

ON THE WORK OF JORGE LEWOWICZ ON EXPANSIVE SYSTEMS

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ABSTRACT. We will try to give an overview of one of the landmark results of Jorge Lewowicz: his classification of expansive homeomorphisms of surfaces. The goal will be to present the main ideas with the hope of giving evidence of the deep and beautiful contributions he made to dynamical systems. We will avoid being technical and try to concentrate on the tools introduced by Lewowicz to obtain these classification results such as Lyapunov functions and the concept of persistence for dynamical systems. The main contribution that we will try to focus on is his conceptual framework and approach to mathematics reflected by the previously mentioned tools and fundamentally by the delicate interaction between topology and dynamics of expansive homeomorphisms of surfaces he discovered in order to establish his result.

The value of a person resides in his major contribution
Arab proverb freely translated¹.

1. INTRODUCTION

Among the contributions of Lewowicz to mathematics, it is hard to ignore what I believe to be his major one: The creation of a school of dynamical systems in Montevideo. This school is also highly influenced by his way of looking at mathematics which I hope will be illustrated in this brief note. The main point is that it is not only the people who work in expansive systems that has been influenced by him.

As a disclaimer, I mention that I am by far not the best qualified to write about Lewowicz's work and that this note does not pretend to be a summary of all of his contributions to mathematics. However, as a member of the above mentioned school, and having been strongly influenced by him, I happily accepted this task and will try to give a panorama of the results of Lewowicz concerning expansive homeomorphisms. I recommend reading [Sam] for a global panorama of Lewowicz's contributions.

Let us start with a simple and elementary definition:

Definition (Expansive homeomorphism). Let $f : M \rightarrow M$ be a homeomorphism of a compact metric space M . We say that f is *expansive* if there exists $\alpha > 0$ such that given $x \neq y \in M$ there exists $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) \geq \alpha$. The largest possible constant α is called the *expansivity constant* of f for the metric d .

Expanded version of a talk given by the author in the conference Dynamical Systems in Montevideo held at Montevideo from 13 to 17 of August 2012. The author was partially supported by CSIC group 618/2010.

¹Translated from a phrase in the entrance of the Institut du Monde Arab in Paris , France.



It is important to remark that although the definition depends on the metric, the notion of expansivity is purely topological and can be stated for general topological spaces by demanding that points outside the diagonal $\Delta \subset M \times M$ escape by iteration of $f \times f$ from a fixed neighborhood² of Δ .

There are many well known examples of expansive homeomorphisms: for example subshifts of finite type (as well as hyperbolic sets of Smale's theory) are expansive; and the dynamics restricted to the minimal set of a Denjoy counter-example is also expansive. We will focus mainly on other kind of examples, those whose phase space is a manifold.

Quoting Lewowicz himself in [L₃]

(...) expansivity means, from the topological point of view, that any point of the space M has a distinctive dynamical behavior. Therefore, a stronger interaction between the topology of M and the dynamics could be expected.

Examples of expansive homeomorphisms on manifolds are given by Anosov and quasi-Anosov diffeomorphisms (see [Fr, FR]) as well as the well known pseudo-Anosov maps introduced by Thurston ([Th]). Of course, products of expansive homeomorphisms are expansive. In [OR] it is proved that every surface of positive genus admits an expansive homeomorphism.

We are now ready to state a landmark result of the work of Lewowicz ([L₃):

Theorem. *There are no expansive homeomorphisms on the two-dimensional sphere S^2 .*

This theorem is highly non-trivial, yet, its statement is completely simple. Let us remark that there is an independent proof of this result and the rest of the results in [L₃] by Hiraide ([H]).

The concrete purpose of this note is to explain the main ideas behind this result as well as the classification theorem of expansive homeomorphisms on surfaces obtained by Lewowicz in [L₃]. Other contributions will be covered by Ruggiero, specially those concerning geodesic flows and quotient dynamics (see also [Ru]), of course, both presentations will have substantial overlap.

It would not be faire to write about Lewowicz's work without giving motivations for the study of expansive homeomorphisms. We will review some motivations in the first sections by reviewing some of Lewowicz's previous work. Other motivations can be found along the literature, in particular [L₄] has a chapter devoted to that.

2. LYAPUNOV FUNCTIONS AND TOPOLOGICAL STABILITY

We start with a quotation from the introduction of [L₁] “*This paper contains some results on topological stability (see [2,3]) that generalize those obtained in [2] much in the*

²In more technical wording, that Δ is a locally maximal set for $f \times f$.

same way as Lyapunov's direct theorem generalizes the asymptotic stability results of the hyperbolic case: if at a critical point, the linear part of a vector field has proper values with negative real parts, the point is asymptotically stable and the vector field has a quadratic Lyapunov function; however, asymptotic stability may also be proved for vector fields with non-hyperbolic linear approximations, provided they have a Lyapunov function. In a way this is what we do here, letting Anosov diffeomorphisms play the role of the hyperbolic critical point and replacing stability by topological stability; we get this time a class of topological stable diffeomorphisms wider than the class of Anosov diffeomorphisms."

Lyapunov functions, introduced by Lewowicz in [L₁] play the role of a metric, which in the case of expansive homeomorphisms is a type of adapted metric which allows to distinguish the stable and unstable parts of the points which are nearby a given orbit. Other kinds of adapted metrics have been then proposed (see [Re, Fa]) but we will focus on Lyapunov functions that are present transversally in much of Lewowicz's work (and also in some of his students, see [Gr₁, Gr₂]). One important tool introduced in order to construct Lyapunov functions is that of quadratic forms (or infinitesimal Lyapunov functions) which have had a strong impact in different directions well beyond expansive systems as we will explain below.

Definition (Lyapunov function). A continuous function $V : U \rightarrow \mathbb{R}$ from a neighborhood U of the diagonal in $M \times M$ is said to be a Lyapunov function for $f : M \rightarrow M$ iff:

- $V(x, x) = 0$ for every $x \in M$.
- $V(f(x), f(y)) - V(x, y) > 0$ for every $x \neq y$.

