}}{\binom{x_k^{p,q}+k}{p-r_1}}\; F^{r_1} e_p \otimes F^{x_k^{p,q} - r_1} e_q.$$

$$v_p^{a,b}=\iota_p^{a,b}e_p=\sum_{r_2=0}^{x_p^{a,b}}(-1)^{r_2}\frac{\binom{x_p^{a,b}}{r_2}}{\binom{x_p^{a,b}+p}{a-r_2}}\;F^{r_2}e_a\otimes F^{x_p^{a,b}-r_2}e_b,$$

$$v_q^{b,c}=\iota_q^{b,c}e_q=\sum_{r_3=0}^{x_q^{b,c}}(-1)^{r_3}\frac{\binom{x_q^{b,c}}{r_3}}{\binom{x_q^{b,c}+q}{b-r_3}}\;F^{r_3}e_b\otimes F^{x_q^{b,c}-r_3}e_c.$$

Thus

$$\phi(e_k)=\mu\sum_{r_1=0}^{x_k^{p,q}}(-1)^{r_1}\frac{\binom{x_k^{p,q}}{r_1}}{\binom{x_k^{p,q}+k}{p-r_1}}\;F^{r_1}.\Big(\sum_{r_2=0}^{x_p^{a,b}}(-1)^{r_2}\frac{\binom{x_p^{a,b}}{r_2}}{\binom{x_p^{a,b}+p}{a-r_2}}\;F^{r_2}e_a\otimes F^{x_p^{a,b}-r_2}e_b\Big)$$
$$\otimes F^{x_k^{p,q}-r_1}.\Big(\sum_{r_3=0}^{x_q^{b,c}}(-1)^{r_3}\frac{\binom{x_q^{b,c}}{r_3}}{\binom{x_q^{b,c}+q}{b-r_3}}\;F^{r_3}e_b\otimes F^{x_q^{b,c}-r_3}e_c\Big).$$

Applying Leibniz's rule to compute the action of F^{r_1} and $F^{x_k^{p,q}-r_1}$ on tensors yields

$$\phi(e_k)=\sum(-1)^{r_1+r_2+r_3}\frac{\binom{r_1}{r_4}\binom{x_k^{p,q}-r_1}{r_5}\binom{x_k^{p,q}}{r_1}\binom{x_p^{a,b}}{r_2}\binom{x_q^{b,c}}{r_3}}{\binom{x_k^{p,q}+k}{p-r_1}\binom{x_p^{a,b}+p}{a-r_2}\binom{x_q^{b,c}+q}{b-r_3}}$$
$$\times F^{r_2+r_4}e_a\otimes\mu\Big(F^{x_p^{a,b}-r_2+r_1-r_4}e_b\otimes F^{r_3+r_5}e_b\Big)\otimes F^{x_q^{b,c}-r_3+x_k^{p,q}-r_1-r_5}e_c.$$

Here the sum runs over all (r_1,r_2,r_3,r_4,r_5) allowed by the binomial coefficients in the numerator. In what follows, all the sums will run over the indicated indices with the restriction that the factorial numbers involved are non-negative. In order to compute μ we need to consider the case when

$$x_p^{a,b}-r_2+r_1-r_4+r_3+r_5=b$$

and thus

$$\phi(e_k)=(-1)^{x_p^{a,b}}\sum_{r_1,r_2,r_3,r_4}(-1)^{r_4+r_3}\frac{\binom{r_1}{r_4}\binom{x_k^{p,q}-r_1}{b-x_p^{a,b}-r_1+r_2-r_3+r_4}\binom{x_k^{p,q}}{r_1}\binom{x_p^{a,b}}{r_2}\binom{x_q^{b,c}}{r_3}}{\binom{x_k^{p,q}+k}{p-r_1}\binom{x_p^{a,b}+p}{a-r_2}\binom{x_q^{b,c}+q}{b-r_3}}$$
$$\times F^{r_2+r_4}e_a\otimes F^{x_k^{a,c}-r_2-r_4}e_c.$$

We know that $\phi(e_k)=\lambda v_k^{a,c}$. Comparing coefficients at $e_a\otimes F^{x_k^{a,c}}e_c$ yields

$$(-1)^{x_p^{a,b}}\sum_{r_1,r_3}(-1)^{r_3}\frac{\binom{x_k^{p,q}-r_1}{b-x_p^{a,b}-r_1-r_3}\binom{x_k^{p,q}}{r_1}\binom{x_q^{b,c}}{r_3}}{\binom{x_k^{p,q}+k}{p-r_1}\binom{x_p^{a,b}+p}{a}\binom{x_q^{b,c}+q}{b-r_3}}=\frac{\lambda}{\binom{x_k^{a,c}+k}{a}}.$$

Replacing x_3 by $x_q^{b,c}-r_3$ and using $\binom{s}{t}=\binom{s}{s-t}$ at appropriate places gives

$$(-1)^{x_p^{a,b}+x_q^{b,c}}\sum_{r_1,r_3}(-1)^{r_3}\frac{\binom{x_k^{p,q}-r_1}{x_k^{a,c}-r_3}\binom{x_k^{p,q}}{r_1}\binom{x_q^{b,c}}{r_3}}{\binom{x_k^{p,q}+k}{p-r_1}\binom{x_q^{b,c}+q}{c-r_3}}=\lambda\frac{\binom{x_p^{a,b}+p}{a}}{\binom{x_k^{a,c}+k}{a}}.$$

We next concentrate on the inner part of the above double sum.

