Systems of polynomial equations defining hyperelliptic d-osculating covers

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Abstract

Let X denote a fixed smooth projective curve of genus 1, defined over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic $\boldsymbol{p} \neq 2$. For any positive integer n, we will consider the moduli space $H(X, n)$, of degree-n finite separable covers of X by a hyperelliptic curve, marked at a triplet of Weierstrass points. We parameterize $H(X, n)$ by a suitable space of rational fractions, and apply it to studying the (finite) subset of degree-n *hyperelliptic tangential covers* of X. We find a polynomial characterization for the corresponding rational fractions and deduce a square system of polynomial equations, whose solutions parameterize the latter covers. Furthermore, we also obtain non-square systems parameterizing *hyperelliptic d-osculating covers*, for any $d > 1$.

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1 Introduction

Let $\mathbb{P}^1 := \mathbb{K} \cup \{\infty\}$ and X denote, the projective line and a fixed smooth projective curve of genus 1, both defined over an algebraically closed field $\mathbb{K}$, of arbitrary characteristic $\boldsymbol{p} \neq 2$. We will also fix a degree-2 projection, $\varphi_X : X \to \mathbb{P}^1$, ramified over $\{\infty, 0, 1, \lambda\}$.

We choose $\omega_o := \varphi_X^{-1}(\infty)$ as its origin, and let $[-1] : X \to X$ denote the corresponding inverse homomorphism, fixing the four half-periods $(\omega_o, \omega_1, \omega_2, \omega_3) := \varphi_X^{-1}\big((\infty, 0, 1, \lambda)\big)$. We will consider all finite separable triply marked morphisms $\pi : (\Gamma, p, p', p'') \to X$, called henceforth *hyperelliptic covers*, satisfying, either (1) and (2), or (3) below:

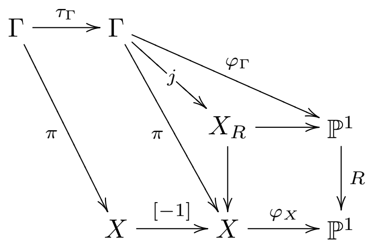
1. Γ is a hyperelliptic curve (c.f. **2.1.**(1)) and $\pi(p) = \omega_o$;

2. the hyperelliptic involution $\tau_\Gamma : \Gamma \to \Gamma$ fixes $\{p, p', p''\}$;

3. $\pi(p) = \omega_o$ and there exists a unique degree-2 projection $\varphi_\Gamma : \Gamma \to \mathbb{P}^1$, ramified at $\{p, p', p''\}$ and such that $\varphi_\Gamma\big((p, p', p'')\big) = (\infty, 0, 1)$.

We start constructing a one-to-one correspondence between $H(X, n)$, the moduli space of degree-n *hyperelliptic covers*, and a family of degree-n rational fractions, as sketched hereafter.

Given any *hyperelliptic* cover $\pi : (\Gamma, p, p', p'') \to X$, we first prove the equality $[\text{-}1] \circ \pi = \pi \circ \tau_\Gamma$, implying that π can be pushed down to a marked projection $R : (\mathbb{P}^1, \infty) \to (\mathbb{P}^1, \infty)$, having odd ramification index at $\{\infty, 0, 1\}$ and fitting in the following commutative diagram:



where X_R is the fiber product of $R : \mathbb{P}^1 \to \mathbb{P}^1$ and $\varphi_X : X \to \mathbb{P}^1$, while $j : \Gamma \to X_R$ is the desingularization of X_R.

Conversely, consider a rational fraction $R := \frac{P}{Q}$ such that:

1. P and Q are coprime and $degP \text{-} degQ > 0$ is odd;

2. R has odd ramification index at $\{0, 1\}$ and $R(0), R(1) \in \{\infty, 0, 1, \lambda\}$,

The latter property is equivalent to:

(2′) the product $PQ(P \text{-} Q)(P \text{-} \lambda Q)$ has odd vanishing order at $\{0, 1\}$.

We let then X_R denote the fiber product of $\varphi_X : X \to \mathbb{P}^1$ with $R : \mathbb{P}^1 \to \mathbb{P}^1$, $j : \Gamma \to X_R$ its desingularization and $(p, p', p'') := \varphi_\Gamma^{-1}\big((\infty, 0, 1)\big)$. The curve Γ is naturally equipped with a marked projection $\pi : (\Gamma, p, p', p'') \to X$ (fitting in a commutative diagram as above), such that π is a *hyperelliptic cover* (**3.1.**).

A similar diagram has already popped up in [7], for the genus-2 case, and the corresponding morphism $R : \mathbb{P}^1 \to \mathbb{P}^1$ been called a *Frey-Kani covering* (see also [4],[5] & [6]). More recently, as kindly pointed to us by Victor Enolskii, A.B.Bogatyrev also considered a correspondence between a class of ramified covers of an elliptic curve, and suitable rational fractions ([2]). Our approach was inspired instead, by earlier H.Langes's work, as explained in [1].

We next focus on a natural stratification, loosely defined hereafter.

Given any genus-g *hyperelliptic cover* $\pi : (\Gamma, p, p'p'') \to X$, we have, on one hand the canonical Abel embedding of Γ into its *Jacobian*, $A_p : \Gamma \to Jac\,\Gamma$, and the flag $\{0\} \subsetneq V^1_{\Gamma,p} \dots \subsetneq V^g_{\Gamma,p} = H^1(\Gamma, O_\Gamma)$, of hyperosculating spaces to $A_p(\Gamma)$ at $A_p(p) \in Jac\,\Gamma$ (cf. [9]). On the other hand, we also have the homomorphism $\iota_\pi : X \to \text{Jac}\,\Gamma$, obtained by dualizing π: $\forall \omega \in X$, $\iota_\pi(\omega)$ is the class of the line bundle $O_\Gamma(\pi^*(\omega - \omega_o))$.

