Rice Formula for processes with jumps and applications

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Abstract

We extend Rice Formula for a process that is the sum of a smooth process and a pure jump process. We obtain formulas for the mean number of both, continuous and discontinuous crossings through a fixed level on a compact time interval. An application to the study of the behavior of the tail of the distribution function of the maximum of the process over a compact time interval is considered. Further, we give a generalization, to the non-stationary case, of Borovkov-Last's Rice Formula for Piecewise Deterministic Markov Processes.

1 Introduction

Counting the number of crossings through a fixed level u by a stochastic process and studying the behavior of the maximum of the process on a time interval are classical, related, problems in Probability Theory. Nevertheless, few is known, in general, about the distribution of the random variable *number of crossings*. Therefore, Rice Formulas, which give expressions for the moments of the number of crossings, are very important and useful.

Roughly speaking, Rice Formulas express the moments of the number of crossings in an integral form involving the parameters and local properties of the process. It is worth to remark that it reduces a global property as the number of crossings to local ones, as the finite-dimensional distribution at a finite number of points in the parameter space of the process, and its derivatives.

Different cases of Rice Formula have been known and used for a long time, see [12], the first one, due to Rice in 1944 [16, 17], involve the Gaussian stationary case. The major part of the literature is devoted to the Gaussian case, and in particular to the stationary case. Extensions for general Gaussian, non-Gaussian processes and fields were stated afterwards, see [3, 10]. Some recent extensions to non-Gaussian processes include the Shot Noise processes [5] and the Generalized Hyperbolic Process [1].

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The applications of Rice Formulas include telecommunications and signal processing [16, 17], reliability theory in engineering [18], oceanography (the height of sea waves) [4, 13], physics and astronomy (random mechanics) [11], (the Shot Noise processes) [5] and (microlensing) [14], random systems of polynomial equations [8, 2], etc.

All these works consider smooth processes of class at least C^1. More recently, Borovkov and Last in [6], see also [7], studied the continuous crossings through a fixed level of a class of processes with jumps named Piecewise Deterministic Markov Processes. In spite of including jumps, the structure of these processes is quite simple in the sense that randomness only is included at the jump instants and magnitudes.

On the present work, we consider a class of stochastic processes with finite intensity of jumps and smooth (stochastic) evolution between these jumps (excepting Section 4). More precisely, we consider a process $\mathcal{X}$ which can be written in the form $\mathcal{X} = \mathcal{Z} + \mathcal{J}$ where $\mathcal{Z}$ is a process with continuously differentiable paths and $\mathcal{J}$ is a pure jump process, independent from $\mathcal{Z}$. That is, $\mathcal{Z}$ describes the continuous evolution of $\mathcal{X}$ and $\mathcal{J}$ describes the jumps of $\mathcal{X}$.

Such a process can cross the level u in a continuous way but it also can cross u at one of the jumps. Our interest relies on both, continuous and discontinuous crossings through the level u. Up to our knowledge, the discontinuous crossings have not been considered in the literature. This work began under the initiative of Mario Wschebor, and is largely inspired by his fundamental contribution in the field, masterly exposed in [3].

We obtain formulas for the mean number of continuous and discontinuous crossings through a fixed level $u \in \mathbb{R}$ by such processes on a compact time interval, and apply them to the study of the distribution function of the maximum of the process. We also include a generalization of Borovkov-Last's Rice-type formula to the non-stationary case

The paper is organized as follows. In section 2 we give some definitions, introduce the processes we deal with and present the main result. In Section 3 we present two examples and the application to the study of the maximum (on these examples). Section 4 contains the generalization of Borovkov-Last's formula. Finally, Section 5 contains the proofs of the main results.

2 Preliminaries and Main Result

Let $T > 0$ and $\mathcal{X} = (X(t) : t \in [0, T])$ be a stochastic process defined on the interval $[0, T]$. We assume that $\mathcal{X}$ can be written in the form $\mathcal{X} = \mathcal{Z} + \mathcal{J}$ where $\mathcal{Z}$ and $\mathcal{J}$ are independent processes on $[0, T]$ as described below.

We assume some regularity conditions on the process $\mathcal{Z}$, see [3], namely:

*A*1 the paths of $\mathcal{Z}$ are C^1, almost surely.

*A*2 We assume that the density $p_{Z(t)}(x)$ is jointly continuous for $t \in [0, T]$ and x in a neighborhood of u. Further, assume that for every $t, t' \in [0, T]$ the joint distribution of $(Z(t), \dot{Z}(t'))$ has a density $p_{Z(t),\dot{Z}(t')}(x, x')$ which is continuous w.r.t. t (t', x, x' fixed) and w.r.t. x at u (t, t', x' fixed).

*A*3 for every $t \in [0,T]$ there exists a continuous version of the conditional expectation

$$\mathbb{E}\,(\dot{Z}(t)|X(t)=x),$$

for x in a neighborhood of u.

*A*4 the modulus of continuity of $\dot{Z}$ tends to 0 if $\delta \to 0$:

$$w(\dot{Z},\delta) := \sup_{0\le s<t\le T,\,|t-s|<\delta} |\dot{Z}(t)-\dot{Z}(s)| \underset{\delta\to 0}{\to} 0.$$

The jump process $\mathcal{J}=(J_t : t\in[0,T])$ is based on a general point process $(\tau_n,\xi_n)_{n\in\mathbb{N}}$ on $[0,\infty)\times\mathbb{R}$, and is constructed via a family of Markov kernels $(P^{(n)}_{x,t})$, $(\pi^{(n)}_{x,t}) : n\in\mathbb{N}$ as follows, see [9]: set $\tau_0=0$ and draw ξ_0 according to the distribution π_0, then, conditioned on the resulting value of ξ_0, say x_0, draw τ_1 with (conditional) distribution

$$\mathbb{P}\,(\tau_1\in\cdot\mid \xi_0=x_0)=P^{(1)}_{x_0}(\cdot).$$

Similarly, conditioned on the values of ξ_0,τ_1, say x_0,t_1, draw ξ_1 with distribution $\pi^{(1)}_{x_0,t_1}(\cdot)$. Then, conditioned on the preceding values and on $\xi_1=x_1$ draw τ_2 with distribution $P^{(2)}_{(x_0,x_1),t_1}(\cdot)$ and so on. Finally, for $\tau_n\le t<\tau_{n+1}$ let $\nu_t=n$ and

$$J(t)=\sum_{k=0}^{n}\xi_k.$$

Hence τ_n represents the n-th jump instant and ξ_n represents the n-th jump magnitude (or increment) of the process $\mathcal{J}$.

Equivalently, the kernels $(P^{(n)}_{x,t})$ can be used to obtain the actual value of the process at the jump instants, in that case we set $J(\tau_n)=\xi_n$.

We can identify the marked point process $(\tau_n,J(\tau_n))$ with the associated random counting measure (RCM for short) $\sum_{n=0}^{\nu_T}\delta_{(\tau_n,J(\tau_n))}$ on $[0,T]\times\mathbb{R}$, where $\delta_{(t,x)}$ is Dirac Delta measure concentrated at (t,x). There exists a predictable random process Λ on $[0,\infty)\times\mathbb{R}$ such that:

$$\mathbb{E}\int f d\mu=\mathbb{E}\int fL(dt,dy)$$

for any predictable function f; here L stands for the (random) measure induced by Λ. Usually L is called the compensating measure of the RCM μ.

Under quite general conditions (for example the absolute continuity of the kernels and the finiteness of the intensity of jumps), the compensating measure L can be written in terms of ordinary Lebesgue integrals.

We turn now to the definition of a crossing through a fixed level by a function. Consider a càdlàg, C^1 between jumps, function $f:[0,T]\to\mathbb{R}$. We say that f has a continuous crossing through the level u at $s\in(0,T)$ if f is continuous at s, $f(s)=u$ and $f'(s)\neq 0$; and a discontinuous crossing if $(f(s^-)-u)(f(s)-u)<0$. If $f'(s)>0$ in the continuous case, or $f(s^-)<u<f(s)$ in the discontinuous one, we say that f has an up-crossing at s, otherwise the crossing is a down-crossing.

By the preceding assumptions on the process $\mathcal{X}$, we can apply these definitions to almost all of its paths. Denote N^c_u (resp. N^d_u) the (random) number of continuous (resp. discontinuous) crossings through the level u by the process $\mathcal{X}$ on the interval $[0,T]$ and by N_u the total number of crossings. Analogously U_u, U_u^c, U_u^d (D_u, D_u^c, D_u^d) denote respectively the total number, continuous, and discontinuous up-crossings (down-crossings).

It is clear that $N_u = N_u^c + N_u^d$, then

$$\mathbb{E}\, N_u = \mathbb{E}\, N_u^c + \mathbb{E}\, N_u^d. \tag{1}$$

The two terms of the r.h.s. of (1) are treated separately and by different methods. For the continuous crossings we recall that classical Rice Formula is based on Kac Counting Formula for the number of crossings of a C^1 function, and observe that when the function has jumps (and the value u is not one of the lateral limits) Kac Formula counts the number of continuous crossings through the level, ignoring the discontinuous ones. On the other hand, for the discontinuous crossings we use techniques from point processes theory.