Lemma 11.1.

$$\sum_{r_1}\frac{\binom{x_k^{p,q}-r_1}{x_k^{a,c}-r_3}\binom{x_k^{p,q}}{r_1}}{\binom{x_k^{p,q}+k}{p-r_1}}=\frac{x_k^{p,q}+k+1}{x_k^{p,q}+k+1-p}\frac{\binom{x_k^{p,q}}{x_k^{a,c}-r_3}}{\binom{x_k^{a,c}+k+1-r_3}{x_k^{p,q}+k+1-p}}.$$

Proof. Note that if x, y, z are non-negative integers and $y \geq z$ then

$$\sum_{r=0}^{z} \binom{x+r}{r}\binom{y-r}{z-r} = \binom{x+y+1}{z}. \tag{11.3}$$

This is easily seen by induction on y by repeatedly using $\binom{s}{t} = \binom{s-1}{t} + \binom{s-1}{t-1}$. Using (11.3) in the equivalent form

$$\sum_{r=0}^{z} \frac{(x+r)!\,(y-r)!}{r!\,(z-r)!} = \frac{x!\,(y-z)!\,(x+y+1)!}{z!\,(x+y-z+1)!}$$

with $x = x_k^{p,q} + k - p$, $y = p$, $z = x_k^{p,q} - x_k^{a,c} + r_3$ yields

$$\begin{aligned}
\sum_{r_1} \frac{\binom{x_k^{p,q}-r_1}{x_k^{a,c}-r_3}\binom{x_k^{p,q}}{r_1}}{\binom{x_k^{p,q}+k}{p-r_1}} &= \sum_{r_1} \frac{(x_k^{p,q})!\,(x_k^{p,q}+k-p+r_1)!\,(p-r_1)!}{(x_k^{a,c}-r_3)!\,(x_k^{p,q}-r_1-x_k^{a,c}+r_3)!\,r_1!\,(x_k^{p,q}+k)!} \\
&= \frac{(x_k^{p,q})!}{(x_k^{a,c}-r_3)!\,(x_k^{p,q}+k)!} \sum_{r_1} \frac{(x_k^{p,q}+k-p+r_1)!\,(p-r_1)!}{r_1!\,(x_k^{p,q}-x_k^{a,c}+r_3-r_1)!} \\
&= \frac{(x_k^{p,q})!}{(x_k^{a,c}-r_3)!\,(x_k^{p,q}+k)!} \\
&\qquad \times \frac{(x_k^{p,q}+k-p)!\,(p-x_k^{p,q}+x_k^{a,c}-r_3)!\,(x_k^{p,q}+k+1)!}{(x_k^{p,q}-x_k^{a,c}+r_3)!\,(k+x_k^{a,c}-r_3+1)!} \\
&= \frac{x_k^{p,q}+k+1}{x_k^{p,q}+k+1-p} \frac{\binom{x_k^{p,q}}{x_k^{a,c}-r_3}}{\binom{x_k^{a,c}+k+1-r_3}{x_k^{p,q}+k+1-p}}
\end{aligned}$$

as we wanted to prove. □

From the above lemma we obtain

$$(-1)^{x_p^{a,b}+x_q^{b,c}} \frac{x_k^{p,q}+k+1}{x_k^{p,q}+k+1-p} \sum_{r_3} (-1)^{r_3} \frac{\binom{x_k^{p,q}}{x_k^{a,c}-r_3}\binom{x_q^{b,c}}{r_3}}{\binom{x_k^{a,c}+k+1-r_3}{x_k^{p,q}+k+1-p}\binom{x_q^{b,c}+q}{c-r_3}} = \lambda \frac{\binom{x_p^{a,b}+p}{a}}{\binom{x_k^{a,c}+k}{a}}.$$

We refer the reader to §12 for the definition and basic properties of the $6j$-symbol as well as the meaning of Δ and R as used below.

