SINGULAR SETS OF PLANAR HYPERBOLIC BILLIARDS ARE REGULAR

GIANLUIGI DEL MAGNO AND ROBERTO MARKARIAN

Abstract. Many planar hyperbolic billiards are conjectured to be ergodic. This paper represents a first step towards the proof of this conjecture. The Hopf argument is a standard technique for proving the ergodicity of a smooth hyperbolic system. Under additional hypotheses, this technique applies as well to certain hyperbolic systems with singularities, including hyperbolic billiards. The supplementary hypotheses regard the subset of the phase space where the system fails to be C^2 differentiable. In this work, we give a detailed proof of one of these hypotheses for a large collection of planar hyperbolic billiards. Namely, we prove that the singular set and each of its iterations consist of a finite number of compact curves of class C^2 with finitely many intersection points.

1. Introduction

A planar billiard is the mechanical system consisting of a point-particle moving freely inside a domain $Q \subset \mathbb{R}^2$ with piecewise differentiable boundary, and being reflected off ∂Q so that the angle of reflection equals the angle of incidence.

This paper concerns hyperbolic billiards, i.e., billiards without vanishing Lyapunov exponents. The study of hyperbolic billiards was started by Sinai. In his seminal paper [21], he proved that if the boundary of a toral domain consists of convex outward (dispersing) arcs, then the corresponding billiard is hyperbolic and K-mixing. Later, Gallavotti and Ornstein showed that Sinai billiards enjoy the Bernoulli property as well [14]. More or less at the same time, Bunimovich proved that billiards in some domains with boundary formed by convex inward (focusing) arcs and line segments are also hyperbolic [1, 2]. The most celebrated example of a Bunimovich billiard is probably the stadium, the region bounded by two semi-circles connected by two parallel segments. The only admissible focusing components in Bunimovich billiards are the arcs of circles. This limitation was eventually overcome by Wojtkowski, Markarian, Donnay and Bunimovich. Using new techniques for establishing the positivity of Lyapunov exponents [19, 23], they proved independently that many other focusing arcs can be used to construct hyperbolic billiards [4, 13, 19, 23].

Besides Sinai and Bunimovich billiards, only special cases of the remaining hyperbolic billiards were proved to be ergodic [3, 9, 10, 11, 18, 20, 22]. Our long-term goal is to demonstrate that many more planar hyperbolic billiards are ergodic.

Date: May 6, 2013.

1991 *Mathematics Subject Classification.* Primary 37D50; Secondary 37A25, 37D25, 37N05.

Key words and phrases. Hyperbolic Billiards, Ergodicity.

G. Del Magno was supported by Fundação para a Ciência e a Tecnologia through the Program POCI 2010 and the Project 'Randomness in Deterministic Dynamical Systems and Applications' (PTDC-MAT-105448-2008). R. Markarian acknowledges Grupo de Investigación 'Sistemas Dinámicos' (CSIC, UR, Uruguay). Both authors thank IMERL and CEMAPRE, where parts of this work were written, for financial support and hospitality. The authors also thank N. Chernov and D. Szász for helpful comments.

A central step in the proof of the ergodicity of a hyperbolic billiard is to show that it is locally ergodic, i.e., all its ergodic components of positive measure are open (mod 0). Results of this type are often called *Local Ergodic Theorems* (LET's). Their proofs usually rely on improved versions of the argument devised by E. Hopf to demonstrate that a geodesic flow on a compact surface of negative curvature is ergodic [15]. Sinai was the first to succeed in adapting the Hopf argument to billiards [21]. To obtain a complete and rigorous proof that hyperbolic billiards are locally ergodic, the plan is to use a LET for general hyperbolic simplectomorphisms with singularities that we proved recently [12].

One of the hypotheses of our LET concerns the regularity of the singular set of a billiard. The singular set of order $k = \pm 1, \pm 2, \ldots$ of a billiard in the domain Q is the subset of the billiard phase space where the kth iterate of the billiard map fails to be defined or C^2 differentiable. This subset corresponds to the trajectories hitting a corner of ∂Q or having a tangential collision with ∂Q before the $(|k|+1)$th collision in the past if $k < 0$, and in the future if $k > 0$. One of the hypotheses of our LET requires that the singular sets of all orders be finite unions of compact curves of class C^2 with finitely may intersections. This property of the singular sets is called regularity. All the assumption in the regularity condition – the finiteness of the number of curves and the intersections, the compactness of the curves and their C^2 differentiability – are essential for the proof of LET. In this paper, we provide a detailed proof of this condition for the billiards described in Section 5. A complete proof of the local ergodicity of these billiards will appear in a forthcoming paper.

It is worth mentioning that the regularity of the singular sets for planar billiards is sometimes taken for granted in the literature. However, to the best of our knowledge, this property has been proved in detail only for dispersing billiards, certain classes of semi-dispersing billiards and Bunimovich billiards [7, Sections 4.8 and 8.10]. In particular, no proof of the regularity of the singular sets has been given for the billiards with focusing boundary components studied in [4, 13, 19].

The paper is organized as follows. Section 2 contains some background material on billiards and the formulation of the main result of the paper. In Section 3, we review the notion of focusing times and the definition of absolutely focusing arcs. In Section 4, we provide the basic definitions concerning invariant cone fields, and introduce a cone field for planar billiards. In Section 5, we first give a detailed description of the billiard considered in this paper, and then prove that their cone field is eventually strictly invariant. This property, which implies the hyperbolicity of the billiards, is used to prove the regularity of the singular sets in Section 6. Finally, the Appendix is devoted to the proof of some technical results extending [13, Theorem 4.4] (see also Theorem 4.7 of this paper) to certain class arcs, including those introduced by Wojtkowski [23] and Markarian [19].

2. Generalities and main result

Let S^1 denote the unit circle of $\mathbb{R}^2$. A subset $\Gamma \subset \mathbb{R}^2$ is called an *arc of class* C^k with $k \geq 1$ if Γ is the image of a C^k embedding $\gamma\colon [0,1] \to \mathbb{R}^2$. Given an arc Γ, we define $\partial\Gamma = \gamma(0) \cup \gamma(1)$. A subset $\Gamma \subset \mathbb{R}^2$ is called a *closed curve* if Γ is homemorphic to S^1. A subset $\Gamma \subset \mathbb{R}^2$ is called a *closed curve of class* C^3 if Γ is the image of a C^3 transformation $\gamma\colon [0,1] \to \mathbb{R}^2$ such that γ is injective on $(0,1)$, $\gamma'(t) \neq 0$ for $t \in [0,1]$ and $\gamma(0) = \gamma(1)$. Given a closed curve Γ of class C^3, we define $\partial\Gamma = \emptyset$ when Γ is C^3 diffeomorphic to S^1, and $\partial\Gamma = \gamma(0)$ otherwise.

Let Q be an open bounded connected subset of $\mathbb{R}^2$. We assume that ∂Q is an union of finitely many disjoint closed curves. When ∂Q consists of more than one closed curve, the interior of one of these curves necessarily contains all the other curves, which can be thought of as the boundary of some obstacles contained in Q. Furthermore, we assume that there exist $n > 0$ curves $\Gamma_1, \ldots, \Gamma_n$ that are closed curves and arcs both of class C^3 such that

(1) the curvature of Γ_i is strictly negative, strictly positive or identically equal to zero (with respect to the orientation of Γ_i chosen below),
(2) $\Gamma_i \cap \Gamma_j \subset \partial\Gamma_i \cap \partial\Gamma_j$ for $i \neq j$,
(3) $\partial Q = \bigcup_{i=1}^n \Gamma_i$,
(4) $\bigcup_{i=1}^n \partial\Gamma_i$ is set of points where ∂Q is not C^3.

The arcs $\Gamma_1, \ldots, \Gamma_n$ are called the *components* of ∂Q. A component of ∂Q is called *dispersing*, *focusing* or *flat* according as its curvature is negative, positive or zero. Note that a flat component is just a straight segment. A point of $\bigcup_{i=1}^n \partial\Gamma_i$ is called a *corner* of ∂Q. Note that our assumptions imply that the curvature and the length of every component of ∂Q are bounded.

The *billiard* in Q is the dynamical system generated by the uniform motion of a point-particle inside Q with elastic reflection at ∂Q so that the angle of reflection equals the angle of incidence. Such a system can be described either by a flow or a map. In this paper, we will be concerned with the billiard map, whose definition is given in the next sections. For the relationship between the ergodic properties of the billiard map and the billiard flow, we refer the reader to the book [7].

2.1. **Billiard map.** Denote by $|\Gamma_i|$ the length of the component Γ_i. Let $I_i = [0, |\Gamma_i|]$, and let $\gamma_i : I_i \to \mathbb{R}^2$ be the parametrization of Γ_i by arc-length such that the domain Q stays on the left of $\gamma_i'(s)$. If Γ_i is a closed curve, then $\gamma_i(0) = \gamma_i(|\Gamma_i|)$. For each $i = 1, \ldots, n$, define

$$M_i = I_i \times [0, \pi],$$

with the elements $(0, \theta)$ and $(|\Gamma_i|, \theta)$ identified for every $\theta \in [0, \pi]$ when Γ_i is C^3 diffeomorphic to S^1. Hence, M_i is either a rectangle or a cylinder. Let M be the disjoint union of $M_1, \ldots, M_n$, i.e.,

$$M = \bigsqcup_{i=1}^n M_i.$$

Accordingly, an element $x \in M$ is the ordered pair $(i, (s, \theta))$ with $1 \leq i \leq n$. The set M is a smooth manifold with boundary. We define $s(x) = s$ and $\theta(x) = \theta$ for $x = (i, (s, \theta)) \in M$. An element $x \in M$ will be often called a *state* or a *collision*.

We denote by M_+, M_-, M_0 the subsets of M obtained by taking the disjoint unions of sets M_i with Γ_i being focusing, dispersing or flat, respectively.

Given $x = (i, (s, \theta)) \in M$, let $q(x) = \gamma_i(s) \in \Gamma_i$, and let $v(x)$ be the unit vector of $\mathbb{R}^2$ such that the oriented angle between $\gamma_i'(s)$ and $v(x)$ is equal to θ. Also, let $\kappa(x)$ be the curvature of Γ_i. Then $r(x) = 1/|\kappa(x)|$ is the radius of curvature of Γ_i. Next, for every $x \in M$ and every $\tau > 0$, denote by $(q(x), q(x) + \tau v(x))$ the open segment of $\mathbb{R}^2$ with endpoints $q(x)$ and $q(x) + \tau v(x)$, and define

$$\rho(x) = \{\tau > 0 \colon (q(x), q(x) + \tau v(x)) \subset Q\}.$$

Define $t \colon M \to [0, +\infty)$ and $q_1 \colon M \to \mathbb{R}^2$ as follows:

$$t(x) = \begin{cases} \sup \rho(x) & \text{if } \rho(x) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$q_1(x) = q(x) + t(x)v(x).$$

Note that $q_1(x) = q(x)$ whenever the following conditions are simultaneously satisfied: Γ_i is flat, $q(x) \notin \partial\Gamma_i$ and $\theta(x) \in \{0, \pi\}$. Also, note that $q_1(x) = q(x)$ whenever

$v(x)$ is pointing outside Q. Let

$$A_1 = \{x \in M : q(x) \text{ is a corner of } \partial Q\},$$
$$A_2 = \{x \in M : \theta(x) \in \{0, \pi\}\},$$
$$A_3 = \{x \in M : q_1(x) \text{ is a corner of } \partial Q\}.$$

Given $x \in M \setminus A_3$, there exists a unique $i_1(x)$ such that $q_1(x)$ belongs to the interior of $\Gamma_{i_1(x)}$. Let $s_1(x) = \gamma^{-1}_{i_1(x)}(q_1(x))$, and define

$$v_1(x) = -v(x) + 2(\gamma'_{i_1(x)}(s_1(x)) \cdot v(x))\gamma'_{i_1(x)}(s_1(x)).$$

Let $\theta_1(x) \in [0, \pi]$ be the oriented angle between $\gamma'_{i_1(x)}(s_1(x))$ and $v_1(x)$. The *billiard map* $T : M \setminus A_3 \to M$ for the domain Q is given by

$$T(x) = (i_1(x), (s_1(x), \theta_1(x))) \qquad \text{for } x \in M \setminus A_3.$$

The map T is not invertible, because there are pairs of points $x = (i, (s, \theta)) \in M \setminus A_3$ and $y = (j, (\bar{s}, \bar{\theta})) \in M \setminus A_3$ with $i \neq j$ and $q(x) = q(y)$ being a corner of Q such that $T(x) = T(y)$. Note however that the restriction of T to $M \setminus (A_1 \cup A_3)$ is invertible. Thus, even though T^{-1} is a multivalued transformation, the restriction of T^{-1} to $T(M \setminus (A_1 \cup A_3))$ is a well defined transformation.

An important property of T is of being time reversible. This means that there exists an involution $\mathcal{I} : M \to M$ given by $\mathcal{I}(i, (s, \theta)) = (i, (s, \pi - \theta))$ such that $\mathcal{I} \circ T = T^{-1} \circ \mathcal{I}$ on $M \setminus (A_1 \cup A_3)$. For notational purposes, we write $-x$ instead of $\mathcal{I}(x)$. Thus, if A is a subset of M, then $-A$ will denote the set $\{-x : x \in A\}$.

Finally, let

$$A_4 = \{x \in M \setminus A_3 : \theta_1(x) \in \{0, \pi\}\}.$$

2.2. **Singular sets.** The boundary of M is given by $\partial M = A_1 \cup A_2$. Let

$$S_1^+ = A_3 \cup A_4 \qquad \text{and} \qquad S_1^- = -S_1^+.$$

Also, define

$$R_1^+ = \partial M \cup S_1^+ \quad \text{and} \quad R_1^- = -R_1^+.$$

Note that $R_1^- = \partial M \cup S_1^-$. The sets $S_1^+, S_1^-, R_1^+, R_1^-$ are all compact. Indeed, for the billiards considered in this paper, the sets R_1^+ and R_1^- consist of finitely many arcs [16, Theorem 6.1, Part V] (see also [7, Theorem 9.29]), and the transformation $T : M \setminus R_1^+ \to M \setminus R_1^-$ is a C^2 diffeomorphism [16, Corollaries 4.1 and 4.4, Part V].

For every $m \geq 1$, define

$$R_{m+1}^+ = R_m^+ \cup T^{-1}(R_m^+) \quad \text{and} \quad R_{m+1}^- = R_m^- \cup T(R_m^-). \tag{1}$$

The set R_m^+ contains the points of M where the map T^m is either not defined or not differentiable. The same is true for R_m^- with T^m replaced by T^{-m}. The sets R_m^+ and R_m^- are called the *singular sets* of T^m and T^{-m}, respectively. By the time-reversibility of T, we have $R_m^- = -R_m^+$. It is easy to see that R_m^+ and R_m^- are compact.