Our main theorem includes the case of non-Gaussian continuous processes $\mathcal{Z}$.

Theorem 1. *Let $\mathcal{Z}$ and $\mathcal{J}$ be two independent processes on $[0,T]$ such that:*

- *$\mathcal{Z}$ verifies the conditions A1, A2, A3 and A4,*
- *$\mathcal{J}$ is a pure jump process, as described above, with finite intensity.*

Then, the mean number of continuous and discontinuous crossings through the level u by the process $\mathcal{X}$ on the interval $[0,T]$ are given, respectively, by:

$$\mathbb{E}\, N_u^c = \int_{[0,T]} \mathbb{E}\left(|\dot{X}(t)| \mid X(t) = u\right) p_{X(t)}(u)dt$$
$$= \int_{[0,T]} \mathbb{E}\left(|\dot{Z}(t)| \mid X(t) = u\right) p_{X(t)}(u)dt,$$

and

$$\mathbb{E}\, N_u^d = \mathbb{E} \iint_{[0,T]\times\mathbb{R}} \mathbf{1}\{(X(t^-) - u)(X(t^-) + y - u) < 0\} L(dt, dy),$$

where $\mathbf{1}A$ is the indicator function of the set A and L is the compensating measure of the random counting measure generated on $[0,T]\times\mathbb{R}$ by the jump process $\mathcal{J}$. Similar formulas hold for the number of up and down crossings.

Some remarks are in order. First, note that when the jump process $\mathcal{J}$ vanishes, that is, if $J(t) = 0$ almost surely for all $t \in [0,T]$, Theorem 1 reduces to Classical Rice Formula for the process $\mathcal{Z}$.

Next, observe that the random variable $J(t)$ does not need to have a density for each t, but, in case it does we have a more explicit result.

Corollary 1. *If $J(t)$ has a continuous density $p_{J(t)}(x)$ for $t \in [0,T]$ and $x \in \mathbb{R}$, we can also write*

$$\mathbb{E}\, N_u^c = \int_{[0,T]} dt \int_{\mathbb{R}} \mathbb{E}\left(|\dot{Z}(t)| \mid Z(t) = v\right) p_{Z(t)}(v) p_{J(t)}(u - v)dv.$$

Finally, a careful analysis of the proof of Theorem 1 in Section 5 shows that the result in Theorem 1 holds true whenever the law of the process $\mathcal{Z}$, restricted to the subintervals $[\tau_i, \tau_{i+1}]$, conditioned to the paths of the jump process verifies the hypothesis $A1$ - $A4$. Besides, $A3$ can be weakened assuming that the product of the conditional expectation and the density of $X(t)$ (or $Z(t)$ in Corollary 1) is continuous.

We end this section specializing these results to the case where $\mathcal{Z}$ is a Gaussian process, here, the hypothesis of continuity of the densities and of the conditional expectation may be released since they follow from the conditions of non-degeneracy of the distribution of $Z(t)$, $t \in [0, T]$, and on the regularity of the paths. Besides, the ingredients in the formulas are computable explicitly.

Corollary 2. *Let $\mathcal{Z}$ be a Gaussian process with C^1 paths such that for every t the distribution of $Z(t)$ is non-degenerated, assume further that $\mathcal{J}$ is a pure jump process independent from $\mathcal{Z}$ with finite intensity of jumps. Then the result of Theorem 1 hold true.*

Corollary 3. *Under the hypothesis of Corollary 2, if $\mathcal{Z}$ has constant variance, in particular if it is stationary, then the formula for the continuous crossings reduces to:*

$$\mathbb{E}\, N_u^c = \mathbb{E}\, |\dot{Z}(0)| \int_0^T p_{X(t)}(u) dt.$$

Here we used the well known fact that for a centered Gaussian process, having constant variance implies the independence of the process and its derivative at each point.

Remark 1. *Let us define the density*

$$p(u) := \frac{1}{T} \int_0^T p_{X(t)}(u)\, dt.$$

Then, we can write

$$\mathbb{E}\, N_u^c = T\, \mathbb{E}\, |\dot{Z}(0)|\, p(u).$$

Thus, the mean number of continuous crossings through the level u by $\mathcal{X}$ is proportional to a probability density function evaluated at u. This expression is similar to Rice's original result, but it involves a different density function.

Furthermore, observe that p is a mixture of the density of $X(t)$ for $t \in [0, T]$. In particular, if $\mathcal{X}$ is a stationary process, the density p reduces to that of $X(0)$, just as in the original formula due to Rice.

3 Examples

In this section we present two examples of the computation of the formulas in Theorem 1 and their application to the study of the distribution of the maximum of these processes in a compact time interval. Furthermore, we compare which kind of crossings predominate as the level u tends to infinity.

3.1 Stationary processes with 1-dimensional Gaussian distribution

In this example we assume that $\mathcal{Z} = (Z(t) : t \in [0, T])$ is a stationary, centered Gaussian process with C^1 paths. Let Γ be the covariance function of $\mathcal{Z}$, i.e: $\Gamma(\tau) = \mathbb{E}\, Z(0)Z(\tau)$, assume also that $\Gamma(0) = 1/2$.

Let $\mathcal{J} = (J(t) : t \in [0, T])$ be a pure jump process constructed in the following way. Given a sequence of independent and identically distributed random variables $(\xi_n : n \in \mathbb{N})$ with common centered normal distribution with variance $1/2$, and $\rho \in \mathbb{R}$ such that $|\rho| < 1$, define

$$A_0 = \xi_0, \text{ and for } n \geq 1 : A_n = \rho A_{n-1} + \sqrt{1-\rho^2}\, \xi_n.$$

It is easy to see that the sequence $(A_n : n \in \mathbb{N})$ is centered, stationary Gaussian with variance $1/2$ and covariance between A_0 and A_n given by $\rho^n/2$.

Further, consider a Poisson process $\nu = (\nu_t : t > 0)$ with intensity $0 < \lambda < \infty$, independent from (ξ_n) and from $\mathcal{Z}$, and let

$$J(t) = A_{\nu_t}.$$

Remark 2. *To define this process in terms of kernels, let $P^{(n)}_{x_{n-1},\tau_{n-1}}$ be the exponential distribution with intensity λ (regardless of x_{n-1}, τ_{n-1}). Define $\nu_t = \max\{n : \tau_n \leq t\}$. Besides, let $\pi^{(n)}_{x_{n-1},\tau_n}$, representing the increments at the jump instant τ_n, be the normal distribution centered at $(\rho - 1)x_{n-1}$ with variance $(1 - \rho^2)/2$.*

We call such a process a Poisson-auto-regressive process (of order 1, covariance ρ and intensity λ).

Proposition 1. *A Poisson-auto-regressive process $\mathcal{J}$ is wide-sense stationary. Besides, the random variable $J(t)$ has a centered normal distribution with variance $1/2$ for every t. The covariance between $J(t)$ and $J(t + \tau)$ is given by $e^{\lambda(1-\rho)\tau}/2$.*

Proof. We compute the distribution and the covariances of $J(t)$ conditioning on the number of jumps of $\mathcal{J}$:

$$\mathbb{P}\left(J(t) \leq x\right) = \sum_{n=0}^{\infty} p_{\nu_t}(n)\mathbb{P}\left(A_n \leq x\right) = \mathbb{P}\left(A_0 \leq x\right),$$

since A_n has centered normal distribution with variance $1/2$ for each n, hence, so does $J(t)$ for all t. In particular, $J(t)$ is centered for all t.

For the second moments we have:

$$\begin{aligned}
&\mathbb{E}\, J(t)J(t+\tau) \\
&\quad = \sum_{n,k=0}^{\infty} p_{\nu_t}(n)\mathbb{P}\left(\nu(t, t+\tau] = k\right)\mathbb{E}\left(J(t)J(t+\tau) \mid \nu_t = n, \nu(t, t+\tau] = k\right) \\
&\quad = \sum_{n,k=0}^{\infty} p_{\nu_t}(n)p_{\nu_\tau}(k)\mathbb{E}\left(A_n A_{n+k}\right) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{k=0}^{\infty} e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}\frac{\rho^k}{2} \\
&\qquad = \frac{1}{2}e^{-\lambda\tau}e^{\lambda\rho\tau},
\end{aligned}$$

where $\nu(t, t+\tau]$ denotes the number of jump instants in the interval $(t, t+\tau]$. $\square$

Next theorem gives the mean number of continuous and discontinuous crossings through the level u by the process $\mathcal{X}$ on the interval $[0,T]$. For the sake of notational simplicity we consider only the up-crossings, but the case of down-crossings is completely analogous.