◇

It can be seen as a function which "sees" the expansivity in one step. It is proved in [L₃] (Theorem 1.3) that these functions characterize expansive homeomorphisms (see [Fa] for a different approach):

Theorem 2.1. *A homeomorphism of a compact metric space is expansive if and only if it admits a Lyapunov function.*

Lyapunov functions also provide a way of establishing topological stability of diffeomorphisms (see [Wa] for the Anosov case) which may not be Anosov.

We recall the definition of topological stability. We say that a homeomorphism $f : M \rightarrow M$ of a compact manifold M is *topologically stable* if there exists $\varepsilon > 0$ such that for every homeomorphism $g : M \rightarrow M$ which is at C^0 -distance smaller than ε of f there exists a continuous surjective map $h : M \rightarrow M$ which semiconjugates f and g , that is:

$$f \circ h = h \circ g$$

Thurston’s pseudo-Anosov maps (see [OR, Th]) do admit Lyapunov functions (see [L₂] Lemma 3.4 or apply the previous theorem³), however, they are not topologically stable: One can make perturbations of a pseudo-Anosov map making that some points have their orbit going “across” the singularities and which will not be shadowed by no orbit of the pseudo-Anosov map. See [L₂] or look at the figures in [L₄] (Figure 2 in page 11) or [LC₁] (Figure 3).

Therefore, the existence of Lyapunov functions alone is not enough to get topological stability. One must add a new hypothesis which can be thought of as a weak topological version of hyperbolicity (see [L₁] Section 5 for a more general and precise definition):

Definition. We say that a Lyapunov function $V : U \rightarrow \mathbb{R}$ is *non-degenerate* if for every $x \in M$ there exists a splitting $T_x M = S_x \oplus U_x$ such that if $C_S(x)$ (resp. $C_U(x)$) is a cone around S_x (resp. U_x) then $V(\cdot, x)$ is positive (resp. negative) in $\hat{C}_S(x)$ (resp. $\hat{C}_U(x)$), the projection of $C_S(x)$ (resp. $C_U(x)$) by the exponential map in a small neighborhood.

◇

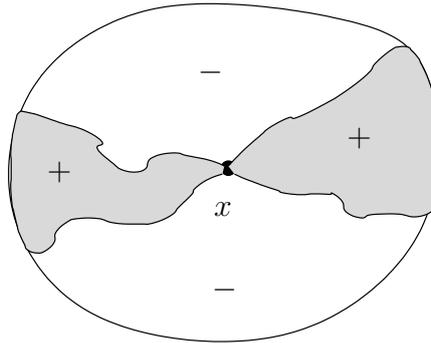


FIGURE 1. Positive and negative regions of $V(\cdot, x)$ when V is non-degenerate in dimension 2.

In a nutshell, the requirement is that the positive and negative regions of the Lyapunov function in a neighborhood of a point resemble topologically to the positive and negative part of a quadratic function (see ⁴ Figure 1). This is exactly what forbids pseudo-Anosov maps to have non-degenerate Lyapunov functions in neighborhood of their singularities.

The following theorem is the main theorem of [L₁]:

Theorem 2.2. *Let f be a C^1 -diffeomorphism with a non-degenerate Lyapunov function, then f is topologically stable.*

In [L₁] a characterization of Anosov diffeomorphisms in terms of quadratic forms is also given. With this approach Lewowicz is able to recover classical results on structural

³In fact Lewowicz uses his construction of a Lyapunov function to obtain an alternative proof of expansiveness of pseudo-Anosov maps.

⁴The lines in my drawings are all crooked on purpose in order to show the topological nature of the objects, :).

stability of Anosov and characterization of Anosov systems in terms of cone-families. Quadratic forms turn out to be, in some applications, better suited for the study of the tangent map dynamics than cone-fields as we will try to explain in the next subsection.

2.1. Quadratic forms, Lyapunov functions and Pesin's theory. In [L₁] the following example of diffeomorphism of \mathbb{T}^2 which is not Anosov and yet admits a non-degenerate Lyapunov function is proposed:

$$F_c(x, y) = \left(2x - \frac{c}{2\pi} \sin(2\pi x) + y, x - \frac{c}{2\pi} \sin(2\pi x) + y \right)$$

For $c < 1$ the diffeomorphism is Anosov (being linear for $c = 0$), for $c = 1$ however, there is no invariant splitting by the differential in the tangent space of the fixed point $(0, 0)$. However, it can be proved that F_1 admits a non-degenerate Lyapunov function, it is volume preserving and also ergodic.

In [CE] it is proven that F_1 as well as many other examples in the boundary of Anosov diffeomorphisms of \mathbb{T}^2 are ergodic and non-uniformly hyperbolic. The proof of non-uniform hyperbolicity relies on the existence of certain quadratic forms which instead of verifying that their first difference is everywhere positive, they verify this almost-everywhere extending the results of [L₁]. The following result was stated without a complete proof in [LL] and the proof was completed in the appendix of [Mar₁]:

Theorem 2.3. *Let f be a volume preserving diffeomorphism admitting a continuous quadratic form $B : TM \rightarrow \mathbb{R}$ such that the quadratic form $f^\sharp(B) - B$ is definitely positive almost everywhere. Then, f is non-uniformly hyperbolic, i.e. Lyapunov exponents are almost-everywhere non-vanishing.*

Here, we denote $f^\sharp(B)_x(v) = B_{f(x)}(D_x f v)$.

This Theorem was later extended in [Mar₂, K] and is quite related to a cone-criterium ([Wo]) but works better in some situations (see [Mar₁, K] and references therein). We will not enter into details about these important results, but we refer the reader to [CM] for further developments and applications to billiard systems. Let us just mention that in the spirit of Lewowicz phrase in the introduction of [L₁] and quoted above, the work of [Mar₂] proves a reciprocal statement to the quadratic form criterium, giving a parallelism with Lyapunov method and Massera's theorem (an important mentor for Lewowicz), see [Mas].

Let us close this section by mentioning a problem which Lewowicz has always insisted on:

Question (Problem 10.3 of [LC₂]). *For $c > 1$ does the Pesin region of F_c have positive measure?*

The latter is a typical coexistence question which has always interested many mathematicians. The maps F_c proposed by Lewowicz are similar to those of [Pry] (see also [Li]).

