DYNAMICAL COHERENCE OF PARTIALLY HYPERBOLIC DIFFEOMORPHISMS OF TORI ISOTOPIC TO ANOSOV

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Abstract. We show that partially hyperbolic diffeomorphisms of d-dimensional tori isotopic to an Anosov diffeomorphism, where the isotopy is contained in the set of partially hyperbolic diffeomorphisms, are dynamically coherent. As a consequence, we obtain intrinsic ergodicity and measure equivalence for partially hyperbolic diffeomorphisms with one-dimensional center direction that are isotopic to Anosov diffeomorphisms.

Keywords: Partial hyperbolicity, Dynamical coherence, Measures of maximal entropy

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1. Introduction

A fundamental problem in dynamical systems is classifying dynamical phenomena and describing the spaces that support these actions. By the 1970s there was a good classification of smooth systems that are uniformly hyperbolic. This is seen in the well known Franks-Manning classification result of Anosov diffeomorphisms of tori.

Recently, there is a great deal of interest in understanding the dynamical properties of partially hyperbolic diffeomorphisms, precise definitions are given in Section 1.3. In this paper we consider partially hyperbolic diffeomorphisms of the d-torus isotopic to Anosov diffeomorphisms. We start with an informal presentation of our results followed by a more precise formulation.

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1.1. Dynamical coherence.

It is well known that the stable and unstable bundles of an Anosov diffeomorphism are integrable. This extends to the stable and unstable bundles of a partially hyperbolic diffeomorphism [HPS], but the integrability of the center bundle is a subtler issue, see for instance [BuW_1]. When the center bundle is integrable the partially hyperbolic diffeomorphism is *dynamically coherent*.

Main Theorem. *Let* $f : \mathbb{T}^d \to \mathbb{T}^d$ *be a partially hyperbolic diffeomorphism which is isotopic to a linear Anosov automorphism along a path of partially hyperbolic diffeomorphisms. Then,* f *is dynamically coherent.*

We establish dynamical coherence without the usual restrictions on the dimension of the center bundle, the strength of the domination, or the geometry of the strong foliations in the universal cover. This is one of the first result on dynamical coherence without restriction on the dimension of the center bundle which holds in "large" open sets (whole connected components of partially hyperbolic diffeomorphisms) and the center fibers are noncompact.

1.2. Maximizing measures.

Another motivation for this paper grew from an attempt to extend the results of [BFSV]. In that paper it is shown that a well known example of partially hyperbolic diffeomorphism, known as Mañé's example (see [M_1] or [BDV] Chapter 7), has a unique measure of maximal entropy. In fact, it is shown there that using the measure of maximal entropy that Mañé's example as a measure preserving transformation is isomorphic to the measure preserving transformation given by a linear Anosov automorphism of $\mathbb{T}^3$ and Haar measure.

This result was extended in [BF] to certain diffeomorphisms that are C^0 close to hyperbolic toral automorphisms, but not partially hyperbolic. In this case the diffeomorphisms satisfy a weak version of hyperbolicity called a dominated splitting.

A further extension was obtained by Ures [U] to all *absolutely* partially hyperbolic diffeomorphisms of $\mathbb{T}^3$ isotopic to Anosov as well as other higher dimensional cases under the further assumption of quasi-isometry of the strong foliations (in order to be able to use results of

[Br, H]). For $\mathbb{T}^3$, under the assumption of pointwise partial hyperbolicity, this result can be weakened to cover all (not necessarily absolute) partially hyperbolic diffeomorphisms of $\mathbb{T}^3$ isotopic to Anosov thanks to the results in [Pot], see [HP] section 6.1.

Let us briefly comment on the idea of the proof of the existence and uniqueness of maximal entropy measures for partially hyperbolic diffeomorphisms with one dimensional center isotopic to Anosov. For such diffeomorphisms there always exists a continuous semiconjugacy to their linear part, and the main point in the proof consists in showing the following properties:

- The fibers of the semiconjugacy are connected arcs of bounded length (and thus carry no entropy).
- The image of the set of points on which the semiconjugacy is 1 to 1 has total Lebesgue measure in $\mathbb{T}^d$.

These two results together with properties of topological and measure theoretic entropy give the desired result (see Section 7 for more details). The main point is to obtain dynamical coherence and use the fact that fibers of the semiconjugacy are contained in center manifolds, this is to be expected since one expects the semiconjugacy to be injective along strong manifolds. This is why in [U] the hypothesis of quasi-isometry and absolute partial hyperbolicity are used.

We prove that partially hyperbolic diffeomorphisms (not necessarily absolute) which are isotopic to the linear Anosov diffeomorphisms along a path of partially hyperbolic diffeomorphisms with one-dimensional center bundle have a unique measure of maximal entropy, see Corollary 1.2 bellow. We remark that even for absolute partially hyperbolic diffeomorphisms this result was not known without further hypothesis on the geometry of the strong foliations.

We remark that in [NY, BF] it was shown that there are systems with a unique measure of maximal entropy and whose topological entropy is C^1 locally constant even if the center bundles have dimension 2. In [NY] the situation is a partially hyperbolic diffeomorphism that is dynamically coherent with 2-dimensional center fibers, and in [BF] there are two transverse foliations each 2-dimensional and tangent to the dominated splitting. In both of these cases the diffeomorphisms can be chosen to be isotopic to Anosov. Moreover, in [BFSV] it shown that there are partially hyperbolic diffeomorphisms isotopic to Anosov (through a path of partially hyperbolic ones) having bidimensional center and having a unique measure of maximal entropy (and whose topological entropy is also C^1 locally constant). This example can be extended to higher dimensional center. Thus, another reason to establish dynamical coherence in the Main Theorem is that under certain additional hypothesis one may be able to establish there is a unique measure of maximal entropy and constant topological entropy for systems isotopic to Anosov diffeomorphisms without the restriction of the center bundle being 1-dimensional (although of course one cannot expect that this holds in the entire connected component in this case).

1.3. **Precise Setting.** We say that $f : M \to M$ is *partially hyperbolic* if there exists a Df-invariant splitting $TM = E^{ss}_f \oplus E^c_f \oplus E^{uu}_f$ such that there exists $N > 0$ and $\lambda > 1$ verifying that for every $x \in M$ and unit vectors $v^\sigma \in E^\sigma_f(x)$ ($\sigma = ss, c, uu$) we have

(i) $\lambda \|Df^N_x v^{ss}\| < \|Df^N_x v^c\| < \lambda^{-1}\|Df^N_x v^{uu}\|$, and
(ii) $\|Df^N_x v^{ss}\| < \lambda^{-1} < \lambda < \|Df^N_x v^{uu}\|$.

We will assume throughout that $N = 1$ due to results in [Gou]. We remark that the bundles can be trivial.

The definition we have used of partial hyperbolicity is the weakest one appearing in the literature. It is sometimes referred to as *pointwise* partial hyperbolicity as opposed to *absolute* partial hyperbolicity [1]. The absolute partial hyperbolicity sometimes simplifies proofs of dynamical coherence (see [Br]) but is quite artificial as it does not capture the real nature of domination (this becomes clear for example when more bundles are involved). We remark that there are different results in the study of dynamical coherence depending on the definition used, see [BBI$_2$, RHRHU, Pot].

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[1] For absolute partial hyperbolicity it is required that the inequalities hold for unit vectors that may belong to the bundles of different points.

We denote

$$\mathsf{PH}(\mathbb{T}^d) = \{f : \mathbb{T}^d \to \mathbb{T}^d \text{ partially hyperbolic}\}.$$

Let $A \in SL(d,\mathbb{Z})$ be a linear Anosov automorphism admitting a dominated splitting of the form $E^{ss}_A \oplus E^{ws}_A \oplus E^{wu}_A \oplus E^{uu}_A$. We denote as $E^s_A = E^{ss}_A \oplus E^{ws}_A$, $E^c_A = E^{ws}_A \oplus E^{wu}_A$ and $E^u_A = E^{wu}_A \oplus E^{uu}_A$. There may be many possibilities for the dimensions of E^{ss}_A and E^{ws}_A (respectively for E^{wu}_A and E^{uu}_A).