Theorem 2. *Let $\mathcal{X} = \mathcal{Z} + \mathcal{J}$ with $\mathcal{Z}, \mathcal{J}$ independent processes such that $\mathcal{Z}$ a stationary, centered Gaussian process with $\Gamma(0) = 1/2$ and C^1 paths and $\mathcal{J}$ a Poisson-auto-regressive process, then*

$$\mathbb{E}\, U_u = T\sqrt{\frac{\lambda_2}{2\pi}}\, \varphi(u) + \lambda T\, p_\rho(u),$$

where φ stands for the standard Gaussian density function, λ_2 is the second spectral moment of $\mathcal{Z}$, and $p_\rho(u)$ is the probability that a two dimensional, centered, Gaussian vector with unit variances and covariance $(1+\rho)/2$ belongs to the set $(-\infty, u) \times (u, \infty)$.

Proof. We begin considering the mean number of continuous crossings, as $\mathcal{Z}$ is Gaussian and stationary, we can apply Corollary 3. Besides, by Proposition 1, the density $p_{X(t)} = p_{X(0)} = \varphi$ for all t, therefore

$$\mathbb{E}\, U_u^c = T\, \mathbb{E}\, |\dot{Z}(0)|\, \varphi(u).$$

Finally, an elementary computation shows that $\mathbb{E}\, \dot{Z}(0)^+ = \sqrt{\lambda_2/2\pi}$. This gives the first term of $\mathbb{E}\, U_u$.

Let us consider now the discontinuous up-crossings. The compensating measure of the point process $(\tau_n, \Delta A_n)_n$, $\Delta A_n = A_n - A_{n-1}$, is $\lambda dt\, F(dy)$, where F is the normal distribution centered at $(\rho-1)A_{\nu_{t^-}}$ and with variance $(1-\rho^2)/2$, see [9, eq. 4.64]. Hence

$$\begin{aligned}\mathbb{E}\, U_u^d = \mathbb{E} \int_0^T \int_{\mathbb{R}} \mathbf{1}\{X(t^-) < u, X(t^-) + y > u\} \lambda dt\, F(dy) \\ = \lambda \mathbb{E} \int_0^T dt \int_{\mathbb{R}} \mathbf{1}\{X(t^-) < u, X(t^-) + y > u\} F(dy).\end{aligned}$$

Actually F is the distribution of ΔA_{ν_t} conditioned on the random vector $(Z(t), A_{\nu_{t^-}})$, so

$$\begin{aligned}\mathbb{E}\, U_u^d &= \lambda \mathbb{E} \int_0^T \mathbb{P}\left(X(t^-) < u, X(t^-) + \Delta A_{\nu_t} > u \mid Z(t), A_{\nu_{t^-}}\right) dt \\ &= \lambda \mathbb{E} \int_0^T \sum_{n=0}^{\infty} p_{\nu_t}(n) \mathbb{P}\left(Z(t) + A_n < u, Z(t) + A_n + \Delta A_{n+1} > u \mid Z(t), A_n\right) dt \\ &= \lambda \int_0^T \sum_{n=0}^{\infty} p_{\nu_t}(n) \mathbb{E}\, \mathbb{P}\left(Z(t) + A_n < u, Z(t) + A_n + \Delta A_{n+1} > u \mid Z(t), A_n\right) dt \\ &= \lambda \int_0^T \sum_{n=0}^{\infty} p_{\nu_t}(n) \mathbb{P}\left(Z(t) + A_n < u, Z(t) + A_n + \Delta A_{n+1} > u\right) dt\end{aligned}$$

where we have conditioned on the number of jumps and used Fubini's theorem. By the stationarity of $\mathcal{Z}$ and $\mathcal{J}$, the latter probability does not depend on t and n, hence

$$\begin{aligned}\mathbb{E}\, U_u^d &= \lambda \mathbb{P}\left(Z(0)+A_0<u, Z(0)+A_0+\Delta A_1>u\right) \int_0^T \sum_{n=0}^{\infty} p_{\nu_t}(n)dt \\ &= \lambda T \mathbb{P}\left(Z(0)+A_0<u, Z(0)+A_0+\Delta A_1>u\right).\end{aligned}$$

Now, observe that $A_0+\Delta A_1 = A_1 = \rho\xi_0 + \sqrt{1-\rho^2}\xi_1$ and that the vector $(Z(0)+A_0, Z(0)+A_1)$ has centered normal distribution with variances one and covariance $(1+\rho)/2$, therefore, the latter probability equals $p_\rho(u)$. This gives the second term and completes the proof. □

Corollary 4. *As $u \to \infty$, we have*

$$\lim \frac{\mathbb{E}\, U_u^d}{\mathbb{E}\, U_u^c} = 0.$$

Thus, the continuous crossings predominate, in mean, for high levels.

Proof. Let (X, Y) be a two dimensional centered Gaussian vector with $\mathbb{E}\, X^2 = \mathbb{E}\, Y^2 = 1/2$ and $\mathbb{E}\, XY = (1+\rho)/2$, then

$$p_\rho(u) = \mathbb{P}\,(X<u, Y>u) \leq \mathbb{P}\,(Y>u).$$

Then

$$p_\rho(u) \leq 1-\Phi(u) \leq \frac{1}{u}\varphi(u) = o(\varphi(u)).$$

The result follows. □

Remark 3. *The processes $\mathcal{Z}$ and $\mathcal{J}$ contribute in $1/2$ to the variance of $\mathcal{X}$, more generally we can define*

$$\mathcal{X} = \alpha\mathcal{Z} + \sqrt{1-\alpha^2}\mathcal{J}$$

for $\alpha \in [0,1]$ and $\mathcal{Z}, \mathcal{J}$ independent, centered, stationary with variance one. Hence,

$$\begin{aligned}\mathbb{E}\, N_u^c(\mathcal{X}) &= T\,\mathbb{E}\,|\dot{X}(0)|\, p_{X(0)}(u) = T\sqrt{\frac{2\lambda_2(\alpha Z(0))}{\pi}}\varphi(u) \\ &= \alpha T\sqrt{\frac{2\lambda_2(Z(0))}{\pi}}\varphi(u) = \alpha T\,\mathbb{E}\,|\dot{Z}(0)|\,\varphi(u) \\ &= \alpha\,\mathbb{E}\, N_u^c(\mathcal{Z}).\end{aligned}$$

Therefore, the mean number of continuous crossings of $\mathcal{X}$ is proportional to that of the continuous process $\mathcal{Z}$, the constant of proportionality being α, that is, the standard deviation of the first summand $\alpha\mathcal{Z}$.

We now use Rice Formula to get upper and lower bounds for the tail of the distribution (i.e: for the overlife function) of the maximum of the process $\mathcal{X}$ with Gaussian stationary continuous part and Poisson-auto-regressive jump part.

Thus, we consider the maximum random variable of $\mathcal{X}$,

$$M(T) = \max\{X(s) : 0 \leq s \leq T\}.$$

These bounds are based on the elementary relation

$$\{M(T) > u\} = \{X(0) > u\} \biguplus \{X(0) < u, U_u \geq 1\},$$

where $\uplus$ denotes the disjoint union. It follows that

$$\mathbb{P}\left(M(T) > u\right) \leq \mathbb{P}\left(X(0) > u\right) + \mathbb{P}\left(U_u \geq 1\right) \leq \mathbb{P}\left(X(0) > u\right) + \mathbb{E}\, U_u, \tag{2}$$

and

$$\begin{aligned} \mathbb{P}\left(M(T) > u\right) &= \mathbb{P}\left(X(0) > u\right) + \mathbb{P}\left(U_u \geq 1\right) - \mathbb{P}\left(U_u \geq 1, X(0) > u\right) \\ &\geq \mathbb{P}\left(X(0) > u\right) + \mathbb{E}\, U_u - \frac{1}{2}\mathbb{E}\, U_{u.[2]} - \mathbb{P}\left(U_u \geq 1, X(0) > u\right), \end{aligned} \tag{3}$$

where $a_{[2]} = a(a-1)$ is the Pochammer symbol and $U_{u.[2]} = (U_u)_{[2]}$

The following theorem contains the upper bound for the tail of the distribution of the maximum of $\mathcal{X}$ on the interval $[0, T]$.

Theorem 3. *As $u \to \infty$ the overlife probability of the maximum verifies*

$$\mathbb{P}\left(M(T) > u\right) \leq 1 - \Phi(u) + T\sqrt{\frac{\lambda_2}{2\pi}}\varphi(u) + \lambda T p_\rho(u), \tag{4}$$

where $p_\rho(u)$ is defined in Theorem 2.

Proof. Is a direct consequence of Theorem 2 and formula (2). □

Remark 4. *Furthermore, if we denote the r.h.s. of (4) by $rhs(u)$, by Corollary 4 we have:*

$$rhs(u) \underset{u\to\infty}{\sim} T\sqrt{\frac{\lambda_2}{2\pi}}\,\varphi(u).$$

The goal of the rest of this section is to show that this upper bound is sharp. In order to do that, we use the lower bound given in equation (3) for the overlife function of the maximum $M(T)$. So we have to deal with the second moment of the number up-crossings.