It can be easily checked that $A_1 \cup T(\partial M) = A_1 \cup S_1^-$. Hence, $\partial M \cup T(\partial M) = \partial M \cup S_1^- = R_1^-$. Note that the analogous equality with T, S_1^-, R_1^- replaced by T^{-1}, S_1^+, R_1^+ does not hold, because the transformation T^{-1} is not defined on A_1. If we define $R_0^- = \partial M$, then the relation on right-hand side of (1) holds for every $m \geq 0$. Iterating, we obtain

$$R_m^- = \partial M \cup T(\partial M) \cup \cdots \cup T^m(\partial M). \tag{2}$$

2.3. **Notation and differential of T.** The transformation $x \mapsto (s(x), \theta(x)) \in \mathbb{R}^2$ is a coordinate map, and the pair $(s, \theta) \in \mathbb{R}^2$ forms a system of local coordinates for M. To simplify the notation, we identify each $x \in M$ with the corresponding pair (s, θ), hence dropping the index i from the representation $x = (i, (s, \theta))$. When we need to specify i, we will write '$x \in M_i$'. Also, we identify the tangent space $T_x M$ with $\mathbb{R}^2$ so that if $u \in T_x M$, then $u = (u_s, u_\theta)$. Finally, we identify the differential $D_x T$ with the differential of the transformation $(s, \theta) \mapsto (s_1, \theta_1)$. For every $x \in M \setminus R_1^+$, the matrix of the differential $D_x T$ reads as [7]

$$D_x T = \begin{pmatrix} \dfrac{\kappa(x)t(x) - \sin\theta(x)}{\sin\theta(T(x))} & \dfrac{t(x)}{\sin\theta(T(x))} \\ \dfrac{\kappa(T(x))\kappa(x)t(x) - \kappa(T(x))\sin\theta(x)}{\sin\theta(T(x))} - \kappa(x) & \dfrac{\kappa(T(x))t(x)}{\sin\theta(T(x))} - 1 \end{pmatrix}. \quad (3)$$

2.4. **Main result.** Recall the notion of arc introduced at the beginning of this section.

Definition 2.1. *A set $X \subset M$ is called* regular *if X is an union of finitely many arcs of class C^2 that can intersect only at their endpoints. The collection of such arcs is called a* decomposition *of X.*

We now restrict our attention to a class of billiard domains characterized by two conditions called B1 and B2. A detailed description of these conditions is given in Section 5. Roughly speaking, they require the focusing components of ∂Q to be of a special type called absolutely focusing (see Definition 3.2), and the distance between a focusing component and another boundary component to be sufficiently large. Billiards satisfying conditions B1 and B2 have non-vanishing Lyapunov exponents. This is proved for a subclass of these billiards in [13], but the proof can be easily modified to cover the general case.

The following theorem is the main result of this paper. It concerns of the singular sets R_m^- and R_m^+.

Theorem 2.2. *Suppose that the domain Q satisfies Conditions B1 and B2. Then the singular sets R_m^- and R_m^+ of the billiard map T for the domain Q are regular for every $m > 0$.*

3. Focusing times and absolutely focusing arcs

We now introduce the concept of focusing times of an infinitesimal family of trajectories. This notion is borrowed from geometrical optics, and permits to obtain an intuitive and convenient description of the action of the derivative of the billiard map on the projective line.

For notational purposes, given a submanifold A of M and a point $x \in A$, we write $T_x^* A$ for $T_x A \setminus \{0\}$.

3.1. **Focusing times.** Consider a vector $u \in T_x^* M$ with $x \in M$, and let $a \mapsto \varphi(a) \in M$ be a differentiable curve such that $\varphi(0) = x$ and $\varphi'(0) = u$. We can associate a family ℓ_+ of lines of $\mathbb{R}^2$ to φ by setting

$$a \mapsto \ell_+(a) = \{q(\varphi(a)) + tv(\varphi(a)) \colon t \in \mathbb{R}\}.$$

We can associate a second family ℓ_- of lines of $\mathbb{R}^2$ to φ by replacing the curve φ in the definition of ℓ_+ with the curve $\mathcal{I} \circ \varphi$. In geometrical terms, the family ℓ_- is obtained by reflecting the lines of ℓ_+ in the boundary ∂Q. All the lines $\ell_+(\ell_-)$ around $\ell_+(0)(\ell_-(0))$ intersect in linear approximation at a point $q_+ \in \ell_+(0)(q_- \in \ell_-(0))$. If the derivative of $v(\varphi(a))$ at $a = 0$ vanishes, then the lines $\ell_+(\ell_-)$ around $\ell_+(0)(\ell_-(0))$ are parallel in linear approximation, and we define both q_+ and q_-

to be the point at infinity. Note that q_+ and q_- depend only on the vector u and not on the choice of the curve φ. The point $q_+(q_-)$ is called the *forward(backward) focal point* of u. For more details, see [23, Section 2].

We denote the distance between $q(x)$ and $q_+(q_-)$ by $\tau^+(x,u)(\tau^-(x,u))$, and called it the *forward(backward) focusing time* of u. Now, suppose that $u=(u_s,u_\theta)$ in coordinates (s,θ). Let $m(u)=u_\theta/u_s\in\hat{\mathbb{R}}$, where $\hat{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$ is the one-point compactification of $\mathbb{R}$. A straightforward computation (for example, see [23, Section 2]) shows that

$$\tau^\pm(x,u)=\begin{cases}\frac{\sin\theta(x)}{\kappa(x)\pm m(u)} & \text{if } m(u)\neq\mp\kappa(x),\\ \infty & \text{otherwise.}\end{cases} \tag{4}$$

Hence, $\tau^+(x,\cdot)$ and $\tau^-(x,\cdot)$ are functions from T^*_xM to $\hat{\mathbb{R}}$. We can identify T^*_xM with $\mathbb{R}^2\setminus\{0\}$. Since $\mathbb{R}^2\setminus\{0\}$ and $\hat{\mathbb{R}}$ are both identified with the real projective line $\mathbb{P}(\mathbb{R}^2)$, we see that $\tau^+(x,\cdot)$ and $\tau^-(x,\cdot)$ are projective transformations for every $x\notin A_2$.

Remark 3.1. *Let Σ be a 1-dimensional differentiable submanifold of M, and suppose that $\Sigma\ni x\mapsto u(x)\in\mathbb{R}^2\setminus\{0\}$ is a non-zero continuous vector field. Since the functions $(x,u)\mapsto\tau^+(x,u)$ and $(x,u)\mapsto\tau^-(x,u)$ are continuous on $(M\setminus A_2)\times(\mathbb{R}^2\setminus\{0\})$, it follows immediately that the $\Sigma\ni x\mapsto\tau^+(x,u(x))$ and $\Sigma\ni x\mapsto\tau^-(x,u(x))$ are continuous at each point $x\notin A_2$.*

For the proof of the following properties of the focusing times, we refer the reader to Wojtkowski's paper [23]. The arithmetic involved in the formulae below is the one of $\hat{\mathbb{R}}$.

Let $d(x)=\sin\theta(x)/\kappa(x)$. The reciprocals of $\tau^+(x,u)$ and $\tau^-(x,u)$ are related by the well known *Mirror Equation* from geometrical optics [7]

$$\frac{1}{\tau^+(x,u)}+\frac{1}{\tau^-(x,u)}=\frac{2}{d(x)}. \tag{5}$$

Many arguments regarding billiards simplify considerably if one considers the projectivization of DT, i.e., the action of DT on the projective real line. Since $\tau^+(x,\cdot)$ is a projective transformation for $x\notin A_2$, this action can be effectively obtained by using the transformation

$$\tau^+(x,u)\mapsto\tau^+(T(x),D_xTu)\qquad\text{for } x\in M\setminus R_1^+.$$

Using the fact that $\tau^-(T(x),D_xTu)=t(x)-\tau^+(x,u)$ and the Mirror Formula, we obtain

$$\tau^+(T(x),D_xTu)=\frac{d(T(x))(t(x)-\tau^+(x,u))}{2(t(x)-\tau^+(x,u))-d(T(x))}\qquad\text{for } x\in M\setminus R_1^+. \tag{6}$$

From (6), one can easily deduce that the focusing times of two tangent vectors $u_1,u_2\in T^*_xM$ with $x\in M\setminus R_1^+$ satisfy the monotonicity relation:

$$\begin{aligned}&-\infty<\tau^+(x,u_1)<\tau^+(x,u_2)<t(x)-\frac{d(T(x))}{2}\\ &\iff\quad\frac{d(T(x))}{2}<\tau^+(T(x),D_xTu_1)<\tau^+(T(x),D_xTu_2)<+\infty.\end{aligned} \tag{7}$$

3.2. **Absolutely focusing arcs.** As already mentioned, the billiards considered in this paper are characterized by two conditions (Conditions B1 and B2 in Section 5). The first condition requires the focusing components of ∂Q to be absolutely focusing arcs. These arcs were introduced independently by Donnay and Bunimovich [4, 13]. The formulation of the absolutely focusing property presented here, it is a rephrasing of Donnay's definition in more quantitative terms.

Definition 3.2. *Suppose that Γ_i is a focusing arc of ∂Q. We say that Γ_i is* absolutely focusing *if there exists a function $\zeta\colon M_i \to (0,1)$ such that*

(1) $\int_0^l \kappa(s)ds \le \pi$,

(2) if $\{x, T(x), T^2(x)\} \subset M_i$ (i.e., if $x, T(x), T^2(x)$ are three consecutive collisions with Γ_i), then

$$\frac{1}{(1-\zeta(x))t(x)} + \frac{1}{\zeta(T(x))t(T(x))} \le \frac{2}{d(T(x))},$$

(3) if $\{x, T(x)\} \subset M_i$, then

$$\max\left\{\frac{d(x)}{2\zeta(x)}, \frac{d(T(x))}{2(1-\zeta(x))}\right\} < t(x).$$

Condition (1) of Definition 3.2 means that the tangents to Γ_i at the endpoints of Γ_i form an angle that is not larger than π. This implies that an absolutely focusing component is never closed. The next lemma describes an important consequence of Condition (1). For its proof, see [13, Lemma 1.1].

Lemma 3.3. *Suppose that Γ_i is an absolutely focusing arc of ∂Q. With the possible exception of the periodic trajectories connecting the endpoints of Γ_i, every trajectory hitting Γ_i eventually leaves Γ_i.*

The geometrical meaning of Conditions (2) and (3) is revealed in the following lemma, which can be easily obtained from Conditions (2) and (3) of Definition 3.2 using relations (5) and (7).

Lemma 3.4. *Let Γ_i be an absolutely focusing arc of ∂Q. If we assume that $\{x, T(x), \ldots, T^n(x)\} \subset M_i$ with $n > 0$, and $u \in T_xM_i$ such that $d(x)/2 < \tau^+(x,u) \le \zeta(x)t(x)$, then*

$$0 < \frac{\zeta(x)t(x)d(x)}{2\zeta(x)t(x) - d(x)} \le \tau^-(x,u) < +\infty, \tag{8}$$

$$\frac{d(T^k(x))}{2} < \tau^+(T^k(x), D_xT^ku) \le \zeta(T^k(x))t(T^k(x)) \qquad \text{for } 1 \le k < n, \tag{9}$$

$$\begin{aligned}\frac{d(T^n(x))}{2} &< \tau^+(T^n(x), D_xT^nu) \\ &\le \frac{d(T^n(x))t(T^{n-1}(x))(1-\zeta(T^{n-1}(x)))}{2t(T^{n-1}(x))(1-\zeta(T^{n-1}(x))) - d(T^n(x))} < +\infty.\end{aligned} \tag{10}$$

Combining together Lemmas 3.3 and 3.4, we obtain the following corollary, which gives a characterization of an absolutely focusing arc corresponding to that one in [13].

Corollary 3.5. *Any infinitesimal family $\mathcal{F}$ of parallel rays hitting an absolutely focusing arc Γ_i and not being a variation of the periodic trajectory connecting the endpoints of Γ_i has the following properties: i) $\mathcal{F}$ leaves Γ_i after a finite number of collisions with Γ_i, and ii) $\mathcal{F}$ focuses in linear approximation between every two consecutive reflections with Γ_i and after leaving Γ_i.*

We now present two major examples of absolutely focusing arcs.

Definition 3.6. *A focusing arc Γ_i of ∂Q is called a W-arc (or a convex scatterer) if for every pair of consecutive collisions $\{x, T(x)\} \subset M_i$, we have*

$$d(x) + d(T(x)) \le t(x). \tag{11}$$

It is easy to check that W-arcs satisfy Conditions (2) and (3) of Definition 3.2 with $\zeta(x) = d(x)/t(x)$. Arcs of this type were introduced by Wojtkowski, who also showed that an arc of class C^4 is a W-arc if and only if $d^2r/ds^2 \le 0$ [23]. Examples of W-arcs are arcs of circles, arcs of cardioids, arcs of logarithmic spirals and the elliptical arcs $\{(x,y) \in \mathbb{R}^2 \colon x^2/a^2 + y^2/b^2 = 1 \text{ and } |x| \le a/\sqrt{2}\}$ with $a, b > 0$.

Definition 3.7. *A focusing arc Γ_i of ∂Q is called an M-arc if for every triple of consecutive collisions $\{x, T(x), T^2(x)\} \subset M_i$, we have*

$$\frac{1}{t(x)} + \frac{1}{t(T(x))} \le \frac{1}{d(T(x))}, \tag{12}$$

and for every pair of consecutive collisions $\{x, T(x)\} \subset M_i$, we have

$$\max\{d(x), d(T(x))\} < t(x). \tag{13}$$

M-arcs satisfy Conditions (2) and (3) of Definition 3.2 with $\zeta(x) = 1/2$. M-arcs were introduced by the second author of this paper [19]. Examples of M-arcs that are not W-arcs are the arcs of ellipses $\{(x,y) \in \mathbb{R}^2 \colon x^2/a^2 + y^2/b^2 = 1 \text{ and } |x| \ge b^4/(a^2+b^2)\}$ with $a, b > 0$.

We mention that the set of W-arcs and M-arcs does not comprise all absolutely focusing arcs. An example of an absolutely focusing arc that is not a W-arc or an M-arc is the half-ellipse $\{(x,y) \in \mathbb{R}^2 \colon x^2/a^2 + y^2/b^2 = 1 \text{ and } x \ge 0\}$ with $a/b < \sqrt{2}$ [13, Theorem 7.1].

In this paper, we impose the condition that each focusing component of ∂Q has to be an absolutely focusing arc of class C^6 or a arc of type W or M satisfying Condition (1) of Definition 3.2. All these arcs share the following important property: if Γ is one of these arcs, then every family of parallel rays hitting Γ focuses in linear approximation after leaving Γ at a distance from Γ that is bounded above by a constant depending only on Γ (see Theorem 4.7 for absolutely focusing arcs of class C^6 and Corollary A.8 for arcs of type W and M).