See also the work of Pesin ([Pes]) on the coexistence problem which is one of the central problems in dynamics.

3. PERSISTENCE

3.1. Persistence vs Topological stability. The concept of persistency was introduced by Lewowicz in [L₂] in order to study some robust properties of certain expansive homeomorphisms under perturbations. In a certain way, it is a property which can be thought of as a dual property to shadowing.

If an expansive homeomorphism has the shadowing property then it is topologically stable (see [L₄]); nevertheless, not every expansive homeomorphism is topologically stable as we have already seen. All known expansive homeomorphisms do verify though this weaker notion of stability which is called *persistence* (or *semi-persistence*) a term coined by Lewowicz in [L₂] (see also [L₅]).

Definition (Persistence). We say that $f : M \rightarrow M$ is *persistent* if for every $\varepsilon > 0$ there exists a C^0 -neighborhood \mathcal{U} of f such that for every $g \in \mathcal{U}$ and $x \in M$ there exists $y \in M$ such that

$$d(f^n(x), g^n(y)) \leq \varepsilon \quad \forall n \in \mathbb{Z}$$

◇

In Lewowicz words ([L₂]):

“(...)roughly, the dynamics of f may be found in each g close to f in the C^0 -topology; however, these g may present dynamical features with no counterpart in f .”

In his paper [L₂] Lewowicz proves some results concerning persistence such as persistence for pseudo-Anosov maps and more generally, for those expansive diffeomorphisms having a dense set of hyperbolic periodic points with codimension one. Those results can be thought of as the germ of further developments in higher dimensions such as [V₁, V₂, V₃, ABP].

In particular, he shows that a small C^1 -perturbation of a pseudo-Anosov map preserving the singularities must be conjugated to the original map; a kind of structural stability result for pseudo-Anosov maps. See also [Ha] for further developments.

Before we continue with the results of [L₂] and some of the consequences found by Lewowicz and coauthors, we are tempted to add another quote of [L₂]:

“We believe that, apart from such applications, there is another reason for studying these persistence properties: it seems plausible to think that if a theory of asymptotic behavior is possible, then semi-persistence (i.e. persistence of positive or negative semi-trajectories) should hold on big subsets of M for large classes of dynamical systems”.

See [L₅] for advances in that hope. He posed a precise question about this problem:

Question (Problem 10.2 of [LC₂]). *For an expansive homeomorphism is every semitrajectory persistent in the future (in the past)?*

Another question which is motivated by this persistency property can be stated as follows:

Question. *Does an expansive homeomorphism minimize the entropy in its isotopy class?*

This is true for every known example, and it is true after Lewowicz classification result for surface homeomorphisms (see also [Ha]). If every expansive homeomorphism is persistent, then nearby homeomorphisms should have at least the same topological entropy. However, it seems that the answer of this fundamental question is at present far out of reach.

3.2. Expansive systems and stable points. Probably the first interaction found by Lewowicz between the topology of the phase space and expansive dynamics is the fact that a non-trivial compact connected and locally connected set admitting an expansive homeomorphism cannot have Lyapunov stable points. If connectedness is not required, this is clearly false as can be seen by considering an homoclinic orbit between two fixed points. A more delicate example, where the phase space is connected can be found in [RR].

As a way to pave the way of some results in low dimensions which required hyperbolic periodic points to have codimension one in dimensions 2 and 3, Lewowicz proved in [L₂] the following result:

Theorem 3.1 (No Stable Points). *Let $f : M \rightarrow M$ be an expansive homeomorphism of a non-trivial compact connected and locally connected metric space, then f has no Lyapunov stable points.*

Recall that a point x is Lyapunov stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $d(f^n(x), f^n(y)) < \varepsilon$ for every $n \geq 0$.

We give here a sketch of the proof of this important result:

SKETCH Let $\alpha > 0$ be the expansivity constant of f . The proof is divided into 3 steps:

Step 1: For $\varepsilon < \alpha$, if

$$S_\varepsilon(x) = \{y : d(f^n(x), f^n(y)) \leq \varepsilon \quad n \geq 0\}$$

then we have that the diameter of $f^n(S_\varepsilon(x))$ converges to zero uniformly on x and n .

Step 2: If x is a Lyapunov stable point, and $\varepsilon > 0$, there exists $\sigma > 0$ such that for $n \geq 0$ we have that $f^{-n}(S_\varepsilon(x))$ contains the ball of radius σ of $f^{-n}(x)$.

Step 3: The previous step implies that every point in the α -limit of x is Lyapunov stable. One can prove using this fact and the first step that the α -limit set must consist

of periodic attractors which are only α -limit points of their own orbit. This gives a contradiction, since it implies that the whole space is a periodic orbit, and being connected a unique point (contradicting that the space was non-trivial).

The hardest step is Step 2 and it is where local connectedness is used in an essential way. Roughly, using local connectedness, if the uniform ball cannot be obtained, one finds a sequence of points x_n, y_n such that they are at distance larger than δ and remain at distance less than ε for all future iterates and for arbitrarily large number of iterates in the past. Taking limits, one contradicts expansivity.

Let us explain briefly how to find such pair of points: If when iterating backwards the δ -ball of x there is no uniform ball, given $n > 0$ one can choose an arc γ (or a connected set) with length smaller than $1/n$ and containing $f^{-k_n}(x)$ (where k_n must necessarily tend to $+\infty$ as $n \rightarrow +\infty$) such that $f^{k_n}(\gamma)$ is not contained in $B_\delta(x)$. By connectedness there exist a point y_n and a backward iterate $x_n = f^{-m_n}(x)$ at distance larger than δ and such that $d(f^j(x_n), f^j(y_n)) \leq \varepsilon$ for every $j \geq -k_n + m_n$.

Since the points x_n and y_n are at distance larger than δ and its $k_n - m_n$ backward iterate sends them at distance less than $1/n$ we get that $k_n - m_n$ also goes to $+\infty$ as $n \rightarrow \infty$. Taking convergent subsequences of x_n and y_n one obtains different points whose orbits remain at less than ε for all iterates contradicting expansivity.

