

# PARTIAL HYPERBOLICITY AND FOLIATIONS IN $\mathbb{T}^3$

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**ABSTRACT.** We prove that dynamical coherence is an open and closed property in the space of partially hyperbolic diffeomorphisms of  $\mathbb{T}^3$  isotopic to Anosov. Moreover, we prove that strong partially hyperbolic diffeomorphisms of  $\mathbb{T}^3$  are either dynamically coherent or have an invariant two-dimensional torus which is either contracting or repelling. We develop for this end some general results on codimension one foliations which may be of independent interest.

**Keywords:** Partial hyperbolicity (pointwise), Dynamical Coherence, Global Product Structure, Codimension one Foliations.

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## 1. INTRODUCTION

1.0.1. It is well known that robust dynamical properties have implications on the existence of  $Df$ -invariant geometric structures (see for example [M, DPU, BDP]). In particular, partial hyperbolicity plays a fundamental role in the study of robust transitivity and stable ergodicity (see [BDV] chapters 7 and 8).

In dimension 2 it is possible to characterize  $C^1$ -robustly transitive diffeomorphisms (i.e. having a  $C^1$ -neighborhood such that every diffeomorphism in the neighborhood has a dense orbit). Mañé has shown in [M] that on any 2-dimensional manifold, a  $C^1$ -robustly transitive diffeomorphism must be Anosov. By classical results of Franks ([F]) we know that an Anosov diffeomorphism of a surface must be (robustly) conjugated to a linear Anosov automorphism of  $\mathbb{T}^2$ . Since these are transitive, we obtain that:

**Theorem** (Mañe-Franks). *If  $M$  is a closed two dimensional manifold, then a diffeomorphism  $f$  is  $C^1$ -robustly transitive if and only if it is Anosov and conjugated to a linear hyperbolic automorphism of  $\mathbb{T}^2$ .*

In a certain sense, this result shows that in order to obtain a robust dynamical property out of the existence of an invariant geometric structure it may be a good idea to develop some theory on the possible topological properties such a diffeomorphism must have.

More precisely, we identify this result as relating the following three aspects of a diffeomorphism  $f$ :

- Robust dynamical properties (in this case, transitivity).
- $Df$ -invariant geometric structures (in this case, being Anosov).
- Topological properties (in this case,  $M$  is the two-torus and  $f$  is isotopic to a linear Anosov automorphism).

1.0.2. In higher dimensions, the understanding of this relationship is quite less advanced, and we essentially only have results in the sense of showing the existence of  $Df$ -invariant geometric structures when certain robust dynamical properties are present (see [DPU, BDP]).

In this paper, we intend to develop some topological implications of having  $Df$ -invariant geometric structures in the case the manifold is  $\mathbb{T}^3$  and we are interested in giving conditions under which the  $Df$ -invariant bundles are integrable.

As in the hyperbolic (Anosov) case, invariant foliations play a substantial role in the understanding of the dynamics of partially hyperbolic systems (even if many important results manage to avoid the use of the existence of such foliations, see for example [BuW<sub>2</sub>, BI, C]). A still (essentially) up to date survey on the results on invariant foliations is [HPS] (see also [B]). We say that a strong partially hyperbolic diffeomorphism  $f$  is *dynamically coherent* if it admits  $f$ -invariant foliations tangent to its  $Df$ -invariant distributions (see Section 2 for precise definitions).

1.0.3. In principle, dynamical coherence may not be neither an open nor closed property among partially hyperbolic dynamics. There are some hypothesis that guarantee openness which are not known to hold in general and are usually hard to verify (see [HPS, B]). We prove here the following result (see Section 3 for complete statement of the results in this paper):

**Theorem.** *Dynamical coherence is an open and closed property<sup>1</sup> among partially hyperbolic diffeomorphisms of  $\mathbb{T}^3$  isotopic to linear Anosov automorphisms (with splitting of the form  $T\mathbb{T}^3 = E^{cs} \oplus E^u$ ). Moreover, the unique obstruction for a strong partially hyperbolic diffeomorphism of  $\mathbb{T}^3$  (i.e. with splitting of the form  $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$ )*

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<sup>1</sup>In any  $C^r$  topology with  $r \geq 1$ .

*to be dynamically coherent is the existence of a contracting or repelling periodic two-dimensional torus whose dynamics is isotopic to Anosov.*

The second part of this result responds in the affirmative for  $\mathbb{T}^3$  to a conjecture made by M.A. Rodriguez Hertz, F.Rodriguez Hertz and R. Ures in general 3 dimensional manifolds (see [RHRHU<sub>2</sub>]). It is important to remark that they have constructed examples showing that dynamical coherence does not hold for every strong partially hyperbolic diffeomorphism. As a consequence of our results, we obtain:

**Corollary.** *Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be a strong partially hyperbolic diffeomorphism such that  $\Omega(f) = \mathbb{T}^3$  then  $f$  is dynamically coherent.*

1.0.4. Recent results by Brin, Burago and Ivanov (see [BBI, BI]) allow one to obtain certain topological descriptions from the existence of invariant foliations.

We remark that recently, [BBI<sub>2</sub>] have shown that absolute strong partially hyperbolic diffeomorphisms of  $\mathbb{T}^3$  are dynamically coherent using a criterium of Brin ([Br]) which relies in this stronger version of partial hyperbolicity in an essential way. This has been used by Hammerlindl in [H] to obtain leaf-conjugacy results for this kind of systems. The proof of our results owes a lot to both of these results.