Definition 1.1.

For any hyperelliptic cover $\pi : (\Gamma, p, p'p'') \to X$*, there exists a smallest positive integer* d *such that the tangent line to* $\iota_\pi(X)$ *is contained in the* d*-dimensional hyperosculating space* $V^d_{\Gamma,p}$*. We call it the osculating order of* π*, and* π *a hyperelliptic* d*-osculating cover (see [9]).*

The $d = 1$ case, for example, occurs when the images of X and Γ are tangent at their common point. The latter, also called *hyperelliptic tangential covers*, have been extensively studied and make up a finite subset of $H(X, n)$ (e.g.: [11], [8] and all the references in both articles). It should also be emphasized that the set of *hyperelliptic tangential covers*, of fixed genus g but arbitrary degree, makes up a dense subset of H_g, the moduli space of genus-g smooth hyperelliptic curves (cf. [3]).

More numerical invariants are yet to be fixed in order to proceed. Besides the ramification index of the *hyperelliptic cover* π at p, say ρ, which must be odd, we will also look at the $2g + 1$ other (so-called Weierstrass) points, fixed by the hyperelliptic involution τ_Γ. Their distribution over each half-period of (X, ω_o), called *Weierstrass type* of π, is given by $(m_i) \in \mathbb{N}^4$, satisfying $m_o + 1 \equiv m_1 \equiv m_2 \equiv m_3 \equiv n(\text{mod}.2)$ and $\sum_i m_i = 2g + 1$. All these invariants are related as follows (cf. **2.3.** & **4.1.**, and [10] **4.3.**, **4.4.**):

$$\rho < 2d + 1 \le 2g + 1 \le 4n - \rho\,, \quad \sum_{i=o}^{3} m_i^2 \le 2(2d - 1)(n - 1) + 4 - \rho^2\,,$$

and if $\boldsymbol{p} > 2$

$$2g + 1 \le \boldsymbol{p}(2d - 1).$$

For given $n, d, g, \rho \in \mathbb{N}^*$ and $(m_i) \in \mathbb{N}^4$ as above, the latter make the sub-moduli space $H^\rho_{(m_i)}Os^d(X, n) \subset H(X, n)$. All together they give a stratification $H(X, n) = \cup_{(m_i)} \cup_{d,\rho} H^\rho_{(m_i)}Os^d(X, n)$.

Having identified $H(X, n)$ in terms of suitable rational fractions (**3.2.**), we obtain a polynomial characterization of each strata $H^\rho_{(m_i)}Os^d(X, n)$ (**4.6**). We immediately deduce a system of N polynomial equations in $N+\frac{1}{2}(2d-1-\rho)$ variables $\big(N := \frac{1}{2}(5n+4+m_1)\big)$, whose solutions parameterize $H^{\rho}_{(m_i)}Os^d(X,n)$.

This fits with already known results, e.g.: $H^1_{(m_i)}Os^1(X,n)$ is finite ([11]) and, for any $1 < d \leq g$, all known irreducible components of $H^1_{(m_i)}Os^d(X,n)$ have dimension $d-1$ (cf. [9]).

At last, we test our approach on $H^1_{(2,1,1,1)}Os^1(X,3)$, known to be, either empty (if $\boldsymbol{p} = 3$) or else, to contain a unique element (for a generic elliptic curve (X,ω_o)). Maple solving the latter system gives indeed a unique solution (if $\boldsymbol{p} \neq 3$), which we actually construct in the Appendix. We also check, in the complex case, that it is none other than the spectral curve, found by B. Dubrovin & S. Novikov, and giving rise to the first examples of non-stationary KdV solutions, doubly-periodic in x.

2 Hyperelliptic covers - General properties

Let $\mathbb{P}^1$ and X denote, respectively, the projective line and a fixed smooth projective curve of genus 1, both defined over $\mathbb{K}$. Choosing an arbitrary point $\omega_o \in X$ as its origin, the pair (X,ω_o) becomes an elliptic curve, having an inverse homomorphism $[-1] : X \to X$ fixing ω_o, as well as three other half-periods, $\{\omega_1,\omega_2,\omega_3\} \subset X$.

The quotient curve is isomorphic to $\mathbb{P}^1$, and $\varphi_X : X \to \mathbb{P}^1$ will denote the corresponding degree-2 projection, sending the triplet $(\omega_o,\omega_1,\omega_2)$ onto $\{\infty,0,1\} \subset \mathbb{P}^1$.

The remaining half-period projects onto $\lambda := \varphi_X(\omega_3) \neq \infty, 0, 1$.

The projection φ_X is classically represented, in affine coordinates, as follows.

The equation $y^2 = x(x-1)(x-\lambda)$ defines a smooth affine plane cubic, which can be compactified inside $\mathbb{P}^1 \times \mathbb{P}^1$, by adding the unibranch singular point (∞,∞). Up to desingularizing it at (∞,∞), the resulting curve is isomorphic to (X,ω_o), with $(\omega_o,\omega_1,\omega_2)$ and $\big((\infty,\infty),(0,0),(1,0)\big)$ identified, and φ_X represented as follows:

$$\big\{(x,y) \in \mathbb{K}^2,\, y^2 = x(x-1)(x-\lambda)\big\} \longrightarrow \mathbb{K}, \quad (x,y) \in \mathbb{K}^2 \mapsto x, \quad \omega_o \mapsto \infty.$$

Definition 2.1.