Theorem 4. *Let the processes $\mathcal{Z}, \mathcal{X}$ and $\mathcal{J}$ be as in Corollary 3. If in addition $\mathcal{J}$ is a Poisson-auto-regressive process, and $\mathcal{Z}$ verifies that $\Gamma(\tau) \neq \pm 1/2$ for all τ and the Geman condition:*

$$\int \frac{\theta'(\tau)}{\tau^2} d\tau \ \textit{converges at } \tau = 0, \tag{5}$$

where θ is defined by $\Gamma(\tau) = 1 - \lambda_2\tau^2/2 + \theta(\tau)$. Then

$$\mathbb{P}\left(M(t) > u\right) = 1 - \Phi(u) + T\sqrt{\frac{\lambda_2}{2\pi}}\,\varphi(u) + o(\varphi(u))$$

Proof. It suffices to show that the additional terms in (3), w.r.t. (2), are $o(\varphi(u))$. Since

$$U_{u,[2]} = U_u(U_u - 1) = U^c_{u,[2]} + U^d_{u,[2]} + 2\, U^c_u U^d_u,$$

the proof is divided in several steps, considering separately each one of the resulting terms.

Claim 1. *We have* $\mathbb{E}\, U^c_{u,[2]} = o(\varphi(u))$.

Step 1. We begin with an upper bound for the second moment of U_u.

$$\begin{aligned}\mathbb{E}\, U^c_{u.[2]} & \\ &\leq \int_0^T \int_0^T \sum_{m,n=0}^{\infty} \mathbb{E}\left[\dot{Z}^+(s)\dot{Z}^+(t) \mid X(s) = X(t) = u, \nu_s = m, \nu_{t-s} = n\right] \\ &\qquad \cdot p_{X(s),X(t),\nu_s,\nu_{t-s}}(u,u,m,n) ds dt. \qquad (6)\end{aligned}$$

This bound is enough for our current purposes, but one can easily obtain the equality by the method of approximation by polygonals.

We adapt the proof of the Rice formula for the factorial moments in [3, Theorem 3.2]. Let C_u be the set of continuous up-crossings of $\mathcal{X}$ in $[0,T]$, $C_u^2 = C_u \times C_u$ and for any Borel set J in $[0,T]^2$ let $\mu(J) = \#(C_u^2 \cap J)$. It follows that $U^c_{u.[2]} = \mu([0,T]^2 \setminus \Delta)$, where Δ is the diagonal, that is, $\Delta = \{(s,t) \in [0,T]^2 : s = t\}$.

Take J_1 and J_2 disjoint intervals in $[0,T]$ and let $J = J_1 \times J_2 (\subset [0,T]^2 \setminus \Delta)$, then

$$\begin{aligned}&\mu(J) = U_u(J_1) \cdot U_u(J_2) \\ &= \lim_{\delta \to 0} \frac{1}{(2\delta)^2} \int_{J_1} \dot{Z}^+(s) \mathbf{1}\{|X(s) - u| < \delta\} ds \cdot \int_{J_2} \dot{Z}^+(t) \mathbf{1}\{|X(t) - u| < \delta\} dt \\ &\qquad = \lim_{\delta \to 0} \frac{1}{(2\delta)^2} \int\int_J \dot{Z}^+(s)\dot{Z}^+(t) \mathbf{1}\{|X(s) - u| < \delta, |X(t) - u| < \delta\} ds dt,\end{aligned}$$

where we applied Kac Formula on each interval, and noted that for δ small enough the quantity in the limit becomes constant, so we can use the same mute variable δ in both limits.

Now, we take expectation on both sides and apply Fatou's Lemma and Fubini's Theorem to pass the expectation inside the integral sign. Then, we condition on the number of jumps of the process in the intervals $[0,s]$, $[0,t]$ and on the values of the process $\mathcal{X}$ at these points, that is (if $s < t$):

$$\begin{aligned}&\mathbb{E}\, \dot{Z}^+(s)\dot{Z}^+(t) \mathbf{1}\{|X(s) - u| < \delta, |X(t) - u| < \delta\} = \\ &\int_{u-\delta}^{u+\delta} \int_{u-\delta}^{u+\delta} \sum_{m,n=0}^{\infty} \mathbb{E}\left[\dot{Z}^+(s)\dot{Z}^+(t) \mid X(s) = x, X(t) = y, \nu_s = m, \nu_t = m+n\right] \\ &\qquad p_{X(s),X(t)|\nu_s = m, \nu_t - \nu_s = n}(x,y) p_{\nu_s}(m) p_{\nu_t}(m+n) dx dy\end{aligned}$$

Observe that, under these conditions $X(s) = Z(s) + A_m$ and $X(t) = Z(t) + A_{m+n}$ are jointly Gaussian, thus the conditional expectation is well defined (and may be computed) via regression. Besides, this fact yields the regularity conditions needed for the integrand. In fact, the conditional expectation and the joint density function are continuous for $t \in [0,T]$ and x, y in a neighborhood of u. Hence, we can pass to the limit w.r.t. u inside the integral sign.

So far we have stated that $\mathbb{E}\,\mu(J)$ is bounded by the integral in the r.h.s. of (6) for any interval $J \subset [0,T]^2 \setminus \Delta$. As both sides represent measures on $[0,T]^2 \setminus \Delta$, the result follows by the standard arguments of Measure Theory.

Step 2. Actually we can take inequality (6) a little further:

$$\mathbb{E}\,U^c_{u.[2]} \leq 2\int_0^T (T-\tau) \sum_{n=0}^{\infty} \mathbb{E}\left[\dot{Z}_0^+ \dot{Z}_\tau^+ \mid Y_0(0) = Y_n(\tau) = u\right] \cdot \\ \cdot\, p_{Y_0(0),Y_n(\tau)}(u,u) p_{\nu_\tau}(n) d\tau,$$

where we set $Y_k(t) = Z(t) + A_k$, for $t \in [0,T]$ and $k \in \mathbb{N}$.

In effect, conditioned on $\nu_s = m, \nu_{t-s} = n$ (if $s < t$; similarly on the other case) we have $X(s) = Z(s) + A_m =: Y_m(s)$ and $X(t) = Z(t) + A_{m+n} =: Y_{m+n}(t)$. It is easy to see that the vector $(Y_m(s), Y_{m+n}(t))$ is independent from ν and has centered normal distribution with variances 1 and covariance $\Gamma_{t-s} + \rho^n/2$, in particular, this law does not depend on s, t but on the difference $t - s$, neither it depends on m. Therefore, the conditional expectation in inequality (6) reduces to that in the r.h.s. of the claimed bound.

Then, we factorize the density function as:

$$\begin{aligned} p_{X(s),X(t),\nu_s,\nu_{t-s}}(u,u,m,n) &= p_{X(s),X(t)|\nu_s,\nu_{t-s}}(u,u) p_{\nu_s}(m) p_{\nu_{t-s}}(n) \\ &= p_{Y_m(s),Y_{m+n}(t)}(u,u) p_{\nu_s}(m) p_{\nu_{t-s}}(n) \\ &= p_{Y_0(0),Y_n(\tau)}(u,u) p_{\nu_s}(m) p_{\nu_\tau}(n). \end{aligned}$$

Clearly $\sum_m p_{\nu_s}(m) = 1$. Finally we make the change of variables $(s,t) \mapsto (s, \tau = t - s)$, and obtain the desired inequality.

In the next two steps we bound each factor in the integrand.

Step 3. A standard regression shows that if $\nu_\tau = n$, then:

$$\mathbb{E}\,(\dot{Z}(0) \mid C) = -\mathbb{E}\,(\dot{Z}(\tau) \mid C) = -\frac{\Gamma'(\tau)u}{1 - (\Gamma(\tau) + \rho^n/2)}.$$

and

$$var(\dot{Z}(0) \mid C) = var(\dot{Z}(\tau) \mid C) = \lambda_2 - \frac{\Gamma'(\tau)^2}{1 - (\Gamma(\tau) + \rho^n/2)^2},$$

where $C = \{X_0(0) = X_n(\tau) = u\}$. Therefore, using the inequalities $a^+b^+ \leq (a+b)^2/4$ and $(a+b)^2 \leq 2(a^2+b^2)$ it follows that $\mathbb{E}\,(\dot{Z}^+(0)\dot{Z}^+(\tau) \mid C) \leq \frac{1}{2}(var(\dot{Z}(0) \mid C) + var(\dot{Z}(\tau) \mid C))$. Then:

$$\mathbb{E}\,(\dot{Z}^+(0)\dot{Z}^+(\tau) \mid C) \leq \lambda_2 - \frac{\Gamma'(\tau)^2}{1 - (\Gamma(\tau) + \rho^n/2)^2}.$$

Note that for $n \geq 1$, as $\rho < 1$, there is no problem when $\tau \to 0$, in effect $1 - (\Gamma(\tau) + \rho^n/2) \to 1/2 - \rho^n/2 > 0$.