Absolute partial hyperbolicity covers many important classes of examples and this makes its study very important. However, the definition of absolute domination is somewhat artificial and does not capture the results about robust dynamical properties (such as the results of [DPU, BDP]). Absolute domination can be compared with a pinching condition on the spectrum of the differential much like the conditions used by Brin and Manning to classify Anosov systems ([BM]). See Section 2 for precise definitions.

1.0.5. The ideal scenario would be to have certain models (as in the 2-dimensional case) which later would contribute to the understanding of the robust dynamical behavior. The following conjecture (concerning strong partially hyperbolic systems) was posed by Pujals according to [BW]:

**Conjecture** (Pujals' Conjecture). *Let  $f$  be a transitive strong partially hyperbolic diffeomorphism of a 3-dimensional manifold  $M$  then, (modulo finite lifts)  $f$  is leaf conjugate to one of the following:*

- *A linear Anosov automorphism of  $\mathbb{T}^3$ .*
- *A skew-product over an Anosov of  $\mathbb{T}^2$  (and thus  $M = \mathbb{T}^3$  or a nilmanifold).*
- *A time one map of an Anosov flow.*

We will not define here the concept of leaf-conjugacy (see [HPS, H]) but as the name suggests it is related with the existence of foliations and dynamical coherence.

The hypothesis on  $f$  being transitive is necessary due to the recent example of [RHRHU<sub>2</sub>]. In [BW] important progress was made towards this conjecture without any assumption on the topology of the manifold.

In [H, H<sub>2</sub>] the conjecture was proved for absolutely dominated strong partially hyperbolic diffeomorphisms of  $\mathbb{T}^3$  and nilmanifolds. In a forthcoming paper ([HP]), with A. Hammerlindl we use the techniques here as well as the ones developed in [H, H<sub>2</sub>] in order to give a proof of Pujals' conjecture on 3-manifolds with almost nilpotent fundamental group.

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## 2. DEFINITIONS AND SOME KNOWN RESULTS

### 2.1. Partial hyperbolicity.

2.1.1. We shall introduce some definitions related to  $Df$ -invariant subbundles.

Given a  $C^1$ -diffeomorphism  $f : M \rightarrow M$  and a  $Df$ -invariant subbundle  $E \subset TM$  we say that  $E$  is *uniformly contracting* (resp. *uniformly expanding*) if there exists  $n_0 > 0$  (resp.  $n_0 < 0$ ) such that:

$$\|Df^{n_0}|_E\| < \frac{1}{2}$$

Given two  $Df$ -invariant subbundles  $E, F$  of  $TM$ , we shall say that  $F$  *dominates*  $E$  if there exists  $n_0 > 0$  such that for every  $x \in M$  and any pair of unit vectors  $v_E \in E(x)$  and  $v_F \in F(x)$  we have that:

$$\|Df^{n_0}v_E\| < \frac{1}{2}\|Df^{n_0}v_F\|$$

It is important to remark here that this notion of domination is weaker than the one appearing in other literature. Sometimes this concept is called *pointwise* (or *relative*) *domination* in contraposition to *absolute domination* (see [HPS]).

We say that  $F$  *absolutely dominates*  $E$  if there exists  $n_0 > 0$  and  $\lambda > 0$  such that for every  $x \in M$  and any pair of unit vectors  $v_E \in E(x)$  and  $v_F \in F(x)$  we have that:

$$\|Df^{n_0}v_E\| < \lambda < \|Df^{n_0}v_F\|$$

The pointwise definition is more suitable in the context of studying robust transitivity and stable ergodicity since it is the one given by the results of [DPU, BDP].

2.1.2. Consider a diffeomorphism  $f : M \rightarrow M$  such that  $TM = E_1 \oplus \dots \oplus E_k$  is a  $Df$ -invariant splitting of the tangent bundle on  $k \geq 2$  non trivial invariant subbundles such that  $E_i$  dominates  $E_j$  if  $i > j$ . Following [BDP] (see also [BDV] appendix B) we say that:

- $f$  is *partially hyperbolic* if either  $E_1$  is uniformly contracting or  $E_k$  is uniformly expanding.
- $f$  is *strongly partially hyperbolic* if both  $E_1$  is uniformly contracting and  $E_k$  is uniformly expanding.

We add the word *absolutely* before this concepts when the domination involved is of absolute nature. We warn the reader about the zoo of different names that have been used throughout time and the literature for these same (and slightly varied) concepts.

These properties are  $C^1$ -robust, the following proposition can be found in [BDV] appendix B:

**Proposition 2.1.** *If  $f$  is partially hyperbolic (resp. strongly partially hyperbolic) then there exist a  $C^1$ -neighborhood  $\mathcal{U}$  of  $f$  such that every  $g \in \mathcal{U}$  is partially hyperbolic (resp. strongly partially hyperbolic).*

We shall group the bundles which are uniformly contracting and call them  $E^s$ , as well as the uniformly expanding ones as  $E^u$ .