Theorem 11.2. *Let λ be defined as in the proof of Theorem 9.1. Then*

$$\lambda = C \begin{Bmatrix} \frac{q}{2} & \frac{k}{2} & \frac{p}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{Bmatrix},$$

where

$$C = \frac{(-1)^{x_k^{a,c}+b+k}(p+q+k+2)(a+b+p+2)(b+c+q+2)\Delta(\frac{a}{2},\frac{b}{2},\frac{p}{2})\,\Delta(\frac{p}{2},\frac{q}{2},\frac{k}{2})\,\Delta(\frac{b}{2},\frac{c}{2},\frac{q}{2})}{4(a+c+k+2)\,\Delta(\frac{a}{2},\frac{c}{2},\frac{k}{2})}.$$

Proof. From the identity above the theorem we have that

$$\lambda = (-1)^{x_p^{a,b}+x_q^{b,c}} \frac{x_k^{p,q}+k+1}{x_k^{p,q}+k+1-p} \frac{(x_k^{a,c}+k)!\,(x_p^{a,b}+p-a)!}{(x_k^{a,c}+k-a)!\,(x_p^{a,b}+p)!} \sum_{r_3} (-1)^{r_3} \frac{\binom{x_k^{p,q}}{x_k^{a,c}-r_3}\binom{x_q^{b,c}}{r_3}}{\binom{x_k^{a,c}+k+1-r_3}{x_k^{p,q}+k+1-p}\binom{x_q^{b,c}+q}{c-r_3}}.$$

If we replace r_3 by $x_k^{a,c}-t$, the above sum is

$$\sum_{r_3}(-1)^{r_3}\frac{\binom{x_k^{p,q}}{x_k^{a,c}-r_3}\binom{x_q^{b,c}}{r_3}}{\binom{x_k^{a,c}+k+1-r_3}{x_k^{p,q}+k+1-p}\binom{x_q^{b,c}+q}{c-r_3}}=\frac{x_k^{p,q}!\,x_q^{b,c}!\,(x_k^{p,q}+k+1-p)!}{(x_q^{b,c}+q)!}$$
$$\times\sum_{r_3}\frac{(-1)^{r_3}(x_k^{a,c}-x_k^{p,q}+p-r_3)!\,(c-r_3)!\,(x_q^{b,c}+q-c+r_3)!}{(x_k^{a,c}-r_3)!\,(x_k^{p,q}-x_k^{a,c}+r_3)!\,r_3!\,(x_q^{b,c}-r_3)!\,(x_k^{a,c}+k+1-r_3)!}$$
$$=(-1)^{x_k^{a,c}}\frac{x_k^{p,q}!\,x_q^{b,c}!\,(x_k^{p,q}+k+1-p)!}{(x_q^{b,c}+q)!}$$
$$\times\sum_{t}\frac{(-1)^{t}(-x_k^{p,q}+p+t)!\,(c-x_k^{a,c}+t)!\,(x_q^{b,c}+q-c+x_k^{a,c}-t)!}{t!\,(x_k^{p,q}-t)!\,(x_k^{a,c}-t)!\,(x_q^{b,c}-x_k^{a,c}+t)!\,(k+1+t)!}.$$

Define j_1, j_2, j_3, j_4, j_5, j_6 by

$$a=2j_1,\qquad p=2j_2,\qquad b=2j_3,$$
$$q=2j_4,\qquad c=2j_5,\qquad k=2j_6.$$

Then the above sum together with (12.2) and $a+c\equiv k \mod 2$ yield

$$\sum_{r_3}(-1)^{r_3}\frac{\binom{x_k^{p,q}}{x_k^{a,c}-r_3}\binom{x_q^{b,c}}{r_3}}{\binom{x_k^{a,c}+k+1-r_3}{x_k^{p,q}+k+1-p}\binom{x_q^{b,c}+q}{c-r_3}}$$
$$=(-1)^{j_1+j_5-j_6}\frac{(j_2+j_4-j_6)!\,(j_3-j_4+j_5)!\,(-j_2+j_4+j_6+1)!}{(j_3+j_4+j_5)!}$$
$$\times\sum_{t}\frac{(-1)^{t}(j_2-j_4+j_6+t)!\,(-j_1+j_5+j_6+t)!\,(j_1+j_3+j_4-j_6-t)!}{t!\,(j_2+j_4-j_6-t)!\,(j_1+j_5-j_6-t)!\,(-j_1+j_3-j_4+j_6+t)!\,(2j_6+1+t)!}$$
$$=(-1)^{j_2+j_4+j_6}\frac{(j_2+j_4-j_6)!\,(j_3-j_4+j_5)!\,(-j_2+j_4+j_6+1)!}{(j_3+j_4+j_5)!}\frac{R^1_{5,6}R^4_{2,6}}{R^1_{2,3}R^4_{3,5}}\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}.$$