Step 4. The vector $(Y_0(0), Y_n(\tau))$ is normally distributed with variances 1 and covariance $\Gamma(\tau) + \rho^n/2$, therefore, the exponential in the density is

$$\exp\left\{-\frac{u^2}{1+\Gamma(\tau)+\rho^n/2}\right\}.$$

For $n \geq 1$ we can bound $\rho^n \leq |\rho| < 1$, hence

$$\exp\left\{-\frac{u^2}{1+\Gamma(\tau)+\rho^n/2}\right\} \leq \exp\left\{-\frac{u^2}{1+\Gamma(\tau)+|\rho|/2}\right\} = o(\varphi(u)).$$

For $n \geq 1$, replacing in the inequality (6) of Step 1 we have:

$$\begin{aligned}
&\mathbb{E}\, U^c_{u.[2]} \leq 2T \\
&\cdot \int_0^T \sum_{n=0}^{\infty} p_{\nu_\tau}(n) \frac{\lambda_2(1-(\Gamma(\tau)+\rho^n/2)^2) - \Gamma'(\tau)^2}{(1-(\Gamma(\tau)+\rho^n/2)^2)^{3/2}} \exp\left\{-\frac{u^2}{1+\Gamma(\tau)+|\rho|/2}\right\} d\tau \\
&\qquad \leq 2T^2 \frac{\lambda_2}{\sqrt{1-((1+|\rho|)/2)^2}} \exp\left\{-\frac{u^2}{(3+|\rho|)/2}\right\} \\
&\qquad = o(\varphi(u)).
\end{aligned}$$

The case $n = 0$, when there are no jumps in $[0, \tau]$, is treated as in [3, Proposition 4.2], in particular, we need Geman condition to ensure the convergence of the integral.

Claim 2. *We have* $\mathbb{E}\, U^d_{u,[2]} = o(\varphi(u))$.

By the arguments in Corollary 4 it suffices to show that

$$\mathbb{E}\, U^d_{u,[2]} \leq c\, p_{|\rho|}(u),$$

for some constant c and u large enough. We can write

$$U^d_u = \sum_{n=0}^{\nu_T} \sum_{k=0}^{n} \mathbf{1}\{X(\tau_k^-) < u, X(\tau_k) > u\},$$

hence, making the product and taking expectation, we have:

$$\begin{aligned}
&\mathbb{E}\, U^d_{u,[2]} = \sum_{n=2}^{\infty} \sum_{1=k<\ell}^{n-1} p_{\nu_T}(n) \\
&\cdot \mathbb{P}\left(Z(\tau_k) + A_{k-1} < u; Z(\tau_k) + A_k > u; Z(\tau_\ell) + A_{\ell-1} < u; Z(\tau_\ell) + A_\ell > u\right) \\
&\qquad \leq \sum_{n=2}^{\infty} \sum_{k<\ell=1}^{n-1} p_{\nu_T}(n) \mathbb{P}\left(Z(\tau_k) + A_{k-1} < u; Z(\tau_\ell) + A_\ell > u\right). \quad (7)
\end{aligned}$$

Besides,

$$\begin{aligned}
&\mathbb{P}\left(Z(\tau_k) + A_{k-1} < u; Z(\tau_\ell) + A_\ell > u\right) \\
&\quad = \int_0^T ds \int_s^T dt p_{\tau_k,\tau_\ell|\nu_T=n}(s,t) \mathbb{P}\left(Z(s) + A_{k-1} < u; Z(t) + A_\ell > u\right) \\
&\quad = \int_0^T ds \int_s^T dt p_{\tau_k,\tau_\ell|\nu_T=n}(s,t) \mathbb{P}\left(Z(0) + A_{k-1} < u; Z(t-s) + A_\ell > u\right),
\end{aligned}$$

in the latter equality we used the stationarity of the process $\mathcal{Z}$. The vector $(Z(0) + A_{k-1}; Z(t-s) + A_\ell)$ is centered Gaussian with variances 1/2 and covariance $\Gamma(\tau) + \rho^{\ell-k+1}/2$. It is easy to check that $-\Gamma(\tau) - \rho^{\ell-k+1}/2 < 1/2(1+|\rho|)$ which is the covariance of the vectors $Z+S$ and $Z+|\rho|S+\sqrt{1-\rho^2}V$, therefore, by the Plackett-Slepian inequality, see [3, Section 2.1], we have

$$\mathbb{P}\,(Z(0)+A_{k-1} < u; Z(t-s)+A_\ell > u) \leq \mathbb{P}\,(Z+S < u; Z+|\rho|S+\sqrt{1-\rho^2}V > u).$$

This bound does not depend on s, t, hence

$$\begin{aligned}
&\mathbb{P}\,(Z(\tau_k) + A_{k-1} < u; Z(\tau_\ell) + A_\ell > u) \\
&\quad\leq \mathbb{P}\,(Z + S < u; Z + |\rho|S + \sqrt{1-\rho^2}V > u) \int_0^T ds \int_s^T dt p_{\tau_k,\tau_\ell|\nu_T=n}(s,t) \\
&\qquad\qquad = \mathbb{P}\,(Z + S < u; Z + |\rho|S + \sqrt{1-\rho^2}V > u) \\
&\qquad\qquad\qquad\qquad = p_{|\rho|(u)},
\end{aligned}$$

since $\tau_k, \tau_\ell \mid \nu_T = n$ is concentrated on $[0,T]^2$. Finally, replacing in the equation (7) we have

$$\mathbb{E}\,U^d_{u,[2]} \leq \sum_{n=2}^{\infty} \sum_{1=k<\ell}^{n-1} p_{|\rho|(u)} = \frac{(\lambda T)^2}{2} p_{|\rho|(u)},$$

and the result follows.

Claim 3. *We have* $\mathbb{E}\,U_{u,[2]} = o(\varphi(u))$.

In effect

$$\begin{aligned}
\mathbb{E}\,U_{u,[2]} &= \mathbb{E}\,(U^c_u + U^d_u)(U^c_u + U^d_u - 1) \\
&= \mathbb{E}\,U^c_{u,[2]} + \mathbb{E}\,U^d_{u,[2]} + 2\mathbb{E}\,U^c_u U^d_u \\
&\leq \mathbb{E}\,U^c_{u,[2]} + \mathbb{E}\,U^d_{u,[2]} + 2\sqrt{\mathbb{E}\,(U^c_u)^2 \mathbb{E}\,(U^d_u)^2} \\
&= \mathbb{E}\,U^c_{u,[2]} + \mathbb{E}\,U^d_{u,[2]} + 2\sqrt{(\mathbb{E}\,U^c_{u,[2]} + \mathbb{E}\,U^c_u)\mathbb{E}\,(U^d_{u,[2]} + \mathbb{E}\,U^d_u)},
\end{aligned}$$

where we used Cauchy-Schwarz inequality. The first two terms in the r.h.s. are treated in the previous lemmas, under the square root sign the first factor is equivalent to $\varphi(u)$ and the second one is $o(\varphi(u))$. Thus, $\mathbb{E}\,U_{u,[2]} = o(\varphi(u))$.

Claim 4. *We have* $\mathbb{P}\,(X(0) > u, U_u \geq 1) = O(\varphi((1+\delta)u))$.

We follow the proof of the analogue assertion in [3, Proposition 4.2]. The key fact is that the distribution of the process $\mathcal{J}$ remains unchanged under time reversal $t \mapsto T - t$.

In effect, let us condition on the number of jumps $\nu_T = n$, then it is easy to check that the (conditional) distribution of $(A_1, A_2, \ldots, A_n) \mid \nu_T = n$ is the same as the distribution of $(A_n, A_{n-1}, \ldots, A_1) \mid \nu_T = n$. Besides, the distribution of $\tau_1, \tau_2, \ldots, \tau_n \mid \nu_T = n$ is that of an uniform (ordered) sample of size n, so it looks the same from 0 and from T. Since the construction of the process $\mathcal{J}$ depends on these elements and there is no difference if we start at 0 or at T the claim follows.

In conclusion,

$$\mathbb{P}\left(M(t)>u\right) \geq 1-\Phi(u)+T\sqrt{\frac{\lambda_2}{2\pi}}\,\varphi(u)+\lambda T\,p_\rho(u)+O(\varphi((1+\delta)u))$$
$$=1-\Phi(u)+T\sqrt{\frac{\lambda_2}{2\pi}}\,\varphi(u)+o(\varphi(u)).$$

Taking into account (4), this completes the proof. □

3.2 Stationary Gaussian continuous process plus CPP

In this example we consider a centered, stationary, Gaussian process $\mathcal{Z}$ with $\Gamma(0)=1$ and C^1 paths and an independent Compound Poisson Process (CPP), $\mathcal{J}$ with finite intensity λ and standard Gaussian jumps.