1. *We will call hyperelliptic any projective curve of genus $g \geq 2$, having an involution $\tau_\Gamma : \Gamma \to \Gamma$, such that the quotient curve Γ/τ_Γ is isomorphic to $\mathbb{P}^1$. The corresponding degree-2 projection $\varphi_\Gamma : \Gamma \to \Gamma/\tau_\Gamma \approx \mathbb{P}^1$ is ramified at the $2g+2$ fixed points of τ_Γ. The latter make up the subset $W_\Gamma \subset \Gamma$ of (so-called) Weierstrass points.*

2. *We obtain a (so-called Rosenheim) affine equation for φ_Γ as follows: choose a triplet of Weierstrass points (p,p',p'') and identify Γ/τ_Γ with*

$\mathbb{P}^1$, by projecting (p,p',p'') onto $(\infty,0,1)$. The equation $v^2=t(t-1)\Pi_j(t-\alpha_j)$, where $\{\alpha_j\}$ are the projections of the remaining $2g-3$ Weierstrass points, defines an affine curve which can be compactified inside $\mathbb{P}^1\times\mathbb{P}^1$, by adding the unibranch singular point (∞,∞). Up to desingularizing it at (∞,∞), the resulting curve is isomorphic to Γ, with (p,p',p'') and $\big((\infty,\infty),(0,0),(1,0)\big)$ identified, and φ_Γ represented as follows:

$$\big\{(t,v)\in\mathbb{K}^2,\, v^2=t(t-1)\Pi_j(t-\alpha_j)\big\}\longrightarrow\mathbb{K}\quad (t,v)\mapsto t,\quad p\mapsto\infty.$$

3. *We will call hyperelliptic cover any finite separable triply marked morphism $\pi:(\Gamma,p,p',p'')\to X$, such that Γ is a hyperelliptic curve and $p,p',p''\in W_\Gamma$. For any $n\geq 1$, H(X,n) will denote the moduli space of degree-n hyperelliptic covers over X.*

Proposition 2.2.

Any hyperelliptic cover $\pi:(\Gamma,p,p',p'')\to X$ satisfies $[-1]\circ\pi=\pi\circ\tau_\Gamma$, and can be pushed down to a unique separable morphism $R:\infty\in\mathbb{P}^1\to\infty\in\mathbb{P}^1$, fitting in the following commutative diagram:

$$\begin{array}{ccccc}
p\in\Gamma & \xrightarrow{\tau_\Gamma} & p\in\Gamma & \xrightarrow{\varphi_\Gamma} & \infty\in\mathbb{P}^1\\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle R}\\
\omega_o\in X & \xrightarrow{[-1]} & \omega_o\in X & \xrightarrow{\varphi_X} & \infty\in\mathbb{P}^1
\end{array}$$

Proof. Let us first mention that there is a unique isomorphism between Γ/τ_Γ and $\mathbb{P}^1$, sending $\varphi_\Gamma\big((p,p',p'')\big)$ to $(\infty,0,1)$. Recall now that for all $\omega\in X$, its inverse with respect to the group structure of (X,ω_o), denoted $[-1](\omega)$, is the unique point such that the divisors $[-1](\omega)-\omega_o$ and $\omega_o-\omega$ are linearly equivalent. Recall also that for any $r\in\Gamma$ the divisor $r+\tau_\Gamma(r)$ is linearly equivalent to $2p$. Hence, $\pi(r)+\pi\big(\tau_\Gamma(r)\big)$ is linearly equivalent to $2\omega_o$, implying that for all $r\in\Gamma$, $\pi\big(\tau_\Gamma(r)\big)=[-1]\big(\pi(r)\big)$ as asserted. It follows from classical results that π can be pushed down to a canonical (separable) morphism between the quotients, defining the unique morphism $R:\mathbb{P}^1\to\mathbb{P}^1$. We can also construct $R(\alpha)$, for any $\alpha\in\Gamma/\tau_\Gamma$, as (the unique point in) $\varphi_X\big(\pi\big(\varphi_\Gamma^{-1}(\alpha)\big)\big)$. ■

Corollary 2.3.

Let $\pi:(\Gamma,p,p',p'')\to X$ be a degree-n hyperelliptic cover, of genus g and ramification index ρ at p. Let W_Γ denote its set of Weierstrass points, and for any $i=0,\dots,3$, let $m_{\pi,i}$ denote the number of Weierstrass points, other than *p, lying over the half-period ω_i. Then:*

1. *π has odd ramification index at any $w \in W_\Gamma$ (hence, ρ is odd);*

2. *the ramification divisor Ram_π is τ_Γ-invariant and has degree $2g - 2$;*

3. *$\pi(W_\Gamma) \subset \{\omega_i\}$ and $m_{\pi,0} + 1 \equiv m_{\pi,1} \equiv m_{\pi,2} \equiv m_{\pi,3} \equiv n(\text{mod}.2)$;*

4. *the genus and degree of π satisfy* $\rho < 2g + 1 = \sum_{i=o}^{3} m_i \leq 4n - \rho$.

Proof.

(1) Choose local coordinates λ and z at $r \in \Gamma$ and $\pi(r) \in X$ respectively, such that $\tau_\Gamma^*(\lambda) = -\lambda$ and $[-1]^*(z) = -z$, and let $s := ind_\pi(r)$. Then, there exists a non-zero constant c such that $\pi^*(z) = c\lambda^s\big(1 + O(\lambda)\big)$. Applying the latter involutions, which satisfy the equality $\pi \circ \tau_\Gamma = [-1] \circ \pi$, we obtain

$$c(-1)^s\lambda\big(1 + O(\lambda)\big) = \tau_\Gamma^* \circ \pi^*(z) = \pi^* \circ [-1]^*(z) = \pi^*(-z) = -c\lambda\big(1 + O(\lambda)\big).$$

Hence, $(-1)^s = -1$ and s is odd.