*Remark 2.2.* In dimension 3 we may have the following forms of partial hyperbolicity:  $TM = E^s \oplus E^c \oplus E^u$  (which will be the strong partially hyperbolic case),  $TM = E^s \oplus E^{cu}$  and  $TM = E^{cs} \oplus E^u$  (the partially hyperbolic case). For simplicity and by the symmetry of the problem, we shall focus only on the cases  $TM = E^{cs} \oplus E^u$  and  $TM = E^s \oplus E^c \oplus E^u$ .

◇

## 2.2. Invariant and almost invariant foliations.

2.2.1. Along this paper, we shall understand by *foliation* a  $C^0$ -foliation with  $C^1$ -leaves which is tangent to a continuous distribution (foliations of class  $C^{1,0+}$  according to the notation of [CC], Definition I.1.2.24). For a foliation  $\mathcal{F}$  in a manifold  $M$ , we shall denote as  $\mathcal{F}(x)$  to the leaf of  $\mathcal{F}$  containing  $x$ .

It is a well known result that there always exist an invariant foliation tangent to  $E^u$ :

**Theorem 2.3** (Unstable foliation [HPS]). *Let  $f : M \rightarrow M$  be a partially hyperbolic diffeomorphism of the form  $TM = E^{cs} \oplus E^u$ . Then, there exist a foliation  $\mathcal{F}^u$  (the*

unstable foliation) tangent to  $E^u$  which is  $f$ -invariant (i.e. we have that  $f(\mathcal{F}^u(x)) = \mathcal{F}^u(f(x))$ ). Also, if  $y \in \mathcal{F}^u(x)$  then  $d(f^{-n}(x), f^{-n}(y)) \rightarrow 0$  exponentially as  $n \rightarrow +\infty$ .

In the strong partially hyperbolic setting, this result can also be applied to  $f^{-1}$  giving rise to a strong stable foliation  $\mathcal{F}^s$ . The question of the existence of an  $f$ -invariant foliation tangent to the center direction  $E^c$  or to the center-stable one  $E^{cs}$  in the general partially hyperbolic setting is the main concern of this paper.

2.2.2. We say that a partially hyperbolic diffeomorphism  $f$  with splitting  $TM = E^{cs} \oplus E^u$  is *dynamically coherent* if there exists an  $f$ -invariant foliation  $\mathcal{F}^{cs}$  tangent to  $E^{cs}$ .

When  $f$  is strongly partially hyperbolic, we say that  $f$  is *dynamically coherent* when both  $E^s \oplus E^c$  and  $E^c \oplus E^u$  are tangent to  $f$ -invariant foliations  $\mathcal{F}^{cs}$  and  $\mathcal{F}^{cu}$  respectively. See [BuW<sub>1</sub>] for more discussion: in particular, this implies the existence of a  $f$ -invariant foliation  $\mathcal{F}^c$  tangent to  $E^c$  (obtained as the intersection of the foliations).

In general, a partially hyperbolic diffeomorphism may not be dynamically coherent, and even if it is, it is not known in all generality if being dynamically coherent is an open property (see [HPS, B]). However, all the known examples in dimension 3 verify the following property which is clearly  $C^1$ -open (we show in Proposition 4.5 that the property is also closed):

**Definition 2.4** (Almost dynamical coherence). We say that  $f : M \rightarrow M$  partially hyperbolic of the form  $TM = E^{cs} \oplus E^u$  is *almost dynamically coherent* if there exists a foliation  $\mathcal{F}$  transverse to the direction  $E^u$ .

◇

The introduction of this property was motivated by [BI] where it is shown that strong partially hyperbolic systems in dimension 3 verify this property.

### 2.3. Isotopy class of a diffeomorphism of $\mathbb{T}^3$ .

2.3.1. Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be a  $C^1$ -diffeomorphism and  $\tilde{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  a lift to the universal cover.

One can define  $f_*$  as the matrix given by the automorphism  $(\tilde{f}(\cdot) - \tilde{f}(0)) : \mathbb{Z}^3 \rightarrow \mathbb{Z}^3$ . This matrix does not depend on the chosen lift of  $f$ . It is direct to check that  $f_*$  is represented by a matrix of integer coefficients and determinant equal to 1 or  $-1$  (since one can see that  $(f_*)^{-1} = (f^{-1})_*$ ).

It is well known that  $f_*$  also represents the action of  $f$  on the fundamental group of  $\mathbb{T}^3$  (which coincides with the first homology group since it is abelian).

We will sometimes view  $f_*$  as a diffeomorphism of  $\mathbb{R}^3$  (which projects into a diffeomorphism of  $\mathbb{T}^3$  since it is a matrix in  $GL(3, \mathbb{Z})$  with determinant of modulus 1).

2.3.2. By compactness, we have that there exists  $K_0 > 0$  such that for every  $x \in \mathbb{R}^3$ :

$$d(\tilde{f}(x), f_*(x)) < K_0$$

We say that  $f$  is *isotopic to Anosov* if  $f_*$  has no eigenvalues of modulus equal to 1.

In fact, it is well known that two diffeomorphisms  $f, g$  of  $\mathbb{T}^3$  are isotopic if and only if  $f_* = g_*$  and this is why we use this terminology, but we will not use this fact.