□

3.3. Analytic models of pseudo-Anosov maps. In his paper [LL] with E. Lima de Sa, they provide a new construction of analytic models of pseudo-Anosov maps that had been obtained by Gerber ([Ge]) based on previous work by Gerber with Katok ([GeK]).

The idea is to replace their conditional stability results by the structural stability theorem of Lewowicz ([L₂]) for pseudo-Anosov maps involving the concept of persistence.

It is important to remark that constructing analytic (even smooth) models of pseudo-Anosov maps is not easy since by a change of coordinates which is C^1 out of a neighborhood of the singularities one cannot obtain a smooth model (this was shown in [GeK]), so a more global modification must be made.

The idea involves “slowing down” in a neighborhood of the singularities (much as one does if one wants to smooth the parametrization of a curve having a corner in its image without altering the image) and then approximating by analytic maps which preserve the singularities as well as some r -jets of the derivative of the map in the singularity. This allows to use the mentioned Lewowicz’s results on persistence ([L₂]).

To show how this creation of models is far from being trivial, let me state an open problem which we are far from understanding. This question was strongly motivated by discussions with Jorge Lewowicz and his constant insistence on the lack of understanding we have of the role of the dynamics of the tangent map (see also the next subsection for related problems):

Question. *Let $f : M \rightarrow M$ be a topological Anosov (i.e. A homeomorphism of M which preserves two topologically transverse foliations one of which contracts distances uniformly and the other one contracts them for backward iterations). Does there exist a smooth model for f ? And analytic?. Assuming the previous questions have positive answers, can these models be made Anosov?.*

◇

The question admits a positive answer both in the codimension one case and in the case where M is a nilmanifold due to the fact that the classification results of Newhouse-Franks-Manning only use the fact that the map is a topological Anosov. However, the question is completely independent a priori of the classification of Anosov systems.

One of the main contributions of [LL], though lateral to the paper has already been explained in this note, and has to do to the way they prove that the resulting approximation maps is still Bernoulli with respect to Lebesgue measure (which can be thought of as the counterpart of the second part of the question above). To do this, they use quadratic forms and that is the germ of further results on non-uniform hyperbolicity as we have already mentioned.

3.4. The C^0 -boundary of Anosov diffeomorphisms. In this section we state a result obtain by Lewowicz in coloboration with J. Tolosa about the C^0 -boundary of codimension one Anosov diffeomorphisms (see [LT]).

They prove:

Theorem 3.2. *Let f be an expansive homeomorphism in the C^0 -boundary of Anosov diffeomorphisms of codimension one in \mathbb{T}^d . Then, f is conjugated to an Anosov.*

With his classification result for expansive homeomorphisms of the torus, this can be further improved to get:

Theorem 3.3. *Let $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be an expansive homeomorphism. Then f is contained in the C^0 -closure of the set of Anosov diffeomorphisms of \mathbb{T}^2 .*

PROOF. Consider $h : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ a homeomorphism isotopic to the identity such that $f = h \circ A \circ h^{-1}$ where A is a linear Anosov automorphism. The existence of such an h is given by Theorem 4.1 below.

Then there exists a sequence of diffeomorphisms h_n converging to h in the C^0 -topology and such that h_n^{-1} also converges to h^{-1} . Since conjugating an Anosov diffeomorphism by a diffeomorphism gives an Anosov diffeomorphism we get that f is approximated in the C^0 -topology by Anosov diffeomorphisms.

□

An important open question that is motivated by this result is the following:

Question (Problem 10.1 of [LC₂]). *Does the C^1 -closure of Anosov diffeomorphisms contains all expansive diffeomorphisms of \mathbb{T}^2 ?*

Notice also that Mañé has proved that the C^1 -interior of expansive diffeomorphisms consists of Quasi-Anosov ones⁵, in particular in \mathbb{T}^2 of Anosov ones ([Ma₁]).

Notice that the set of Anosov diffeomorphisms in an given isotopy class of \mathbb{T}^2 forms a connected set (see [FG]).

4. CLASSIFICATION THEOREM IN SURFACES

It can be shown easily that the only closed one dimensional manifold, namely the circle, admits no expansive homeomorphisms. This can be proved using the Poincaré's classification of homeomorphisms of the circle by discussing depending on the rotation number. Other than that, some examples and some results on the non-existence of expansive homeomorphisms of other one dimensional continua, nothing was known about the existence or structure of expansive homeomorphisms. It is to be remarked that Mañé proved ([Ma₂]) that if a compact metric space admits an expansive homeomorphism, then it must have finite topological dimension.

Examples in every orientable surface different from the sphere were already known ([OR]), but there was no clue for example on which isotopy classes admitted them. The classification of expansive homeomorphisms of surfaces was thus meant to be started from scratch and that was what Lewowicz did ([L₃): He gained an impressive understanding of their dynamics and their relation with the topology of the phase space and one of the most striking aspects of his study is that he relied only on some well known and almost elementary properties of plane topology. Of course, once he got a classification of expansive homeomorphisms in terms of their dynamics and local behavior the final form of the result, giving conjugation to already known models, used some less elementary techniques ([Fr, Th]).

The starting point was the non existence of stable points proved by him in [L₂] and reviewed in the previous section. In this section we will give an overview of the classification results for expansive homeomorphisms of surfaces and the main ideas involved in the proof. We recall that as we said in the introduction, these results were obtained independently by Hiraide [H].

What we will provide is far from a complete proof of this classification result, but we hope that the outline here can be used as a guide to read the original paper [L₃] and to obtain some insight on the proof.

4.1. Statement of the result. Along this section, S will denote an orientable closed (compact, connected, without boundary) surface. It is well known that these surfaces are

⁵Is in this paper that Mañé introduces the concept of *dominated splitting*.

well characterized by their Euler characteristic, and consist of the sphere S^2 , the torus \mathbb{T}^2 and the higher genus surfaces S_g with $g \geq 2$.