2) Its τ_Γ-invariance stems directly from the equality $[-1] \circ \pi = \pi \circ \tau_\Gamma$.

3) For any $r \in W_\Gamma$, the divisors $2r$ and $2p$ are linearly equivalent. Hence $2\pi(r) - 2\omega_o$ is linearly equivalent to 0. Therefore $\pi(r) \in \{\omega_i, i = 0, \cdots, 3\}$ as asserted. Each fiber $\pi^{-1}(\omega_i)$ being τ_Γ-invariant, its subset of non-Weierstrass points is made of pairs of points. Thus $n - m_{\pi,i} - \rho\delta_{i,0} \equiv n - m_{\pi,i} - \delta_{i,0} \equiv 0(\text{mod}.2)$.

4)Recall first that W_Γ has cardinal $\sharp W_\Gamma = 2g + 2$; hence $\sum_i m_i = 2g + 1$. On the other hand, we know that $\rho - 1 \leq deg(Ram_\pi) = 2g - 2 = -3 + \sum_{i=o}^{3} m_{\pi,i}$, as well as $m_{\pi,i} + \rho\delta_{i,0} \leq n$, for any $i = 0, \cdots, 3$. Hence $\rho \leq 2g - 1 \leq 4n - \rho$. ∎

3 Polynomial approach to hyperelliptic covers

Given the elliptic curve (X, ω_o) and the degree-2 cover $\varphi_X : X \to \mathbb{P}^1$, we have associated in **2.2.**, to any $\pi \in H(X, n)$, a particular rational fraction $R = \frac{P}{Q}$.

Conversely, we have the following result.

Proposition 3.1.

Given any degree-n separable morphism $R : \infty \in \mathbb{P}^1 \to \infty \in \mathbb{P}^1$, with odd ramification indices at $(\infty, 0, 1)$, and such that $R(0), R(1) \in \{\infty, 0, 1, \lambda\}$, there exists a unique $\pi \in H(X,n)$, satisfying $R \circ \varphi_\Gamma = \varphi_X \circ \pi$.

Sketch of proof. Choosing a projection $R : \infty \in \mathbb{P}^1 \to \infty \in \mathbb{P}^1$ as above, is equivalent to choosing a rational fraction $R(t) = \frac{P(t)}{Q(t)}$, such that $degP - degQ$ is an odd positive integer, and the product $PQ(P-Q)(P-\lambda Q)$ has odd vanishing multiplicity at $\{0,1\}$. Replacing the variable x by the rational fraction $R(t)$ in the equation $y^2 = x(x-1)(x-\lambda)$, multiplying it by $Q(t)^4$ and making the birational change of variable $w = yQ(t)^2$, gives the affine equation of the fiber product of $R : \mathbb{P}^1 \to \mathbb{P}^1$ with $\varphi_X : X \to \mathbb{P}^1$, i.e.: $w^2 = P(t)Q(t)\big(P(t)-Q(t)\big)\big(P(t)-\lambda Q(t)\big)$. The corresponding completion in $\mathbb{P}^1 \times \mathbb{P}^1$, say Γ, comes with a degree-2 cover $\varphi_\Gamma : (t,w) \in \Gamma \mapsto t \in \mathbb{P}^1$, ramified at the triplet $(p,p',p'') = \big((\infty,\infty),(0,0),(1,0)\big)$, as well as the projection $\pi : (t,w) \in \Gamma \mapsto (x,y) = (R(t), \frac{w}{Q(t)^2}) \in X$. The corresponding involution $\tau_\Gamma : (t,w) \mapsto (t,-w)$, fixes the triplet (p,p',p'') of unibranch points, and $\varphi_\Gamma\big((p,p',p'')\big) = (\infty,0,1)$. Hence, up to desingularizing Γ, we obtain the unique *hyperelliptic cover*, $\pi : (\Gamma,p,p',p'') \to X$, fitting in a commutative diagram as above, and such that $\varphi_\Gamma\big((p,p',p'')\big) = (\infty,0,1)$. ■

Corollary 3.2.

The moduli space $H(X,n)$ is canonically identified with the variety made of all pairs of polynomials (P,Q), satisfying the following conditions:

1. *P and Q are coprime and Q is monic;*

2. *$degP = n$ and $degP - degQ$ is an odd positive integer;*

3. *the product $PQ(P-Q)(P-\lambda Q)$ has odd vanishing order at $\{0,1\}$.*

Remark 3.3.

1. Given such a pair (P,Q), the product $P(t)Q(t)\big(P(t)-Q(t)\big)\big(P(t)-\lambda Q(t)\big)$ can be uniquely factored as $t(t-1)A(t)B(t)^2$, where $B(t)$ is monic, A has odd degree and $t(t-1)A(t)$ has no multiple root. It follows that the affine curve $\{(t,v) \in \mathbb{K}^2, v^2 = t(t-1)A(t)\}$, completed as explained in **2.1.**(2), and equipped with the projection $(t,v) \mapsto (x,y) := \big(\frac{P(t)}{Q(t)}, \frac{vB(t)}{Q(t)^2}\big)$, gives the smooth *hyperelliptic cover* of X, uniquely associated to (P,Q).

2. According to **2.3.**, π has odd ramification index at the $2g+2$ *Weierstrass points*, while Ram_π has degree $2g-2$. Hence, there must be at least $g+3$ ones with $ind_\pi = 1$. In particular, we may choose the above triplet $(p,p',p'') \in W^3_\Gamma$ without ramification, or equivalently, restrict to pairs of

polynomials (P, Q) as above, such that $degQ = n - 1$ and 0 and 1 are simple roots of $P(t)Q(t)\big(P(t) - Q(t)\big)\big(P(t) - \lambda Q(t)\big)$.