We consider $\mathsf{PH}_{A,s,u}(\mathbb{T}^d) \subset \mathsf{PH}(\mathbb{T}^d)$ the subset of those which are isotopic to A and whose splitting verifies $\dim E^{ss}_f = \dim E^{ss}_A = s$ and $\dim E^{uu}_f = \dim E^{uu}_A = u$. In order to simplify notation we will denote $\mathsf{PH}_{A,s,u}(\mathbb{T}^d)$ as $\mathsf{PH}_A(\mathbb{T}^d)$ leaving the dimensions of E^σ_A ($\sigma = ss, uu$) implicit from the context (we will leave them fixed throughout the paper).

For $X \subset \mathbb{T}^d$ we let $\widetilde{X}$ denote the lift of X to $\mathbb{R}^d$. Similarly, for $f : \mathbb{T}^d \to \mathbb{T}^d$ a diffeomorphism we let $\widetilde{f} : \mathbb{R}^d \to \mathbb{R}^d$ be the lift of f.

Given $f \in \mathsf{PH}_A(\mathbb{T}^d)$ we know from [Fr] there exists $H_f : \mathbb{R}^d \to \mathbb{R}^d$ a continuous and surjective map such that

$$A \circ H_f = H_f \circ \widetilde{f}.$$

Moreover, $H_f(x+\gamma) = H_f(x) + \gamma$ for every $\gamma \in \mathbb{Z}^d$.

Remark 1. The map H varies continuously with f in the C^0-topology. This is a general fact which does not require f to be partially hyperbolic. This means that given $\varepsilon > 0$ there exists a neighborhood $\mathcal{U}$ of f in the C^0-topology such that $d(H_f(x), H_g(x)) < \varepsilon$ for every $x \in \mathbb{R}^d$ and $g \in \mathcal{U}$.

◊

If there exist f-invariant foliation $\mathcal{W}^{cs}_f$ and $\mathcal{W}^{cu}_f$ tangent respectively to $E^s_f \oplus E^c_f$ and $E^c_f \oplus E^u_f$, then there exists an invariant center foliation $\mathcal{W}^c_f$ tangent to E^c_f and f is dynamically coherent.

We say that a dynamically coherent $f \in \mathsf{PH}_A(\mathbb{T}^d)$ is *center-fibered* if $H_f^{-1}(E^c_A + H_f(x)) = \widetilde{\mathcal{W}^c_f}(x)$. This means that by the semiconjugacy H_f different leaves of the center foliation map surjectively to different translates of E^c_A.

We denote $\mathsf{PH}^0_A(\mathbb{T}^d)$ to be the connected components of $\mathsf{PH}_A(\mathbb{T}^d)$ containing a dynamically coherent center-fibered[2] Anosov diffeomorphism. Notice that the linear Anosov diffeomorphism A is center-fibered so that $\mathsf{PH}^0_A(\mathbb{T}^d)$ is a non-empty open set with at least one connected component. Notice also that the space of Anosov diffeomorphisms may not be connected [FG], so that the set $\mathsf{PH}^0_A(\mathbb{T}^d)$ is potentially larger than the connected component containing A.

1.4. **Precise Statement of results.**

Theorem 1.1. *Every $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$ is dynamically coherent and center fibered.*

We prove some intermediary results in more generality. Also, the theorem can be applied even in the case where E^{ss}_A or E^{uu}_A are zero dimensional (if both are trivial, the theorem itself is trivial).

We deduce the following consequence:

Corollary 1.2. *If $\dim E^c_f = 1$ then there exists a unique maximal entropy measure with equal entropy to the linear part. Furthermore, the unique measure of maximal entropy is a hyperbolic measure.*

See Section 7 for definitions and the proof of the Corollary.

It is possible that our results can be applied in the case of partially hyperbolic diffeomorphisms isotopic to Anosov diffeomorphisms in nilmanifolds. However, this has to be done with some care since even the initial Anosov diffeomorphism may not be dynamical coherent (see [BuW$_1$] for possible problems). It may then be the case that every partially hyperbolic diffeomorphism isotopic to such Anosov through partially hyperbolic diffeomorphisms will not be dynamically coherent (extending a construction announced by Gourmelon [BuW$_1$]), but we have not checked this in detail.

It is also possible that our techniques shed light in studying the case of partially hyperbolic diffeomorphisms of $\mathbb{T}^d$ isotopic to linear partially hyperbolic automorphisms even if these are not Anosov. This is

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[2] We remark that this implies that by the semiconjugacy, which in this case is a conjugacy, the center stable and center unstable foliations map onto the center stable and center unstable foliations of the linear part.

because there are some types of semiconjugacies when the linear part is partially hyperbolic, and under some (possibly more restrictive) hypothesis one expects that our techniques could be adapted to that case. Notice that the non-dynamical coherent examples given by [RHRHU] are not isotopic to their linear representative through partially hyperbolic systems.

2. Definitions and preliminaries:

2.1. **First remarks.** For $f \in \mathrm{PH}(\mathbb{T}^d)$ there exist f-invariant foliations $\mathcal{W}^{ss}_f$ and $\mathcal{W}^{uu}_f$ tangent to E^{ss}_f and E^{uu}_f respectively that we call the *strong foliations*. We let $\widetilde{\mathcal{W}}^\sigma(x)$ denote the associated σ foliation for $\widetilde{f}$ (where $\sigma = ss, uu$ or when they exist $\sigma = cs, cu, c$).

In general, we have $H_f(\widetilde{\mathcal{W}}^{uu}_f(x)) \subset E^{wu}_A \oplus E^{uu}_A + H_f(x)$. Similarly, for $\widetilde{\mathcal{W}}^{ss}_f$ we have $H_f(\widetilde{\mathcal{W}}^{ss}_f(x)) \subset E^{ws}_A \oplus E^{ss}_A + H_f(x)$.

We now introduce some notation. Let

$$B^\sigma_R(x) = B_R(x) \cap (E^\sigma_A + x)$$

for $\sigma = ss, uu, c, s, u, ws, wu$. For $f \in \mathsf{PH}(\mathbb{T}^d)$ we let

$$D^\sigma_{R,f}(x) = \{y \in \widetilde{\mathcal{W}}^\sigma(x) \ : \ d_{\mathcal{W}^\sigma}(x,y) < R\}$$

where $d_{\mathcal{W}^\sigma}(\cdot,\cdot)$ denotes the metric inside the leaves induced by restricting the metric of $\mathbb{R}^d$ to a Riemannian metric in the leaves.

From the continuous variation on compact parts of the strong manifolds one has the following classical result [HPS].

Proposition 2.1. *For every $R > 0$ and $\varepsilon > 0$ there exists $\mathcal{U}$ a C^1-neighborhood of f and $\delta > 0$ such that for every $g \in \mathcal{U}$ and every $x, y \in \mathbb{R}^d$ with $d(x,y) < \delta$ one has*

$$d_{C^1}(D^\sigma_{R,g}(x), D^\sigma_{R,f}(y)) < \varepsilon$$

for $\sigma = ss, uu$.

Remark 2. For $f \in \mathsf{PH}(\mathbb{T}^d)$ there exist constants $1 < \lambda_f < \Delta_f$ such that in a C^1-neighborhood $\mathcal{U}$ of f we have

$$D^{uu}_{(\lambda_f R),g}(\widetilde{g}(x)) \subset \widetilde{g}(D^{uu}_{R,g}(x)) \subset D^{uu}_{(\Delta_f R),x}(\widetilde{g}(x))$$

for every $g \in \mathcal{U}$, $x \in \mathbb{R}^d$ and $R > 0$. A similar result holds for D^{ss} by applying $\widetilde{g}^{-1}$. This follows from the fact that the derivative of f restricted to the unstable bundle is always larger than λ_f and the global derivative of f is smaller than Δ_f (from compactness). Therefore, one can also show for g C^1-close to f that one has the same estimates for the derivative of g in any vector lying in a small cone around the unstable direction of f, so that the estimates hold for disks tangent to a cone close to the unstable direction.