The main result of [L₃] is the following:

Theorem 4.1 (Classification of expansive homeomorphisms of surfaces). *Let $f : S \rightarrow S$ an expansive homeomorphism. Then, $S \neq S^2$ and:*

- *If $S = \mathbb{T}^2$ then f is conjugate to a linear Anosov automorphism.*
- *If $S = S_g$ then f is conjugate to a pseudo-Anosov map ([Th]).*

As we mentioned, Lewowicz result has two parts, first, he gives a complete dynamical classification of expansive homeomorphisms by a detailed study of the stable and unstable sets of all the points in S , obtaining for them a local product structure outside some finite set of “singularities” which have a local behavior much like those of pseudo-Anosov maps. Then, by using global arguments and shadowing results he obtains the desired conjugacy.

Lewowicz result can be thought of in a now very fashionable way called *rigidity*: Rigidity results (or non-existence results) are those which give strong restrictions from a priori very weak ones. In the words of Frederic Le Roux in [Ler]: “(...) a simple dynamical property can imply a strong rigidity. The most striking result here is probably Hiraide-Lewowicz theorem that an expansive homeomorphism on a compact surface is conjugate to a pseudoAnosov homeomorphism”.

4.2. Stable and unstable sets. This and the next will be the more technical sections of this note. However, we will try to first give a statement which will be proved in these two sections which will allow the reader to continue. Then, we will enter in some details.

Let $f : S \rightarrow S$ be an expansive homeomorphism with expansivity constant equal to α . Consider the following sets:

$$S_\varepsilon(x) = \{y \in S : d(f^n(x), f^n(y)) \leq \varepsilon \quad n \geq 0\}$$

$$U_\varepsilon(x) = \{y \in S : d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon \quad n \geq 0\}$$

Expansivity can be reformulated as $S_\alpha(x) \cap U_\alpha(x) = \{x\}$ for every $x \in S$. We call $S_\varepsilon(x)$ (resp. $U_\varepsilon(x)$) the ε -stable set of x (resp. ε -unstable set of x).

As we mentioned in the previous section, it can be easily proved that the diameter of $f^n(S_\varepsilon(x))$ converges to zero uniformly independently of x if $\varepsilon < \alpha$. The key technical result in the classification of expansive homeomorphisms of surfaces can be stated in terms of these sets:

Theorem 4.2 (Classification Theorem Local Version). *Let $f : S \rightarrow S$ be an expansive homeomorphism. Then, there exists a finite set F (possibly empty) such that for every $x \in S \setminus F$ we have that there exists $\varepsilon > 0$ such that $S_\varepsilon(x)$ is a continuous arc having x in its interior. Moreover, there exists a neighborhood U of x having local product structure. For*

$x \in F$ we have that the sets $S_\varepsilon(x) \setminus \{x\}$ and $U_\varepsilon(x) \setminus \{x\}$ are both a finite number (≥ 3) of arcs which are alternated and in each angle they form, there is also local product structure.

We must explain some of the terminology appearing in the statement (see also Figure 2 for a visual explanation).

Local product structure means the following: We say that in an open set U centered in x there is *local product structure* if there is a homeomorphism

$$h : [-1, 1] \times [-1, 1] \rightarrow \bar{U}$$

such that $h(0, 0) = x$ and $h(\{t_0\} \times [-1, 1])$ is contained in a stable set $S_\varepsilon(h(t_0, 0))$ and $h([-1, 1] \times \{s_0\})$ is contained in an unstable set $U_\varepsilon(h(0, s_0))$.

In a similar way, given a point x , we can consider a connected component L^s of $S_\varepsilon(x) \setminus \{x\}$ and a connected component L^u of $U_\varepsilon(x) \setminus \{x\}$. If U is a neighborhood of x and A is a connected component of $U \setminus (L^s \cup L^u \cup \{x\})$ which does not intersect $S_\varepsilon(x) \cup U_\varepsilon(x)$ we say that A is an *angle*. We say that the angle has local product structure if a similar property as above holds except that $h : [0, 1] \times [0, 1] \rightarrow \bar{A}$ and it sends $h(0, 0) = x$ with the rest of the properties being equal (see Figure 2).

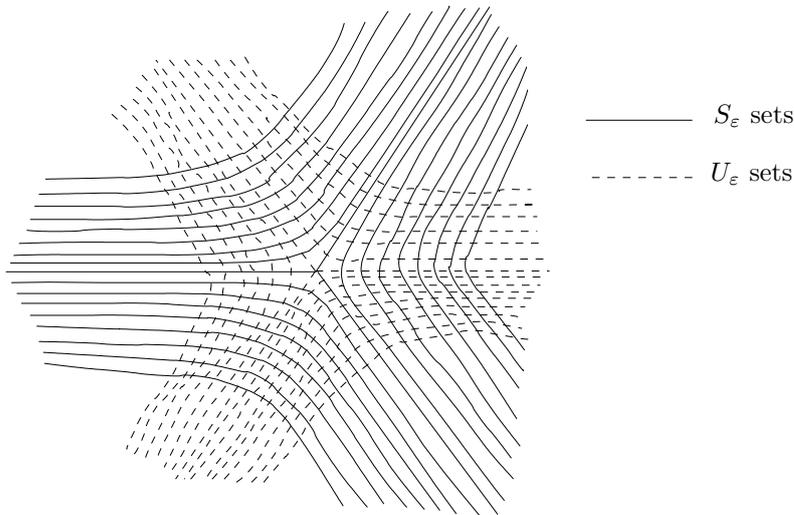


FIGURE 2. Local picture at a singular point with $p = 3$ “legs”.

Before we continue with a sketch of the proof of this result, let us make some comments on some existing extensions. First, similar properties have been obtained for expansive flows in dimension 3 ([Pat₁]). Also, by assuming the existence of a dense set of topologically hyperbolic periodic points these results can be extended to any dimension ([V₁, ABP]), except that the behavior in the singularities is not well understood⁶ except in dimension 3 or in the codimension one case ([V₂, ABP]) where one can show that they do not exist. With some differentiability assumptions, the hypothesis of the existence

⁶It seems that we lack examples of “genuine” pseudo-Anosov maps in higher dimensions.

of periodic points can be removed, at least in dimension 3 ([V₃]). This has also been extended to the plane under certain conditions of the behavior at infinity ([Gr₁, Gr₂]).