Therefore

$$\lambda=(-1)^{j_1+2j_3+j_5+j_6}\frac{j_2+j_4+j_6+1}{-j_2+j_4+j_6+1}\frac{(j_1+j_5+j_6)!\,(-j_1+j_2+j_3)!}{(-j_1+j_5+j_6)!\,(j_1+j_2+j_3)!}$$
$$\times\frac{(j_2+j_4-j_6)!\,(j_3-j_4+j_5)!\,(-j_2+j_4+j_6+1)!}{(j_3+j_4+j_5)!}\frac{R^1_{5,6}R^4_{2,6}}{R^1_{2,3}R^4_{3,5}}\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}$$
$$=(-1)^{j_1+2j_3+j_5+j_6}\frac{(j_2+j_4+j_6+1)(j_1+j_2+j_3+1)(j_3+j_4+j_5+1)}{(j_1+j_5+j_6+1)}\frac{\Delta_{123}\Delta_{246}\Delta_{345}}{\Delta_{156}}\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}.$$

The theorem now follows from the symmetry $\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}=\begin{Bmatrix} j_4 & j_6 & j_2\\ j_1 & j_3 & j_5\end{Bmatrix}$. □

This, together with the accompanying definitions given §12, concludes the proof of Theorem 9.1.

12. Appendix. The Clebsch-Gordan coefficients and the 6j-symbol

In this appendix we recall the basic facts about the 6j-symbol the we needed in this paper. We will mainly follow [VMK].

Let $2j_1$, $2j_2$ and $2j_3$ be three non-negative integers and define (see [VMK, §8.2, eq.(1)])

$$\Delta(j_1,j_2,j_3)=\sqrt{\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!}}$$

if $(2j_1, 2j_2, 2j_3)$ satisfies the triangle condition (see Definition 9.4); otherwise set $\Delta(j_1, j_2, j_3) = 0$.

If $2m_1$, $2m_2$ and $2m_3$ are three integers such that $|m_i| \leq j_i$ for $i = 1, 2, 3$, we recall that the corresponding *Clebsch-Gordan coefficient* is zero, if $m_1 + m_2 \neq m_3$, and otherwise is (see [VMK, §8.2, eq.(3)])

$$\begin{aligned} C^{j_3,m_3}_{j_1,m_1;\, j_2,m_2} &= \Delta(j_1, j_2, j_3)\sqrt{(2j_3+1)} \\ &\quad \times \sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)!} \\ &\quad \times \sum_r \frac{(-1)^r}{r!(j_1+j_2-j_3-r)!(j_1-m_1-r)!(j_2+m_2-r)!(j_3-j_2+m_1+r)!(j_3-j_1-m_2+r)!}, \end{aligned}$$

where the sum runs over all r such that all the numbers under the factorial symbol are non-negative.

Let $a = 2j_1$, $b = 2j_2$ and $k = 2j$. If we define (see [VMK, §3.1.1])

$$\mathcal{M}_{j,m} = \sqrt{\frac{(j+m)!}{(j-m)!}}\, F^{j-m} e_k$$

then (see [VMK, §8.2, eq.(10)])

$$\left\{ \{\mathcal{M}_{j_1} \otimes \mathcal{M}_{j_2}\}_{j,m} := \sum_{m_1+m_2=m} C^{j,m}_{j_1,m_1;\, j_2,m_2}\, \mathcal{M}_{j_1,m_1} \otimes \mathcal{M}_{j_2,m_2} : m = -j, -j+1, \dots, j \right\}$$

is a basis of the unique $\mathfrak{sl}(2)$-submodule of $V(a) \otimes V(b)$ isomorphic to $V(k)$, and in fact, the map

$$\begin{aligned} V(k) &\to V(a) \otimes V(b) \\ \mathcal{M}_{j,m} &\mapsto \{\mathcal{M}_{j_1} \otimes \mathcal{M}_{j_2}\}_{j,m} \end{aligned}$$

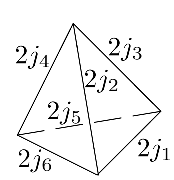
is exactly $\frac{1}{\Delta(j_1,j_2,j)}\sqrt{\frac{2j+1}{j_1+j_2+j+1}}\ \iota^{a,b}_k$.

The (classical) *Racah-Wigner* $6j$*-symbol* is a real number $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$ associated to six non-negative half-integer numbers j_1, j_2, j_3, j_4, j_5 and j_6. The $6j$-symbol plays a central role in angular momentum theory since they describe the recoupling of three angular momenta. Some classical references to them, other than [VMK], are [CFS], [Ed], [RBMW], etc. Let us recall its definition in terms of the representation theory of $\mathfrak{sl}(2)$.