◇

2.2. **Strong Almost Dynamical Coherence.**

The following definitions are motivated by the ones introduced in [Pot] but slightly adapted to our needs.

Definition 1 (Almost parallel foliations)**.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be foliations of $\mathbb{T}^d$. Then they are *almost parallel* if there exists $R > 0$ such that for every $x \in \mathbb{R}^d$ there exists y_1 and y_2 such that:

- $\widetilde{\mathcal{F}}_1(x) \subset B_R(\widetilde{\mathcal{F}}_2(y_1))$ and $\widetilde{\mathcal{F}}_2(y_1) \subset B_R(\widetilde{\mathcal{F}}_1(x))$
- $\widetilde{\mathcal{F}}_2(x) \subset B_R(\widetilde{\mathcal{F}}_1(y_2))$ and $\widetilde{\mathcal{F}}_1(y_2) \subset B_R(\widetilde{\mathcal{F}}_2(x))$.

◇

Being almost parallel is an equivalence relation (see [HP] Appendix B). Notice that the condition can be stated in terms of Hausdorff distance by saying that for every $x \in \mathbb{R}^d$ there exists y_1 and y_2 such that the Hausdorff distance between $\widetilde{\mathcal{F}}_1(x)$ and $\widetilde{\mathcal{F}}_2(y_1)$ is smaller than R and the Hausdorff distance between $\widetilde{\mathcal{F}}_2(x)$ and $\widetilde{\mathcal{F}}_1(y_2)$ is smaller than R.

Definition 2 (Strong Almost Dynamical Coherence)**.** Let $f \in \mathsf{PH}_A(\mathbb{T}^d)$ we say it is *strongly almost dynamically coherent* (SADC) if there exists foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ (not necessarily invariant) which are respectively transverse to E^{uu}_f and E^{ss}_f and are almost parallel to the foliations $E^{ss}_A \oplus E^c_A$ and $E^c_A \oplus E^{uu}_A$ respectively.

◇

The next result is proved in [Pot, Proposition 4.5].

Proposition 2.2. *Being SADC is an open and closed property in* $\mathsf{PH}_A(\mathbb{T}^d)$*. In particular, every* $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$ *verifies this property.*

The idea of the proof is that open is trivial since the same foliation works by the continuous variation of the E^{ss} and E^{uu} bundles. If $f_n \to f$ one can choose n large enough so that the bundles are close. By choosing the foliation $\mathcal{F}^{cs}_n$ for f_n and iterating backwards by f_n a finite number of times one gets a foliations which works for f. Notice that since f_n is isotopic to A it fixes the class of foliations almost parallel to any A-invariant hyperplane.

3. σ-PROPERNESS

We define Π^σ_x to be the projection of $\mathbb{R}^d$ onto $E^\sigma_A + x$ along the complementary subbundles of A, we will usually omit the subindex x.

Definition 3 (σ-properness)**.** For $\sigma = ss, uu$ we say that $f \in \mathsf{PH}_A(\mathbb{T}^d)$ is *σ-proper* if the map $\Pi^\sigma \circ H_f|_{\widetilde{\mathcal{W}}^\sigma}$ is (uniformly) proper. More precisely, for every $R > 0$ there exists $R' > 0$ such that, for every $x \in \mathbb{R}^d$ we have

$$(\Pi^\sigma \circ H_f)^{-1}(B^\sigma_R(H_f(x))) \cap \widetilde{\mathcal{W}}^\sigma_f(x) \subset D^\sigma_{R',f}(x).$$

$$\diamondsuit$$

Lemma 3.1. *Let $f \in \mathsf{PH}_A(\mathbb{T}^d)$ be such that there exists $R_1 > 0$ verifying that for every $x \in \mathbb{R}^d$ we have*

$$(\Pi^\sigma \circ H_f)^{-1}(B^\sigma_1(H_f(x))) \cap \widetilde{\mathcal{W}}^\sigma_f(x) \subset D^\sigma_{R_1,f}(x)$$

then, f is σ-proper.

Proof. We consider the case $\sigma = uu$ the other is symmetric. Since A is Anosov and expands uniformly along E^{uu}_A we know that given $R > 0$ there exists $N > 0$ such that for every $z \in \mathbb{R}^d$ we have $B^{uu}_R(A^N(z)) \subset A^N(B^{uu}_1(z))$.

Consider $R > 0$ and $R' = \Delta^N_f R_1$ with N as defined above.

Let

$$y \in (\Pi^\sigma \circ H_f)^{-1}(B^{uu}_R(H_f(x))) \cap \widetilde{W}^\sigma_f(x).$$

Then, we can see that $\widetilde{f}^{-N}(y) \in D^{uu}_{R_1}(\widetilde{f}^{-N}(x))$. Indeed, since

$$\Pi^{uu}(H_f(y)) \in B^{uu}_R(H_f(x))$$

and $A^{-N}(H_f(y)) = H_f(\tilde{f}^{-N}(y))$ we have that

$$\Pi^{uu}(A^{-N}(H_f(y))) \in B_1^{uu}(A^{-N}(x))$$

from how we chose N.

Then, from the hypothesis of the Lemma we know that $\widetilde{f}^{-N}(y) \in H_f^{-1}(A^{-N}(H_f(y)))$ is contained in $D^{uu}_{R_1,f}(\widetilde{f}^{-N}(x))$.

Using Remark 2 we deduce that $y \in D^{uu}_{R',f}(x)$ as desired.

□

In the remainder of this section we will show the equivalence between σ-properness and the following conditions:

(I^σ) The function $\Pi^\sigma \circ H_f$ is injective when restricted to each leaf of $\widetilde{\mathcal{W}}^\sigma_f$.

(S^σ) The function $\Pi^\sigma \circ H_f$ is onto $E^\sigma_A + H_f(x)$ when restricted to each leaf of $\widetilde{\mathcal{W}}^\sigma_f(x)$.

Lemma 3.2. *If $f \in \mathsf{PH}_A(\mathbb{T}^d)$ is σ-proper, then it verifies both (I^σ) and (S^σ).*

Proof. First we show the injectivity of $\Pi^\sigma \circ H_f$ along leaves of $\widetilde{\mathcal{W}}^\sigma_f$. Assume by contradiction that y belongs to the leaf $\widetilde{\mathcal{W}}^\sigma_f(x)$ of $\widetilde{\mathcal{W}}^\sigma_f$ and that $\Pi^\sigma \circ H_f(x) = \Pi^\sigma \circ H_f(y)$ where $y \neq x$. Since $y \neq x$ there exists a $\delta > 0$ such that $y \neq D^\sigma_{\delta,f}(x)$. Using Remark 2 we know that given $R_1 > 0$ there exists $N \in \mathbb{Z}$ such that $\widetilde{f}^N(y) \notin D^\sigma_{R_1,f}(\widetilde{f}^N(x))$.

Consider R_1 given by σ-properness applied to $R = 1$. Then, we know that

$$(\Pi^\sigma \circ H_f)^{-1}(B^\sigma_1(H_f(z))) \subset D^\sigma_{R_1,f}(z)$$

for every $z \in \mathbb{R}^d$. However, we have

$$(\Pi^\sigma \circ H_f)^{-1}(B^\sigma_1(H_f(\widetilde{f}^N(x)))$$

contains $\widetilde{f}^N(y)$, and $\widetilde{f}^N(y)$ is not contained in $D^\sigma_{R_1,f}(\widetilde{f}^N(x))$, a contradiction.