4. Cone fields

For the definition of a symplectic cone field and the associated quadratic form for general symplectic maps with singularities, we refer the reader to [11, 18]. In the 2-dimensional setting, the definitions reduce to the following.

Definition 4.1. *Consider a 2-dimensional vector space V, and let X_1 and X_2 be two linearly independent vectors of V.*

- *The set $C(X_1, X_2) = \{a_1 X_1 + a_2 X_2 \colon a_1 a_2 \ge 0\} \subset V$ is called the* cone generated *by X_1 and X_2.*
- *The set $\operatorname{int} C = \{a_1 X_1 + a_2 X_2 \colon a_1 a_2 > 0\} \cup \{0\}$ is called the* interior *of the cone $C(X_1, X_2)$.*
- *The cone $C'(X_1, X_2) = C(X_1, -X_2)$ is called the* complementary cone *of $C(X_1, X_2)$.*

A cone C can be identified with a closed interval I of the projective space $\mathbb{P}(V)$. Accordingly, $\operatorname{int} C$ and C' can be identified with the interior of I and the closed interval $\mathbb{P}(V) \setminus \operatorname{int} I$, respectively.

Definition 4.2. *Let U be an open set of M, and let X_1 and X_2 be two measurable vector fields on U such that $X_1(x)$ and $X_2(x)$ are linear independent for every $x \in U$.*

- *A* cone field *C on U, which is denoted by (U, C), is a family of cones $\{C(x)\}_{x \in U}$ given by $C(x) = C(X_1(x), X_2(x)) \subset T_x M$ for every $x \in U$. A cone field (U, C) is called* continuous *if the vector fields X_1 and X_2 are continuous on U.*
- *A cone field (U, C) is called* invariant *if $x \in U$ and $T^k(x) \in U$ with $k > 0$ implies that $D_x T C(x) \subset C(T(x))$. A cone field (U, C) is called* eventually strictly invariant *if it is invariant, and for almost every $x \in U$, there*

exists an integer $k(x) > 0$ *such that* $T^{k(x)}(x) \in U$ *and* $D_x T^{k(x)} C(x) \subset \operatorname{int} C(T^{k(x)}(x))$.

Remark 4.3. *Let* C *be a cone field on* $U \subset M \setminus A_2$*. Since the focusing times* $\tau^+(x, \cdot)$ *and* $\tau^-(x, \cdot)$ *are projective transformations for* $x \in U \setminus A_2$*, the cone* $C(x) \in T_x M$ *can be conveniently described as*

$$C(x) = \{u \in T_x^* M \colon \tau^+(x, u) \in I_+(x)\} \cup \{0\}$$

or

$$C(x) = \{u \in T_x^* M \colon \tau^+(x, u) \in I_-(x)\} \cup \{0\}$$

for proper closed intervals $I_-(x)$ *and* $I_+(x)$ *of* $\hat{\mathbb{R}}$.

4.1. **Cone fields for billiards.** The cone fields described in this subsection play a key role in the proof of the regularity of the singular sets. They also play a prominent role in the proof of other properties, like the hyperbolicity and the ergodicity. We start this subsection with some preliminary definitions.

Definition 4.4. *Given a curved component* Γ_i *of* ∂Q*, define*

$$E_i = \{x \in M_i \colon q_1(-x) \notin \Gamma_i\}$$

Since $q_1(-x) \notin \Gamma_i$ *for every* $x \in E_i$*, we say that* x *is an* entering state *of* Γ_i *or simply that* x enters Γ_i*. The union of all the sets* E_i *is denoted by* E.

Definition 4.5. *Suppose that* Γ_i *is a curved component* Γ_i *of* ∂Q*. For every* $x \in M_i \setminus A_2$*, the number*

$$n(x) = \max\{k \geq 0 \colon T^j(x) \in M_i \text{ for } 0 \leq j \leq k \text{ and } q_1(T^k(x)) \notin \Gamma_i\}$$

denotes the number of collisions of x *with* Γ_i *before leaving* Γ_i*. If* $x \in M_i$ *is a periodic point of period two whose trajectory coincides with the segment joining the endpoints of* Γ_i*, then we define* $n(x) = 1$.

Remark 4.6. *Note that* $E_i \setminus A_2$ *is open, and that* $-E_i$ *consists of states leaving* Γ_i*. According to definition 4.5, we have* $n(x) = 0$ *whenever* $x \in -E_i$.

Billiard cone fields are usually defined almost everywhere on the entire set M. However, as it will be clear from our arguments, it suffices to define them only on the set $\overline{E}$. Accordingly, we will consider cone fields of the form (U_x, C_x) with $U_x \subset M$ being a neighborhood of $x \in \overline{E}$. To facilitate the comprehension of the geometrical meaning of these cones, we provide several equivalent definitions, using the slope m and the focusing times τ^+ and τ^- of tangent vectors. We write $U_x^* = U_x \setminus A_2$ and $C_x^*(y) = C_x(y) \setminus \{0\}$ for $y \in U_x$. To define the cone $C_x(y)$, it is enough to define $C_x^*(y)$.

4.1.1. *Dispersing components.* Suppose that Γ_i is a dispersing component of ∂Q. Then, for every $x \in \overline{E}_i$, we choose $U_x = M_i$, and set

$$C_x^*(y) = \left\{u \in T_y^* M_i \colon m(u) \leq \kappa(x)\right\} \qquad \text{for all } y \in U_x.$$

Using relation (4), we see that if $y \in U_x^*$, then $C_x^*(y)$ can be equivalently written as

$$C_x^*(y) = \left\{u \in T_y^* M_i \colon 0 \leq \tau^-(y, u) \leq +\infty\right\},$$

or

$$C_x^*(y) = \left\{u \in T_y^* M_i \colon d(y)/2 \leq \tau^+(y, u) \leq 0\right\}.$$

This cone field (U_x, C_x) is clearly continuous. From the geometrical point of view, the cone $C_x(y)$ consists of tangent vectors u focusing inside the osculating disk with radius $|r(y)|/4$ of Γ_i at the point $q(y)$. A similar cone fields for dispersing components was first introduced in [23].

4.1.2. *Focusing components.* The cone field (U_x, C_x) for focusing components is borrowed from [13]. Since its construction is very well detailed in Sections 4 and 5 of Donnay's paper, we limit ourselves here to summarize its main properties in the following theorem. Note that in this theorem, the focusing components are arcs of class C^6.

Theorem 4.7. *Suppose that Γ_i is an absolutely focusing component of class C^6. Then there exist a family $\{(U_x, C_x)\}_{x\in\overline{E}_i}$ of continuous cone fields with $U_x \subset M_i$ being a neighborhood of x, a family $\{g_x\}_{x\in\overline{E}_i}$ of continuous functions $g_x : U_x \to \mathbb{R}$ and two positive constants t_i^- and t_i^+ such that if $x \in \overline{E}_i$, then*

(1) for every $y \in U_x^*$, we have

$$C_x^*(y) = \left\{u \in T_y^* M_i \colon g_x(y) \le m(u) \le \kappa(y)\right\},$$

(2) for every $y \in U_x^*$, $u \in C_x^*(y)$ and $0 \le k < n(y)$, we have

$$\frac{d(T^k(y))}{2} \le \tau^+(T^k(y), D_y T^k u) < t(T^k(y)) - \frac{d(T^{k+1}(y))}{2},$$

(3) we have

$$\sup_{y\in U_x^*} \inf_{u\in C_x^*(y)} \tau^-(y,u) \le t_i^-,$$

and

$$\sup_{y\in U_x^*} \sup_{u\in C_x^*(y)} \tau^+(T^{n(y)}(y), D_y T^{n(y)} u) \le t_i^+.$$

Proof. We explain where the proof of the single conclusions can be found in [13]. The cone $C_x(y)$ described in Part (1) of the theorem corresponds the cone defined in (4.7) of [13]. The cone field C_x is constructed in [13, Theorem 4.4] (see also [13, Lemma 5.9]). The continuity of the cone field C_x is not explicitly formulated in Donnay's paper, but it is an obvious consequence of its construction in the proof of [13, Theorem 4.4], and it follows from the continuity of the function g_x (called m_l by Donnay) on a sufficiently small neighborhood of x, which we denote by U_x. Part (2) follows from the construction of C_x, namely from relations 4.3-4.5 and [13, Proposition 4.1]. Part (3) is one of the conclusion of [13, Theorem 4.4]. □

Remark 4.8. *The following remarks clarify the conclusion of Theorem 4.7.*

- *If $y \in U_x^*$, then the cone $C_x^*(y)$ can be equivalently defined as*

$$C_x^*(y) = \left\{u \in T_y^* M_i \colon d(y)/2 \le \tau^+(y,u) \le G_x^+(y)\right\},$$

or

$$C_x^*(y) = \left\{u \in T_y^* M_i \colon G_x^-(y) \le \tau^-(y,u) \le +\infty\right\},$$

where

$$G_x^{\pm}(y) = \frac{\sin\theta(x)}{(\kappa(x) \pm g_x(y))}$$

are the forward and backward focusing times of vectors of $C_x^*(y)$ with slope $g_x(y)$. Note also that the forward and backward focusing times of the vectors of $C_x^*(y)$ with slope $\kappa(x)$ are $d(y)/2$ and ∞, respectively.

- *Part (2) of the theorem means that each vector of $C_x^*(y)$ with $y \in U_x^*$ focuses between two consecutive collisions of the trajectory of y along Γ_i.*
- *The two inequalities of Part (3) have the following interpretation. The first inequality amounts to saying that every infinitesimal family of trajectories entering Γ_i with a backward focusing time equal to or greater than t_i^- corresponds to a tangent vector of the cone field (C_x, U_x). The second inequality can be rephrased by saying that each infinitesimal family of trajectories, specified by a tangent vector $u \in C_x^*(y)$ with $y \in U_x$, focuses is*

linear approximation when it leaves the Γ_i, and its forward focusing time is uniformly bounded above by a constant t_i^+ depending only on Γ_i.
- *The constants t_i^- and t_i^+ depend continuously on Γ_i in the C^6 topology (see* [13, Theorem 4.4]*).*

In Corollary A.8, we extend Theorem 4.7 to W-arcs and M-arcs.

5. Conditions on the billiard domains

In this section, we complete the description of the hypotheses of (i.e., Conditions B1 and B2) Theorem 2.2. We also show that under these hypotheses, the cone fields introduced in Subsection 4.1 are strictly invariant along every piece of orbits starting and ending at elements of E. This property is used in the proof of Theorem 2.2.

B1 (Non-polygonal domains): The domain Q is not a polygon, and the components of ∂Q are of the following types: straight segments, dispersing arcs of class C^3, absolutely focusing arcs of class C^6, W-arcs and M-arcs of class C^3 satisfying Condition (1) of Definition 3.2.

B2 (Distance between boundary components): We assign a pair of numbers (τ_i^+, τ_i^-) to each curved component Γ_i as follows: set $\tau_i^+ = \tau_i^- = 0$ when Γ_i is dispersing, and $\tau_i^+ = t_i^+$ and $\tau_i^- = t_i^-$ with t_i^+ and t_i^- as in Theorem 4.7 when Γ_i is absolutely focusing. Then, given two curved components Γ_i and Γ_j, denote by $t_{ij} \geq 0$ the infimum of the Euclidean length in $\mathbb{R}^2$ of all finite billiard orbits $\{x_0, \ldots, x_n\}$ with $n > 0$ such that $x_0 \in M_i$ and $x_n \in M_j$. We assume that

(1) there exists a real $\tau > 0$ such that if Γ_i and Γ_j are curved components, and one of the them is focusing, then
$$t_{ij} \geq \tau_i^+ + \tau_j^- + \tau,$$
(2) the distance between every focusing component Γ_i and every corner of ∂Q not belonging to Γ_i is greater than $\max\{\tau_i^-, \tau_i^+\}$.

Remark 5.1. *Some comments on Conditions B1 and B2 are in order.*

- *Condition B2 has a couple of obvious consequences for the geometry of Q: i) the internal angle between a focusing component and an adjacent curved component is greater than π, and ii) the internal angle between a focusing component and an adjacent flat component is greater than $\pi/2$.*
- *Conditions B1 and B2 allow ∂Q to have cusps formed by two dispersing components or a dispersing and a flat component.*
- *If the focusing components are W-arcs or M-arcs, then the first part of condition B2 has a simple geometric formulation in terms of the relative position of the circles of semi-curvature of distinct focusing components* [2, 23]*, or in terms of the distance of the circles of curvature of the focusing components from the other components* [19].
- *To our knowledge, all the 2-dimensional hyperbolic billiards constructed so far satisfy these conditions with the exception of two cases* [5, 6].
- *We are not able to prove Theorem 2.2 without Part (2) of Condition B2. This assumption is specifically used to prove that the tangent spaces of the sets R_k^- and R_k^+ are contained inside the cones C_x and C'_x, respectively. This property is called alignment and permits to show that R_1^+ and R_k^- are transversal. The alignment it also one of the hypotheses of the Local Ergodic Theorem* [11, 18].

In the next lemma, we show that the cone field $\{(U_x, C_x)\}_{x\in\overline{E}}$ is strict invariant along a piece of an orbit starting and ending at an element of E. This property plays an important role in the proof of Proposition 6.2.