If one of following four triples

$$(2j_1, 2j_2, 2j_3),\ (2j_1, 2j_5, 2j_6),\ (2j_4, 2j_2, 2j_6),\ (2j_4, 2j_5, 2j_3)$$

does not satisfy the triangle condition then $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$ is zero by definition. If all the above four triples do satisfy the triangle condition, which may be depicted by the following tetrahedron,



let

$$a=2j_1,\qquad b=2j_2,\qquad p=2j_3,$$
$$c=2j_4,\qquad k=2j_5,\qquad q=2j_6. \tag{12.1}$$

The $6j$-symbols $\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}$ are the coefficients needed to express the $\mathfrak{sl}(2)$-module homomorphism

$$\big(\iota_p^{a,b}\otimes 1\big)\circ\iota_k^{p,c}: V(k)\to V(p)\otimes V(c)\to V(a)\otimes V(b)\otimes V(c)$$

as a linear combination of the $\mathfrak{sl}(2)$-module homomorphisms

$$\big(1\otimes\iota_q^{b,c}\big)\circ\iota_k^{a,q}: V(k)\to V(a)\otimes V(q)\to V(a)\otimes V(b)\otimes V(c)$$

as q varies. More precisely, the following sets

$$\Big\{\big(\iota_p^{a,b}\otimes 1\big)\circ\iota_k^{p,c} : p\in\mathbb{Z}\text{ and }\big(\iota_p^{a,b}\otimes 1\big)\circ\iota_k^{p,c}\neq 0\Big\}$$
$$\Big\{\big(1\otimes\iota_q^{b,c}\big)\circ\iota_k^{a,q} : q\in\mathbb{Z}\text{ and }\big(1\otimes\iota_q^{b,c}\big)\circ\iota_k^{a,q}\neq 0\Big\}$$

are two different bases of $\mathrm{Hom}_{\mathfrak{sl}(2)}\big(V(k),V(a)\otimes V(b)\otimes V(c)\big)$ and the $6j$-symbol describe the transition matrix between these two bases, that is $\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}$ is implicitly defined by the following identity (see (12.1))

$$\frac{(-1)^{j_1-j_2-j_4+j_5}}{(j_1+j_2+j_3+1)(j_3+j_4+j_5+1)\Delta(j_1,j_2,j_3)\Delta(j_3,j_4,j_5)}\big(\iota_p^{a,b}\otimes 1\big)\circ\iota_k^{p,c}$$
$$=\sum_q\frac{(-1)^q(q+1)}{(j_2+j_4+j_6+1)(j_1+j_6+j_5+1)\Delta(j_2,j_4,j_6)\Delta(j_1,j_6,j_5)}\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}\big(1\otimes\iota_q^{b,c}\big)\circ\iota_k^{a,q}.$$

which is equivalent to say that

$$\frac{(-1)^{j_1-j_2-j_4+j_5}}{\sqrt{2j_3+1}}C^{j_3,m_3}_{j_1,m_1;\,j_2,m_2}C^{j_5,m_5}_{j_3,m_3;\,j_4,m_4}$$
$$=\sum_{j_6}(-1)^{2j_6}\sqrt{2j_6+1}\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}C^{j_6,m_6}_{j_2,m_2;\,j_4,m_4}C^{j_5,m_5}_{j_1,m_1;\,j_6,m_6}$$

for all m_i, $i=1,\dots,6$, such that $|m_i|\le j_i$ and $m_1+m_2=m_3$, $m_3+m_4=m_5$, $m_2+m_4=m_6$ and $m_1+m_6=m_5$. This identity is derived from [VMK, §8.7.5, eq.(36)] and the symmetry properties of the Clebsch-Gordan coefficients [VMK, §8.4.2, eq.(5)].

If we set $\Delta_{x,y,z}=\Delta(j_x,j_y,j_z)$, then the $6j$-symbol can be explicitly expressed as (see [VMK, §9.2.1, eq.(1)])

$$\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}=\Delta_{1,2,3}\Delta_{3,4,5}\Delta_{2,4,6}\Delta_{1,5,6}$$
$$\times\sum_t\frac{(-1)^t(t+1)!}{(t-\alpha_0)!\,(t-\alpha_1)!\,(t-\alpha_2)!\,(t-\alpha_3)!\,(\beta_1-t)!\,(\beta_2-t)!\,(\beta_3-t)!}$$

where t runs from $\max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}$ to $\min\{\beta_1,\beta_2,\beta_3\}$ and

$$\begin{aligned}
\alpha_0&=j_1+j_2+j_3, & &\\
\alpha_1&=j_1+j_5+j_6, & \beta_1&=j_2+j_3+j_5+j_6,\\
\alpha_2&=j_4+j_2+j_6, & \beta_2&=j_1+j_3+j_4+j_6,\\
\alpha_3&=j_4+j_5+j_3, & \beta_3&=j_1+j_2+j_4+j_5.
\end{aligned}$$

Also, if

$$R^z_{x,y} = \sqrt{\tfrac{(j_x+j_y-j_z)!}{(j_x-j_y+j_z)!\,(-j_x+j_y+j_z)!\,(j_x+j_y+j_z+1)!}}.$$

then (see [VMK, §9.2.1, eq.(5)])