Now, we shall show surjectivity of $\Pi^\sigma \circ H_f$ along leaves of $\widetilde{\mathcal{W}}^\sigma_f$ onto E^{uu}_A. In the argument we will use the injectivity property established above.

We claim first that injectivity of $\Pi^\sigma \circ H_f$ implies that there exists a $\delta > 0$ such that

$$\Pi^\sigma \circ H_f(\partial D^\sigma_{1,f}(x)) \cap B^\sigma_\delta(H_f(x)) = \emptyset.$$

Indeed, otherwise there would exist a pair of sequences x_n, y_n such that $y_n \in \partial D^\sigma_{1,f}(x_n)$ and that

$$\Pi^\sigma \circ H_f(y_n) \in B^\sigma_{1/n}(H_f(x_n)).$$

Taking a subsequence and composing with deck transformations we can assume that both sequences converge to points x, y. We have that $y \in \partial D^\sigma_{1,f}(x)$ in particular $y \neq x$ but we know by continuity of H_f that $H_f(x) = H_f(y)$ contradicting injectivity.

From injectivity and Invariance of Domain's (see for instance [Hat] Theorem 2B.3), we know that for every $z \in \mathbb{R}^d$ we have $S_z = \Pi^\sigma \circ H_f(\partial D^\sigma_{1,f}(x))$ is a $(\dim E^{uu}_f - 1)$-dimensional sphere embedded in $E^{uu}_A + H_f(x)$. Using Jordan's Separation Theorem ([Hat] Proposition 2B.1) and the fact that $\dim E^{uu}_f = \dim E^{uu}_A$ we deduce that S_z separates $E^{uu}_A + H_f(x)$ into two components. Moreover, the image by $\Pi^\sigma \circ H_f$ of $D^\sigma_{1,f}(x)$ is the bounded component and contains $H_f(x)$. From the above remark it also contains $B^\sigma_\delta(H_f(x))$.

Now, fix $R > 0$, then there exists $N \in \mathbb{Z}$ such that

$$B^\sigma_R(z) \subset A^N(B^\sigma_\delta(A^{-N}(z))).$$

Using the semiconjugacy we see that

$$B^\sigma_R(H_f(x)) \subset \Pi^\sigma \circ H_f(\widetilde{f}^N(D^\sigma_{1,f}(\widetilde{f}^{-N}(x)))).$$

Since this holds for any R we know $\Pi^\sigma \circ H_f$ verifies (S^σ) as desired.

$\square$

Lemma 3.3. *If $f \in \mathsf{PH}_A(\mathbb{T}^d)$ verifies (I^σ) and (S^σ) then f is σ-proper.*

Proof. The fact that f has properties (I^σ) and (S^σ) implies that for every $x \in \mathbb{R}^d$ we know

$$\Pi^\sigma \circ H_f : \widetilde{\mathcal{W}}^\sigma_f(x) \to E^\sigma_A + H_f(x)$$

is a homeomorphism for every $x \in \mathbb{R}^d$. In particular, we deduce that

$$(\Pi^\sigma \circ H_f)^{-1}(B^\sigma_1(H_f(x))) \cap \widetilde{\mathcal{W}}^\sigma(x)$$

is bounded for every $x \in \mathbb{R}^d$.

Consider the function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\varphi(x)$ is the infimum of the values of R such that

$$(\Pi^\sigma \circ H_f)^{-1}(B_1^\sigma(H_f(x))) \cap \widetilde{\mathcal{W}}^\sigma(x) \subset D^\sigma_{R,f}(x).$$

From Lemma 3.1 we know that if φ is uniformly bounded in $\mathbb{R}^d$ then f is σ-proper. Since φ is $\mathbb{Z}^d$-periodic, it is enough to control its values in a fundamental domain that is compact. Thus, it is enough to show that if $x_n \to x$ then $\limsup \varphi(x_n) \leq \varphi(x)$.

To show this, notice that $\Pi^\sigma \circ H_f(D^\sigma_{\varphi(x),f}(x))$ contains $B_1^\sigma(H_f(x))$. Since it is a homeomorphism we deduce that for every ε, there exists δ such that

$$B^\sigma_{1+\delta}(H_f(x)) \subset \Pi^\sigma \circ H_f(D^\sigma_{\varphi(x)+\varepsilon,f}(x)).$$

Using the continuous variation of the $\widetilde{\mathcal{W}}^\sigma$ leaves (Proposition 2.1) and continuity of $\Pi^\sigma \circ H_f$ we deduce that for n large enough that $\Pi^\sigma \circ H_f(D^\sigma_{\varphi(x)+\varepsilon,f}(x_n))$ contains $B_1^\sigma(H_f(x_n))$ showing that $\limsup \varphi(x_n) \leq \varphi(x) + \varepsilon$ and this holds for every $\varepsilon > 0$.

$\square$

4. Dynamical coherence

We now state a criteria for integrability of the bundles of a partially hyperbolic diffeomorphisms. This criteria generalizes the one given in [Pot] for dimension 3 (though it requires stronger hypothesis).

We recall that two transverse foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathbb{T}^d$ have a *global product structure* if for any two points $x, y \in \mathbb{R}^d$ the leaves $\widetilde{\mathcal{F}}_1(x)$ and $\widetilde{\mathcal{F}}_2(y)$ intersect in a unique point.

Theorem 4.1. *Assume that $f \in \mathsf{PH}_A(\mathbb{T}^d)$ verifies the following properties:*

- *f is SADC.*
- *f is uu-proper.*

Then, the bundle $E^{ss}_f \oplus E^c_f$ is integrable into an f-invariant foliation $\mathcal{W}^{cs}_f$ that verifies

$$H_f^{-1}((E^{ss}_A \oplus E^c_A) + H_f(x)) = \widetilde{\mathcal{W}}^{cs}_f(x).$$

Moreover, we know $\widetilde{\mathcal{W}}^{cs}_f$ *has a global product structure with* $\widetilde{\mathcal{W}}^{uu}_f$*.*

Proof. We know $\{H^{-1}(E^s_A \oplus E^c_A + y)\}_{y\in\mathbb{R}^d}$ is an $\widetilde{f}$-invariant partition of $\mathbb{R}^d$ that is invariant under deck transformations. This follows as a direct consequence of the semiconjugacy relation and the fact that H is $\mathbb{Z}^d$-periodic. We shall show that under the assumptions of the theorem that $\{H^{-1}(E^s_A \oplus E^c_A + y)\}_{y\in\mathbb{R}^d}$ is a foliation.

Let $\mathcal{F}^{cs}$ be a foliation given by the SADC property. Since it is almost parallel to the linear foliation induced by the subspace $E^{ss}_A \oplus E^c_A$ and H_f is a bounded distance from the identity we know $H(\widetilde{\mathcal{F}}^{cs}(x))$ is a bounded Hausdorff distance of (a translate of) $E^{ss}_A \oplus E^c_A$ for every $x \in \mathbb{R}^d$.

From the properties (I^{uu}) and (S^{uu}) we deduce that there is a global product structure between $\widetilde{\mathcal{F}}^{cs}$ and $\widetilde{\mathcal{W}}^{uu}$. Indeed, consider $x, y \in \mathbb{R}^d$, we shall first show that $\widetilde{\mathcal{F}}^{cs}(x)$ intersects $\widetilde{\mathcal{W}}^{uu}(y)$. To do this, consider the set $Q = \mathbb{R}^d \backslash \widetilde{\mathcal{F}}^{cs}(x)$. By a Jordan Separation like result one deduces that the $d - cs - 1$-homology of Q is non-trivial where $cs = \dim E^{ss}_A + \dim E^c_A$. For a proof see for example Lemma 2.1 of [ABP].