### 3. PRECISE STATEMENT OF RESULTS AND ORGANIZATION OF THE PAPER

#### 3.1. Statement of results.

3.1.1. We are now in conditions to state precisely our results:

**Theorem A.** *Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  an almost dynamically coherent partially hyperbolic diffeomorphism with splitting of the form  $T\mathbb{T}^3 = E^{cs} \oplus E^u$  and with  $\dim E^u = 1$ . Assume that  $f$  is isotopic to Anosov, then:*

- $f$  is (robustly) dynamically coherent and has a unique  $f$ -invariant foliation  $\mathcal{F}^{cs}$  tangent to  $E^{cs}$ .
- There exists a global product structure between the lift of  $\mathcal{F}^{cs}$  to the universal cover and the lift of  $\mathcal{F}^u$  to the universal cover.
- If  $f_*$  has two eigenvalues of modulus larger than 1 then they must be real and different.

Given two transversal foliations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  in a simply connected manifold  $\tilde{M}$  we say that they have *global product structure* if for every  $x, y \in \tilde{M}$  we have that the intersection between  $\mathcal{F}_1(x)$  and  $\mathcal{F}_2(y)$  has exactly one point.

In section 6 we obtain some general results on global product structure for codimension one foliations which may be of independent interest.

As a consequence of the fact that almost dynamical coherence is an open and closed property (see Proposition 4.5 below) we obtain:

**Corollary.** *Dynamical coherence is an open and closed property among partially hyperbolic diffeomorphisms of  $\mathbb{T}^3$  isotopic to Anosov.*

Notice that we are not assuming that the dimension of  $E^{cs}$  is the same as the number of stable eigenvalues of the linear part  $f_*$  of  $f$ . Although this is hidden in the statement, it can be seen in the proof that in the case where the dimension of  $E^{cs}$  coincides with the number of stable eigenvalues of  $f_*$  the proof is simpler.



3.1.2. In the strong partially hyperbolic case we are able to give a stronger result partly based on results of [BI] showing that strongly partially hyperbolic diffeomorphisms of 3-manifolds are almost dynamically coherent.

**Theorem B.** *Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be a strong partially hyperbolic diffeomorphism, then:*

- *Either there exists a unique  $f$ -invariant foliation  $\mathcal{F}^{cs}$  tangent to  $E^s \oplus E^c$  or,*
- *There exists a periodic two-dimensional torus  $T$  tangent to  $E^s \oplus E^c$  which is repelling.*

*Remark 3.1.* Indeed, it is not hard to show that in the case there is a repelling torus, it must be an *Anosov tori* as defined in [RHRHU<sub>1</sub>] (this follows from Proposition 2.4 of [BBI]). In the example of [RHRHU<sub>2</sub>] it is shown that the second possibility is not empty.

◇

A diffeomorphism  $f$  is *chain-recurrent* if there is no open set  $U$  such that  $f(\overline{U}) \subset U$  (see [C] Chapter 1 for an introduction to this concept in the context of differentiable dynamics). In particular, if  $\Omega(f) = \mathbb{T}^3$  then  $f$  is chain-recurrent.

**Corollary.** *Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  a chain-recurrent strongly partially hyperbolic diffeomorphism. Then,  $f$  is dynamically coherent.*

In the isotopy class of Anosov, one can see that no center-stable nor center-unstable tori can exist. We obtain (see Theorem 8.1):

**Corollary.** *Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  a strong partially hyperbolic diffeomorphism isotopic to a linear Anosov automorphism. Then,  $f$  is dynamically coherent.*

3.1.3. In the strong partially hyperbolic case, when no torus tangent to  $E^s \oplus E^c$  nor  $E^c \oplus E^u$  exists, we deduce further properties on the existence of planes close to the  $f$ -invariant foliations. These results are essential to obtain leaf-conjugacy results (and will be used in [HP]). See Proposition 8.7 and Proposition A.1.

**3.2. Organization of the paper.** This paper is organized as follows: In section 4 we introduce some new concepts, some known results and prove preliminary results which will be used later. In section 5 we characterize Reebless foliations on  $\mathbb{T}^3$  by following the ideas introduced in [BBI<sub>2</sub>]. Later, in section 6 we give conditions under which the lift of some codimension one foliations to the universal cover have global product structure. In particular, we obtain a quantitative version of a known result which we believe may be of independent interest and whose proof may be useful (at least to non-experts in foliations such as the author). Finally, in section 7 we prove Theorem A and in section 8 we prove Theorem B.

If the reader is interested in Theorem A only, she may skip subsections 4.4, 4.6 and subsection 5.3.

If instead the reader is interested in Theorem B only, he may skip subsection 4.2, section 6 and read Appendix A instead of section 7 (after having read section 8).

The author suggest that the reader with some intuition on foliations should skip sections 5 and 6 and come back to them when having understood the core of the proof. We have made an effort to include the necessary background in sections 7 and 8 in order to send the results of sections 5 and 6 into a “black box” containing the used results.

#### 4. PRELIMINARY RESULTS AND DEFINITIONS

In this paper, we will be mainly concerned with foliations and diffeomorphisms of  $\mathbb{T}^3$ . We shall fix the usual euclidean metric as the distance in  $\mathbb{R}^3$ , and this will induce a metric on  $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$  via  $p : \mathbb{R}^3 \rightarrow \mathbb{T}^3$  the canonical quotient by translations of  $\mathbb{Z}^3$ .