Just to show how far we are from obtaining a similar result in higher dimensions let me state the following open question (it is known in the smooth case in \mathbb{T}^3 , see [V₃]):

Question. *Has every expansive homeomorphism of a manifold a periodic point?*

Now, let us discuss the main points of the proof of Theorem 4.2. Let us remark that each of these steps are interesting by themselves, and some of them hold in higher dimensions.

The first step of the proof consists on showing that every point in S has a local stable and unstable set of uniform size.

Proposition 4.1. *For $f : S \rightarrow S$ expansive homeomorphism and $\varepsilon < \alpha$ the expansivity constant, there exists $\delta > 0$ such that for every $x \in S$ we have that the connected component of $S_\varepsilon(x) \cap B_\delta(x)$ containing x intersects $\partial B_\delta(x)$.*

SKETCH The proof of this proposition holds in any dimension. The key point is the non existence of Lyapunov stable (in fact, Lyapunov unstable) points proven in Theorem 3.1.

Once this is obtained, one can construct large connected sets by considering the sets D_n build as the connected component containing x of n -th preimage of the ball of radius ε centered at $f^n(x)$ by f^{-n} . The fact that there are no Lyapunov unstable points allows one to prove that these sets have all diameter bounded from below and allow to construct the desired set as

$$C_\varepsilon^s(x) = \bigcap_N \overline{\bigcup_{n \geq N} D_n}$$

One has to check that this has the desired properties (see [L₃] Lemma 2.1), in particular that the sets D_n have diameter bounded from below. This can be done using Lyapunov functions and the metric they define, or using the metric introduced in [Fa]. Also, it can be done by barehanded arguments (see [L₄]).

□

We remark that the previous result gives a conceptual proof that S^1 does not admit expansive homeomorphisms: On the one hand they cannot have Lyapunov stable points, but on the other hand the stable set of a point must contain a connected set of large diameter, thus, non-empty interior, a contradiction.

We make a remark on stable and unstable sets which is of importance in many steps of the proof. It can be thought of as a “big angles” result. The proof is not difficult (see Lemma 3.3 of [ABP]).

Proposition 4.2 (Big Angles). *Let $f : S \rightarrow S$ be an expansive homeomorphism with expansivity constant α . Given $V \subset U$ neighborhoods of x and $\rho > 0$ small enough, there exists a neighborhood $W \subset V$ of x such that if $y, z \in W$ we have that $d(S_\varepsilon(y) \cap U \setminus V, U_\varepsilon(z) \cap U \setminus V) > \rho$.*

The next step of the proof is probably the deepest and it is really dependent on the two-dimensionality of the problem. Here one sees a clear manifestation of the already quoted phrase of “*a stronger interaction of the topology of M and the dynamics of f could be expected*”.

Theorem 4.3. *For an expansive homeomorphism $f : S \rightarrow S$ with expansivity constant α and $\varepsilon < \alpha/10$, the connected component of $S_\varepsilon(x)$ containing x is locally connected at each of its points and therefore arc-connected.*

SKETCH We will only give a brief outline with an heuristic idea of this subtle proof. We refer the reader to [L₃] pages 119-121 for details (see also [L₄] pages 21-25).

Consider $C_\varepsilon^s(x)$ the connected component of $S_\varepsilon(x)$ containing x . We first show that it is locally connected at x and then a clever argument allows to show local connectedness at every point. Once this is proved, arc-connectedness follows since a compact connected and locally connected set is arc-connected.

The proof is by contradiction. Roughly, the idea is that if it is not locally connected at x we can think that in an arbitrarily small ball of x the set $S_\varepsilon(x)$ is a sequence of connected sets approaching x but connecting to $C_\varepsilon^s(x)$ outside the ball. Using separation properties of the plane (which are extensions of Jordan’s curve theorem) we obtain some point z which is trapped in both sides by connected components of $S_\varepsilon(x)$. Since the unstable set of z has a large connected component containing z , we know it must leave the neighborhood, however, it can intersect $S_\varepsilon(x)$ only once, so we obtain that it leaves forming “small angles” with $S_\varepsilon(x)$ a contradiction with Proposition 4.2.

In fact, there are some subtleties in what we have just said, since the fact that the unstable set of z has a large connected component does not imply that it must have two sides, and there is no problem to have one side going out by intersecting $S_\varepsilon(x)$. To solve this, Lewowicz makes a clever argument that he then repeats several times in his proof and so we partially reproduce it here: He considers an arc joining two different connected components of $C_\varepsilon^s(x)$ locally and he divides the arc depending on which side the unstable set of the points leave the neighborhood: a connectedness argument allows him to conclude that either there is a point whose unstable intersects twice $S_\varepsilon(x)$ (contradicting expansivity) or a point whose unstable leaves forming small angles (also a contradiction). This connectedness argument uses the fact that stable and unstable sets vary semicontinuously⁷.

Now, to get local connectedness at every point, we use local connectedness at the centers at many scales. Consider $y \in C_\varepsilon(x) \subset C_{2\varepsilon}(y)$. Then $C_{2\varepsilon}(y)$ is locally connected at y , so for every $\sigma > 0$ and $z \in C_\varepsilon(x)$ close to y there exists a connected set $C \subset C_{2\varepsilon}(y) \cap B_\sigma(y)$ containing y and z . Since there are no stable points, we know that $C \cup C_\varepsilon(x) \subset C_{2\varepsilon}(y)$

⁷This is a general property that holds for any homeomorphism and it is not hard to check. See for example Lemma 3.2 of [ABP].

cannot separate, so, by an extension of Jordan’s separation theorem we get that $C_\varepsilon(x) \cap C$ is connected and we deduce that $C_\varepsilon(x)$ is locally connected at y .

This finishes the sketch of the proof.

□

We will give an outline of the rest of the proof of Theorem 4.2 in the next subsection. We will omit even more details.