Working locally with the corresponding equations one can deduce the ramification divisor Ram_π, out of Ram_R, even when π and R have wild ramification. In particular, for any $r \in \Gamma$, $ind_\pi(r)$ and $ind_R(\varphi_\Gamma)$ are related as follows.

Lemma 3.4.

Let $\pi : (\Gamma, p, p', p'') \to X$ be the hyperelliptic cover associated to the projection $R : \infty \in \mathbb{P}^1 \to \infty \in \mathbb{P}^1$, and $\varphi_\Gamma : p \in \Gamma \to \infty \in \mathbb{P}^1$ the corresponding degree-2 projection. Then, for any $\alpha \in \mathbb{P}^1$:

1. *if $R(\alpha) \notin \{0, 1, \lambda, \infty\}$, the fiber $\varphi_\Gamma^{-1}(\alpha)$ has two points, say $r \neq \tau_\Gamma(r) \in \Gamma$, and $ind_\pi(r) = ind_\pi\big(\tau_\Gamma(r)\big) = ind_R(\alpha)$;*

2. *if $R(\alpha) \in \{0, 1, \lambda, \infty\}$ and $ind_R(\alpha)$ is even, the fiber $\varphi_R^{-1}(\alpha)$ has two points, say $r \neq \tau_\Gamma(r) \in \Gamma$, and $ind_\pi(r) = ind_\pi\big(\tau_\Gamma(r)\big) = \frac{1}{2} ind_R(\alpha)$;*

3. *if $R(\alpha) \in \{0, 1, \lambda, \infty\}$ and $ind_R(\alpha)$ is odd, there is a unique (Weierstrass) point in $\varphi_R^{-1}(\alpha)$, say $r = \tau_\Gamma(r) \in W_\Gamma$, and $ind_\pi(r) = ind_R(\alpha)$.*

Definition 3.5.

Given $\pi : (\Gamma, p, p', p'') \to X \in H(X, n)$ and $i = 0, \cdots, 3$, we let $m_{\pi,i}$ denote the number of Weierstrass points, other than p, projecting onto ω_i. We will call $(m_{\pi,i})$ the Weierstrass type of π.

We gather herafter basic relations and (in-)equalities between all data associated to π (cf. **2.2.** & **2.3.**, and [10] **4.3.**, **4.4.**).

Proposition 3.6.

Let $\pi : (\Gamma, p, p', p'') \to X$ be a degree-n hyperelliptic cover of genus g, with Weierstrass type equal to $(m_{\pi,i})$, ramification index ρ at p and $\pi(p) = \omega_o$.

Let $R = \frac{P}{Q}$ denote its associated rational fraction (**3.1.**) *and consider the factorizations*

$$P = cA_1B_1^2,\ \ Q = A_oB_o^2,\ \ P - Q = cA_2B_2^2, \qquad P - \lambda Q = cA_3B_3^2\,,$$

where $c \in \mathbb{K}^*$, and $\forall i = 0, \cdots, 3$, A_i and B_i are monic, and A_i has simple roots. Then,

1. $degP = n$ *and* $\rho = degP - degQ$;

2. $\Gamma \setminus \{p\}$ *is isomorphic to the affine curve* $\{(t,v) \in \mathbb{K}^2,\, v^2 = c^3 \Pi_i A_i(t)\}$;

3. *outside* $\{\pi^{-1}(\omega_o)\}$, *the projection* $\pi : \Gamma \to X$ *is isomorphic to*

   $(t,v) \mapsto (x,y) := \big(\frac{P(t)}{Q(t)}, v\frac{\Pi_i B_i(t)}{Q(t)^2}\big)$;

4. $(m_{\pi,i}) = (degA_i)$, *hence* $m_{\pi,i} + \delta_{o,i} \equiv n(mod.2)$ *for any* $i = 0, \cdots, 3$;

5. $2g + 1 = \sum_i degA_i = \sum_i m_i$ *and* $\sum_{i=o}^{3} m_i^2 \leq 2(2d-1)(n-1) + 4 - \rho^2$.

4 Hyperelliptic covers and polynomial equations

We will consider hereafter a finer stratification of $H(X,n)$, obtained by fixing the ramification index $\rho := ind_\pi(p)$, the *osculating order* d, as well as the *Weierstrass type* instead of the genus g, and satisfying the appropriate relations (cf. **2.3.** & **4.1.**, and [10] **4.3.**, **4.4.**).

Definition 4.1.

For any $n, g, d, \rho \in \mathbb{N}^*$ *and* $(m_i) \in \mathbb{N}^4$ *such that*

1. ρ *is odd and* $\rho < 2d + 1 \leq 2g + 1 = \sum_{i=o}^{3} m_i \leq 4n - \rho$,

2. $m_0 + 1 \equiv m_1 \equiv m_2 \equiv m_3 \equiv n(mod.2)$,

3. $\sum_{i=o}^{3} m_i^2 \leq 2(2d-1)(n-1) + 4 - \rho^2$,

4. *and* $2g + 1 \leq \boldsymbol{p}(2d-1)$ *if* $\boldsymbol{p} := char(\mathbb{K}) > 2$,

we let $H^\rho_{(m_i)}Os^d(X,n) \subset H^\rho_g Os^d(X,n)$ *denote the moduli subspace made of those degree-n, hyperelliptic d-osculating covers, with Weierstrass type* (m_i) *such that* $\sum_i m_i = 2g+1$ *and ramification index* ρ *at the first marked point. Taking the union over all possible geni g, indices* ρ *and corresponding Weierstrass types, we obtain the following stratification:*

$$\cup_{\rho,d} \cup_{(m_i)} H^\rho_{(m_i)}Os^d(X,n) = \cup_{\rho,d} \cup_g H^\rho_g Os^d(X,n) = H(X,n).$$