We shall always denote as  $B_\varepsilon(\mathcal{C})$  to the  $\varepsilon$ -neighborhood of a set  $\mathcal{C}$  with the metric we have just defined. Usually  $\tilde{X}$  will denote the lift of  $X$  to the universal cover (whatever  $X$  is).

##### 4.1. Some generalities on foliations.

4.1.1. As we mentioned, for us, a *foliation* will mean a  $C^0$ -foliation with  $C^1$ -leaves which are tangent to a continuous distribution.

We will mention some classical results on codimension one foliations. We refer the reader to [CC, HH] for more information and proofs.

*Remark 4.1.* With our definition of foliation, it is easy to prove that for every codimension one foliation  $\mathcal{F}$  on a manifold  $M$  there exists a transverse foliation  $\mathcal{F}^\perp$ . Indeed, it suffices to consider a  $C^1$ -line field  $X$  transverse to the continuous distribution tangent to  $\mathcal{F}$ . This line field will integrate (uniquely) into a one dimensional foliation transverse to  $\mathcal{F}$ .

◇

4.1.2. When one has two transverse foliations  $\mathcal{F}$  (of codimension one) and  $\mathcal{F}^\perp$  in a compact manifold  $M$  of dimension  $d$  there always exists a *uniform local product structure*. This means that there exists  $\delta > 0$  such that every point  $x$  has a neighborhood  $U$  which contains the  $\delta$ -neighborhood of  $x$  and a homeomorphism  $\varphi : U \rightarrow \mathbb{R}^d$  (a *foliation chart*) sending connected components of leaves of  $\mathcal{F}$  in  $U$  to sets of the form  $\mathbb{R}^{d-1} \times \{t\}$  and connected components of leaves of  $\mathcal{F}^\perp$  in  $U$  to sets of the form  $\{x\} \times \mathbb{R}$ .

We will sometimes abuse notation and call a *local leaf* of a foliation  $\mathcal{F}$  to a connected component of the leaf in a local product structure neighborhood without explicitly mentioning which neighborhood we are referring to.

4.1.3. As defined above, we say that two transversal foliations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  in a simply connected manifold  $\tilde{M}$  have *global product structure* if for every  $x, y \in \tilde{M}$  we have that the intersection between  $\mathcal{F}_1(x)$  and  $\mathcal{F}_2(y)$  has exactly one point.

For codimension one foliations, the fact that leaves cannot intersect in more than one point is usually proved by using the following result due to Haefliger (see [So, HH] for the  $C^0$ -version):

**Proposition 4.2** (Haefliger's argument). *Let  $\mathcal{F}$  be a codimension one foliation of a simply connected (not necessarily compact) manifold  $\tilde{M}$  such that every leaf of  $\mathcal{F}$  is simply connected and let  $\mathcal{F}^\perp$  be a transverse foliation, then, no leaf of  $\mathcal{F}^\perp$  can intersect a leaf of  $\mathcal{F}$  twice.*

4.1.4. We close this subsection by recalling some facts about *holonomy* (see [HH, CC] for more information).

We say that a foliation  $\mathcal{F}$  is *without holonomy* if no leaf of  $\mathcal{F}$  has non-trivial holonomy. We will not define this, but we give two conditions under which a codimension one foliation  $\mathcal{F}$  of a compact manifold is without holonomy:

- If every leaf of  $\mathcal{F}$  is simply connected.
- If every leaf of  $\mathcal{F}$  is closed and every two leaves are homotopic.

Reeb's stability theorem (see [HH, CC]) has the following consequences which we will use later:

**Proposition 4.3.** *Let  $\mathcal{F}$  be a codimension one foliation of a 3-dimensional manifold  $M$ :*

- (i) *If there is a leaf homeomorphic to  $S^2$  then  $M$  is finitely covered by  $S^2 \times S^1$ .*
- (ii) *If a foliation  $\mathcal{F}$  of  $\mathbb{T}^3$  has a dense set of leaves homeomorphic to two-dimensional torus which are all homotopic, then, every leaf of  $\mathcal{F}$  is a torus homotopic to the ones in the dense set. In particular,  $\mathcal{F}$  is without holonomy.*

4.1.5. In [BW] the following criterium for obtaining a foliation out of a dense subset of leaves was given:

**Proposition 4.4** ([BW] Proposition 1.6 and Remark 1.10). *Let  $E$  be a continuous codimension one distribution on a manifold  $M$  and  $S$  a (possibly non connected) injectively immersed submanifold everywhere tangent to  $E$  which contains a family of disks of fixed radius and whose set of midpoints is dense in  $M$ . Then, there exists a foliation  $\mathcal{F}$  tangent to  $E$  which contains  $S$  in its leaves.*

## 4.2. Almost dynamical coherence is an open and closed property.

4.2.1. Almost dynamical coherence is not a very strong requirement. From its definition and the continuous variation of the unstable bundle under perturbations it is clear that it is an open property. With the basic facts on domination we can show:

**Proposition 4.5.** *Let  $\{f_n\}$  a sequence of almost dynamically coherent partially hyperbolic diffeomorphisms converging in the  $C^1$ -topology to a partially hyperbolic diffeomorphism  $f$ . Then  $f$  is almost dynamically coherent.*

PROOF. Let us call  $E_n^{cs} \oplus E_n^u$  to the splitting of  $f_n$  and  $E^{cs} \oplus E^u$  to the splitting of  $f$ . We use the following well known facts on domination (see [BDV] appendix B):

- The subspaces  $E_n^{cs}$  and  $E_n^u$  converge as  $n \rightarrow \infty$  towards  $E^{cs}$  and  $E^u$ .
- The angle between  $E^{cs}$  and  $E^u$  is larger than  $\alpha > 0$ .