4.3. Singularities. The purpose of this section is to outline the rest of the proof of Theorem 4.2. We will not enter in details here, we will only explain the main steps of the proof.

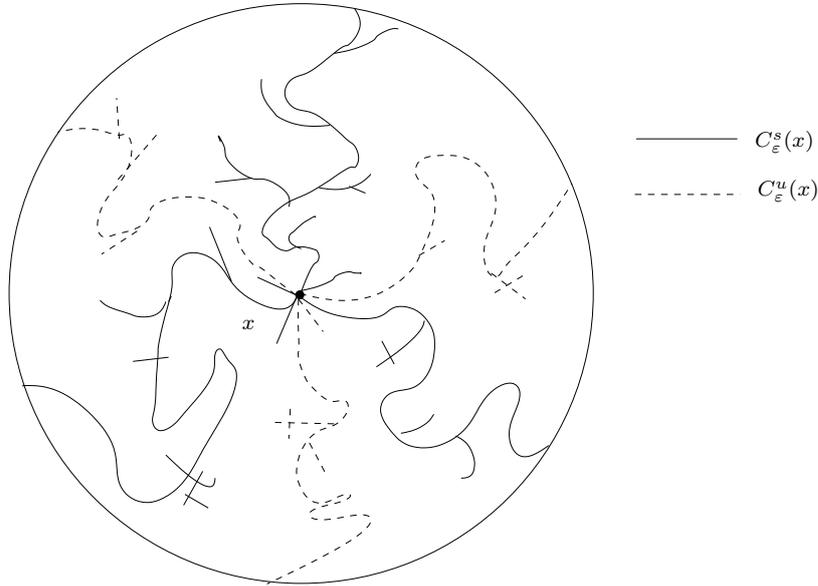


FIGURE 3. Structure of local stable and unstable sets. They are arc connected but may a priori be still “ugly”.

Pick a point $x \in S$. As we have already seen, $S_\varepsilon(x)$ has at least one connected component intersecting $\partial B_\delta(x)$. If we consider the connected component $C_\varepsilon^s(x)$ of $S_\varepsilon(x) \cap B_\delta(x)$ and the connected component $C_\varepsilon^u(x)$ of $U_\varepsilon(x) \cap B_\delta(x)$ we know that there are arcs joining x to $\partial B_\delta(x)$ contained in those sets. It is possible to make an equivalence relation between these arcs that identify arcs which start at x and then bifurcate near the boundary of $B_\delta(x)$. By using this, the big angles property and connected arguments similar to the ones used in the previous section, Lewowicz shows:

Lemma 4.1. *The number of (equivalence classes of) arcs in $C_\varepsilon^s(x)$ and $C_\varepsilon^u(x)$ joining x to the boundary of $B_\delta(x)$ is the same, finite, and moreover, they are alternated in the order of $\partial B_\delta(x)$.*

This result together with the invariance of domain theorem and further application of the previous arguments give the following property around points which is almost the end of the proof of Theorem 4.2.

Proposition 4.3. *For every $x \in S$, there exists a neighborhood U such that every point y in $U \setminus \{x\}$ has a neighborhood with local product structure.*

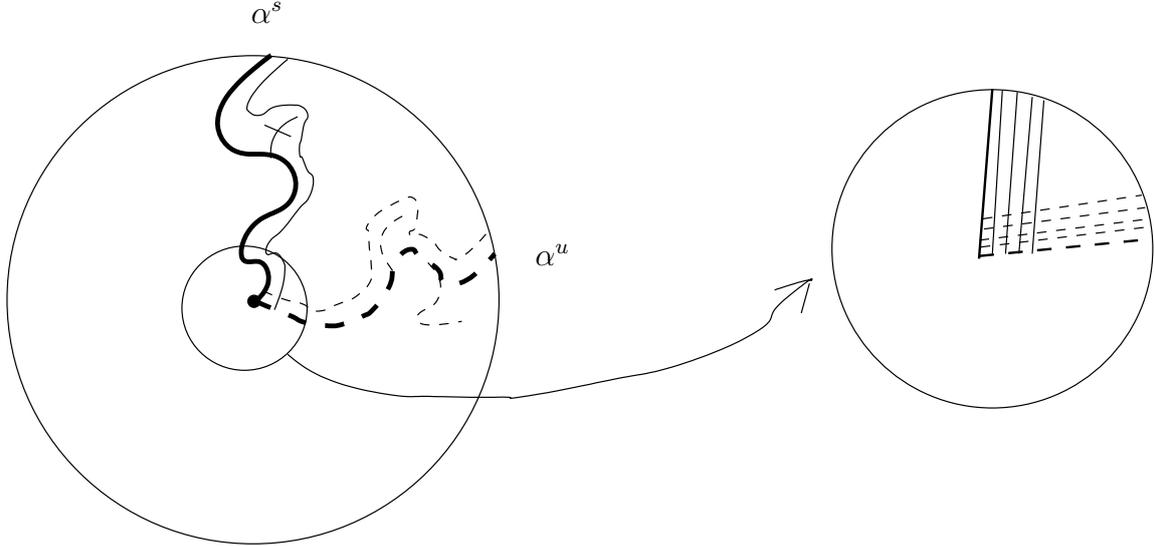


FIGURE 4. How to obtain local product structure.

This is proved as follows: Consider an arc α^s of $C_\varepsilon^s(x)$ and a consecutive one α^u of $C_\varepsilon^u(x)$. Now, for points of α^u close to x we have that using semicontinuous variation⁸ of stable sets and the “big angles” property (Proposition 4.2) that the stable set of the points near x goes out of $B_\delta(x)$ near α^s . The same happens for points in α^s near x and their unstable sets. This allows to find a continuous and injective (due to expansivity) map from a neighborhood of x in α^s times a neighborhood of x in α^u into S . By the invariance of domain theorem this map is open and thus every point in this “angle” has local product structure. This can be done in all the angles formed by the stable and unstable arcs of x (see Figure 4).