(12.2)

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = (-1)^{j_1+j_2+j_4+j_5} \frac{R^1_{2,3} R^4_{3,5}}{R^1_{5,6} R^4_{2,6}}$$
$$\times \sum_t \frac{(-1)^t\,(-j_1+j_5+j_6+t)!\,(j_2-j_4+j_6+t)!\,(j_1+j_3+j_4-j_6-t)!}{t!\,(j_1+j_5-j_6-t)!\,(j_2+j_4-j_6-t)!\,(-j_1+j_3-j_4+j_6+t)!\,(2j_6+1+t)!}.$$

We need the following three properties of the $6j$-symbol (see [VMK, §9.4.2]):

(i) The $6j$-symbol is invariant under the permutation of any two columns.
(ii) The $6j$-symbol is invariant if upper and lower arguments are interchanged in any two columns.
(iii) If all the triples

$$(2j_1, 2j_2, 2j_3),\ (2j_1, 2j_5, 2j_6),\ (2j_4, 2j_2, 2j_6),\ (2j_4, 2j_5, 2j_3)$$

satisfy the triangle condition, but one of them is a degenerate triangle, then $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \neq 0$. Indeed, if one of the above triples corresponds to a degenerate triangle, then (i) and (ii) imply that we may assume $j_6 = j_1 + j_5$. Now it follows from (12.2) that

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = (-1)^{j_1+j_2+j_4+j_5} \frac{R^1_{2,3} R^4_{3,5}}{R^1_{5,6} R^4_{2,6}}$$
$$\frac{(-j_1+j_5+j_6)!\,(j_2-j_4+j_6)!\,(j_1+j_3+j_4-j_6)!}{(j_2+j_4-j_6)!\,(-j_1+j_3-j_4+j_6)!\,(2j_6+1)!} \neq 0.$$

The following lemma shows that, under certain additional conditions, other $6j$-symbols are non-zero.

Lemma 12.1. *Let j_1, j_2, j_3, j_4, j_5 and j_6 be non-negative half-integer such that $j_6 = j_1 + j_5$, $j_2 = j_3$ and all the triples*

$$(2h, 2j_2, 2j_3),\ (2h, 2j_5, 2j_6),\ (2j_4, 2j_2, 2j_6),\ (2j_4, 2j_5, 2j_3)$$

satisfy the triangle condition for $h = j_1$ and $h = j_1 + 1$. If $\begin{Bmatrix} j_1+1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = 0$ then $\begin{Bmatrix} j_1+2 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \neq 0$ and $\begin{Bmatrix} j_1+3 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \neq 0$.

Proof. The Biedenharn-Elliott identity yields, in particular, the following three-term recurrence relation (see [SG, pag. 1963])

(12.3)

$$i_1 E(i_1+1) \begin{Bmatrix} i_1+1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \end{Bmatrix} + F(i_1) \begin{Bmatrix} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \end{Bmatrix} + (i_1+1) E(i_1) \begin{Bmatrix} i_1-1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \end{Bmatrix} = 0$$

where

$$\begin{aligned}F(i_1)=(2i_1+1)\big(&i_1(i_1+1)(-i_1(i_1+1)+i_2(i_2+1)+i_3(i_3+1))\\&+i_5(i_5+1)(i_1(i_1+1)+i_2(i_2+1)-i_3(i_3+1))\\&+i_6(i_6+1)(i_1(i_1+1)-i_2(i_2+1)+i_3(i_3+1))\\&-2i_1(i_1+1)i_4(i_4+1)\big)\end{aligned}$$

and

$$E(i_1)=\sqrt{\big(i_1^2-(i_2-i_3)^2\big)\big((i_2+i_3+1)^2-i_1^2\big)\big(i_1^2-(i_5-i_6)^2\big)\big((i_5+i_6+1)^2-i_1^2\big)}.$$

If we fix $(i_2,i_3,i_4,i_5,i_6)=(j_2,j_3,j_4,j_5,j_6)$ we obtain

$$\begin{aligned}E(i_1)&=\sqrt{i_1^2\big((2j_2+1)^2-i_1^2\big)\big(i_1^2-j_1^2\big)\big((j_1+2j_5+1)^2-i_1^2\big)}\\F(i_1)&=-(2i_1+1)i_1(i_1+1)\\&\qquad\times(i_1(i_1+1)-2j_2(j_2+1)-j_5(j_5+1)-j_6(j_6+1)+2j_4(j_4+1)),\end{aligned}$$

and we point out that the triangle conditions satisfied by $(2(j_1+1),2j_5,2j_6)$ and $(2(j_1+1),2j_2,2j_3)$ imply that

$$E(j_1+1)\neq 0.$$

We also claim that $F(j_1+2)\neq 0$, and this will be proved later by considering separately the cases $j_1=0$ and $j_1>0$.