The process $\mathcal{J}$ is defined by the kernels $P^{(n)}_{x_{n-1},\tau_{n-1}}$, the exponential distribution of intensity $\lambda>0$ and $\pi^{(n)}_{x_{n-1},\tau_n}$, representing the increments, the standard normal distribution. We can also define $\mathcal{J}$ in terms of random variables as follow, let $(\tau_n)_{n\in\mathbb{N}}$ be independent random variables with exponential distribution with intensity $0<\lambda$ and $(\xi_n)_{n\in\mathbb{N}}$ be independent standard Gaussian random variables independent from (τ_n); then for $t\geq 0$ define:

$$\nu_t := \max\{n : \tau_n < t\}; \quad J(t) := \sum_{n=1}^{\nu_t} \xi_n.$$

Thus, $J(t)$ is the sum of a random number of standard normal random variables.

It is well known, see [9] for instance, that the compensating measure of the CPP is $\lambda dt\,\Phi(dx)$, where Φ is the standard Gaussian distribution. In particular, the compensating measure is deterministic.

Denote by φ_n the centered Gaussian density with variance n, in particular, $\varphi_1=\varphi$ is the standard normal density. Note that $\varphi_n * \varphi_m = \varphi_{n+m}$, where $*$ stands for the convolution.

Next theorem gives the mean number of up-crossings through u by $\mathcal{X}$. The mean number of down-crossings is analogue.

Theorem 5. *For the process defined above we have*

$$\mathbb{E}\,U_u = T\sqrt{\frac{\lambda_2}{2\pi}}\,p(u)+\lambda T\int_{-\infty}^{u}\overline{\Phi}(u-x)\,p(x)dx,$$

where λ_2 is the second spectral moment of $\mathcal{Z}$ and $p=\sum_{n=1}^{\infty}p_n\varphi_n$ with $p_n := \frac{1}{\lambda T}\mathbb{P}\left\{\nu_T\geq n\right\}$.

Remark 5. *In spite of the notation, the first term, which corresponds to the continuous up-crossings, does depend on λ through the density function p, which has expectation 0 and variance $\lambda T/2$.*

Remark 6. *Observe that the mean number of continuous crossings, the first term, has the usual form, see Remark 1.*

Proof. We begin with the continuous crossings. Using the formula in Remark 1, we have

$$\mathbb{E}\, U_u^c = \int_0^T dt \int_{-\infty}^{\infty} \mathbb{E}\left(\dot{Z}^+(t) \mid Z(t) = v\right) p_{Z(t)}(v) p_{J(t)}(u - v) dv$$
$$= \mathbb{E}\, \dot{Z}^+(0) \int_0^T dt \int_{-\infty}^{\infty} p_{Z(0)}(v) p_{J(t)}(u - v) dv,$$

where we used the stationarity of $\mathcal{Z}$ as in Corollary 3. Recall that $\mathbb{E}\, \dot{Z}^+(0) = \sqrt{\lambda_2/2\pi}$. Now, by Lemma 1 we decompose $p_{J(t)}$, so

$$\mathbb{E}\, U_u^c = \sqrt{\frac{\lambda_2}{2\pi}} \int_0^T dt \sum_{n=0}^{\infty} p_{\nu_t}(n) \int_{-\infty}^{\infty} p_{Z(0)}(v) \varphi_n(u - v) dv$$
$$= \sqrt{\frac{\lambda_2}{2\pi}} \int_0^T dt \sum_{n=0}^{\infty} p_{\nu_t}(n) (p_{Z(0)} * \varphi_n)(u)$$
$$= \sqrt{\frac{\lambda_2}{2\pi}} \sum_{n=0}^{\infty} \varphi_{n+1}(u) \int_0^T p_{\nu_t}(n) dt.$$

This integral is computed in the second item of Lemma 2, we get

$$\mathbb{E}\, U_u^c = T \sqrt{\frac{\lambda_2}{2\pi}} \sum_{n=0}^{\infty} \frac{1}{\lambda T} \mathbb{P}\left\{\nu_T \geq n + 1\right\} \varphi_{n+1}(u)$$
$$= T \sqrt{\frac{\lambda_2}{2\pi}} \sum_{n=0}^{\infty} p_{n+1} \varphi_{n+1}(u) = T \sqrt{\frac{\lambda_2}{2\pi}} \sum_{n=1}^{\infty} p_n \varphi_n(u)$$
$$= T \sqrt{\frac{\lambda_2}{2\pi}}\, p(u)$$

which gives the first term of the result.

Now, we turn to the discontinuous crossings

$$\mathbb{E}\, U_u^d = \mathbb{E} \int_0^T \int_{-\infty}^{\infty} \mathbf{1}\{X(t^-) < u; X(t^-) + y > u\} L(dt, dy)$$
$$= \mathbb{E} \int_0^T \int_{-\infty}^{\infty} \mathbf{1}\{X(t^-) < u; X(t^-) + y > u\} \lambda dt\, \Phi(dy),$$

where, as we said before, Φ is the distribution of the jumps. Since the compensating measure is deterministic we have

$$\mathbb{E}\, U_u^d = \lambda \int_0^T \int_{-\infty}^{\infty} \mathbb{E}\, \mathbf{1}\{X(t^-) < u; X(t^-) + y > u\} dt\, \Phi(dy)$$
$$= \lambda \int_0^T \int_{-\infty}^{\infty} \mathbb{P}\left(X(t^-) < u; X(t^-) + y > u\right) dt\, \Phi(dy)$$
$$= \lambda \int_0^T \mathbb{P}\left(X(t^-) < u; X(t^-) + \xi > u\right) dt,$$

being ξ a standard normal variable independent from $X(t^-)$. Now, we condition on $X(t^-)$, since, for fixed, t almost surely $J(t^-) = J(t)$ we may use the same computations as in the computation of $\mathbb{E}\, N_u^c$:

$$\mathbb{E}\, U_u^d = \lambda \int_0^T \int_{-\infty}^u \mathbb{P}\left(\xi > u - x \mid X(t^-) = x\right) p_{X(t^-)}(x) dx dt$$
$$= \lambda \int_{-\infty}^u \overline{\Phi}(u-x) dx \int_0^T p_{X(t^-)}(x) dt$$
$$= \lambda T \int_{-\infty}^u \overline{\Phi}(u-x) p(x) dx.$$

This completes the proof. □

Next, we compare the mean numbers of continuous and discontinuous up-crossings through the level u by $\mathcal{X}$ as $u \to \infty$.

Corollary 5. *As $u \to \infty$ the mean numbers of continuous and discontinuous up-crossings through the level u are of the same order. More precisely*

$$\lim_{u\to\infty} \frac{\lambda}{\sqrt{\lambda_2}}\, \mathbb{E}\, U_u^c \le \lim_{u\to\infty} \mathbb{E}\, U_u^d \le \lim_{u\to\infty} \lambda \sqrt{\frac{2\pi}{\lambda_2}}\, \mathbb{E}\, U_u^c$$

Proof. We bound from above and from below $\mathbb{E}\, U_u^d$. First, we have

$$\int_{-\infty}^u \overline{\Phi}(u-y) p(y) dy = \int_0^\infty \overline{\Phi}(y) p(u-y) dy$$
$$= \int_0^{2u} \overline{\Phi}(y) p(u-y) dy + \int_{2u}^\infty \overline{\Phi}(y) p(u-y) dy$$
$$\ge \int_0^{2u} \overline{\Phi}(y) p(u-y) dy \ge p(u) \int_0^{2u} \overline{\Phi}(y) dy,$$

where we used that p is even and decreasing on $[0, \infty)$. Furthermore

$$\lim_{u\to\infty} \int_0^{2u} \overline{\Phi}(y) dy = \int_0^\infty \overline{\Phi}(y) dy = \mathbb{E}\, \xi^+ = \sqrt{\frac{2}{\pi}}$$

Therefore

$$\mathbb{E}\, U_u^d \ge \lambda T\, p(u) \int_0^\infty \overline{\Phi}(y) dy \underset{u\to\infty}{\sim} \frac{\lambda T}{\sqrt{2\pi}} p(u).$$

On the other hand,

$$\mathbb{E}\, U_u^d = \lambda T \sum_{n=1}^\infty p_n \int_0^\infty \overline{\Phi}(u-y) \varphi_n(y) dy \le \lambda T \sum_{n=1}^\infty p_n \int_{-\infty}^\infty \overline{\Phi}(u-y) \varphi_n(y) dy.$$