Now, consider  $f_n$  such that the angle between  $E_n^{cs}$  and  $E^u$  is everywhere larger than  $\alpha/2$ . Let  $\mathcal{F}_n$  be the foliation transverse to  $E_n^u$  given by the fact that  $f_n$  is almost dynamically coherent.

Since  $f_n$  is a diffeomorphism, we can define a foliation by iterating backwards by  $f_n$  any foliation in  $M$ . Moreover, the invariant distribution tangent to such a foliation will be transformed by  $Df_n^{-m}$  when iterated backwards.

We can choose  $m > 0$  sufficiently large such that  $f_n^{-m}(\mathcal{F}_n)$  is tangent to a small cone (of angle less than  $\alpha/2$ ) around  $E_n^{cs}$ .

This implies that  $f_n^{-m}(\mathcal{F}_n)$  is a foliation transverse to  $E^u$  and this gives that  $f$  is almost dynamically coherent as desired.

□

4.2.2. Consider  $\mathcal{PH}^1(M)$  the set of partially hyperbolic  $C^1$ -diffeomorphisms of  $M$ . Let  $\mathcal{P}$  be a connected component of  $\mathcal{PH}^1(M)$ ; if  $\mathcal{P}$  contains an almost dynamically coherent diffeomorphism, then every diffeomorphism in  $\mathcal{P}$  is almost dynamically coherent.

In [BI] (Key Lemma 2.1) it is proved that every strong partially hyperbolic diffeomorphism of a 3-dimensional manifold is almost dynamically coherent. It is important to remark that it is a mayor problem to determine whether partially hyperbolic diffeomorphisms in 3-dimensional manifolds are almost dynamically coherent<sup>2</sup>.

The author is not aware of whether the following question is known or still open:

**Question 1.** *Are there any examples of partially hyperbolic diffeomorphisms of  $\mathbb{T}^3$  isotopic to a linear Anosov automorphism which is not isotopic to the linear Anosov automorphism through a path of partially hyperbolic diffeomorphisms?*

The hypothesis of almost dynamical coherence is essential in the proof of Theorem A, but if the previous question admits a negative answer, from what we have proved in this subsection it would be a superfluous hypothesis.

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<sup>2</sup>Which with the arguments of [BI] would prove the non-existence of robustly transitive diffeomorphisms on the 3-dimensional sphere which is an old open problem.

### 4.3. Consequences of Novikov's Theorem.

4.3.1. We shall state some results motivated by the work of Brin, Burago and Ivanov ([BBI, BI, BBI<sub>2</sub>]) on strong partial hyperbolicity. They made the beautiful remark that a foliation transverse to the unstable foliation cannot have Reeb components since that would imply the existence of a closed unstable curve which is impossible. We shall not define Reeb component here, but we refer the reader to [CC] chapter 9 for information on this concepts. Reeb components in invariant foliations for partially hyperbolic diffeomorphisms were also considered in [DPU] (Theorem H) but in the context of robust transitivity and with different arguments.

The remark of Brin, Burago and Ivanov can be coupled with the  $C^0$  version of the classical Novikov's theorem (see [CC] Theorems 9.1.3 and 9.1.4 and the Remark on page 286):

**Theorem 4.6** (Novikov [So, CC]). *Let  $\mathcal{F}$  be a (transversally oriented) codimension one foliation on a 3-dimensional compact manifold  $M$  and assume that one of the following holds:*

- *There exist a positively oriented closed loop transverse to  $\mathcal{F}$  which is nullhomotopic, or,*
- *there exist a leaf  $S$  of  $\mathcal{F}$  such that the fundamental group of  $S$  does not inject on the fundamental group of  $M$  (i.e. the leaf  $S$  is not incompressible).*

*Then,  $\mathcal{F}$  has a Reeb component.*

4.3.2. As a corollary of Novikov's theorem, we obtain the following properties as in [BI, BBI<sub>2</sub>] (many of the statements here also hold for general 3-dimensional manifolds, see for example [Pw] for related results):

**Corollary 4.7.** *Let  $f$  be a partially hyperbolic diffeomorphism of  $\mathbb{T}^3$  of the form  $T\mathbb{T}^3 = E^{cs} \oplus E^u$  ( $\dim E^{cs} = 2$ ) which is almost dynamically coherent with foliation  $\mathcal{F}$ . Let  $\tilde{\mathcal{F}}$  and  $\tilde{\mathcal{F}}^u$  the lifts of the foliations  $\mathcal{F}$  and the unstable foliation  $\mathcal{F}^u$  to  $\mathbb{R}^3$ . Then:*