Since $\widetilde{\mathcal{F}}^{cs}(x)$ is a bounded Hausdorff distance from $E^{ss}_A \oplus E^c_A$ one deduces that there is a non-trivial cycle of $H_{d-cs-1}(Q)$ inside E^{uu}_A. Choosing this cycle sufficiently far away from $\widetilde{\mathcal{F}}^{cs}(x)$, and using properties (I^{uu}) and (S^{uu}) one deduces the existence of a non-trivial cycle contained in $\widetilde{\mathcal{W}}^{uu}_f(y)$. This gives the intersection point (for more details see the proof of Proposition 3.1 in [ABP]).

Now we must prove that the intersection point between $\widetilde{\mathcal{W}}^{uu}(x)$ and $\widetilde{\mathcal{F}}^{cs}(y)$ is unique. For this, it is enough to show that given a leaf $\widetilde{\mathcal{F}}^{cs}(x)$ of $\widetilde{\mathcal{F}}^{cs}$ there is no leaf of $\widetilde{\mathcal{W}}^{uu}_f$ intersecting $\widetilde{\mathcal{F}}^{cs}(x)$ more than once. We will use the following easy facts that follow from the hypothesis we have made on f:

1. H_f is injective along leafs of $\widetilde{\mathcal{W}}^{uu}_f$. Moreover, for every $y \in \mathbb{R}^d$ we have $H_f(\mathcal{W}^{uu}_f(y))$ intersects $E^{ss}_A \oplus E^c_A$ in a unique point.
2. The image $L = H_f(\widetilde{\mathcal{F}}^{cs}(x))$ is contractible and at bounded Hausdorff distance from $E^{ss}_A \oplus E^c_A$.

Property (1) and the continuity of $\widetilde{\mathcal{W}}^{uu}_f$ and H_f allow us to define a continuous map $\varphi : L \to E^{ss}_A \oplus E^c_A$ that is onto by what we have already proved. Local product structure and property (2) imply that φ must be a covering and consequently a homeomorphism. Using again that H_f is injective along $\widetilde{\mathcal{W}}^{uu}_f$ we conclude uniqueness of the intersection point as desired.

To finish the proof of the theorem we argue as in Theorem 7.2 of [Pot]. Let us sketch the main points since in this case the proof becomes simpler. Since $\widetilde{\mathcal{F}}^{cs}$ is uniformly transverse to E^{uu}_f, there are uniform local product structure boxes in $\mathbb{R}^d$. Inside each local product structure box, by choosing suitable coordinate systems, one can look at the leaves of the foliations $\widetilde{\mathcal{F}}_n = \widetilde{f}^{-n}(\widetilde{\mathcal{F}}^{cs})$ as uniformly bounded graphs from a disk of dimension $cs = \dim E^{ss}_f + \dim E^c_f$ to a disk of dimension $uu = \dim E^{uu}_f$. These family of graphs are precompact in the C^1-topology (see for instance [HPS] or [BuW$_2$, Section 3]). The key point, whose proof is identical as the one of the first claim in the proof of Theorem 7.2 of [Pot] is that the image by H_f of any of these limit graphs (which are C^1-manifolds tangent to $E^{ss}_f \oplus E^c_f$) is contained in the corresponding translate of $E^{ss}_A \oplus E^c_A$. Now, using the fact that H_f is injective along strong unstable manifolds, one deduces that such limits are unique, and so the limit graphs form a well defined foliation with the desired properties (see Theorem 7.2 of [Pot] for more details).

Since the foliation $\widetilde{\mathcal{W}}^{cs}_f$ has the same properties of $\widetilde{\mathcal{F}}^{cs}$ we get global product structure exactly as above.

□

A symmetric statement holds for f being ss-proper, so we obtain the next corollary.

Corollary 4.2. *If $f \in \mathsf{PH}_A(\mathbb{T}^d)$ verifies the SADC property and is both uu-proper and ss-proper, then f is dynamically coherent and center fibered.*

To prove our main theorem the goal will be to show that having the SADC property and being σ-proper for $\sigma = uu, ss$ are open and closed properties among partially hyperbolic diffeomorphisms of $\mathbb{T}^d$ isotopic to linear Anosov automorphisms.

5. Opennes and Closedness of σ-properness

In this section we prove that being σ-proper is an open and closed property among diffeomorphisms in $\mathsf{PH}_A(\mathbb{T}^d)$ having the SADC property. Without the SADC property it is not hard to show that it is an open property, however, our proof of closedness uses Theorem 4.1 so we need the SADC property (which we already know is open and closed by Proposition 2.2).

Proposition 5.1. *Being σ-proper is a C^1-open property in $\mathsf{PH}_A(\mathbb{T}^d)$.*

Proof. From Lemma 3.1 it is enough to show that there exists a C^1-neighborhood $\mathcal{U}$ of f such that for each $g \in \mathcal{U}$ we know there exists R_1 such that for every $x \in \mathbb{R}^d$ we have

$$(\Pi^\sigma \circ H_g)^{-1}(B_1^\sigma(H_g(x))) \cap \widetilde{\mathcal{W}}_g^\sigma(x) \subset D_{R_1,g}^\sigma(x).$$

Since f is σ-proper we know from Lemma 3.2 that $\Pi^\sigma \circ H_f$ is a homeomorphism from $\widetilde{\mathcal{W}}_f^\sigma(x)$ onto $E_A^\sigma + H(x)$. We can choose R_1 such that

$$\Pi^\sigma \circ H_f((D_{R_1,f}^\sigma(x))^c) \cap B_2^\sigma(H_f(x)) = \emptyset.$$

Let $A_{R_1,R_2,g}^\sigma(x)$ be the annulus $D_{R_2,g}^\sigma(x) \setminus D_{R_1,g}^\sigma(x)$ for any $R_2 > R_1$. For $R_2 > \Delta_f R_1$ we have

$$\Pi^\sigma \circ H_f(A_{R_1,R_2,f}^\sigma(x)) \cap B_2^\sigma(H_f(x)) = \emptyset.$$

Choose $\mathcal{U}$ a neighborhood of f such that

(i) the constant Δ_f holds for every $g \in \mathcal{U}$ (see Remark 2), and
(ii) for every $g \in \mathcal{U}$ we have that $\Pi^\sigma \circ H_g(A_{R_1,R_2,g}^\sigma(x)) \cap B_1^\sigma(H_1(x)) = \emptyset$ (this can be done due to Remark 1 and Proposition 2.1).

This implies that

$$(\Pi^\sigma \circ H_g)^{-1}(B_1^\sigma(H_g(x))) \cap \widetilde{\mathcal{W}}_g^\sigma(x) \subset D_{R_1,g}^\sigma(x).$$

Indeed, otherwise there exists $y \in \widetilde{\mathcal{W}}_g^\sigma(x)$ such that $\Pi^\sigma \circ H_g(y) \in B_1^\sigma(H_g(x))$ but such that $y \notin D_{R_2,g}^\sigma(x)$. From the choice of Δ_f we know that there exists $n \in \mathbb{Z}$ such that $\widetilde{g}^n(y) \in A_{R_1,R_2}^\sigma(\widetilde{g}^n(x))$ (moreover $n > 0$ for $\sigma = ss$ and $n < 0$ for $\sigma = uu$) and one knows

$$\Pi^\sigma \circ H_g(\widetilde{g}^n(y)) \in B_1^\sigma(H_g(\widetilde{g}^n(x)))$$

which contradicts (ii) above.

□

Notice that the proof shows that the σ-properness is indeed uniform in the whole neighborhood of f.

The following is the hardest part of the proof of the theorem.

Proposition 5.2. *Being σ-proper and SADC is a C^1-closed property in $\mathsf{PH}_A(\mathbb{T}^d)$.*

Proof. Consider $f_k \to f$ such that f_k are σ-proper and SADC. From Proposition 2.2 we know that f is also SADC. We will use k instead of f_k in the subscripts to simplify the notation. Let us assume that $\sigma = uu$. Notice that the diffeomorphisms f_k are in the hypothesis of Theorem 4.1 so that for every $k > 0$ there exist an f_k-invariant foliation $\mathcal{W}^{cs}_k$ tangent to $E^{ss}_k \oplus E^c_k$ which verifies that $\widetilde{\mathcal{W}}^{cs}_k(x) = H_k^{-1}((E^{ss}_A \oplus E^c_A) + H_k(x))$.