Note that this is a convolution formula, for the overlife function, of two (independent) random variables, say $Z \sim \Phi$ and $V_n \sim \varphi_n$. Then we can write the latter integral as $\mathbb{P}\left(Z + V_n > u\right)$. Furthermore, since Φ is the Gaussian standard distribution and φ_n is the Gaussian density with zero mean and variance n, this probability equals $\mathbb{P}(V_{n+1} > u) = \overline{\Phi}(u/\sqrt{n+1})$. Thus

$$\mathbb{E}\, U_u^d \leq \lambda T \sum_{n=1}^{\infty} p_n \overline{\Phi}\left(\frac{u}{\sqrt{n+1}}\right) \leq \lambda T \sum_{n=1}^{\infty} p_n (n+1)\varphi_{n+1}(u) = \lambda T(p(u) + \delta(u)),$$

where $\delta(u) := \sum_{n=1}^{\infty} p_n(n+1)\varphi_{n+1}(u) - p(u)$. In order to obtain the desired result, it suffices to prove that $\delta(u) \to 0$ as $u \to \infty$. In effect

$$|\delta(u)| \leq \left| \sum_{n=1}^{\infty} \left[p_n(n+1) - p_{n+1}\right] \varphi_{n+1}(u) \right| + |p_1 \varphi_1(u)|$$

It is clear that the second term of the r.h.s tends to zero when $u \to \infty$. Let us look at the first term, note that $\sum_{n=1}^{\infty} p_n(n+1)$ is, roughly speaking, the second factorial moment of the Poisson distribution, in effect

$$\sum_{n=1}^{\infty} p_n(n+1) = 1 + \frac{1}{\lambda T} \sum_{n=1}^{\infty} n \mathbb{P}(\nu_T \geq n) = 1 + \frac{1}{\lambda T} \sum_{n=1}^{\infty} n \sum_{k=n}^{\infty} p_{\nu_T}(k)$$
$$= 1 + \frac{1}{\lambda T} \sum_{k=1}^{\infty} \sum_{n=1}^{k} n p_{\nu_T}(k) = 1 + \frac{1}{2\lambda T} \sum_{k=1}^{\infty} k(k-1) p_{\nu_T}(k) < \infty$$

Thus, the first term is a mixture of Gaussian densities φ_n (times a constant), hence it tends to zero. Then $\delta(u) \to 0$ as $u \to \infty$ and

$$\mathbb{E}\, U_u^d \leq \lambda T(p(u) + \delta(u)) \underset{u \to \infty}{\sim} \lambda T p(u).$$

Now, the result follows joining the two obtained bounds for $\mathbb{E}\, U_u^d$ and using Theorem 5. □

Remark 7. *Note that in this case the discontinuous crossings through the level u are not negligible w.r.t. the continuous crossings when u tends to infinity, in contrast with the situation in the example of the previous section.*

With the usual procedure, i.e. equation (2), we obtain the following result.

Corollary 6. *As $u \to \infty$ the tail of the distribution of the maximum is bounded from above by*

$$\mathbb{P}(M(T) > u) \leq \mathbb{P}(X(0) > u) + T\sqrt{\frac{\lambda_2}{2\pi}} p(u) + \lambda \int_{-\infty}^{u} \overline{\Phi}(u-y) p(y) dy.$$

4 Generalization of Borovkov-Last's formula

Borovkov and Last in [6] are interested in the continuous crossings through a level u by a stationary Piecewise Deterministic Markov Process. A process $\mathcal{X}$ of this kind, starts at a random position, then jumps a random quantity at random times but moves deterministically between jumps.

Such a process is described by a general point process $(\tau_n, \xi_n)_n$, as described in Section 2, and a (non-random) rate function $\mu : \mathbb{R} \to \mathbb{R}$. More precisely, the process $\mathcal{X}$ has jumps at the points (τ_n), the magnitude of the jump is ξ_n and on the interval $[\tau_n, \tau_{n+1})$, $X(t)$ follows the integral curve of μ with initial condition $X(\tau_n) = X(\tau_n^-) + \xi_n$.

Note that the jump part of the process is not independent of the continuous one.

Let $D_\mu = \{u : \mu(u) = 0\}$. Observe that if $\mu(u) > 0$ (resp. $<$), the continuous crossings of level u can only be up-crossings (resp. down-crossings).

Next theorem extends Borovkov-Last Formula to the non-stationary case.

Theorem 6. *Let $u \notin D_\mu$ and assume that μ and $p_{X(t)}$ are continuous w.r.t. x in a neighborhood of u and $t \in [0, T]$. Then:*

$$\mathbb{E}\, N_u^c = |\mu(u)| \int_0^T p_{X(t)}(u) dt$$

Proof. For the levels $u \notin D_\mu$ we can apply Kac counting formula pathwise for almost all paths of $\mathcal{X}$. In effect, the continuity of μ implies that the paths are of class C^1 between the jumps and that $\dot{X}(t) = \mu(X(t))$ for almost all $t \in [0, T]$. Since $X(t)$ has a density, the value u is not taken at the extremes of the interval neither at the jump points almost surely. Furthermore, by the continuity of this density, there are not tangencies at level u.

Now we take expectation on both sides of Kac Counting Formula and observe that the number of continuous crossings of the level $u \notin D_\mu$ is bounded by the number of jumps $+1$ of $\mathcal{X}$ in $[0, T]$. In effect, the sign of $\mu(u)$ determines the direction of the continuous crossings of u, so, between two continuous crossings must be a discontinuous one in the opposite direction, thus there is at most one continuous crossing at each one of the intervals of the partition $\tau_0, \dots, \tau_{\nu_T}, T$. Then, since ν_T in integrable, we may pass to the limit under the expectation sign:

$$\mathbb{E}\, N_u^c = \lim_{\delta \downarrow 0} \frac{1}{2\delta} \int_0^T \mathbb{E}\, \left[|\dot{X}(t)| \mathbf{1}_{\{|X(t)-u|<\delta\}} \right] dt.$$

Now, $\dot{X}(t)$ is a deterministic function of $X(t)$, namely $\dot{X}(t) = \mu(X(t))$, so the integrand is simply the expectation of a function of $X(t)$, therefore

$$\mathbb{E}\, N_u^c = \lim_{\delta \downarrow 0} \frac{1}{2\delta} \int_0^T \int_{u-\delta}^{u+\delta} |\mu(x)| p_{X(t)}(x) dx dt.$$

By the continuity of the integrand and the compactness of the domain we can pass the limit inside the integral w.r.t. t. Then, the result follows by the mean value theorem. □

As a corollary, when $\mathcal{X}$ is stationary, we obtain Borovkov-Last's Formula.

Corollary 7. *If in addition to the conditions of Theorem 6, the process $\mathcal{X}$ is stationary, then*

$$\mathbb{E}\, N_u^c = |\mu(u)| p(u),$$

where $p = p_{X(0)}$.

Remark 8. *Note that on this case the (net) number of discontinuous crossings, namely the difference of down and up crossings $D_u^d - U_u^d$, is related to the number of continuous crossings, actually we have:*

$$\mathbb{E}\, D_u^d - \mathbb{E}\, U_u^d = \mathbb{E}\, N_u^c + \operatorname{sgn}(\mu(u))(\mathbb{P}\,(X(T) < u) - \mathbb{P}\,(X(0) < u))$$

5 Proofs

First, we prepare some auxiliary results.

Lemma 1. *Let T, X, Y, Z be random variables, then*

$$\mathbb{E}\,(T \mid X = x, Y = y) = \int_{-\infty}^{\infty} \mathbb{E}\,(T \mid X = x, Y = y, Z = z) p_{Z|X=x,Y=y}(z) dz.$$

and

$$\begin{aligned} &\mathbb{E}\,(T \mid X = x, Z = z) p_{X|Z=Z}(x) \\ &\qquad = \int_{-\infty}^{\infty} \mathbb{E}\,(T \mid X = x, Y = y, Z = z) p_{X,Y|Z=z}(x, y) dy. \end{aligned}$$

Proof. These equalities follow directly from the properties of conditional expectation, see [15, Page 215]. For example, for the first one take $F_2 = (X, Y)$ and $F_3 = (X, Y, Z)$. □

Lemma 2 (Auxiliaries)**.** *Consider a CPP $(J(t))$ with standard Gaussian jumps, then*

1. *the density of the CPP can be written as:*

$$p_{J(t)}(x) = \sum_{n=1}^{\infty} p_{\nu_t}(n) \varphi_n(x).$$

2. *The integral*

$$\int_0^T p_{\nu_t}(n) dt = \frac{1}{\lambda} \mathbb{P}\,(\nu_T \geq n + 1) = \frac{1}{\lambda} \mathbb{P}\,(\tau_{n+1} \leq T)$$

3. *If $p_n = \frac{1}{\lambda T} \mathbb{P}\,(\nu_T \geq n)$, then $\sum_{n=1}^{\infty} p_n = 1$*

As usual we denote the time epochs by τ_n and the number of jumps in $[0, t]$ by ν_t.

Proof. 1. Conditioning on the value of ν_t we have:

$$F_{J(t)}(x) = \mathbb{P}\,(J(t) \leq x) = \sum_{n=0}^{\infty} p_{\nu_t}(n) \mathbb{P}\,(J(t) \leq x \mid \nu_t = n).$$

Now, if $\nu_t = n$, $J(t)$ is the sum of n independent standard Gaussian random variables, hence, the conditional probability in the r.h.s. of the latter equation is the distribution of centered normal random variable with variance n. The result follows taking derivatives on both sides.