Assume that $\begin{Bmatrix}j_1{+}1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}=0$. Since $j_6=j_5+j_1$ it follows that $\begin{Bmatrix}j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}\neq 0$ (see (iii) above) and $\begin{Bmatrix}j_1{-}1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}=0$ (the triple $(2(j_1{-}1),2j_5,2j_6)$ does not satisfy the triangle condition). Since $E(j_1+1)\neq 0$, it follows, from the recurrence relation (12.3) applied to $i_1=j_1+1$, that $\begin{Bmatrix}j_1{+}2 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}\neq 0$.

Now, accepting that $F(j_1+2)\neq 0$, the recurrence relation (12.3) applied to $i_1=j_1+2$ implies that $\begin{Bmatrix}j_1{+}3 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}\neq 0$ as we wanted to prove.

It remains to be proved that $F(j_1+2)\neq 0$. From the recurrence relation (12.3) applied to $i_1=j_1$ we obtain that $F(j_1)=0$ and, if $j_1>0$, it follows that

$$j_1(j_1+1)-2j_2(j_2+1)-j_5(j_5+1)-j_6(j_6+1)+2j_4(j_4+1)=0,$$

which implies that $F(j_1+2)\neq 0$. If $j_1=0$ then

$$\begin{aligned}0&=\begin{Bmatrix}j_1{+}1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}\\&=\begin{Bmatrix}1 & j_2 & j_2\\ j_4 & j_6 & j_6\end{Bmatrix}\\&=(-1)^{1+j_2+j_4+j_6}\,\frac{R^1_{2,3}R^4_{3,5}}{R^1_{5,6}R^4_{2,6}}\\&\quad\times\sum_{t=0}\frac{(-1)^t\,(-1+2j_6+t)!\,(j_2-j_4+j_6+t)!\,(1+j_2+j_4-j_6-t)!}{t!\,(1-t)!\,(j_2+j_4-j_6-t)!\,(-1+j_2-j_4+j_6+t)!\,(2j_6+1+t)!}.\end{aligned}$$

and hence

$$\frac{(-1+2j_6)!\,(j_2-j_4+j_6)!\,(1+j_2+j_4-j_6)!}{(j_2+j_4-j_6)!\,(-1+j_2-j_4+j_6)!\,(2j_6+1)!} - \frac{(2j_6)!\,(j_2-j_4+j_6+1)!\,(j_2+j_4-j_6)!}{(j_2+j_4-j_6-1)!\,(j_2-j_4+j_6)!\,(2j_6+2)!} = 0,$$

or

$$(j_2-j_4+j_6)(1+j_2+j_4-j_6)(j_6+1)-(j_2-j_4+j_6+1)(j_2+j_4-j_6)j_6 = 0,$$

which implies $j_2(j_2+1)+j_6(j_6+1)-j_4(j_4+1)=0$ and therefore

$$F(2) = -30\big(6-2j_2(j_2+1)-j_5(j_5+1)-j_6(j_6+1)+2j_4(j_4+1)\big) = -180.$$

□

References

[BH] K. Bongartz and B. Huisgen-Zimmermann, *The geometry of uniserial representations of algebras II. Alternate viewpoints and uniqueness*, J. Pure Appl. Algebra **157** (2001) 23–32

[BL] L. C. Biedenharn and J. D. Louck, *Angular Momentum in Quantum Physics*, The Racah-Wigner Algebra in Quantum Theory, 9; Encyclopedia of Mathematics and Its Applications, cd. G. C. Rota (Addison-Wesley, Reading, MA, 1981).

[Bl] L.M. Blumenthal, *Theory and applications of distance geometry*, Chelsea, 1970.

[BCFM] A. Bianchi, V. Chari, G. Fourier, A. Moura, *On multigraded generalizations of Kirillov-Reshetikhin modules*, arXiv:1208.3236, to appear in Algebras and Representation theory.

[Ca] P. Casati, *Irreducible $\mathfrak{sl}_{n+1}$-representations remain indecomposable restricted to some Abelian subalgebras*, J. Lie Theory **20** (2010) 393-407.

[CKR] V. Chari, A. Khare, T. Ridenour *Faces of polytopes and Koszul algebras*, J. Pure Appl. Algebra **216** (2012), no. 7, 1611–1625.

[CFS] S. Carter, D.E. Flath, M. Saito, *The Classical and Quantum 6j-symbols*, Mathematical Notes 43, Princeton University Press, 1995.

[CG] V. Chari, J. Greenstein, *Minimal affinizations as projective objects*, J. Geom. Phys. **61** (2011) 594–609.

[CM] V. Chari, A. Moura, *The restricted Kirillov-Reshetikhin modules for the current and twisted current algebras*, Commun. Math. Phys. **266** (2006), 431–454.