Lemma 5.2. *Suppose that the billiard in Q satisfies Conditions B1 and B2. Also, suppose that there exist $x_1 \in \overline{E}_{i_1}$, $x_2 \in \overline{E}_{i_2}$ with $i_1 \neq i_2$, and $y \in E_{i_1} \cap U_{x_1} \setminus R^+_k$ for some $k > 0$ such that $T^k(y) \in E_{i_2} \cap U_{x_2}$. Then*

$$D_y T^k C_{x_1}(y) \subset \operatorname{int} C_{x_2}(T^k(y)).$$

Proof. Let $u \in C_{x_1}(y)$. First, suppose that $T^j(y) \in M_0$ for every $n(y) < j < k$ whenever $k - n(y) > 1$. In this case, it is easy to see that $T^k(y) \in E$ and

$$\tau^-(T^k(y), D_{T^k(y)}T^k u) = l - \tau^+(T^{n(y)}(y), D_y T^{n(y)} u),$$

where l is the length of the piece of the billiard trajectory starting at $T^{n(y)}(y)$ and ending at $T^k(y)$. Now, let Γ_{i_1} and Γ_{i_2} be the components of ∂Q such that $x \in M_{i_1}$ and $T^k(x) \in M_{i_2}$. Since $\tau^+(T^{n(y)}(y), D_{T^{n(y)}(x)}T^{n(y)}u) \leq \tau^+_{i_1}$ by the definition of $\tau^+_{i_1}$, and $\inf_{u\in C^*_{x_2}(T^k(y))} \tau^-(T^k(y), u) \leq \tau^-_{i_2} < l - \tau^+_{i_1}$ by Condition B2, it follows that

$$\begin{aligned}\tau^-(T^k(y), D_{T^k y}T^k u) &= l - \tau^+(T^{(n(y))}(y), D_y T^{n(y)} u)\\ &\geq l - \tau^+_{i_1} > \tau^-_{i_2}\\ &\geq \inf_{u\in C^*_{x_2}(T^k(y))} \tau^-(T^k(y), u).\end{aligned}$$

Remembering the definition of $C_{x_2}(T^k(y))$, we can conclude that $D_{T^k y}T^k u \in \operatorname{int} C_{x_2}(T^k(y))$. This clearly implies that

$$D_y T^k C_{x_1}(y) \subset \operatorname{int} C_{x_2}(T^k(y)). \tag{14}$$

Next, we consider the alternative case when $T^j(y) \notin M_0$ for some $n(y) < j < k$ whenever $k - n(y) > 1$, or equivalently when $T^j(y) \in E$ for some $n(y) < j < k$ whenever $k - n(y) > 1$. Let $n(y) < k_1 < \cdots < k_n = k$ be integers such that $T^{k_i}(y) \in E$ and $T^j(y) \notin E$ for $k_i < j < k_{i+1}$ whenever $k_{i+1} - k_i > 1$ and for $0 < i < n$. Using conclusion (14) from the previous case, we obtain

- $D_y T^{k_1} C_{x_1}(y) \subset \operatorname{int} C_{T^{k_1}(y)}(T^{k_1}(y))$,
- $D_{T^{k_i}(y)} T^{k_{i+1}-k_i} C_{T^{k_i}(y)}(T^{k_i}(y)) \subset \operatorname{int} C_{T^{k_{i+1}}(y)}(T^{k_{i+1}}(y))$ for $0 < i < n-1$,
- $D_{T^{k_{n-1}}(y)} T^{k-k_{n-1}} C_{T^{k_{n-1}}(y)}(T^{k_{n-1}}(y)) \subset \operatorname{int} C_{x_2}(T^k(y))$.

From these chain of inequalities, we can easily conclude that

$$D_y T^k C_{x_1}(y) \subset \operatorname{int} C_{x_2}(T^k(y)).$$

□

6. Regularity

This section is devoted to the proof of Theorem 2.2. From now on, we assume that the billiard domain Q satisfies Conditions B1 and B2.

6.1. **Outline of the proof of Theorem 2.2.** Because of the generality of the billiard domains considered, to prove Theorem 2.2, we have to analyze individually many different cases. As a consequence, the proofs of this theorem and its preliminary results contained in this section are quite lengthy. To help the reader understand the proof of Theorem 2.2, we briefly describe here its general structure.

First of all, we observe that it suffices to prove the theorem only for the sets R^-_k. Indeed, the theorem then extends automatically to the set R^+_k by the time reversibility of T. The proof of Theorem 2.2 is by induction. Accordingly, we first assume that a given singular set R_k^- is regular. Then, we show that the intersection of the sets S_1^+ and R_k^- consists of finitely many elements. This fact allows us to write R_k^- as a finite union of arcs $\Sigma_1, \dots, \Sigma_n$ of class C^2 that can intersect each other or S_1^+ only at their endpoints.

Since T is a C^2 transformation on $M \setminus S_1^+$, it follows that each set $T(\Sigma_i)$ is a curve of class C^2. From this, we can easily conclude that $T(R_k^-) = \bigcup_{i=1}^n T(\Sigma_i)$ consists of finitely many curves of class C^2. However, this does not complete the proof, because we really need to show that the closure $\overline{T(\Sigma_i)}$ of each curve $T(\Sigma_i)$ is an arc of class C^2, and that the arcs $\overline{T\Sigma_1}, \dots, \overline{T\Sigma_n}$ can only intersect each other or ∂M at their endpoints. This would follow immediately if the restriction of T to each connected component of $M \setminus R_1^+$ admitted a C^2 extension up to its boundary, but the billiard map T does not have this property for every domains considered in this paper. The central part of the proof of Theorem 2.2 consists exactly in proving that the sets $\overline{T\Sigma_1}, \dots, \overline{T\Sigma_n}$ have the property described above. This is achieved in Propositions 6.4 and 6.7. The difficulty of the proof of Proposition 6.7 is due to two main reasons: the unboundedness of the derivative of T near the set S_1^+, and the generality of the billiard domains considered.

An important part of the proof of Theorem 2.2 consists in showing that the set $R_k^- \cap S_1^+$ is finite. This is obtained by proving that R_k^- and S_1^+ are transversal. The crucial step to prove this property is Proposition 6.2, where we show that the singular sets R_k^- and R_k^+ are properly aligned with respect to the cone field (U_x, C_x) introduced in Section 4.1.

6.2. **Singular arcs.** We now introduce the notion of a singular arc. Roughly speaking, a singular arc is an arc of class C^2 contained in R_m^+ for some $m > 0$.

Definition 6.1. *An arc Σ of class C^2 is called* negatively singular *if either $\Sigma \subset \partial M$ or there exist $j > 0$ arcs $\Sigma_0, \dots, \Sigma_{j-1}$ such that after defining $\Sigma_j = \Sigma$, we have*

- $\Sigma_0 \subset \partial M$,
- $\Sigma_i \cap S_1^+ \subset \partial \Sigma_i$ *for* $0 \le i < j$,
- $\operatorname{int} \Sigma_{i+1} = T(\operatorname{int} \Sigma_i)$ *for* $0 \le i < j$,
- $\Sigma_i \cap \partial M \subset \partial \Sigma_i$ *for* $1 \le i \le j$.

A positively singular *arc is defined similarly by replacing T with T^{-1}.*

In the next lemma, we prove that under proper conditions the vectors tangent to the negatively singular arcs and to the positively singular arcs are contained inside the cones of the field (U_x, C_x) and (U_x, C'_x), respectively.

Proposition 6.2. *Suppose that Σ is a negatively singular arc such that $\Sigma \not\subset \partial M$. Then, for every $z \in \overline{E}$ and every $y \in \overline{E} \cap U_z \cap \Sigma \setminus A_2$ that is an accumulation point of $E \cap U_z \cap \operatorname{int} \Sigma$, we have*

$$T_y \Sigma \subset \operatorname{int} C_z(y).$$

A similar conclusion with $C_z(y)$ replaced by $C'_z(y)$ holds under the same hypotheses when Σ is a positively singular arc.

Proof. We limit ourselves to prove the proposition for a negatively singular arc Σ. The other case can be proved in a similar fashion. Let $\Sigma_0, \dots, \Sigma_j = \Sigma$ be the arcs associated to Σ as in Definition 6.1, and let $z \in \overline{E}$. Note that $j > 0$ because $\Sigma \not\subset \partial M$ by assumption. First, we prove the proposition for $y \in E \cap U_z \cap \operatorname{int} \Sigma$. The assumption of the proposition is clearly satisfied for such an y. Since $y \in \operatorname{int} \Sigma$, there exist $y_0, \dots, y_j$ with $y_i \in \operatorname{int} \Sigma_i$ for each $0 \le i \le j$ and $y_j = y$ such that $T(y_{i-1}) = y_i$ for each $1 \le i \le j$. Note that $y_i \notin \partial M$, because $\Sigma_i \cap \partial M \subset \partial \Sigma_i$ for each $1 \le i \le j$. Let k be the smallest integer in $\{1, \dots, j\}$ such that $y_k \in E$. Such a k exists because $y_j \in E$ by assumption. If $y_1, \dots, y_{k-1} \in M_0$, then define $m = 0$, otherwise, define m to be largest integer in $\{1, \dots, k-1\}$ such that $y_m \notin M_0$. Let Γ_{i_1} and Γ_{i_2} be the components of ∂Q for which $y_m \in M_{i_1}$ and $y_k \in M_{i_2}$. If $m > 0$, then it is not difficult to see that Γ_{i_1} is focusing, that $y_1, \dots, y_m$ are consecutive collisions with Γ_{i_1}, and that $y_m \in -E$ (i.e., the state y leaves a curved component of ∂Q). From this observation, we deduce that $i_1 \neq i_2$ for $m > 0$.

Next, we claim that

$$0 \leq \tau^+(y_m, T^*_{y_m}\Sigma_m) \leq \tau^+_{i_1}. \tag{15}$$

This is obvious for $m = 0$. In fact, $\tau^+(y_0, T^*_{y_0}\Sigma_0) = 0$ as a consequence of the fact that Σ_0 consists of states that are either emerging from the same endpoint of one of the components of ∂Q or tangent to the same dispersing component of ∂Q. For $m > 0$ instead, we have $0 < \tau^+(y_m, T^*_{y_m}\Sigma_m) \leq \tau^+_{i_1}$ by using Part (3) of Theorem 4.7 and inequality (7).

From the definitions of k and m, we see that $y_{m+1}, \dots, y_{k-1} \in M_0$ whenever $k > m+1$. Therefore $\tau^-(y_k, T^*_{y_k}\Sigma_k) = l - \tau^+(y_m, T^*_{y_m}\Sigma_m)$, where l is the length of the piece of trajectory starting at y_m and ending at y_k. By (15) and Condition B2, we obtain $\tau^-(y_k, T^*_{y_k}\Sigma_k) = l > \tau^-_{i_2}$ if $m = 0$, and $\tau^-(y_k, T^*_{y_k}\Sigma_k) \geq l - \tau^+_{i_1} > \tau^-_{i_2}$ otherwise. Hence, in any case, we obtain $\tau^-(y_k, T^*_{y_k}\Sigma_k) > \tau^-_{i_2}$, which, in view of the definition of $C_{y_k}(y_k)$ and $\tau^-_{i_2}$, implies that $T^*_{y_k}\Sigma_k \subset \operatorname{int} C_{y_k}(y_k)$. By applying Lemma 5.2 to $y_k, y \in E$ with $x_1 = y_k$ and $x_2 = z$, we can finally conclude that $T^*_y\Sigma \subset \operatorname{int} C_z(y)$.

We are now ready to prove the proposition in its full generality. So, let $y \in \overline{E} \cap U_z \cap \Sigma \setminus A_2$ be an accumulation point of $E \cap U_z \cap \operatorname{int}\Sigma$. Then there exists a sequence $w_n \in E \cap U_z \cap \operatorname{int}\Sigma$ such that $w_n \to y$ as $n \to +\infty$. Our previous conclusion implies that $T^*_{w_n}\Sigma \subset \operatorname{int} C_z(w_n)$ for $n > 0$. There two cases: $\Sigma \subset M_+$ and $\Sigma \subset M_-$. We discuss only the first case, because the second one can be studied in a similar fashion. So, suppose that $\Sigma \subset M_+$. Then by Condition B2, it follows immediately that

$$\tau^-(w_n, T^*_{w_n}\Sigma) \geq \tau^-_i + \tau \qquad \text{for } n > 0, \tag{16}$$

where Γ_i is the focusing component such that $\Sigma \subset M_i$. Note that the focusing time $\tau^-(w_n, T^*_{w_n}\Sigma)$ is well defined because $w_n \notin \partial M$. By Remark 3.1, we have $\tau^-(y, T^*_y\Sigma) = \lim_{n\to+\infty} \tau^-(w_n, T^*_{w_n}\Sigma)$. Taking the limit as $n \to +\infty$ on the left hand-side in (16), we obtain $\tau^-(y, T^*_y\Sigma) > \tau^-_i$, which gives $T^*_y\Sigma \subset \operatorname{int} C_z(y)$. $\square$

6.3. **Local coordinates for M.** Suppose that Σ_1 is a negatively singular arc such that $\Sigma_1 \cap S^+_1 \subset \partial\Sigma_1$. In this case, each set $q(\Sigma_1)$ and $q_1(\operatorname{int}\Sigma_1)$ is contained in a single component of ∂Q. Let Γ_1 and Γ_2 be the boundary components containing $q(\Sigma_1)$ and $q_1(\operatorname{int}\Sigma_1)$, respectively. Since boundary components are compact, the entire closure of $q_1(\operatorname{int}\Sigma_1)$ is contained in Γ_2. Given $z_1 \in \Sigma_1$, let $p_1 = q(z_1)$, and let p_2 be the intersection point between Γ_2 and the ray emerging from z_1 having the minimum distance from p_1.

To prove Theorem 2.2, we have to study the image of Σ_1 under the billiard map T for all possible choices of Γ_1 and Γ_2. For computational purposes, it is convenient to have an analytic description of the configuration formed by Γ_1 and Γ_2 in a proper Cartesian coordinates system (X, Y) for $\mathbb{R}^2$. In doing so, we also obtain a new system of local coordinates for M. We can choose the Cartesian coordinate system (X, Y) (see Fig. 1) so that

- $p_1 = (0, 0)$ and $p_2 = (\bar{X}, 0)$ with $\bar{X} \leq 0$ in coordinates (X, Y);
- there exist two closed intervals U_1 and U_2 of $\mathbb{R}$ with U_1 containing 0 and U_2 containing $\bar{X}$, and two C^3 functions $f_1 : U_1 \to \mathbb{R}$ and $f_2 : U_2 \to \mathbb{R}$ with $f_1(0) = f_2(\bar{X}) = 0$ such that $\Gamma_1 = \operatorname{graph} f_1$ and $\Gamma_2 = \operatorname{graph} f_2$; moreover, we have either $f_1'' \equiv 0$ (Γ_1 flat) or $f_1'' > 0$ on U_1 (Γ_1 curved), and either $f_2'' \equiv 0$ (Γ_2 flat) or $|f_2''| > 0$ on U_2 (Γ_2 curved).

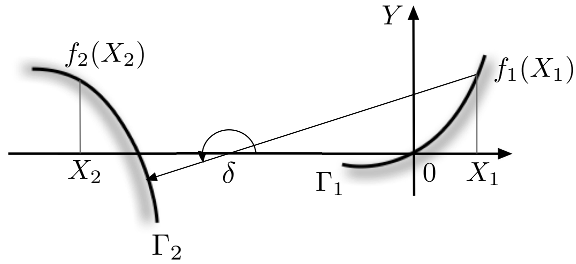


FIGURE 1. Γ_1 and Γ_2 in the Cartesian coordinate system (X, Y)

Given $x \in M_1 \cup M_2$, let X be the X-coordinate of $q(x)$, and let $\delta \in [0, 2\pi)$ be the angle between the X-axis and the vector $v(x)$. Recall that $M_1 \cup M_2$ is equipped with the local coordinates (s, θ). The pair (X, δ) forms another local coordinate system on $M_1 \cup M_2$. In coordinates (X, δ), we clearly have $z_1 = (0, \pi)$. The relation between (s, θ) and (X, δ) on M_1 can be easily derived, and reads as follows

$$\begin{aligned} s &= \int_0^X \sqrt{1 + f_1'^2(\tilde{X})} d\tilde{X} \\ \theta &= \begin{cases} \delta - \tan^{-1} f_1'(X) & \text{if } f_1'(X) \geq 0, \\ -\delta + \tan^{-1} f_1'(X) + 2\pi & \text{if } f_1'(X) < 0. \end{cases} \end{aligned} \tag{17}$$

The transformation $(s, \theta) \mapsto (X, \delta)$ is a C^2 change of coordinates.