Any $\pi \in H^{\rho}_{(m_i)}Os^d(X,n)$ will be equipped hereafter with its associated rational fraction $R(t) := \frac{P(t)}{Q(t)}$, such that $degP - degQ = \rho$, and canonical factorizations $P = cA_1B_1^2,\ Q = A_oB_o^2,\ P - Q = cA_2B_2^2,\ P - \lambda Q = cA_3B_3^2$ (cf. **3.6.**). In particular, the function $v = \frac{yQ^2}{\Pi_i B_i}$ has a pole of order $2g+1$ at p, and simple zeroes at any other Weierstrass point. We will also pick $z := \frac{P}{yQ}$, as local coordinate of X at its origin ω_o.

Recall at last that π is a *hyperelliptic d-osculating cover* if, and only if, there exists a *d-osculating function* for π, i.e.: a τ_Γ-anti-invariant projection $\kappa : \Gamma \to \mathbb{P}^1$, with poles only at $\pi^{-1}(\omega_o)$, such that $\kappa + \frac{1}{z}$ is holomorphic at $\pi^{-1}(\omega_o) \setminus \{p\}$ and has a pole of order $2d - 1$ at $p \in \Gamma$ ([9]).

The following Lemmas pave the way to a *polynomial d-osculating characterization*, and will be instrumental in parameterizing any strata $H^{\rho}_{(m_i)}Os^d(X,n)$ by the solutions of a system of polynomial equations.

Lemma 4.2.

The function $\kappa_o := \frac{v}{A_oB_o} : \Gamma \to \mathbb{P}^1$ is anti-τ_Γ-invariant (i.e.: $\kappa_o \circ \tau_\Gamma = -\kappa_o$), holomorphic outside $\pi^{-1}(\omega_o)$ and has a pole of same order as $\frac{1}{z} = \frac{yQ}{P}$, at any point of $\pi^{-1}(\omega_o) \setminus \{p\}$.

Proof

We just need to check that $\kappa_o z$ is holomorphic at $\pi^{-1}(\omega_o) \setminus \{p\}$, which follows from the equalities, $v = \frac{yQ^2}{\Pi_i B_i}$ and $\kappa_o z = \frac{v}{A_oB_o}\frac{P}{yQ} = \frac{P}{B_1B_2B_3}$. ■

Lemma 4.3.

Any anti-τ_Γ-invariant meromorphic function $\kappa : \Gamma \to \mathbb{P}^1$, holomorphic outside $\pi^{-1}(\omega_o)$, with a pole of same order as $\frac{1}{z} = \frac{yQ}{P}$ at any point of $\pi^{-1}(\omega_o) \setminus \{p\}$, is equal to $\frac{vM}{A_oB_o}$, for a unique polynomial $M(t)$.

Proof

The functions κ and $\frac{v}{A_0B_0}$ have a pole of the same order as $\frac{1}{z}$ at any point of $\pi^{-1}(\omega_o) \setminus \{p\}$. Thus, their quotient is holomorphic there. Outside $\pi^{-1}(\omega_o)$ they are holomorphic and anti-τ_Γ-invariant, hence vanish at any Weierstrass point. On the other hand, v vanishes to order 1 at any Weierstrass point other than p. Hence, their quotient, $\kappa\frac{A_0B_0}{v}$, is τ_Γ-invariant and can only have a pole at p. It follows that $\kappa\frac{A_0B_0}{v}$ is a polynomial, say $M(t)$, as asserted. ■

Lemma 4.4.

Let $\kappa : \Gamma \to \mathbb{P}^1$ *be* τ_Γ*-anti-invariant and holomorphic outside* $\pi^{-1}(\omega_o)$*. Then,* $\kappa + \frac{1}{z}$ *has a pole of order* $2d\text{-}1$ *at* $p \in \Gamma$*, and no other pole in* $\pi^{-1}(\omega_o)$*, if and only if there exists a polynomial* T*, such that* A_1B_1 *divides* $TA_0B_0 + B_2B_3$*. In the latter case,* $\kappa + \frac{1}{z} = \frac{vT}{A_1B_1}$ *and* $degT = \frac{1}{2}(n + degA_1 + 2d\text{-}2g\text{-}2)$*.*

Proof

In case $\kappa + \frac{1}{z}$ is holomorphic at $\pi^{-1}(\omega_o) \setminus \{p\}$, the meromorphic function κ satisfies the conditions of **Lemma 4.3.**. In particular, $\kappa = \frac{vM}{A_oB_o}$, for some polynomial M, implying

$$\kappa + \frac{1}{z} = \frac{vM}{A_oB_o} + \frac{yQ}{P} = \frac{vM}{A_oB_o} + \frac{vB_2B_3}{A_oB_oA_1B_1} = \frac{v}{A_1B_1}\frac{(MA_1B_1 + B_2B_3)}{A_0B_0}.$$

Moreover, A_0B_0 must divide $(MA_1B_1 + B_2B_3)$, in order for $\kappa + \frac{1}{z}$ to be holomorphic at $\pi^{-1}(\omega_o) \setminus \{p\}$. Let T denote their quotient. It follows that $MA_1B_1 = TA_0B_0\text{-}B_2B_3$, hence A_1B_1 divides $TA_0B_0\text{-}B_2B_3$, and $\kappa + \frac{1}{z} = \frac{vT}{A_1B_1}$ as asserted.