First we shall show the following:

Claim 5.3. *For any k large enough $\widetilde{\mathcal{W}}^{uu}_f$ has a global product structure with $\widetilde{\mathcal{W}}^{cs}_k$.*

Proof. Consider a finite covering of $\mathbb{T}^d$ by boxes of local product structure for the bundles of f. We can consider them small enough so that the bundles are almost constant in each box (and by changing the metric, also almost orthogonal to each other). By choosing k_0 sufficiently large we know that for every $k \geq k_0$ the same boxes are also local product structure boxes for f_k. If B is such a box of local product structure we denote by $2B$ and $3B$ the box of double and triple the size, respectively, centered at the same point as B.

We can consider the covering small enough and k_0 sufficiently large so that there exists $\varepsilon > 0$ verifying that for every $k > k_0$ we know

- the boxes $2B$ and $3B$ are also local product structure boxes for all the f_k's in particular
- for every B of the covering and every disk D tangent to a small cone around E^{uu}_f of internal radius ε and centered at a point $x \in B$ we have that D intersects every center-stable plaque of

$\mathcal{W}^{cs}_k$ which intersects $2B$ in a (unique) point contained in $3B$ (see figure 1).

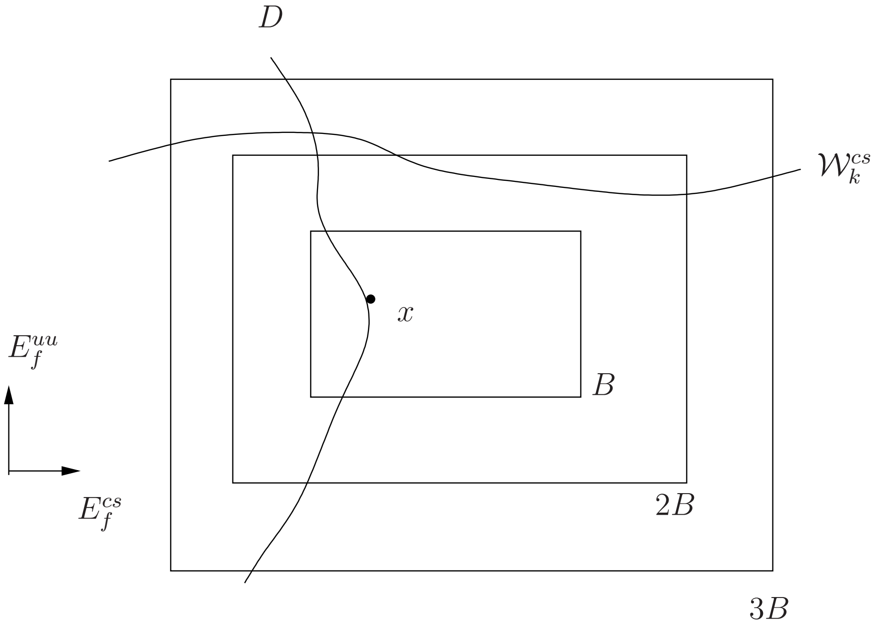

FIGURE 1. The local product structure boxes.

We know that H_k is injective along leaves of $\widetilde{\mathcal{W}}^{uu}_k$ so that we have that given a connected component $2B$ of the lift of local product structure box we have

$$\mathrm{int}(\Pi^{uu} \circ H_k(2B)) \neq \emptyset.$$

Moreover, since there are finitely many such boxes, we know that $\Pi^{uu} \circ H_k(2B)$ contains a uniform ball of radius at least δ which is independent of B. We deduce that every disk D of internal radius ε centered at a point x and tangent to a small cone around E^{uu}_f verifies that $\Pi^{uu} \circ H_k(D)$ contains $B^{uu}_\delta(H_k(x))$.

This implies that for every $x, y \in \mathbb{R}^d$, if we denote as d as the distance between $\Pi^{uu} \circ H_k(x)$ and $\Pi^{uu} \circ H_k(y)$ and let $N_0 > \frac{d}{\delta}$, then

$$D^{uu}_{N_0\epsilon,f}(x) \cap \widetilde{\mathcal{W}}^{cs}_k(y) \neq \emptyset. \tag{1}$$

Indeed, consider the straight segment joining $\Pi^{uu} \circ H_k(x)$ with $\Pi^{uu} \circ H_k(y)$ in $E^{uu} + H_k(x)$. We can cover this segment by N_0 balls $B_1, ..., B_{N_0}$

of radius $\delta/2$ and such that $B_i \cap B_{i+1} \neq \emptyset$. Now, $\Pi^{uu} \circ H_k(D^{uu}_{\epsilon,f}(x))$ contains B_1. Thus, $\Pi^{uu} \circ H_k(D^{uu}_{2\epsilon,f}(x))$ contains $B_1 \cup B_2$ and inductively we have $\Pi^{uu} \circ H_k(D^{uu}_{N_0\epsilon,f}(x))$ contains $B_1 \cup \ldots \cup B_{N_0}$ and $\Pi^{uu} \circ H_k(y)$, this implies (1).

Therefore, for every $x, y \in \mathbb{R}^d$ we have that $\widetilde{\mathcal{W}}^{uu}_f(x)$ intersects $\widetilde{\mathcal{W}}^{cs}_k(y)$. Now, let see that this intersection point is unique. Since $\widetilde{\mathcal{W}}^{uu}_f(x)$ intersects transversally $\widetilde{\mathcal{W}}^{cs}_k(y)$ for any x, y and $H_k(\widetilde{\mathcal{W}}^{cs}_k(y)) = (E^{ss} \oplus E^c_A) \oplus H(y)$ we conclude that $H_k(\widetilde{\mathcal{W}}^{uu}_f(x))$ is topologically transversal to $(E^{ss} \oplus E^c_A) \oplus H(y)$ for any x, y. This implies that

$$\Pi^{uu} : H_k(\widetilde{\mathcal{W}}^{uu}_f(x)) \to E^{uu}$$

is a covering map and since $H_k(\widetilde{\mathcal{W}}^{uu}_f(x))$ is contractible we know it is one-to-one. Thus, we have the global product structure and also that $\Pi^{uu} \circ H_k$ restricted to $\widetilde{\mathcal{W}}^{uu}_f(x)$ is a homeomorphism onto E^{uu}_A.

$\diamondsuit$

We now return to the proof of the proposition. We first show there exists some $R > 0$ such that for every $x \in \mathbb{R}^d$ we have

$$(\Pi^{uu} \circ H_f)^{-1}(B^{uu}_1(H_f(x))) \cap \widetilde{\mathcal{W}}^{uu}_f(x) \subset D^{uu}_{R,f}(x).$$

We will show that for every $x \in \mathbb{R}^d$ there exists some finite $\psi(x)$ such that

$$(\Pi^{uu} \circ H_f)^{-1}(B^{uu}_1(H_f(x))) \cap \widetilde{\mathcal{W}}^{uu}_f(x) \subset D^{uu}_{\psi(x),f}(x)$$

Then, one can conclude by arguing as in the proof of Lemma 3.3 by considering the infimum $\varphi(x)$ of all possible values of $\psi(x)$ satisfying the property which will be a semicontinuous and periodic function which by a compactness argument is enough to complete the proof.