Since the arc Σ_1 is an embedded closed interval, there exist a closed interval $I_1 = [a_1, b_1]$ with $a_1 < b_1$ and two C^2 functions $X_1 \colon I_1 \to \mathbb{R}$ and $\delta_1 \colon I_1 \to \mathbb{R}$ with $(X_1'(a), \delta_1'(a)) \neq (0, 0)$ for all $a \in I_1$ such that $\gamma_1 := (X_1, \delta_1) : I_1 \to \Sigma_1$ is a C^2 diffeomorphism. Choose $\bar{a} \in [a_1, b_1]$ so that $\gamma_1(\bar{a}) = z_1$. The value $\bar{a}$ is fixed throughout this entire section. By the choice of the coordinate system (X, Y), it follows that $X_1(\bar{a}) = 0$ and $\delta_1(\bar{a}) = \pi$.

In the next lemma, we compute the forward focusing time of $\gamma_1'(a)$ in coordinates (X, δ).

Lemma 6.3. *We have*

$$\tau^+(\gamma_1(a), \gamma_1'(a)) = \begin{cases} [\tan \delta_1(a) - f_1'(X_1(a))] \cos \delta_1(a) \dfrac{X_1'(a)}{\delta_1'(a)} & \text{if } \delta_1'(a) \neq 0, \\ \infty & \text{if } \delta_1'(a) = 0. \end{cases} \tag{18}$$

Proof. For every $a \in I_1$, denote by $s_1(a)$ and $\theta_1(a)$ the components of $\gamma_1(a)$ in the coordinates (s, θ). From (17), we obtain

$$s_1'(a) = X_1'(a) \sqrt{1 + f_1'^2(X_1(a))}, \tag{19}$$

$$\theta_1'(a) = \pm \left(\delta_1' - \frac{f_1''(X_1(a)) X_1'(a)}{1 + f_1'^2(X_1(a))} \right), \tag{20}$$

where the sign in (20) is positive if $f_1'(X_1) \geq 0$ and negative otherwise. Note that the curvature of Γ_1 at the point $q(\gamma_1(a))$ is given by

$$\kappa(\gamma_1(a)) = \pm \frac{f_1''(X_1(a))}{[1 + f_1'^2(X_1(a))]^{3/2}}, \tag{21}$$

where the sign follows the same rule as in (20). This rule agrees with our convention on the curvature of the boundary components specified at the beginning of Section 2. By replacing (19)-(21) in Formula (4), we finally obtain (18). $\square$

6.4. **Diffeomorphic extension of T on singular arcs.** It is not true in general that a billiard map admits a homeomorphic extension up to the boundary of each of its continuity domains. This can be already seen for the billiard map of the Bunimovich's stadium. But it is true that the map T admits a homeomorphic extension along every singular arcs. This property is proved in the next lemma.

Proposition 6.4. *Let Σ_1 be a negatively singular arc such that $\Sigma_1 \cap S_1^+ \subset \partial\Sigma_1$. There exist a unique curve Σ_2 and a unique homeomorphism $\bar{T}\colon \Sigma_1 \to \Sigma_2$ such that $\Sigma_2 \cap \partial M \subset \partial\Sigma_2$ and $\bar{T}|_{\operatorname{int}\Sigma_1} = T|_{\operatorname{int}\Sigma_1}$.*

Proof. Since T is a continuous bijection on $M \setminus S_1^+$, and Σ_1 and M are compact, it is enough to show that $T|_{\operatorname{int}\Sigma_1}$ extends to a continuous bijection $\bar{T}$ on the entire Σ_1. This can be achieved by showing that $\lim_{\operatorname{int}\Sigma_1 \ni z \to z_1} T(z)$ exists for all $z_1 \in \partial\Sigma_1$. Indeed, the existence of those limits implies at once that the transformation $\bar{T}$ and the set Σ_2 defined by

$$\bar{T}(z_1) = \begin{cases} T(z_1) & \text{if } z_1 \in \operatorname{int}\Sigma_1, \\ \lim_{\operatorname{int}\Sigma_1 \ni z \to z_1} T(z) & \text{if } z_1 \in \partial\Sigma_1, \end{cases}$$

and

$$\Sigma_2 = \bar{T}(\Sigma_1)$$

have the wanted properties. The uniqueness of $\bar{T}$ and Σ_2 is a direct consequence of their definitions. The inclusion $\Sigma_2 \cap \partial M \subset \partial\Sigma_2$ holds, because $\bar{T}z \in \partial M$ implies $z \in \partial\Sigma_1$.

To complete the proof, it remains to prove that $\lim_{\operatorname{int}\Sigma_1 \ni z \to z_1} T(z)$ exists for every $z_1 \in \partial\Sigma_1$. The only case when the existence of this limit is not trivial is when $q_1(\operatorname{int}\Sigma_1)$ belongs to a flat boundary component, and the flat component is contained in the ray emerging from $z_1 \in \partial\Sigma_1$. Therefore, the rest of this proof is devoted to the study of that case. The remaining cases are trivial, and can be studied similarly.

To deal with the case described above, we proceed as follows. Denote by Γ_1 and Γ_2 the boundary components containing $q(\Sigma_1)$ and $q_1(\operatorname{int}\Sigma_1)$, respectively. Suppose that $z_1 \in \partial\Sigma_1$. Let $p_1 = q(z_1)$, and denote by p_2 the intersection point between Γ_2 and the ray emerging from z_1 having minimum distance from p_1. The possibility $p_1 = p_2$ may be ruled out, because by Condition B2 (see also the first part of Remark 5.1), it can occur only if Γ_1 and Γ_2 are both either focusing or dispersing. Since Σ_1 is singular, we can also rule out the situation when z_1 is a tangent collision and Γ_1 is focusing. This is so because otherwise the number of consecutive collisions along Γ_1 of the negative semi-orbit of an element $z \in \Sigma_1$ would be unbounded for z sufficiently close to z_1, contradicting the fact that Σ_1 is a singular arc.

The situation above corresponds to the configuration with $f_2 \equiv 0$ and $\bar{a} = a_1$ described in Subsection 6.3. In the notation of that subsection, we write $T(\gamma_1(a)) = (X_2(a), \delta_2(a))$ for every $a \in (a_1, b_1)$. Clearly, we have

$$\begin{aligned} X_2(a) &= X_1(a) - \frac{f_1(X_1(a))}{\tan\delta_1(a)}, \\ \delta_2(a) &= \pi - \delta_1(a). \end{aligned}$$

We immediately see that the $\lim_{\operatorname{int}\Sigma_1 \ni z \to z_1} T(z)$ exists if and only if the $\lim_{a \to a_1^+} X_2(a)$ exists. Using l'Hôpital's rule, we obtain

$$\lim_{a \to a_1^+} X_2(a) = -\frac{f_1'(0) X_1'(0)}{\delta_1'(0)} \in U_2.$$

Indeed, $X_1'(0)$ and $\delta_1'(0)$ cannot vanish simultaneously by assumption, and $\delta_1'(0)$ alone cannot vanish, because otherwise X_2 would be unbounded in a neighborhood of a_1, contradicting the fact that $X_2(a) \in U_2$ for every $a \in I_1$. This completes the proof. □

The next lemma is a technical result used in the proofs of Proposition 6.7 and Theorem 2.2.

Lemma 6.5. *Suppose that Σ is a negatively singular arc such that $\Sigma \not\subset \partial M$ and $\Sigma \subset M_+$. Then*

$$\tau^+(y, T_y^*\Sigma) < t(y) \qquad \text{for } y \in \Sigma \setminus A_2. \tag{22}$$

Proof. Since Σ is a singular arc, there exist finitely many states $y_0, \dots, y_j$ with $y_i \in \Sigma_i$ for each $0 \leq i \leq j$ and $y_j = y$ such that $T_{i-1}y_{i-1} = y_i$ for each $1 \leq i \leq j$. Here T_i is the homeomorphic extension of $T|_{\operatorname{int}\Sigma_i}$ to Σ_i as in Proposition 6.4. Also, note that $j > 0$ because $\Sigma \not\subset \partial M$ by assumption.

We first suppose that $\Sigma_0, \dots, \Sigma_j \subset M_k \subset M_+$. In particular, this means that $y_0, \dots, y_j$ are consecutive collisions with the focusing component Γ_k. To prove (29) for this case, we argue similarly as in the proof of inequality (15). Namely, since $\tau^+(y_0, T_{y_0}^*\Sigma_0) = 0$, by using Part (2) of Theorem 4.7 and inequality (7), we obtain an upper bound for $\tau^+(y, T_y^*\Sigma)$. We arrive at (29) by showing that this upper bound is not greater than $t(y)$. This can be done easily by using Parts (2) and (3) of Theorem 4.7 and Condition B2.

Next, suppose that there exist an integer $0 < m \leq j$ such that $\Sigma_{m-1} \not\subset M_k$ and $\Sigma_m, \dots, \Sigma_j \subset M_k$. It is clear that $y_m \in \overline{E} \setminus A_2$, and that y_m is an accumulation point of $E \cap U_{y_m} \cap \operatorname{int}\Sigma_m$, where U_{y_m} is the neighborhood of y_m as in Theorem 4.7. By Proposition 6.2, we can then infer that $T_{y_m}^*\Sigma_m \subset \operatorname{int} C_{y_m}(y_m)$. Using Parts (2) and (3) of Theorem 4.7, we obtain $\tau^+(y, T_y^*\Sigma) < t(y)$ if $y \notin -E$, and $\tau^+(y, T_y^*\Sigma) \leq \tau_k^+$ if $y \in -E$. To complete the proof of (29) for the case under consideration, we need only to observe that $\tau_k^+ < t(y)$ by Condition B2. □

We now go back to the general situation described in Subsection 6.3. The notation here is as in that subsection. The relation between two consecutive collisions, the first occurring at Γ_1 and the second occurring at Γ_2, can be expressed as a simultaneous zero of the functions $F\colon M_1 \times M_2 \to \mathbb{R}$ and $G\colon M_1 \times M_2 \to \mathbb{R}$ defined by

$$\begin{aligned} F(X_1, \delta_1, X_2, \delta_2) &= f_1(X_1) - f_2(X_2) - \tan\delta_1 \cdot (X_1 - X_2), \\ G(X_1, \delta_1, X_2, \delta_2) &= \delta_1 + \delta_2 - 2\alpha_2(X_2) + \pi, \end{aligned}$$

for every $(X_1, \delta_1, X_2, \delta_2) \in M_1 \times M_2$. The function α_2 denotes the angle formed by the X-axis with the normal to Γ_2 at $(X_2, f_2(X_2))$ pointing inside the billiard domain Q. The relation $F(X_1, X_2, \delta_1, \delta_2) = 0$ is the equation of the line having slope equal to $\tan\delta_1$ and passing through the points $(X_1, f_1(X_1))$ and $(X_2, f_2(X_2))$, whereas the relation $G(X_1, \delta_1, X_2, \delta_2) = 0$ describes the equality of the angles of incidence and reflection at $(X_2, f_2(X_2))$.

Recall that $I_1 \ni a \mapsto (X_1(a), \delta_1(a))$ is a parametrization of the singular arc Σ_1. Let $F_1\colon M_2 \times I_1 \to \mathbb{R}$ and $G_1\colon M_2 \times I_1 \to \mathbb{R}$ be the C^2 differentiable functions given by

$$\begin{aligned} F_1(X_2, \delta_2, a) &= F(X_1(a), \delta_1(a), X_2, \delta_2), \\ G_1(X_2, \delta_2, a) &= G(X_1(a), \delta_1(a), X_2, \delta_2) \end{aligned}$$

for every $(X_2, \delta_2, a) \in M_2 \times I_1$. Also, define

$$D_1(X_2, \delta_2, a) = \begin{vmatrix} \partial_{\delta_2} F_1 & \partial_a F_1 \\ \partial_{\delta_2} G_1 & \partial_a G_1 \end{vmatrix}$$
$$= [\tan \delta_1(a) - f_1'(X_1(a))] \, X_1'(a) + \frac{\delta_1'(a)}{\cos^2 \delta_1(a)} \, (X_1(a) - X_2) \quad (23)$$

for every $(X_2, \delta_2, a) \in M_2 \times I_1$. Note that D_1 is well defined because, according to our setting, we always have $\cos \delta_1(a) \neq 0$ for a sufficiently close to $\bar{a}$.

The next lemma elucidates the relation between $D_1(X_2, \delta_2, a)$ and several geometrical quantities associated to the collision $\gamma_1(a)$. Recall that $t(\gamma_1(a))$ denotes the length of the segment connecting $q(\gamma_1(a))$ to $q_1(\gamma_1(a))$.

Lemma 6.6. *We have*

$$D_1(X_2, \delta_2, a) = \begin{cases} [\tau^+(\gamma_1(a), \gamma_1'(a)) - t(\gamma_1(a))] \, \dfrac{\delta_1'(a)}{\cos \delta_1(a)} & \text{if } \delta_1'(a) \neq 0, \\ [\tan \delta_1(a) - f_1'(X_1(a))] \, X_1'(a) & \text{if } \delta_1'(a) = 0. \end{cases} \quad (24)$$

Proof. We have

$$t(\gamma_1(a)) = \frac{X_2 - X_1(a)}{\cos \delta_1(a)}. \quad (25)$$

To obtain Formula (24), we just need to rewrite (23) using (18) and (25). □

The next proposition plays a key role in the proof of Theorem 2.2. Roughly speaking, it states that although the map T is not differentiable everywhere on M, nevertheless it maps singular arcs into singular arcs. The assumption that the arcs are singular is crucial, and we do not know whether it can be dropped.

Proposition 6.7. *Under the same hypotheses of Proposition 6.4, the curve Σ_2 is a negatively singular arc.*

Proof. We only need to show that the curve Σ_2 is an arc. Indeed, from this fact and the construction of Σ_2, it is a simple matter to see that Σ_2 must be a negatively singular arc. By Proposition 6.4, we already know that Σ_2 is connected and compact. Hence, it remains to demonstrate that Σ_2 is a 1-dimensional C^2 submanifold. This amounts to showing that for every $z_2 \in \Sigma_2$, there is a C^2 diffeomorphism γ_{z_2} from an interval I_{z_2} to a subarc of Σ_2 containing z_2. We split the proof into three cases:

(A) Γ_1 and Γ_2 disjoint and Γ_2 curved,
(B) Γ_1 and Γ_2 intersecting and Γ_2 curved,
(C) Γ_2 flat.

Each of these cases is further split into several subcases. The proof for Case (A) follows from Lemmas 6.8-6.12, the proof for Case (B) follows from Lemmas 6.13-6.15, and the proof for Case (C) follows from Lemma 6.16. □

Throughout the rest of this subsection (including the proofs of Lemmas 6.8-6.16), the notation is as in Subsection 6.3.