2. By definition, $p_{\nu_t}(n)$, as a function of t, is equal to the density function of the Gamma distribution, with parameters λ, $n+1$, divided by λ, furthermore, it is well known, [15], that this is the distribution of τ_{n+1}, hence

$$\int_0^T p_{\nu_t}(n)dt = \frac{\mathbb{P}\left(\tau_{n+1} \leq T\right)}{\lambda}.$$

The result follows since the events $\{\tau_{n+1} \leq T\}$ and $\{\nu_T \geq n+1\}$ coincide.

3. It follows directly from the facts that ν_T has Poisson distribution with mean λT and that for a non-negative integer valued random variable X: $\mathbb{E}\, X = \sum_{n=1}^{\infty} \mathbb{P}\left(X \geq n\right)$. $\square$

Proof of Theorem 1. Let us start with the formula for the mean number of continuous crossings, we compute the expectation by conditioning on the paths of the pure-jump part $\mathcal{J}$ and use the proof of the non-Gaussian case on [3].

Then, we condition on the number of jumps, $\nu_T = n$, on the jump instants, $\tau_k = t_k$, thus

$$\begin{aligned}\mathbb{E}\, N_u^c &= \mathbb{E}\left(\mathbb{E}\left[N_u^c\left(\mathcal{X}, [0,T]\right) \mid \nu_T = n; \boldsymbol{\tau} = \mathbf{t}\right]\right) \\ &= \sum_{n=0}^{\infty} p_{\nu_T}(n) \int_{[0,T]^n} p_{\boldsymbol{\tau}}(\mathbf{t})\, \mathbb{E}\left[N_u^c\left(\mathcal{X}, [0,T]\right) \mid \nu_T = n; \boldsymbol{\tau} = \mathbf{t}\right] d\mathbf{t},\end{aligned}$$

where we set $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ and $\mathbf{t} = (t_1, \ldots, t_n)$.

Now, we look at the integrand and, since the number of crossings is additive w.r.t. the interval, we split the interval $[0,T]$ as the union of the intervals $\mathcal{I}_k := [t_{k-1}, t_k)$, then

$$\mathbb{E}\left[N_u^c\left(\mathcal{X}, [0,T]\right) \mid \nu_T = n; \boldsymbol{\tau} = \mathbf{t}\right] = \sum_{k=1}^{n} \mathbb{E}\left[N_u^c\left(\mathcal{X}, \mathcal{I}_k\right) \mid \nu_T = n; \boldsymbol{\tau} = \mathbf{t}\right]$$

Each term can be written, see Lemma (1), as

$$\int_{-\infty}^{\infty} \mathbb{E}\left[N_u^c\left(\mathcal{X}, \mathcal{I}_k\right) \mid \nu_T = n; \boldsymbol{\tau} = \mathbf{t}; J(t_{k-1}) = y\right] p_{J(t_{k-1})|\nu_T = n; \boldsymbol{\tau} = \mathbf{t}}(y) dy.$$

Now, conditionally on $\nu_T = n; \boldsymbol{\tau} = \mathbf{t}$ and $J(\tau_{k-1}) = y$, the process $\mathcal{X}$ can be written as $\mathcal{Z} + y$ on $\mathcal{I}_k$. Since $\mathcal{Z}$ verifies the conditions $A1$, $A2$, $A3$ and $A4$ on $\mathcal{I}_k$ so does the process $\mathcal{Z} + y$, therefore we may apply Rice Formula, see [3], on each interval under these conditions to obtain

$$\begin{aligned}&\mathbb{E}\left[N_u^c\left(\mathcal{X}, \mathcal{I}_k\right) \mid \nu_T = n; \boldsymbol{\tau} = \mathbf{t}; J(t_{k-1}) = y\right] \\ &\qquad = \int_{\tau_{k-1}}^{\tau_k} \mathbb{E}\left[|\dot{Z}(t)| \mid X(t) = u, \nu_T = n; \boldsymbol{\tau} = \mathbf{t}, J(\tau_{k-1}) = y\right] \\ &\qquad\qquad \cdot p_{X(t)|\nu_T = n; \boldsymbol{\tau} = \mathbf{t}, J(\tau_{k-1}) = y}(u) dt\end{aligned}$$

Each one of these expressions should be replaced in the previous one.

Finally, we have to integrate (the conditions), for notational simplicity, let us write $g(n,\mathbf{t},y) = \mathbb{E}\left[|\dot{Z}(t)| \mid X(t) = u, \nu_T = n; \boldsymbol{\tau} = \mathbf{t}, J(\tau_{k-1}) = y\right]$, $g(n,\mathbf{t}) = \mathbb{E}\left[|\dot{Z}(t)| \mid X(t) = u, \nu_T = n; \boldsymbol{\tau} = \mathbf{t}\right]$ and $g(n) = \mathbb{E}\left[|\dot{Z}(t)| \mid X(t) = u, \nu_T = n\right]$. Let us perform the integrals one by one, starting w.r.t. y, (use Fubini) then

$$\begin{aligned}
\int_{\mathbb{R}} g(n,\mathbf{t},y) p_{X(t)|\nu_T=n;\boldsymbol{\tau}=\mathbf{t},J(\tau_k)=y}(u) p_{J(t_k)|\nu_T=n;\boldsymbol{\tau}=\mathbf{t}}(y)dy \\
= \int_{\mathbb{R}} g(n,\mathbf{t},y) p_{(X(t),J(\tau_k))|\nu_T=n;\boldsymbol{\tau}=\mathbf{t}}(u,y)dy \\
= g(n,\mathbf{t}) p_{X(t)|\nu_T=n;\boldsymbol{\tau}=\mathbf{t}}(u),
\end{aligned}$$

Where we used Lemma (1). Now, we sum the integrals over k and integrate w.r.t. $\mathbf{t}$:

$$\begin{aligned}
\int_0^T dt \int_{[0,T]^n} g(n,\mathbf{t}) p_{X(t)|\nu_T=n;\boldsymbol{\tau}=\mathbf{t}}(u) p_{\boldsymbol{\tau}|\nu_T=n}(\mathbf{t}) d\mathbf{t} \\
= \int_0^T dt \int_{[0,T]^n} g(n,\mathbf{t}) p_{(X(t),\boldsymbol{\tau})|\nu_T=n}(u.\mathbf{t}) d\mathbf{t} \\
= \int_0^T g(n) p_{X(t)|\nu_T=n}(u) dt.
\end{aligned}$$

By the same arguments one can remove the condition on ν_T. The result follows.

Now we proceed to the formula for the mean number of discontinuous crossings through level u. For the definitions used below see [9].

Clearly, $\mathcal{X}$ only can have a discontinuous crossing through u at the points $\tau_n; n = 1, \ldots, \nu_T$, the jump of $\mathcal{X}$ at each one of these points is due to the jump of $\mathcal{J}$. Hence, we consider the Marked Point Process $((\tau_k, \xi_k) : k \geq 0)$ associated to $\mathcal{J}$ on $[0,\infty) \times \mathbb{R}$, which defines a Random Counting Measure $\mu(dt,dy)$, in terms of which we can write:

$$\begin{aligned}
N_u^d = \sum_{k=1}^{\nu_T} \mathbf{1}\{(X(\tau_k^-) - u)(X(\tau_k^-) + \Delta X(\tau_k) - u) < 0\} \\
= \sum_{0 \leq t \leq T} \mathbf{1}\{(X(t^-) - u)(X(t^-) + \xi_{\nu_t} - u) < 0\} \\
= \int_{[0,T]\times\mathbb{R}} \mathbf{1}\{(X(t^-) - u)(X(t^-) + y - u) < 0\} \mu(dt,dy)
\end{aligned}$$

Is easy to see that this RCM has a compensating measure denoted by $L(dt,dy)$, see [9] again. Taking expectations on both sides we have:

$$\begin{aligned}
\mathbb{E}\, N_u^d = \mathbb{E}\left[\mathbb{E}\left(N_u^d \mid \mathcal{Z}\right)\right] \\
= \mathbb{E}\left[\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \mathbf{1}\{(X(t^-) - u)(X(t^-) + y - u) < 0\} \mu(dt,dy) \mid \mathcal{Z}\right]\right] \\
= \mathbb{E} \int_{[0,T]\times\mathbb{R}} \mathbf{1}\{(X(t^-) - u)(X(t^-) + y - u) < 0\} L(dt,dy)
\end{aligned}$$

where we used that, conditioned on $\mathcal{Z}$, the integral is done w.r.t. the RCM μ associated with $\mathcal{J}$ which coincides with the integral w.r.t. the compensating measure $L(dt, dy)$ since the integrand is predictable, in fact it is a function of t^-. Finally we integrate with respect to $\mathcal{Z}$ and the result follows. $\square$

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