Furthermore, since $\kappa + \frac{1}{z}$ has a pole of order $2d\text{-}1$, the polynomial T has degree $\frac{1}{2}(n + degA_1 + 2d\text{-}2g\text{-}2)$.

Conversely, let T be such a polynomial and M satisfy $MA_1B_1 = B_2B_3 + TA_0B_0$. Then, the τ_Γ-anti-invariant function $\kappa := \frac{vM}{A_oB_o}$ has the required properties, i.e.: $\kappa + \frac{1}{z} = \frac{vT}{A_1B_1}$ has a pole of order $2d\text{-}1$ at p, and is holomorphic at any other point of $\pi^{-1}(\omega_o)$. In other words, κ is a *d-osculating function* for π. ■

Theorem 4.5. *(d-osculating polynomial criterion)*

A degree-n hyperelliptic cover π *, associated to* $c \in \mathbb{C}^*$ *and a set* $\{A_i, B_i\}$ *of polynomials as above, has osculating order d, i.e.:* $\pi \in H^\rho_{(m_i)}Os^d(X, n)$*, if and only if there exists a polynomial* T*, such that* A_1B_1 *divides* $TA_0B_0 + B_2B_3$*. In the latter case,* $\kappa + \frac{1}{z} = \frac{vT}{A_1B_1}$ *and* $degT = \frac{1}{2}(n + degA_1 + 2d\text{-}2g\text{-}2)$*.*

Proof

According to **4.4.** κ is a *d-osculating function* for $\pi \in H^\rho_{(m_i)}Os^d(X, n)$, if and only if, there exists a polynomial T satisfying the above conditions. ■

Theorem 4.6.

For any $n, d, g, \rho \in \mathbb{N}^*$ *and* $(m_i) \in \mathbb{N}^4$ *as in* ***4.1.****, there exists a polynomial system of* $N := \frac{1}{2}(5n + 4 + m_1)$ *equations, in an open dense subset of* $\mathbb{K}^{N+\frac{1}{2}(2d-1-\rho)}$*, such that its set of solutions parameterizes the strata* $H^\rho_{(m_i)}Os^d(X, n)$*.*

Proof. Consider $c \in \mathbb{K}^*$, two arbitrary sequences of monic polynomials, (A_i) and (B_i), such that $degA_i = m_i$ and $degB_i^2 = n\text{-}m_i\text{-}\rho\delta_{i,o}$ for any $i = 0, \cdots, 3$, and a polynomial T of degree $degT = \frac{1}{2}(n + degA_1 + 2d\text{-}2g\text{-}2)$. These data depend upon

$$1 + \sum_i deg(A_iB_i) + deg(T) + 1 = \frac{1}{2}(5n + 4 + degA_1 + 2d\text{-}1\text{-}\rho)$$

variables, and we ask them to satisfy the following set of $2n + 2 + deg(A_1B_1) = \frac{1}{2}(5n + 4 + m_1)$ equations:

$$cA_1B_1^2\text{-}A_oB_o^2 = cA_2B_2^2, \quad cA_1B_1^2\text{-}\lambda A_oB_o^2 = cA_3B_3^2\ ,$$

$$t(t\text{-}1) \quad \text{divides} \quad \Pi_iA_i \quad \text{and} \quad A_1B_1 \quad \text{divides} \quad TA_oB_o\text{-}B_2B_3\ .$$

Let $P := cA_1B_1^2,\ Q := A_oB_o^2$, and assume further the open conditions

$$c \neq 0\,, \quad disc(\Pi_iA_i) \neq 0 \quad \text{and} \quad resultant(P, Q) \neq 0\,.$$

The degree-n *hyperelliptic cover* π, associated to $R := \frac{P}{Q}$ is isomorphic (outside $\pi^{-1}(\omega_o)$), to

$$(t, v) \in \left\{v^2 = c^3\Pi_iA_i\right\} \quad \mapsto \quad \left(R(t), \frac{v\Pi_iB_i}{Q^2}\right) = (x, y) \in \left\{y^2 = x(x\text{-}1)(x\text{-}\lambda)\right\}.$$

Let also M denote the polynomial satisfying $MA_1B_1 = TA_0B_0\text{-}B_2B_3$. Then, $\kappa := \frac{vM}{A_oB_o}$ is a *hyperelliptic d-osculating function for* π ([9]). In other words, $\pi \in H^{\rho}_{(m_i)}Os^d(X, n)$. Conversely, let $\pi \in H^{\rho}_{(m_i)}Os^d(X, n)$, and consider its associated rational fraction $R := \frac{P}{Q}$, together with the canonical factorizations $P = cA_1B_1^2$, $Q = A_oB_o^2$, $P\text{-}Q = cA_2B_2^2$ and $P\text{-}\lambda Q = cA_3B_3^2$. Recall that for any $i = 0, \cdots, 3$, A_i has simple roots, and A_i and B_i are monic, of degrees m_i and $\frac{1}{2}(n\text{-}m_i\text{-}\rho\delta_{i,o})$, respectively. Needless to say that they satisfy the equations

$$cA_1B_1^2\text{-}A_oB_o^2 = cA_2B_2^2, \quad cA_1B_1^2\text{-}\lambda A_oB_o^2 = cA_3B_3^2$$

and

$$t(t\text{-}1) \quad \text{divides} \quad \Pi_iA_i\,.$$

Moreover, there must exist a polynomial T, such that

$$A_1B_1 \quad \text{divides} \quad TA_oB_o\text{-}B_2B_3 \quad \text{and} \quad degT = \frac{1}{2}(n + degA_1 + 2d\text{-}2g\text{-}2),$$

implying that $\kappa := \frac{vT}{A_1B_1}\text{-}\frac{yQ}{P}$ is a *d-osculating function* for π. In other words, any equivalence class $\pi \in H^{\rho}_{(m_i)}Os^d(X, n))$ corresponds to a unique solution of the latter systems of equations (and open conditions). ■

Appendix

Every complex elliptic curve (X, ω_o) is isomorphic to a torus $(\mathbb{C}/\Lambda, 0)$, for some lattice $\Lambda \subset \mathbb{C}$. The corresponding Weierstrass $\wp$ function is Λ-periodic and satisfies the differential equation $\wp'^2 = 4\Pi_{j=1}^3(\wp - e_j)$, with $\sum_{j=1}^3 e_j = 0$. In particular, $u(x) = 2\wp(x)$ is a stationnary, Λ-periodic, KdV solution.