Let $z_1 \in \Sigma_1$, and let $z_2 = \bar{T} z_1 \in \Sigma_2$, where $\bar{T}$ is the homeomorphic extension of $T|_{\operatorname{int} \Sigma_1}$ as in Proposition 6.4. In the coordinates (X, δ), we have $z_1 = \gamma_1(\bar{a}) = (0, \pi)$ and $z_2 = (\bar{X}, \bar{\delta})$ for some $\bar{\delta} \in [0, 2\pi)$. By the definition of z_1 and z_2, we have

$$F_1(\bar{X}, \bar{\delta}, \bar{a}) = G_1(\bar{X}, \bar{\delta}, \bar{a}) = 0.$$

The strategy to prove Lemmas 6.8-6.12 is as follows. Suppose that we have $D_1(\bar{X}, \bar{\delta}, \bar{a}) \neq 0$. Then, we can use the Implicit Function Theorem to find three

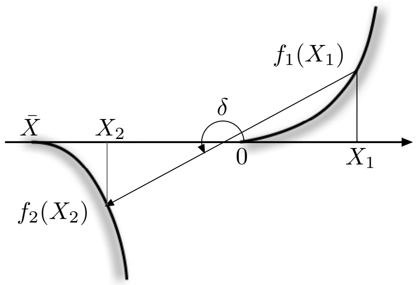


FIGURE 2. Γ_1 focusing and Γ_2 dispersing

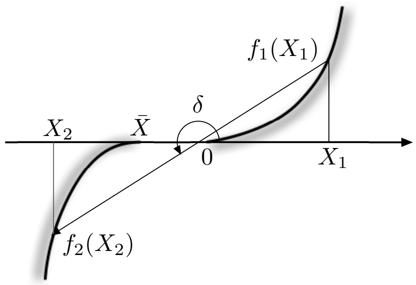


FIGURE 3. Γ_1 and Γ_2 focusing

open intervals I_2, W_1, W_2 containing $\bar{X}, \bar{\delta}, \bar{a}$, respectively, and two C^2 real functions $\phi_1 \colon I_2 \to W_1$ and $\phi_2 \colon I_2 \to W_2$ such that

$$F_1(X_2, \phi_1(X_2), \phi_2(X_2)) = G_1(X_2, \phi_1(X_2), \phi_2(X_2)) = 0 \qquad \text{for } X_2 \in I_2.$$

If we define

$$I_3 = \{X_2 \in I_2 \colon (X_2, f_2(X_2)) \in \Sigma_2\},$$

then the wanted curve γ_{z_2} is given by $\gamma_{z_2}(X_2) = (X_2, \phi_1(X_2))$ for all $X_2 \in I_3$.

Case (A) splits further into several subcases as described in Lemmas 6.8-6.12. To prove these lemmas, we show that $D_1(\bar{X}, \bar{\delta}, \bar{a}) \neq 0$ for each particular subcase.

Lemma 6.8. *Suppose that the boundary components Γ_1 and Γ_2 are disjoint, and that Γ_1 is focusing and Γ_2 dispersing (see Fig. 2). Then under the same hypotheses of Proposition 6.4, the curve Σ_2 is a negatively singular arc.*

Proof. Note first that the ray emerging from z_1 cannot be tangent to Γ_1, because otherwise the number of consecutive collisions of the negative semi-orbit of $\gamma_1(\bar{a})$ with Γ_1 would be unbounded for a sufficiently close to $\bar{a}$, contradicting the fact that Σ_1 is a singular arc. So $z_1 \notin A_2$, and Lemma 6.5 implies that $\tau^+(\gamma_1(\bar{a}), \gamma_1'(\bar{a})) < t(z_1)$. From (18), we then see that $\delta_1'(\bar{a}) \neq 0$. These two facts together with Formula (24) give $D_1(\bar{X}, \bar{\delta}, \bar{a}) \neq 0$. □

Lemma 6.9. *Suppose that the boundary components Γ_1 and Γ_2 are disjoint, and that Γ_1 and Γ_2 are both focusing (see Fig. 3). Then under the same hypotheses of Proposition 6.4, the curve Σ_2 is a negatively singular arc.*

Proof. The proof is exactly as the one of Lemma 6.8. □

Lemma 6.10. *Suppose that the boundary components Γ_1 and Γ_2 are disjoint, and that Γ_1 and Γ_2 are both dispersing (see Figs. 4a and 4b). Then under the same hypotheses of Proposition 6.4, the curve Σ_2 is a negatively singular arc.*

Proof. The only difference between this configuration and Configuration 1 is that $f_1'(0) = 0$ is now allowed. When $f_1'(0) \neq 0$ (see Fig. 4a), to prove that $D_1(\bar{X}, \bar{\delta}, \bar{a}) \neq 0$, we can argue as in the proof of Lemma 6.8. Note however that this time, we have $\tau^+(\gamma_1'(\bar{a})) < 0$, which follows from Proposition 6.2 and the definition of the cone field C_x on M_-.

Now suppose that $f_1'(0) = 0$. A direct inspection of Fig. 4b reveals that

$$\tan \delta_1(a) \geq f_1'(X_1(a)) \qquad \text{for } a \in I_1. \tag{26}$$

Indeed, if this was not the case, the ray emerging from $\gamma_1(a)$ would enter the component Γ_1 rather than leaving it. Without loss of generality, we can assume that $X_1'(\bar{a}) \geq 0$. Since $\delta_1(\bar{a}) = \pi$ and $X_1(\bar{a}) = 0$, from (26), we easily obtain $\delta_1'(\bar{a}) \geq f_1''(0)X_1'(\bar{a})$. Suppose that $\delta_1'(\bar{a}) = 0$. Since $f_1'' > 0$, it follows that $X_1'(\bar{a}) = 0$, and so $\gamma_1'(\bar{a}) = (0, 0)$, which is contradiction. Therefore, we must have $\delta_1'(\bar{a}) \neq 0$. Since $f_1'(0) = 0$, Formula (18) implies that $\tau^+(\gamma_1'(\bar{a})) = 0$. Finally, using the fact that $t(\gamma_1(\bar{a})) > 0$ and Formula (24), we conclude again that $D_1(\bar{X}, \bar{\delta}, \bar{a}) \neq 0$. □

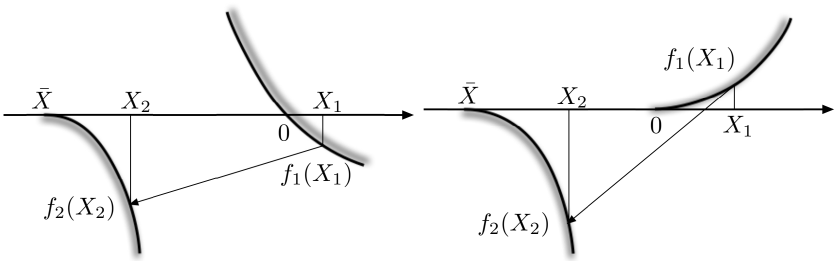
Lemma 6.11. *Suppose that the boundary components Γ_1 and Γ_2 are disjoint, and that Γ_1 is dispersing and Γ_2 is focusing (see Fig. 5). Then under the same hypotheses of Proposition 6.4, the curve Σ_2 is a negatively singular arc.*

Proof. The proof of this case is exactly as the one of Lemma 6.10. □

Lemma 6.12. *Suppose that the boundary components Γ_1 and Γ_2 are disjoint, and that Γ_1 is flat and Γ_2 is curved (see Fig. 6). Then under the same hypotheses of Proposition 6.4, the curve Σ_2 is a negatively singular arc.*

Proof. Suppose first that $\Sigma_1 \subset A_2$. This means that Σ_1 consists of vectors leaving from an endpoint of Γ_1. In this case, we can take $X_1(a) = 0$ and $\delta_1(a) = \pi - a$ with $\bar{a} = 0$ and $b_1 = \epsilon$ for some $\epsilon > 0$ sufficiently small. It follows that $X_1'(\bar{a}) = 0$, $\delta_1'(\bar{a}) = 1$ and $\tau^+(\gamma_1'(\bar{a})) = 0$. By Formula (18), we obtain $D_1(\bar{X}, \bar{\delta}, \bar{a}) = t(\gamma_1(\bar{a})) \neq 0$.

Now, suppose that Σ_1 is not contained in A_2. We claim that this case can be reduced to the previous case or to one of the cases considered in Lemmas 6.8-6.11. Indeed, by the definition of a singular arcs, there exist an integer $k > 0$ and a singular arc Λ contained in $M_- \cup M_+ \cup A_2$ such that $\bar{T}^i \Lambda \subset M_0$ for every $0 < i < k$ and $\bar{T}^k \Lambda = \Sigma_1$, where $\bar{T}^i$ denotes the homeomorphic extension of $T^i|_{\text{int}\,\Lambda}$ to Λ for every $1 \leq i \leq k$ guaranteed by Proposition 6.4. Now, instead of Σ_1 and the map T, we can consider the singular arc Λ and the map T^k, which is either one of the cases considered in Lemmas 6.8-6.11 or the previous case considered in this lemma. □



(A) $\tan \bar{\delta} \neq f_1'(0)$ (B) $\tan \bar{\delta} = f_1'(0)$

FIGURE 4. Γ_1 and Γ_2 dispersing

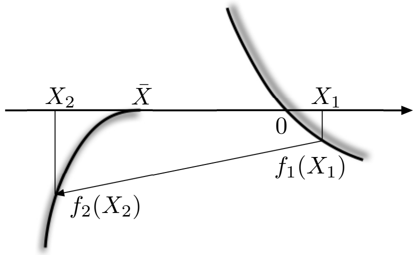


FIGURE 5. Γ_1 dispersing and Γ_2 focusing

We now deal with Case (B) of the proof of Proposition 6.7. This case is split into three subcases analyzed in Lemmas 6.13-6.15. To prove the second lemma, we use a result by Lazutkin, whereas to prove the first and the third lemma, we proceed in the following way.

For the general Case (B), we have $\bar{X} = 0$, and so $t(z_1) = 0$. We can assume that $D_1(\bar{X}, \bar{\delta}, \bar{a}) = 0$, otherwise the Implicit Function Theorem would immediately imply the conclusion as explained in Case (A). Since the function F_1 does not depend on δ_2, we consider F_1 as a function of X_2 and a alone. The point $(\bar{X}, \bar{a})$ is therefore a zero and a critical point of F_1. Now, we assume that $(\bar{X}, \bar{a})$ is non-degenerate (i.e., $\det \operatorname{Hess} F_1(\bar{X}, \bar{a}) \neq 0$). Then, by Morse Lemma, there exist two open intervals J_1 and J_2 of the origin, a neighborhood $V \subset \mathbb{R}^2$ of $(\bar{X}, \bar{a})$ and a C^2 diffeomorphism $\Phi\colon J_1 \times J_2 \to V$ such that $F_1(\Phi(\alpha, \beta)) = \alpha^2 - \beta^2$ for all $(\alpha, \beta) \in J_1 \times J_2$. Write $\Phi(\alpha, \beta) = (\phi_1(\alpha, \beta), \phi_2(\alpha, \beta))$, and define $\psi_i^{\pm}(\alpha) = \phi_i(\alpha, \pm\alpha)$ for every $\alpha \in J_1$ and each $i = 1, 2$.

The previous observation tells us that if $(\bar{X}, \bar{a})$ is a non-degenerate critical point of F_1, then the zeros of F_1 contained in the neighborhood V are given by $(\psi_1^+(\alpha), \psi_2^+(\alpha))$ and $(\psi_1^-(\alpha), \psi_2^-(\alpha))$ for all $\alpha \in J_1$. Now, note that $G_1(X_2, \delta_2, a) = 0$ if and only if $\delta_2 = -\delta_1(a) + 2\alpha_2(X_2) - \pi$, and let $\varphi^{\pm}(\alpha) = -\delta_1(\psi_2^{\pm}(\alpha)) + 2\alpha_2(\psi_1^{\pm}(\alpha)) - \pi$ for every $\alpha \in J_1$. Putting all together, we then see that the simultaneous zeros of F_1 and G_1 in V are given by $(\psi_1^+(\alpha), \varphi^+(\alpha), \psi_2^+(\alpha))$ and $(\psi_1^-(\alpha), \varphi^-(\alpha), \psi_2^-(\alpha))$ for all $\alpha \in J_1$. If we set $J^{\pm} = \{\alpha \in J_1 \colon \psi_2^{\pm}(\alpha) \geq \bar{a}\}$, then in coordinates (X, δ), the wanted curve γ_{z_2} is given by either $\gamma_{z_2}(\alpha) = (\psi_1^+(\alpha), \varphi^+(\alpha))$ for all $\alpha \in J^+$, or by $\gamma_{z_2}(\alpha) = (\psi_1^-(\alpha), \varphi^-(\alpha))$ for all $\alpha \in J^-$.

To finish, it remains to show that $\gamma'_{z_2}(\alpha) \neq 0$ for every α. To prove this property, we make another assumption (the first one was $\det \operatorname{Hess} F_1(\bar{X}, \bar{a}) \neq 0$), namely that $\delta'_1(\bar{a}) \neq 0$. By possibly shrinking the interval J^+, we can assume without loss of generality that $\delta'_1(\psi_2^+(\alpha)) \neq 0$ for every $\alpha \in J^+$. Now, suppose that $\gamma'_{z_2}(\alpha) = (\psi_1^{+'}(\alpha), \varphi^{+'}(\alpha)) = (0, 0)$ for some $\alpha \in J^+$. It is easy to check that $\varphi^{+'}(\alpha) = 0$ implies that $\delta'_1(\psi_2^+(\alpha))\psi_2^{+'}(\alpha) = 0$. But $\delta'_1(\psi_1^+(\alpha)) \neq 0$ so that $\psi_2^{+'}(\alpha) = 0$. Hence, we have $\psi_1^{+'}(\alpha) = \psi_2^{+'}(\alpha) = 0$. However, this cannot happen, because Φ is a diffeomorphism. Hence, $\gamma_{z_2}(\alpha) \neq 0$ for every α.

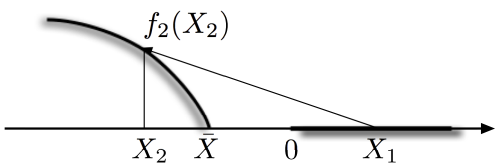


FIGURE 6. Γ_1 flat and Γ_2 curved

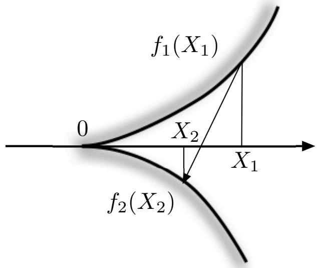


FIGURE 7. Γ_1 and Γ_2 dispersing

The proof of the next lemmas consists precisely in showing that the assumptions $\delta_1'(\bar{a}) \neq 0$ and $\det \operatorname{Hess} F_1(\bar{X}, \bar{a}) \neq 0$ of the argument just described are satisfied for the subcases of Case (B).