We know that $d_{C^0}(H_k, H_f) < K_0$. Also, since $\Pi^{uu} \circ H_k$ restricted to $\widetilde{\mathcal{W}}^{uu}_f(x)$ is a homeomorphism onto E^{uu}_A we know for some $R_1 > 0$ that

$$\Pi^{uu} \circ H_k((D^{uu}_{R_1,f}(x))^c) \cap B^{uu}_{2+2K_0}(H_k(x)) = \emptyset$$

and so

$$\Pi^{uu} \circ H_f((D^{uu}_{R_1,f}(x))^c) \cap B^{uu}_1(H_f(x)) = \emptyset.$$

This implies that

$$(\Pi^{uu} \circ H_f)^{-1}(B_1^{uu}(H_f(x))) \cap \widetilde{\mathcal{W}}_f^{uu}(x) \subset D^{uu}_{R_1,f}(x).$$

Setting $\psi(x) = R_1$ we conclude the proof.

□

6. Proof of Theorem 1.1

From our previous results we obtain the following:

Theorem 6.1. *Let $f \in \mathsf{PH}_A(\mathbb{T}^d)$ be in the same connected component of a partially hyperbolic g which is σ-proper (for $\sigma = ss, uu$) and has the SADC property. Then f is dynamically coherent and center fibered.*

Proof. Propositions 5.1 and 5.2 together with Proposition 2.2 imply that being σ-proper ($\sigma = ss, uu$) and having the SADC property is an open and closed property in $\mathsf{PH}_A(\mathbb{T}^d)$. This implies that every f in the the same connected component of a partially hyperbolic g that is σ-proper (for $\sigma = ss, uu$) and has the SADC property is in the hypothesis of Corollary 4.2.

□

Proof of Theorem 1.1. It is enough to show that if g is an Anosov diffeomorphism that is partially hyperbolic (with the same dimensions of splittings) dynamically coherent and center-fibered, then it must be σ-proper and have the SADC property.

This follows from the following remarks:

- For an Anosov diffeomorphism, the map H is a homeomorphism, so that if it is center-fibered then the map $\Pi^\sigma \circ H$ must be injective along strong manifolds. Thus, we get that g is σ-proper by Lemma 3.3.
- Since g is dynamically coherent, the central stable foliation is a g-invariant foliation uniformly transverse to E_g^{uu} which is mapped by H into the foliation $E_A^{ss} \oplus E_A^c$ by σ-properness and center-fiberness. Since H is at bounded distance from the identity and applying the same argument to g^{-1} we get that SADC property for g.

□

We will now deduce some more additional properties of the systems. We recall that a foliation $\widetilde{\mathcal{F}}$ of $\mathbb{R}^d$ is called *quasi-isometric* if there exist constants $C, D > 0$ such that for any pair of points x, y in the same leaf of $\widetilde{\mathcal{F}}$ one has

$$d_{\widetilde{\mathcal{F}}}(x,y) \leq C d(x,y) + D$$

where as before $d_{\widetilde{\mathcal{F}}}(\cdot,\cdot)$ denotes the leafwise distance between points and $d(\cdot,\cdot)$ the usual distance in $\mathbb{R}^d$. We remark that if the foliation $\widetilde{\mathcal{F}}$ has C^1-leafs, it is possible to change the constants to have $D = 0$.

Proposition 6.2. *If $f : \mathbb{T}^d \to \mathbb{T}^d$ is σ-proper ($\sigma = ss, uu$) then the foliation $\widetilde{\mathcal{W}^\sigma}$ is quasi-isometric.*

Proof. First we choose a metric on $\mathbb{R}^d$ by declaring E^{ss}_A, E^c_A and E^{uu}_A mutually orthogonal, this metric is equivalent to the usual metric on $\mathbb{R}^d$. The proof consists on 3 steps:

(i) For every $K > 0$ there exists C_K such that if $d(x,y) < K$ then $d_\sigma(x,y) < C_K d(x,y)$.

(ii) For every $C_1 > 0$ there exists K such that for every $x \in \mathbb{R}^d$ we have that $\widetilde{\mathcal{W}^\sigma}(x)$ is contained in $B_{K/2}(x) \cup (\mathcal{E}^\sigma_{C_1} + x)$ where $\mathcal{E}^\sigma_{C_1}$ is the cone around E^σ_A of vectors $v = v^\sigma + v^\perp$ satisfying that $\|v^\perp\| < C_1 \|v^\sigma\|$ with $v^\sigma \in E^\sigma_A$ and $v^\perp \in (E^\sigma_A)^\perp$. Notice that $(E^\sigma_A)^\perp = E^{cs}_A$ if $\sigma = uu$ and $(E^\sigma_A)^\perp = E^{cu}_A$ if $\sigma = ss$.

(iii) If $y \in \widetilde{\mathcal{W}^\sigma}(x)$ one can choose points $x = x_1, \dots, x_n = y$ in $\widetilde{\mathcal{W}^\sigma}(x)$ and K such that $d(x_i, x_{i+1}) < K$ and such that
$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq 3 d(x,y)$$
.

Once we have this, putting together properties (i) and (iii) we deduce that

$$d_\sigma(x,y) \leq \sum d_\sigma(x_i, x_{i+1}) \leq C_K \sum d(x_i, x_{i+1}) < 3 C_K d(x,y)$$

showing quasi-isometry.

We first notice that (i) is a direct consequence of σ-properness. In fact, if (i) did not hold we would obtain a sequence x_n, y_n of points at distance smaller or equal to K such that $d_\sigma(x_n, y_n) \geq n$. Using σ-properness we would obtain that $d(H_f(x_n), H_f(y_n)) \to \infty$. On the other hand, since H_f is at bounded distance from the identity and x_n and y_n are at distance smaller than K one gets that $d(H_f(x_n), H_f(y_n))$ must be bounded, a contradiction.

Let us prove (ii). Since H_f is at bounded distance from the identity, to prove (ii) it is enough to show the same property for $H_f(\widetilde{\mathcal{W}}^\sigma_f(x))$.

Assume that it is not true. Then, there exists a cone $\mathcal{E}^\sigma$ and we may find sequences (using σ-properness) $x_n, y_n \in \mathbb{R}^d$ such that $y_n \in H_f(\widetilde{\mathcal{W}}^\sigma_f(x_n))$ with $d(y_n, H_f(x_n)) \to \infty$ and $y_n \notin \mathcal{E}^\sigma + H_f(x_n)$. We assume for simplicity that $\sigma = uu$, the other case is quite similar.

Let $\lambda_c^{-1} = \|A_{/E^{cs}_A}\|$ and let $\lambda_u = \|A^{-1}_{/E^{uu}_A}\|$. Notice that $\lambda_u/\lambda_c < 1$. Notice first that if $\lambda_c > 1$ we know $A_{/E^{cs}_A}$ is contacting so that $H_f : \widetilde{\mathcal{W}}^{uu}_f(x) \to E^u_A + H_f(x)$ is a homeomorphism and property (ii) is immediate. Also, since A is Anosov we can assume that (maybe by considering an iterate) that $\lambda_c \neq 1$. So, in what follows we shall assume that $\lambda_c < 1$.

Let $\epsilon > 0$ and let $m_n = \inf\{m \geq 0 : \lambda_c^m d(y_n, H_f(x_n)) \leq \epsilon\}$. Since $d(y_n, H_f(x_n)) \to \infty$ and $y_n \notin \mathcal{E}^\sigma + H_f(x_n)$ we have that $m_n \to \infty$. And we have that

$$d(A^{-m_n}(y_n), A^{-m_n}(H_f(x_n))) \geq \lambda_c \epsilon.$$

On the other hand

$$\begin{aligned} & d(\Pi^{uu}(A^{-m_n}(y_n)), A^{-m_n}(H_f(x_n))) = \\ = \; & d(A^{-m_n}(\Pi^{uu}(y_n)), A^{-m_n}(H_f(x_n))) \\ \leq \; & \lambda_u^{m_n} \frac{d(y_n, H_f(x_n))}{C_1} \leq \left(\frac{\lambda_u}{\lambda_c}\right)^{m_m} \frac{\epsilon}{C_1} \to_{n\to\infty} 0 \end{aligned}$$

Now, composing with deck transformation we may assume that

$$f^{-m_n}(x_n) \to x$$

and

$$A^{-m_n}(y_n) \to y \in H_f(\widetilde{\mathcal{W}}^\sigma_f(x)),\ y \neq H_f(x).$$

But $\Pi^{uu}(y) = x$, a contradiction with property (I^{uu}) (which follows from uu-properness by Lemma 3.2). Thus, we obtain that (ii) is verified.