In 1974, B.Dubrovin & S.Novikov found the first examples of non-stationnary, Λ-periodic, KdV solutions (cf. Dokl.Akad.Nauk. SSSR, **15**, 1597 - 1601), of the form $u(x,t) = \sum_{i=1}^3 2\wp(x - \alpha_i(t))$. The associated spectral curve can be presented as a degree-3 *hyperelliptic tangential cover* of genus 2, and must have Weierstrass type $(2,1,1,1)$; hence, its equivalence class must belong to $H_{(2,1,1,1)}Os^1(X,3)$, on which we will test our approach.

More generally, we take the equation $\{y^2 = 4\Pi_{j=1}^3(x - e_j)\}$, with $\sum_j e_j = 0$, defined over an algebraically closed field $\mathbb{K}$ of characteristic $\boldsymbol{p} \neq 2, 3$, and let (X, ω_o) denote the corresponding elliptic curve. We then consider $\varphi_X : X \to \mathbb{P}^1$ and $R : \mathbb{P}^1 \to \mathbb{P}^1$, the degree-2 and degree-3 projections $(x,y) \mapsto x$, and $t \mapsto R(t) := \frac{t^3+g_3}{3t^2-g_2}$, where $g_2 := -4(e_1e_2 + e_1e_3 + e_2e_3)$ and $g_3 := 4e_1e_2e_3$. Their fiber product gives rise to a degree-3 projection $\pi : \Gamma \to X$, on which we choose a triplet (p, p', p'') satisfying $\varphi_\Gamma(p) = \infty \in \mathbb{P}^1$ and $\pi^{-1}(\omega_o) = \{p, p', p''\}$. Then:

Proposition A.1.

$H_{(2,1,1,1)}Os^1(X,3)$ is either empty, if $g_2 = 0$, or else, it contains as unique element, the above triply marked cover $\pi : (\Gamma, p, p', p'') \to X$. In case $\mathbb{K} = \mathbb{C}$, the latter is the spectral curve found by Dubrovin & Novikov..

Proof.
The canonical factorizations

$$P = A_1B_1^2, \quad Q = A_0B_0^2, \quad P - Q = A_2B_2^2 \quad \text{and} \quad P - \lambda Q = A_3B_3^2$$

are given as follows:

$$3A_0 = 3t^2 - g_2, \;\; B_0 = 1, \quad \text{and} \quad A_j = t + e_j, \;\; B_j = t - 2e_j \quad \text{for any} \quad j = 1,2,3.$$

In particular, $\Gamma \setminus \{p\}$ is isomorphic to $\{v^2 = (3t^2 - g_2)\Pi_{j=1}^3(t + e_j)\}$, hence isomorphic to Dubrovin & Novikov's one if $\mathbb{K} = \mathbb{C}$. Moreover, $3A_0B_0 = 3t^2 - g_2$, while $A_1B_1 = t^2 - e_1t - 2e_1^2$ and $B_2B_3 = t^2 + 2e_1t + e_2e_3$. It easily follows that $3A_0B_0 = -2A_1B_1 - B_2B_3$. Hence $\pi \in H_{(2,1,1,1)}Os^1(X,3)$ as asserted, because the tangency criterion **4.3.** is satisfied. ■

References

[1] Accola R., Previato E., *Covers of Tori: Genus 2*, Letters in Mathematical Physics, **76**, Numbers 2-3, May 2006 , 135-161(27).

[2] Bogatyrev A.B., *Chebyshev representation for rational functions*, Sbornik Math., **201**:11, May 2010, 1579-1598

[3] Colombo, E., Pirola, G.P., Previato, E., *Density of elliptic solitons*, J. Reine Angew. Math. **451** (1994), 161-169.

[4] Kani E., *The number of curves of genus 2 with elliptic differentials*, J. Reine Angew. Math. **485** (1997) 93-121.

[5] Kani E., *Hurwitz spaces of genus 2 covers of an elliptic curve*, Collect. Math. **54** (2003), 1-51.

[6] Kani E., *The number of genus-*2 *covers of an elliptic curve*, Man. Math. **121** (2006), 51-80.

[7] Shaska T., *Curves of genus 2 with (N,N) decomposable Jacobians*, J. Symb. Comp., **31:5** (2001), 603-617.

[8] Treibich A., *On hyperelliptic tangential covers and elliptic finite-gap potentials*, Uspekhi Mat. Nauk, **56**:6 (2001), 89-136.

[9] Treibich A., *Revêtements hyperelliptiques d-osculateurs et solitons elliptiques de la hiérarchie KdV*, C.R.A.S. Paris, Série I, **345** (2007), 213-218.

[10] Treibich A., *Hyperelliptic d-osculating covers and rational surfaces*, (http://premat.fing.edu.uy/papers/2012/144.pdf).

[11] Treibich A. & Verdier J.-L., *Solitons Elliptiques*; Progress in Math., **88**. (app.by J.Oesterlé); The Grothendieck Festschrift, Volume III, Ed. Birkhäuser (1990), pp.437-479.

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