Lemma 6.13. *Suppose that the boundary components Γ_1 and Γ_2 intersect, and that Γ_1 and Γ_2 are both dispersing (see Fig. 7). Then under the same hypotheses of Proposition 6.4, the curve Σ_2 is a negatively singular arc.*

Proof. First note that we must have $f_2''(0) < 0$ and $f_1'(0) = 0$, because if $f_1'(0) > 0$, then z_1 would leave the table. Next, we have $\delta_1(\bar{a}) = \pi$ by construction (see Subsection 6.3), but one may wonder whether the possibility $\delta_1(\bar{a}) = 0$ should be considered as well. This possibility can be ruled out. In fact, it implies that the rays emerging from Σ_1 leave the corner formed at the origin by Γ_1 and Γ_2, and since in a neighborhood of z_1, there are vectors with negative semi-trajectory having an arbitrarily large number of collisions before reversing their direction of motion and escaping from the corner (see [8, Section 3]), we obtain the contradictory conclusion that Σ_1 is not a singular arc.

Now, since Σ_1 is not contained in ∂M and $\delta_1(\bar{a}) = \pi$, it is not difficult to see that x must come out from a corner formed by two dispersing boundary components or by a dispersing component and a flat component. Arguing exactly as in the second part of the proof of Lemma 6.10, we obtain $\delta_1'(\bar{a}) \geq f_1''(0)X_1'(\bar{a})$, $X_1'(\bar{a}) \geq 0$ and $\delta_1'(\bar{a}) > 0$. Hence,

$$\partial^2_{X_2X_2} F_1(\bar{X}, \bar{a}) = -f_2''(0) > 0,$$
$$\partial^2_{X_2a} F_1(\bar{X}, \bar{a}) = \delta_1'(\bar{a}) > 0,$$
$$\partial^2_{aa} F_1(\bar{X}, \bar{a}) = \left(f_1''(0)X_1'(\bar{a}) - 2\delta_1'(\bar{a})\right) X_1'(\bar{a}) \leq 0,$$

and so $\det \operatorname{Hess} F_1(\bar{X}, \bar{a}) < 0$. □

Lemma 6.14. *Suppose that the boundary components Γ_1 and Γ_2 intersect, and that Γ_1 and Γ_2 are both focusing (see Fig. 8). Then under the same hypotheses of Proposition 6.4, the curve Σ_2 is a negatively singular arc.*

Proof. We must have $f_2 \equiv f_1$. We can assume without loss of generality that $f_1'(0) = 0$. The proof for this case is a direct consequence of a general result by Lazutkin stating that $\tilde{T}$, the restriction of the map T to the set of the collisions of M_1 that do not leave M_1, is at least a C^3 diffeomorphism [17, Proposition 14.2]. In virtue of Lazutkin's result, the wanted curve γ_{z_2} is given by $\gamma_{z_2} = \tilde{T} \circ \gamma_1$. □

Lemma 6.15. *Suppose that the boundary components Γ_1 and Γ_2 intersect, and that Γ_1 is flat and Γ_2 is focusing (see Fig. 9). Then under the same hypotheses of Proposition 6.4, the curve Σ_2 is a negatively singular arc.*

Proof. We consider separately the two cases: $\Sigma_1 \subset A_1$ and $\Sigma_1 \not\subset A_1$. In the first case, we can proceed as in the proof of Lemma 6.12. We then obtain $X_1'(\bar{a}) = 0$ and $\delta_1'(\bar{a}) \neq 0$, and therefore $\det \operatorname{Hess} F_1(\bar{X}, \bar{a}) = -\delta_1'^2(\bar{a}) \neq 0$. The second case can be reduced to the previous case or to one of those studied in Lemmas 6.13 and 6.14. □

The next lemma takes care of Case (C) of Proposition 6.7.

Lemma 6.16. *Suppose that the boundary components Γ_2 is flat (see Fig. 10). Then under the same hypotheses of Proposition 6.4, the curve Σ_2 is a negatively singular arc.*

Proof. We have three possibilities: i) Γ_1 and Γ_2 disjoint and Γ_1 curved, ii) Γ_1 and Γ_2 intersecting and Γ_1 dispersing (Γ_1 cannot be focusing by Condition B2), and iii) Γ_1 is flat and $\Sigma_1 \not\subset A_1$. For these possibilities, the condition $\delta_1'(\bar{a}) \neq 0$ can be proved by arguing as in Lemma 6.8, Lemma 6.13 and Lemma 6.12, respectively. Since $f_2'' \equiv 0$, it follows immediately that $\det \operatorname{Hess} F_1(\bar{X}, \bar{a}) = -\delta_1^2(\bar{a}) \neq 0$. □

6.5. **Proof of Theorem 2.2.** We begin with some technical propositions.

Proposition 6.17. *Suppose that $\Omega \not\subset \partial M$ and $\Xi \not\subset \partial M$ are singular arcs contained in $R_k^- \cap M_-$ for some $k \geq 1$ and $R_1^+ \cap M_-$, respectively. Then the set $\Omega \cap \Xi$ is finite.*

Proof. Case $\Omega \subset M_-$ and $\Xi \subset M_-$. Since $M_- \subset E$, Part (1) of Proposition 6.2 implies that

$$\begin{aligned} T_y^*\Omega &\subset \operatorname{int} C_y(y) \qquad \text{for } y \in \operatorname{int}\Omega, \\ T_y^*\Xi &\subset \operatorname{int} C_y'(y) \qquad \text{for } y \in \operatorname{int}\Xi. \end{aligned} \tag{27}$$

From the definition of the cone field C_y for $y \in M_-$, we see that, in coordinates (s, θ), the arc Ω is a strictly decreasing, and the arc Ξ is strictly increasing. We deduce that the intersection $\Omega \cap \Xi$ cannot contain more than one point. □

Proposition 6.18. *Suppose that $\Omega \not\subset \partial M$ and $\Xi \not\subset \partial M$ are singular arcs contained in $R_k^- \cap M_+$ for some $k \geq 1$ and $R_1^+ \cap M_+$, respectively. Then the set $\Omega \cap \Xi$ is finite.*

Proof. Case $\Omega \subset M_+$ and $\Xi \subset M_+$. We claim that the set $\Omega \cap \Xi$ does not intersect A_2. Suppose instead, that $y \in \Omega \cap \Xi \cap A_2 \neq \emptyset$. Since y is tangent to Γ_{i_1}, we must have $T(y) = y$ or $T^{-1}(y) = y$. If $T(y) = y$, then it is not difficult to see that in order for Ξ to be an arc (and so $\operatorname{int}\Xi \neq \emptyset$) contained in R_1^+ but not in ∂M, the boundary ∂Q must have a component that is tangent to Γ_{i_1} at $q(y)$. In order words, ∂Q has a cusp at $q(y)$ formed by Γ_{i_1} and a dispersing component with the vector y directed inside the billiard domain. In this situation, the function associating to each element of $\operatorname{int}\Omega$ sufficiently close to y the number of consecutive collisions required to leave the cusp along its negative semi-trajectory is unbounded. Hence,

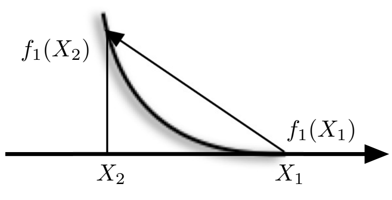


FIGURE 8. Γ_1 and Γ_2 focusing

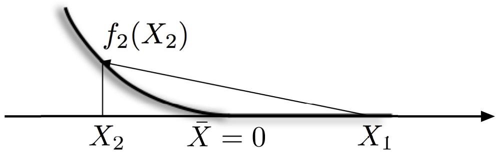


FIGURE 9. Γ_1 flat and Γ_2 focusing

Ω cannot be contained in R_k^-, and we obtain a contradiction[1] It remains to consider the case $T^{-1}y = y$ and $T(y) \neq y$. This case corresponds to a state y attached at one endpoint of the focusing component Γ_{i_1} and leaving Γ_{i_1}. Also in this case, it is fairly easy to see that the number of collisions required by elements of $\operatorname{int}\Omega$ sufficiently close to y to leave the cusp along their negative semi-trajectories is unbounded. Hence, we obtain a contradiction again, and we can conclude that $\Omega \cap \Xi \cap A_2 = \emptyset$.

Let $y \in \Omega \cap \Xi$. Since Ξ consists of states whose next collision with ∂Q takes place at the same endpoint of one of the component of ∂Q or is tangent to the same dispersing component of ∂Q. As a consequence, we have

$$\tau^+(y, T_y^*\Xi) = t(y). \tag{28}$$

Next, by Lemma 6.5, we have

$$\tau^+(y, T_y^*\Omega) < t(y). \tag{29}$$

Comparing (28) and (29), we see that the tangent spaces $T_y\Omega$ and $T_y\Xi$ are linear independent. Since this property holds for every $y \in \Omega \cap \Xi$, we conclude that Ω and Ξ are transversal. This fact together with the compactness of Ω and Ξ implies that the set $\Omega \cap \Xi$ is finite. ☐

Proposition 6.19. *Suppose that $\Omega \not\subset \partial M$ and $\Xi \not\subset \partial M$ are singular arcs contained in $R_k^- \cap M_0$ for some $k \geq 1$ and $R_1^+ \cap M_0$, respectively. Then the set $\Omega \cap \Xi$ is finite.*

Proof. Case $\Omega \subset M_0$ and $\Xi \subset M_0$. As for the case $\Omega \subset M_+$ and $\Xi \subset M_+$, it is enough to prove that Ω and Ξ are transversal. Again, we start by showing that the set $\Omega \cap \Xi$ does not intersect A_2. The proof is similar to the one for the case $\Omega \subset M_+$ and $\Xi \subset M_+$. Suppose that $y \in \Omega \cap \Xi \cap A_2 \neq \emptyset$. Then, as before, it follows that $T(y) = y$ or $T^{-1}(y) = y$. The case $T(y) = y$ can be studied exactly as for $\Omega \subset M_+$ and $\Xi \subset M_+$ so that we turn to the case $T^{-1}(y) = y$ and $T(y) \neq y$. This case corresponds to a state y attached to one endpoint of the flat component Γ_{i_1} and leaving Γ_{i_1}. In order for Ξ to be an arc contained in R_1^- but not in ∂M, we see that $q(\Xi)$ must be contained in a component of ∂Q meeting Γ_{i_1} at $q(y)$. But then Ω and Ξ do not belong to M_{i_1}, obtaining a contradiction. We can then conclude that $\Omega \cap \Xi \cap A_2 = \emptyset$. The fact that $y \notin A_2$ implies immediately that $\tau^+(y, T_y^*\Xi) = t(y) > 0$.

[1]An alternative way to obtain a contradiction would have been to observe that a billiard domain with a boundary ∂Q with a cusp formed by a focusing and a dispersing component is forbidden by Condition B2.

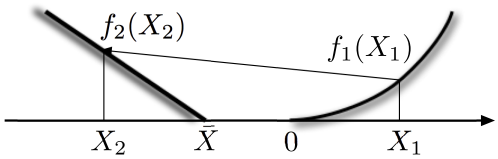


FIGURE 10. Γ_1 focusing and Γ_2 flat

Now, since Ω is a singular arc, we can find $y_0, \dots, y_j \in M$ with $y_i \in \Omega_i$ for each $0 \leq i \leq j$ and $y_j = y$ such that $T_{i-1} y_{i-1} = y_i$ for each $1 \leq i \leq j$, where T_i is the extension of T as in Proposition 6.4. Note that $j > 0$ because $\Omega \not\subset \partial M$. First, suppose that the sets $\Omega_1, \dots, \Omega_j$ are all contained in M_0. Then, it is clear that $\tau^+(y, T_y^*\Omega) = -l < 0$ with l being the length of the piece of the trajectory starting at y_0 and ending at y. Hence, Ω and Ξ are transversal at y. Now, suppose that there exists $0 < k < j$ such that $\Omega_k \not\subset M_0$ and $\Omega_{k+1}, \dots, \Omega_j \subset M_0$. Clearly, we have $y_k \in -E$. If $\Omega_k \subset M_+$, then exactly as we have proved $\tau^+(y, T_y^*\Omega) \leq \tau_{i_1}^+$ when $y \in -E$ for the case $\Omega \subset M_+$ and $\Xi \subset M_+$, we can show that $\tau^+(y_k, T_{y_k}^*\Omega_k) \leq \tau_{i_2}^+$, where Γ_{i_2} is the focusing component such that $\Omega_k \subset M_{i_2}$. By Condition B2, we obtain $\tau^+(y, T_y^*\Omega) < t(y)$. If $\Omega_k \subset M_-$, then we can find a sequence $w_n \in \operatorname{int}\Omega$ such that $w_n \to y$ as $n \to +\infty$. From the definition of a singular arc, it follows that $T^{k-j}(w_n) \in \operatorname{int}\Omega_k$ for every $n > 0$. By Proposition 6.2, we have $T^*_{T^{k-j}(w_n)}\Omega_k \subset \operatorname{int} C_{T^{k-j}(w_n)}(T^{k-j}(w_n))$ for every $n > 0$, and so $\tau^+(T^{k-j}(w_n), T^*_{T^{k-j}w_n}\Omega_k) < 0$ for every $n > 0$. Using relation (6), we see that

$$\tau^+(w_n, T_{w_n}^*\Omega) < 0 \qquad \text{for every } n > 0. \tag{30}$$

We know that $\tau^+(w_m, T_{w_n}^*\Omega) \to \tau^+(y, T_y^*\Omega)$ as $n \to +\infty$ by Remark 3.1. Thus, passing to the limit on the left hand-side of (30), we obtain $\tau^+(y, T_y^*\Omega) < 0$. Thus, the tangent spaces $T_y\Omega$ and $T_y\Xi$ are linear independent. We can then conclude that Ω and Ξ are transversal. □

We are finally ready to prove Theorem 2.2.

Proof of Theorem 2.2. We will prove that the sets R_k^- and R_k^+ have a property that is stronger than the one formulated in the statement of the theorem. Namely that R_k^- and R_k^+ are regular, and admit a decomposition consisting of singular arcs. This property will called Property S. It is enough to prove this property just for the sets R_k^-, because it is then automatically true for the sets R_k^+ as well, in virtue of the fact that $R_k^+ = -R_k^-$.