It is immediate to conclude that:

Corollary 4.1. *There exists a finite set $F \subset S$ such that every point outside of which every point has local product structure. Moreover, for x in F have a neighborhood such that their local stable and unstable sets are $p \geq 1$ (and different from 2 which would imply local product structure around x) arcs starting at x and arriving at the boundary.*

It remains only to discard the possibility of having a unique arc in the stable set of x . This is an important issue since for example S^2 admits diffeomorphisms (even analytic, see [Ge, LL]) that have the local form we have obtained but with points having a singularity with a unique “leg”. Needless to say, those examples are not expansive, since for points very near to x in the stable set, very small horseshoes are created, contradicting

⁸A disclaimer is that to be precise, this argument needs that there are at least two arcs of stable and two arcs of unstable for x . We will ignore this problem and “solve it” afterwards because we believe it gives a better heuristic of the global argument. See [L₃] for a correct proof.

expansivity. Building in this example, and using the arguments developed by Lewowicz for the other parts of the proof, one can give a general proof of the following (see also Figure 5):

Proposition 4.4. *The number p in the above corollary is ≥ 3 for every point in F .*

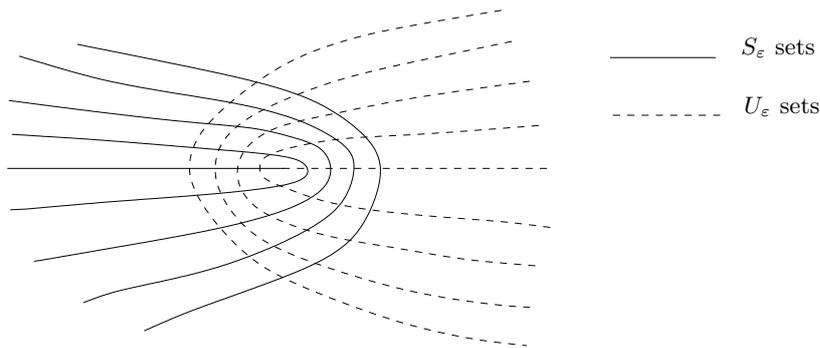


FIGURE 5. One leg implies that local stable and local unstable sets intersect in more than one point contradicting expansivity.

This concludes the outline of the proof of Theorem 4.2.

4.4. Non-existence of expansive homeomorphisms on S^2 . We show here how Theorem 4.2 is enough to show that the two-dimensional sphere S^2 cannot admit expansive homeomorphisms.

The easiest way to see this is using index theory for foliations. The local product structure obtained allows one to see that stable and unstable sets foliate the surface admitting and expansive homeomorphisms giving rise to a continuous foliation with finitely many singularities of prong type. Since for every singularity the number of legs is ≥ 3 we deduce that even if the foliation may be non-orientable then the index of the singularities is always negative (notice that if there were only one leg, then the index is positive and equal to $1/2$ so that one can make one example in S^2 with four such singularities). This implies that S^2 cannot support such a homeomorphism.

If the reader is not comfortable with the use of continuous (and not differentiable) foliations, one can go to [L₃] where a more elementary proof is given using Poincare-Bendixon's like arguments.

4.5. Other surfaces. In the torus case, essentially, due to the work of Franks, it is enough to show that there are no singularities (which is clear by the index argument shown above) and that the map is isotopic to a linear Anosov automorphism. Consider then $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ an expansive homeomorphism.

Although he might have used the already known argument on the growth of periodic points and Lefschetz index, Lewowicz gives a different argument which is very beautiful⁹.

I outline it here: Lift f to the universal cover to obtain

$$\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

Let $A \in GL(2, \mathbb{Z})$ be its linear part (i.e. A is the matrix given by $\tilde{f}(\cdot) - \tilde{f}(0) : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$). If A is not hyperbolic, then it has both eigenvalues of modulus 1 since it has determinant of modulus 1. Then, one obtains that the diameter of a set iterated by A grows at most polynomially. Since \tilde{f} is at bounded distance from A , the same holds for the iterates of a set by \tilde{f} . If J is an unstable arc contained in a local product structure box, one gets that $\text{diam}(\tilde{f}^n(J)) \leq p(n)$ where p is a polynomial.

On the other hand, we know that the length¹⁰ of an unstable arc by \tilde{f} must grow exponentially due to expansivity, so, for the same J we get that the length of $\tilde{f}^n(J)$ is comparable to λ^n with $\lambda > 1$. Moreover, since there are no singularities, a Poincare-Bendixon's like type of argument implies that an arc of unstable cannot intersect the same box of local product structure twice. This implies, via the quadratic growth of volume of \mathbb{R}^2 that the diameter of an arc of unstable of length L is comparable to \sqrt{L} which will still be exponential. This gives a contradiction and completes the proof. See [L₃] Theorem 5.3 for more details.

In the higher genus case the proof is even more delicate. He again stands on previous conjugacy results by Handel [Ha] (improving the results of [L₂]) that state that in the isotopy class of a pseudo-Anosov map there exist certain semiconjugacies. Then, as in the torus case he must prove that the local classification theorem (Theorem 4.2) provides enough tools to show that f is isotopic to pseudo-Anosov. He uses Thurston's classification and shows that no homotopy class of simple curves can be periodic (see Lemma 6.4 of [L₃]) which allows him to conclude.

□

Acknowledgments: My understanding of the theorem of classification on surfaces owes tremendously to conversations with A. Artigue and J. Brum. The writing of this note also benefited from many conversations, exchange and suggestions of: D.Armentano, E. Catsigeras, M. Cerminara, M.Delbracio, N. de Leon, H. Enrich, J. Groisman, P. Lessa, R. Markarian, A. Passeggi, M. Paternain, R. Ruggiero, A. Sambarino, M. Sambarino and J. Vieitez. Of course, I want to thank particularly Jorge Lewowicz, for his work and all the enlightening conversations we had, mathematically and otherwise.

⁹I could not trace a similar argument to before Lewowicz's paper. However, this kind of argument has been rediscovered many times so I do not claim that it is the first time it appeared. I show it in order to stress the continuous search of Lewowicz for understanding and for conceptual and clean arguments.

¹⁰Since it is a continuous arc this is not really well defined. One can measure thus length by "counting" the number of local product structure boxes it intersects.

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