- (i) *For every  $x \in \mathbb{R}^3$  we have that  $\tilde{\mathcal{F}}(x) \cap \tilde{\mathcal{F}}^u(x) = \{x\}$ .*
- (ii) *The leaves of  $\tilde{\mathcal{F}}$  are properly embedded complete surfaces in  $\mathbb{R}^3$ . In fact there exists  $\delta > 0$  such that every euclidean ball  $U$  of radius  $\delta$  can be covered by a continuous coordinate chart such that the intersection of every leaf  $S$  of  $\tilde{\mathcal{F}}$  with  $U$  is either empty or represented as the graph of a function  $h_S : \mathbb{R}^2 \rightarrow \mathbb{R}$  in those coordinates.*
- (iii) *Each closed leaf of  $\mathcal{F}$  is an incompressible two dimensional torus (i.e. such that the inclusion induces an injection of fundamental groups).*
- (iv) *For every  $\delta > 0$ , there exists a constant  $C_\delta$  such that if  $J$  is a segment of  $\tilde{\mathcal{F}}^u$  then  $\text{Vol}(B_\delta(J)) > C_\delta \text{length}(J)$ .*

PROOF. If a foliation  $\mathcal{F}$  on a compact closed 3-dimensional manifold  $M$  has a Reeb component, then, every one dimensional foliation transverse to  $\mathcal{F}$  has a closed leaf (see [BI] Lemma 2.2). Since  $\mathcal{F}^u$  is one dimensional, transverse to  $\mathcal{F}$  and has no closed leaves, we obtain that  $\mathcal{F}$  cannot have Reeb components. To prove (i) and (ii) one can assume that  $\mathcal{F}$  is transversally oriented since this holds for a finite lift and the statement is in the universal cover.

The proof of (i) is the same as the one of Lemma 2.3 of [BI], indeed, if there were two points of intersection, one can construct a closed loop transverse to  $\tilde{\mathcal{F}}$  which descends in  $\mathbb{T}^3$  to a nullhomotopic one. By Novikov's theorem (Theorem 4.6), this implies the existence of a Reeb component, a contradiction.

Once (i) is proved, (ii) follows from the same argument as in Lemma 3.2 in [BBI<sub>2</sub>]. Notice that the fact that the leaves of  $\tilde{\mathcal{F}}$  are properly embedded is trivial after (i), with some work, one can prove the remaining part of (ii) (see also Lemma 7.6 for a more general statement).

Part (iii) follows from the fact that if  $S$  is a closed surface in  $\mathbb{T}^3$  which is not a torus, then it is either a sphere or its fundamental group cannot inject in  $\mathbb{T}^3$  (see [R] and notice that neither a group with exponential growth nor the fundamental group of the Klein bottle can inject in  $\mathbb{Z}^3$ ).

Since  $\mathcal{F}$  has no Reeb components (and the same happens for any finite lift) we obtain that if  $S$  is a closed leaf of  $\mathcal{F}$  then it must be a sphere or an incompressible torus. But  $S$  cannot be a sphere since in that case, the Reeb's stability theorem (see Proposition 4.3) would imply that all the leaves of  $\mathcal{F}$  are spheres and that the foliated manifold is finitely covered by  $S^2 \times S^1$  which is not the case.

The proof of (iv) is as Lemma 3.3 of [BBI<sub>2</sub>]. By (i) there cannot be two points in the same leaf of  $\tilde{\mathcal{F}}^u$  which are close but in different local unstable leaves. We can find  $\epsilon > 0$  and  $a > 0$  such that in a curve of length  $K$  of  $\tilde{\mathcal{F}}^u$  there are at least  $aK$  points whose balls of radius  $\epsilon$  are disjoint (and all have the same volume).

Now, consider  $\delta > 0$  and  $\tilde{\delta} = \min\{\delta, \epsilon\}$ . Let  $\{x_1, \dots, x_l\}$  with  $l > a \text{ length}(J)$  be points such that their  $\tilde{\delta}$ -balls are disjoint. We get that  $U = \bigcup_{i=1}^l B_{\tilde{\delta}}(x_i) \subset B_{\delta}(J)$  and we have that  $\text{Vol}(U) > l \text{Vol}(B_{\tilde{\delta}}(x_i))$ . We obtain that  $C_{\delta} = \frac{4\pi}{3} a \delta^3$  works.

□

4.3.3. Recall from subsection 2.3 that if  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  then  $f_* \in GL(3, \mathbb{Z})$  denotes its action on the first homology group (which has determinant of modulus 1). Using Corollary 4.7 it was proven in [BBI, BI] that:

**Theorem 4.8** (Brin-Burago-Ivanov [BBI, BI]). *Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  an almost dynamically coherent partially hyperbolic diffeomorphism. Then,  $f_*$  has at least an eigenvalue of absolute value larger than 1 and at least one of absolute value smaller than 1.*

4.4. **Branching foliations and Burago-Ivanov's results.** We follow [BI] section 4.

We define a *surface* in a 3-manifold  $M$  to be a  $C^1$ -immersion  $\iota : U \rightarrow M$  of a connected smooth 2-dimensional manifold without boundary. The surface is said to be *complete* if it is complete with the metric induced in  $U$  by the Riemannian metric of  $M$  pulled back by the immersion  $\iota$ .