[CS1] L. Cagliero and F. Szechtman, *Jordan-Chevalley decomposition in finite dimesional Lie algebras*, Proc. Amer. Math. Soc. **139** (2011) 3909-3913.

[CS2] L. Cagliero and F. Szechtman, *Classification of uniserial modules of a family of solvable Lie algebras*, in preparation.

[CS3] L. Cagliero and F. Szechtman, *Indecomposable modules of $\mathfrak{sl}(2)\ltimes V(m)$*, in preparation.

[CMS] P. Casati, S. Minniti, V. Salari, *Indecomposable representations of the Diamond Lie algebra*, Journal of Mathematical Physics **51** (2010), no. 3, 033515, 20pp.

[Dd] A. Douglas and H. de Guise, *Some nonunitary, indecomposable representations of the Euclidean algebra $\mathfrak{e}(3)$* J. Phys. A: Math. Theor. **43** (2010), no. 8, 085204, 13pp.

[DR] A. Douglas, J. Repka, *Embedding of the Euclidean algebra e(3) into sl(4,C) and restriction of irreducible representations of sl(4,C)*, Journal of Mathematical Physics **52** (2011), no. 1, 013504, 9pp.

[DP] A. Douglas and A. Premat, *A class of nonunitary, finite dimensional representations of the euclidean algebra $\mathfrak{e}(2)$*, Communications in Algebra **35** (2007) 1433-1448.

[Ed] A. Edmonds, *Angular Momentum in Quantum Mechanics*, Princeton, New Jersey: Princeton University Press (1957).

[EFK] P. Etingof, I. Frenkel, A. Kirillov, *Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations*, AMS, 1998.

[G] J. Greenstein *Quivers with relations arising from Koszul algebras of g-invariants*, J. Algebra 322 (2009) 4430–4478.

[GP] I.M. Gelfand, V.A. Ponomarev, *Remarks on the classification of a pair of commuting linear transformations in a finite dimensional vector space*, Functional Anal. Appl. **3** (1969) 325-326

[GV] S. Garoufalidis and R. Van Der Veen, *Asymptotics of classical spin networks*, arXiv:0902.3113.

[HZ] B. Huisgen-Zimmermann, *The geometry of uniserial representations of finite dimensional algebras I*, J. Pure Appl. Algebra **127** (1998) 39–72.

[HZ2] B. Huisgen-Zimmermann, *The geometry of uniserial representations of finite dimensional algebras. III: Finite uniserial type*, Transactions of the American Mathematical Society **348** (1996) 4775-4812.

[K] A. Khare, *Category O over a deformation of the symplectic oscillator algebra*, J. Pure Appl. Algebra **195** (2005) 131–166.

[L] J. Louck, *Survey of zeros of 3j and 6j coefficients by Diophantine equation methods*, Group theory and special symmetries in nuclear physics (Ann Arbor, MI, 1991), 268-275, World Sci. Publ., River Edge, NJ, 1992.

[M] A. Moura, *Restricted limits of minimal affinizations*, Pacific J. Math **244** (2010) 359–397.

[Pi] A. Piard, *Sur des représentations indécomposables de dimension finie de $SL(2).R^2$*, Journal of Geometry and Physics, Volume **3**, Issue 1, 1986, 1–53.

[Pr] A. Premat, *Indecomposable representations of the euclidrean algebra*, preprint.

[Rd] J. Repka and H. de Guise, *Some finite-dimensional indecomposable representations of E(2)*, J. Math. Phys. **40** (1999) 6087-6109

[RBMW] M. Rotenberg, R. Bivins, N. Metropolis and J.K. Wooten, *The 3-j and 6-j Symbols*, Technology Press, Cambridge, Massachusetts (1959).

[R] J. Raynal, *On the zeros of 6-j coefficients*, J. Phys. A **27** (1994), no. 9, 3105 –3127.

[WD] K. Wirth and A. Dreiding, *Edge lengths determining tetrahedrons*, Elem. Math. **64** (2009) 160-170.

[Sa] A. Savage, *Quivers and the Euclidean group*, Contemporary Mathematics **478** 2009 177-188.

[SG] K. Schulten and R. G. Gordon. *Exact recursive evaluation of 3J and 6J coefficients for quantum mechanical coupling of angular momenta*, J. Math. Phys. **16** (1975) 1961–1970.

[VMK] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum theory of angular momentum*, World Scientific, Singapore, 1989.

[ZR] L. Zamick and S. Robinson, *Zeros of 6-j symbols and more: Atoms, Nuclei and Bosons*, arXiv:1105.3637.

CIEM-CONICET, FAMAF-Universidad Nacional de Córdoba, Córdoba, Argentina.
E-mail address: `cagliero@famaf.unc.edu.ar`

Department of Mathematics and Statistics, University of Regina, Canada
E-mail address: `fernando.szechtman@gmail.com`