Finally, to prove (iii) we use (ii): We choose $C_1 \leq 1/2$ and K from (ii) and we define the sequence x_i inductively. First, we impose $x_1 = x$. Then, if $d(x_i, y) < K$ we choose $x_{i+1} = y$. Otherwise we pick x_{i+1} as follows. Notice that $d(\Pi^\sigma(y), x_i) \geq \frac{2}{3}K$ and let z_{i+1} be the point in the segment joining x_i and $\Pi^\sigma(y)$ (which is contained in $E^\sigma_A + x_i$) at distance $\frac{2}{3}K$ from x_i.

Now, $(\Pi^\sigma)^{-1}(z_{i+1}) \cap (\mathcal{E}^\sigma + x_i)$ is a disc D_i of radius $\frac{2}{3}C_1K$ in $(E^\sigma_A)^\perp + z_{i+1}$. Since $\Pi^\sigma \circ H_f$ is homomorphism onto $E^\sigma_A + H_f(x_i)$ when restricted to $\widetilde{\mathcal{W}}^\sigma(x_i)$ and H_f is at bounded distance from the identity, we conclude that Π^σ is onto $E^\sigma_A + x_i$ when restricted to $\widetilde{\mathcal{W}}^\sigma_f(x_i)$. By (ii) there is at least one point in $D_i \cap \mathcal{W}^\sigma_f(x_i)$. We set x_{i+1} to be one of these points.

We must now show that the process finishes in finitely many steps. Notice that since $y \in \mathcal{E}^\sigma + x_i$ the straight line segment joining x_i and y intersects D_i and $d(y, D_i) \leq d(x_i, y) - \frac{2}{3}K$. Thus

$$d(x_{i+1}, y) \leq d(x_i, y) - \frac{2}{3}K + \frac{2}{3}C_1K \leq d(x_i, y) - \frac{1}{3}K.$$

So that the process ends in finitely many steps.

Notice also that $d(x_i, x_{i+1}) \leq K$ and so the above inequality also shows that

$$d(x_{i+1}, y) \leq d(x_i, y) - \frac{1}{3}d(x_i, x_{i+1}).$$

Therefore, if we have chosen the sequence $x = x_1, x_2, ..., x_n = y$ we have by induction that

$$d(x_{n-1}, y) \leq d(x, y) - \frac{1}{3}\sum_{i=1}^{n-2} d(x_i, x_{i+1})$$

and so

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq 3d(x, y).$$

□

When the central dimension is one it is possible to use the results of [H] to obtain a property called *leaf conjugacy*. This notion is related with the existence of the semiconjugacy but slightly different, it says that there exists a homeomorphism $h : \mathbb{T}^d \to \mathbb{T}^d$ which sends center leaves of f to center leaves of the linear Anosov diffeomorphism and conjugates the dynamics modulo the center behavior (see [H] for more details).

The results in [H] are proved in the absolute partially hyperbolic setting, but in [HP] it is explained which hypothesis should be added in the pointwise case in order to recover his results.

Proposition 6.3. *Let* $f \in \mathsf{PH}_A(\mathbb{T}^d)$ *with* $\dim E^c_f = 1$ *and verifying SADC property and* σ*-properness for* $\sigma = ss, uu$ *then* f *is leaf conjugate to* A.

Proof. Theorem 3.2 in [HP] states that the following properties of a dynamically coherent partially hyperbolic diffeomorphism with one dimensional center and isotopic to A guarantee leaf conjugacy.

(i) The foliations $\widetilde{\mathcal{W}}^\sigma_f$ $(\sigma = cs, cu)$ are almost parallel to the corresponding linear foliations of A.

(ii) The foliations $\widetilde{\mathcal{W}}^\sigma_f$ are *asymptotic* to E^σ_A (i.e. We have that $\frac{d(\Pi^\sigma(x),\Pi^\sigma(y))}{d(x,y)} \to 1$ as uniformly $d(x, y) \to \infty$ with x, y in the same leaf of $\widetilde{\mathcal{W}}^\sigma$).

(iii) The foliations $\widetilde{\mathcal{W}}^\sigma_f$ $(\sigma = ss, uu)$ are quasi-isometric.

SADC property implies property (i).

It is quite easy to see that using the semiconjugacy with A that conditions (I^σ) and (S^σ) imply property (ii). Recall that σ-properness implied properties (I^σ) and (S^σ) (Lemma 3.2).

The proof then concludes by applying Proposition 6.2 to conclude that (iii) is also satisfied.

$\square$

7. Measures of Maximal Entropy

The variational principle states that if $f : X \to X$ is continuous and X is a compact metric space, then $h_{top}(f) = \sup_\mu h_\mu(f)$ where μ varies among all f-invariant Borel probability measures, see for instance [M_2]. It is thus an interesting question to know whether a given system posses measures that have equal entropy to the topological entropy of the system, and when such measures exist (which are called *measures of maximal entropy*) to know how many of them are there. When there is a unique measure of maximal entropy the system is *intrinsically ergodic*.

Corollary 1.2 states that every diffeomorphism in $\mathsf{PH}^0_A(\mathbb{T}^d)$ with one dimensional center is intrinsically ergodic. In [U] a similar result is proved under the added assumption of absolute partially hyperbolic case and under stronger assumptions on the foliations of f.

Proof of Corollary 1.2. From Theorem 1.1 and Proposition 6.3, let h be the semiconjugacy from f to the hyperbolic toral automorphism we will denote by A, then for each $x \in \mathbb{T}^d$ we know that $[x] = h^{-1}h(x)$ is a point or bounded closed interval in the center fiber containing x.

The Leddrappier-Walters type arguments in [BFSV] allow us to conclude that $h_{\text{top}}(f) = h_{\text{top}}(A)$ and that a lift of the Haar measure, μ, for A is a measure of maximal entropy for f.

From Lemma 4.1 in [U] we know that

$$\mu\{x \in \mathbb{T}^d \,:\, [x] = \{x\}\} = 1.$$

Theorem 1.5 in [BFSV] now applies and we know that f is intrinsically ergodic. Furthermore, the unique measure of maximal entropy can be seen as the unique lift of Haar measure for A and also as the limit of the measures given by the periodic classes as defined in [BFSV].

We now show that the unique measure of maximal entropy is hyperbolic for each $g \in \mathrm{PH}^0_A(\mathbb{T}^d)$. We will assume that for the Anosov diffeomorphism that the center direction is expanding, the other case is symmetrical and we would use the inverse maps. For $g \in \mathrm{PH}^0_A(\mathbb{T}^d)$ we let μ_g be the unique measure of maximal entropy. To show that μ_g is hyperbolic we need only show that the Lyapunov exponent in the center direction is nonzero.

From a refined Pesin-Ruelle in Theorem 3.3 of [HSX] we know

$$h_{\text{top}}(A) = h_{\text{top}}(g) = h_{\mu_g}(g) \leq \lambda_c(g) + \chi_u(g)$$

where λ_c is the Lyapunov exponent in the center direction and $\chi_u(g)$ is the volume growth of the unstable foliation as defined in [HSX]. Since g is isotopic to A we know from [HSX] that $\chi_u(g) = \chi_u(A)$. Lastly, since

$$h_{\text{top}}(A) = \lambda_c(A) + \chi_u(A)$$

we see that $\lambda_c(g) \geq \lambda_c(A) > 0$ so μ_g is hyperbolic.

□

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