We argue by induction. The sets $R_0^- = R_0^+$ clearly satisfy Property S. Next, fix $k > 0$, and assume that all the sets $R_1^-, \dots, R_k^-$ satisfy Property S. By the time-reversal symmetry of the billiard dynamics, the same is true for the sets $R_1^+, \dots, R_k^+$. Now, consider the sets R_k^- and R_1^+, and choose a decomposition consisting of singular arcs for R_k^- and another one for R_1^+. Let Ω be an element from the chosen decomposition of R_k^-, and let $\Xi \not\subset \partial M$ be an element from the chosen decomposition of R_1^+. Clearly, if $q(\Omega)$ and $q(\Xi)$ do not belong to the same component of ∂Q, then $\Omega \cap \Xi$ is empty. Hence, we can assume without loss of generality that $\Omega, \Xi \subset M_{i_1}$ for some i_1.

Our first task is to show that the set $\Omega \cap \Xi$ is finite (possibly empty). This is achieved by studying several cases. First, we consider the case $\Omega \subset \partial M$. From the definition of a singular arc and the assumption $\Xi \not\subset \partial M$, it follows immediately that $\Omega \cap \Xi \subset \partial\Xi$. Thus the set $\Omega \cap \Xi$ cannot contain more than two elements. The remaining cases to analyze are the following: i) $\Omega, \Xi \subset M_-$, ii) $\Omega, \Xi \subset M_+$ and iii) $\Omega, \Xi \subset M_0$. The finiteness of $\Omega \cap \Xi$ for these cases follows from Propositions 6.17, 6.18 and 6.19, respectively.

Now, note that the set S_1^+ introduced in Section 2 coincides with the union of the arcs not contained in ∂M from a decomposition of R_1^+ formed by singular arcs. Hence, the set R_k^- can be written as the union of finitely many arcs $\Omega_1, \dots, \Omega_m$, which are necessarily singular, with the following properties

- $\Omega_i \cap \Omega_j$ if $i \neq j$,
- $\Omega_i \cap S_1^+ \subset \partial\Omega_i$ for every $1 \leq i \leq m$.

It is clear that the arcs $\Omega_1, \dots, \Omega_m$ and the transformation $T|_{\operatorname{int}\Omega_i}$ satisfy the hypotheses of Proposition 6.7. Accordingly, the restriction $T|_{\operatorname{int}\Omega_i}$ admits a homeomorphic extension T_i to the entire arc Ω_i, and $\Sigma_i := T_i\Omega_i$ is a singular arc with the property that $\Sigma_i \cap \partial M \subset \partial\Sigma_i$. Since T is a homeomorphism on $\bigcup_{i=1}^m \operatorname{int}\Omega_i$ and $\Omega_i \cap \Omega_j \subset \partial\Omega_i$ whenever $i \neq j$, we conclude that $\Sigma_i \cap \Sigma_j \subset \partial\Sigma_i$ whenever $i \neq j$. Therefore, $\bigcup_{i=1}^m T(\Omega_i) = \bigcup_{i=1}^m \Sigma_i$.

Since $T(R_k^-) = \bigcup_{i=1}^m T(\Omega_i)$, we have $T(R_k^-) = \bigcup_{i=1}^m \Sigma_i$. It follows that $T(R_k^-)$ is a regular set with a decomposition consisting of singular arcs, i.e, $T(R_k^-)$ satisfies Property S. From $R_{k+1}^- = \partial M \cup T(R_k^-)$ (see (2)), we obtain $R_{k+1}^- = \partial M \cup \bigcup_{i=1}^m \Sigma_i$. Since $\Sigma_i \cap \partial M \subset \partial\Sigma_i$, it is easy to see that R_{k+1}^- satisfies the property S as well. This completes the proof. □

Appendix A

In this appendix, we prove that Theorem 4.7 extends to some absolutely focusing arcs, including W-arcs and M-arcs.

Suppose that Γ_i is an absolutely focusing arc, and let ζ be the function associated to Γ_i as in Definition 3.2. In this appendix, our main assumption is Condition A below. It is precisely this condition that allows us to extend Theorem 4.7 to Γ_i even when Γ_i is not C^6.

Condition A: There exist $\bar\theta \in (0, \pi/2)$ and two constants a_1 and a_2 such that

$$1/4 < a_1 \le \zeta(x) \le a_2 < 3/4$$

for every $x \in M_i$ with $\theta(x) \in (0, \bar\theta) \cup (\pi - \bar\theta, \pi)$.

We start by defining the cone fields $\{(U_x, C_x)\}_{x \in \bar E_i}$ for Γ_i as in Theorem 4.7. It suffices to specify the neighborhood U_x and the function G_x^+ (see Remark 4.8). Let $x \in \bar E_i$. First, suppose that $x \in A_1$ or $n(x) \ge 2$. Let U_x to be any neighborhood of x in M_i such that $n(y) \ge 1$, and define $G_x^+(y) = \zeta(y)t(y)$ for every $y \in U_x$. Now, suppose that $x \notin A_1$ and $n(x) \le 1$. Let U_x to be any neighborhood of x in M_i such that $n(y) \le n(x)$ for every $y \in U_x$. Then define

$$G_x^+(y) = \begin{cases} d(x) & \text{if } n(x) = 0, \\ \zeta(x)t(x) & \text{if } n(x) = 1. \end{cases}$$

Remark A.1. *The situation when $x \in \bar E_i$ is one of the two periodic points corresponding to the periodic trajectory joining the two endpoints of Γ_i belongs to the case $n(x) = 1$ by definition. Also, note that we can choose U_x as explained in virtue of* [13, Proposition 3.6]*, which remains valid in our setting.*

Remark A.2. *We observe that $G_x^+(y) = \zeta(y)t(y)$ corresponds to the function $g_x(y) = \sin\theta(y)/(\zeta(y)t(y)) - \kappa(y)$ as in Theorem 4.7. Also, we observe, that the cone field (U_x, C_x) coincides with the one introduced in* [19, 20] *for $n(x) \ge 2$, but it does not for the remaining values of $x \in \bar E_i$.*

Denote by w_y the vector of $C_x(y)$ with forward focusing time $\tau^+(y, w_y) = G_x^+(y)$.

Lemma A.3. *We have*

$$\inf_{u \in C_x^*(y)} \tau^-(y, u) = \tau^-(y, w_y),$$

$$\sup_{u \in C_x^*(y)} \tau^+(T^{n(y)}(y), D_yT^{n(y)}u) = \tau^+(T^{n(y)}(y), D_yT^{n(y)}w_y).$$

Proof. The two equalities are direct consequences of the definition of the cone field (U_x, C_x) and the monotonicity relation (7). □

Lemma A.4. *The functions $y \mapsto \tau^-(y, w_x)$ and $y \mapsto \tau^+(y, D_y T^l w_x)$ for $l = 0, \dots, n(y)$ are continuous on U_x^* for every $x \in \bar{E}_i$.*

Proof. The claim follows from the continuity of $\tau^-(\cdot, u)$ and $\tau^+(\cdot, u)$ for $m(u) \neq \kappa$ and $m(u) \neq -\kappa$, the continuity of $D.T^l u$, and the upper semi-continuity of n. □

Let u_x^+ (respectively, u_x^-) be the vector of $T_x M_i$ such that $\tau^+(x, u_x^+) = \zeta(x)t(x)$ (respectively, $\tau^-(x, u_x^-) = (1-\zeta(T^{-1}(x))t(T^{-1}x))$ for every $x \in M_i$ with $n(-x) > 0$ (respectively, $n(x) > 0$). Also, let $r_i = \max_{M_i} r(x)$, and denote by $|\Gamma_i|$ the length of Γ_i.

Lemma A.5. *There exists two positive constants b_i^- and b_i^+ only depending on Γ_i such that $\tau^-(x, u_x^+) \leq b_i^-$ for $x \in M_i$ with $n(x) > 0$, and $\tau^+(x, u_x^-) \leq b_i^+$ for $x \in M_i$ with $n(-x) > 0$.*

Proof. From the Mirror Formula (5), we obtain

$$\tau^-(x, u_x^+) = \frac{d(x)}{2 - \dfrac{d(x)}{\zeta(x)t(x)}}, \tag{31}$$

and

$$\tau^+(x, u_x^-) = \frac{d(x)}{2 - \dfrac{d(x)}{[1-\zeta(T^{-1}(x))]t(T^{-1}(x))}}. \tag{32}$$

Since $d(x) \leq r_i$, it is enough to show that the denominators of the expressions on the right hand-side (31) and (32) are both uniformly bounded away from zero.

Choose $\epsilon < \min\{3/2 - 2a_2, 2a_1 - 1/2\}$. Then there exists $0 < \bar{\theta} < \pi/2$ such that $1/2 - \epsilon \leq d(x)/t(x) \leq 1/2 + \epsilon$ and $1/2 - \epsilon \leq d(x)/t(T^{-1}(x)) = d(x)/t(-x) \leq 1/2 + \epsilon$ for every $x \in M_i$ with $\theta(x) \in (0, \bar{\theta}) \cup (\pi - \bar{\theta}, \pi)$ (see [19, Section 3]). This together Condition A and the choice of ϵ gives

$$2 - \frac{d(x)}{\zeta(x)t(x)} \geq \frac{2a_1 - 1/2 - \epsilon}{a_1} > 0,$$

and

$$2 - \frac{d(x)}{(1-\zeta(T^{-1}(x))t(T^{-1}(x))} \geq \frac{3/2 - 2a_2 - \epsilon}{1 - a_2} > 0.$$

To complete the proof, we observe that by Part (3) of Definition 3.2, the denominators are greater than some constant $c > 0$ for every $x \in M_i$ with $\theta(x) \in [\bar{\theta}, \pi - \bar{\theta}]$. □

Proposition A.6. *There exist two positive constants t_i^- and t_i^+ such that for every $x \in \bar{E}_i$, we have*

$$\sup_{y \in U_x^*} \inf_{u \in C_x^*(y)} \tau^-(y, u) \leq t_i^-,$$

and

$$\sup_{y \in U_x^*} \sup_{u \in C_x^*(y)} \tau^+(T^{n(y)}(y), D_y T^{n(y)} u) \leq t_i^+.$$

Proof. By Lemma A.3, it suffices to prove the proposition for

$$\sup_{y \in U_x^*} \tau^-(y, w_y) \qquad \text{and} \qquad \sup_{y \in U_x^*} \tau^+(T^{n(y)}(y), D_y T^{n(y)} w_y).$$

The first inequality follows from Lemma A.5 and a trivial computation when $n(x) = 0$.The second inequality follows from Lemmas A.4-A.5 when $n(x) = 1$, and from (10) and Lemma A.5 when $n(x) \geq 2$. □

Corollary A.7. *Theorem 4.7 extends to every absolutely focusing arc satisfying Condition A.*

Proof. Part (2) of Theorem 4.7 follows by the properties of $\{(U_x, C_x)\}_{x\in\bar{E}_i}$, whereas Part (3) is Proposition A.6. □

Corollary A.8. *Theorem 4.7 extends to W-arcs and M-arcs.*

Proof. For M-arcs, we have $\zeta \equiv 1/2$. For W-arcs, we have

$$2/5 \leq \zeta(x) = d(x)/t(x) \leq 2/3$$

for $\theta(x)$ sufficiently close to 0 or π, which follows from the computations in the proof of Lemma A.5. □

References

[1] L. A. Bunimovich, *Billiards that are close to scattering billiards*, (Russian) Mat. Sb. (N.S.) **94**(136) (1974), 49–73.

[2] L. A. Bunimovich, *On ergodic properties of nowhere dispersing billiards*, Comm. Math. Phys. **65** (1979), 295–312.

[3] L. A. Bunimovich, *A theorem on ergodicity of two-dimensional hyperbolic billiards*, Comm. Math. Phys. **130** (1990), 599–621.

[4] L. A. Bunimovich, *On absolutely focusing mirrors*, Ergodic theory and related topics, III (Güstrow, 1990), Springer (1992), 62–82.

[5] L. A. Bunimovich, G. Del Magno, *Track billiards*, Comm. Math. Phys. **288** (2009), 699–713.

[6] L. Bussolari, M. Lenci, *Hyperbolic billiards with nearly flat focusing boundaries. I*, Phys. D **237** (2008), 2272–2281.

[7] N. I. Chernov and R. Markarian, *Chaotic billiards*, Mathematical Surveys and Monographs *127*, AMS, Providence, 2006.

[8] N. I. Chernov and R. Markarian, *Dispersing billiards with cusps: slow decay of correlations*, Comm. Math. Phys. **270** (2007), 727–758.

[9] N. I. Chernov and S. Troubetzkoy, *Ergodicity of billiards in polygons with pockets*, Nonlinearity **11** (1998), 1095–1102.

[10] G. Del Magno, *Ergodicity of a class of truncated elliptical billiard*, Nonlinearity **14** (2001), 1761–1786.

[11] G. Del Magno and R. Markarian, *Bernoulli Elliptical Stadia*, Comm. Math. Phys. **233** (2003), 211–230.

[12] G. Del Magno and R. Markarian, *A local ergodic theorem non-uniformly hyperbolic symplectic maps with singularities*, Ergodic Theory Dynam. Systems, doi: 10.1017/S0143385712000284, Published online by Cambridge University Press 09 July 2012.

[13] V. Donnay, *Using Integrability to Produce chaos: Billiards with Positive Entropy*, Comm. Math. Phys. **141** (1991), 225–257.

[14] G. Gallavotti and D. S. Ornstein, *Billiards and Bernoulli Schemes*, Comm. Math. Phys. **38** (1974), 83–101.

[15] E. Hopf, *Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung*, Ber. Verh. Sächs. Akad. Wiss. Leipzig **91** (1939), 261–304.

[16] A. Katok and J. -M. Strelcyn, *Invariant manifolds, entropy and billiards; smooth maps with singularities*, Lecture Notes in Mathematics **1222**, Springer, New York, 1986.

[17] V. F. Lazutkin, *KAM Theory and Semiclassical Approximations to Eigenfunctions*, Springer, 1993.

[18] C. Liverani and M. Wojtkowski, *Ergodicity in Hamiltonian Systems*, Dynamics reported, Dynam. Report. Expositions Dynam. Systems (N.S.) **4** Springer, Berlin (1995), 130–202.

[19] R. Markarian, *Billiards with Pesin region of measure one*, Comm. Math. Phys. **118** (1988), 87–97.

[20] R. Markarian, *New ergodic billiards: exact results*, Nonlinearity **6** (1993), 819–841.

[21] Ya. G. Sinai, *Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards*, Russ. Math. Surveys **25** (1970), 137–189.

[22] D. Szász, *On the K-property of some planar hyperbolic billiards*, Comm. Math. Phys. **145** (1992), 595–604.

[23] M. Wojtkowski, *Principles for the design of billiards with nonvanishing Lyapunov exponents*, Comm. Math. Phys. **105** (1986), 391–414.

CEMAPRE, ISEG, 1200 Lisbon, Portugal
E-mail address: delmagno@iseg.utl.pt

Instituto de Matemática y Estadística 'Prof. Ing. Rafael Laguardia' (IMERL), Facultad de Ingeniería, Universidad de la República, Montevideo, Uruguay
E-mail address: roma@fing.edu.uy