Given a point  $x$  in (the image of) a surface  $\iota : U \rightarrow M$  we have that there is a neighborhood  $B$  of  $x$  such that the connected component  $C$  containing  $\iota^{-1}(x)$  of  $\iota^{-1}(B)$  verifies that  $\iota(C)$  separates  $B$ . We say that two surfaces  $\iota_1 : U_1 \rightarrow M, \iota_2 : U_2 \rightarrow M$  *topologically cross* if there exists a point  $x$  in (the image of)  $\iota_1$  and  $\iota_2$  and a curve  $\gamma$  in  $U_2$  such that  $\iota_2(\gamma)$  passes through  $x$  and intersects both connected components of a neighborhood of  $x$  with the part of the surface defined above removed. It is not hard to prove that the definition is indeed symmetric (see [BI] Section 4).

A *branching foliation* on  $M$  is a collection of complete surfaces tangent to a given continuous 2-dimensional distribution on  $M$  such that:

- Every point belongs to (the image of) at least one surface.
- There are no topological crossings between surfaces of the collection.
- The branching foliation is complete in the following sense: If  $x_k \rightarrow x$  and  $L_k$  are (images of) surfaces of the collection through the points  $x_k$ , we have that  $L_k$  converges in the  $C^1$ -topology to (the image of) a surface  $L$  of the collection through  $x$ .

We call the (image of the) surfaces *leaves* of the branching foliation. We will abuse notation and denote a branching foliation as  $\mathcal{F}_{bran}$  and by  $\mathcal{F}_{bran}(x)$  the set of leaves which contain  $x$ .

**Theorem 4.9** ([BI], Theorem 4.1 and Theorem 7.2). *Let  $f : M^3 \rightarrow M^3$  a strong partially hyperbolic diffeomorphism with splitting  $TM = E^s \oplus E^c \oplus E^u$  into non trivial one-dimensional bundles. There exists branching foliations<sup>3</sup>  $\mathcal{F}_{bran}^{cs}$  and  $\mathcal{F}_{bran}^{cu}$  tangent to  $E^{cs} = E^s \oplus E^c$  and  $E^{cu} = E^c \oplus E^u$  respectively which are  $f$ -invariant. For every  $\varepsilon > 0$  there exist foliations  $\mathcal{S}_\varepsilon$  and  $\mathcal{U}_\varepsilon$  tangent to an  $\varepsilon$ -cone around  $E^{cs}$  and  $E^{cu}$  respectively. Moreover, there exist continuous and surjective maps  $h_\varepsilon^{cs}$  and  $h_\varepsilon^{cu}$  at  $C^0$ -distance smaller than  $\varepsilon$  from the identity which send the leaves of  $\mathcal{S}_\varepsilon$  and  $\mathcal{U}_\varepsilon$  to leaves of  $\mathcal{F}_{bran}^{cs}$  and  $\mathcal{F}_{bran}^{cu}$  respectively.*

Since  $E^s$  and  $E^u$  are uniquely integrable (Theorem 2.3) we get that leaves of  $\mathcal{F}^s$  and  $\mathcal{F}^u$  are contained in the leaves of  $\mathcal{F}_{bran}^{cs}$  and  $\mathcal{F}_{bran}^{cu}$  respectively.

*Remark 4.10.* Notice that the existence of the maps  $h_\varepsilon^{cs}$  and  $h_\varepsilon^{cu}$  which are  $\varepsilon$ -close to the identity implies that when lifted to the universal cover, the leaves of  $\mathcal{S}_\varepsilon$  (resp.  $\mathcal{U}_\varepsilon$ ) remain at distance smaller than  $\varepsilon$  from lifted leaves of  $\mathcal{F}_{bran}^{cs}$  (resp.  $\mathcal{F}_{bran}^{cu}$ ).

<sup>3</sup>The fact that one can choose them complete in the sense defined above is proved in Lemma 7.1 of [BI].

◇

We obtain as a corollary the following result we have already announced and which in our terminology can be stated as follows:

**Corollary 4.11** (Key Lemma 2.2 of [BI]). *A strong partially hyperbolic diffeomorphism on a 3-dimensional manifold is almost dynamically coherent.*

Using Corollary 4.11 and Theorem 4.8 we deduce:

**Corollary 4.12** (Main Theorem of [BI]). *Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  a strong partially hyperbolic diffeomorphism. Then,  $f_*$  has an eigenvalue of modulus larger than 1 and an eigenvalue of modulus smaller than 1.*

The proof of Theorem B relies on the following property of branching foliations. We indicate its proof for completeness (it also follows from Proposition 4.4):

**Proposition 4.13.** *If every point of  $M$  belongs to a unique leaf of the branching foliation, then the branching foliation is a true foliation.*

PROOF. Let  $E$  be the two-dimensional distribution tangent to the branching foliation and we consider  $E^\perp$  a transverse direction which we can assume is  $C^1$  and almost orthogonal to  $E$ .

By uniform continuity we find  $\varepsilon$  such that for every point  $p$  in  $M$  the  $2\varepsilon$ -ball centered at  $p$  verifies that it admits a  $C^1$ -chart to an open set in  $\mathbb{R}^3$  which sends  $E$  to an almost horizontal  $xy$ -plane and  $E^\perp$  to an almost vertical  $z$ -line.

Let  $D$  be a small disk in the (unique) surface through  $p$  and  $\gamma$  a small arc tangent to  $E^\perp$  thorough  $p$ . Given a point  $q \in D$  and  $t \in \gamma$  we have that inside  $B_{2\varepsilon}(p)$  there is a unique point of intersection between the curve tangent to  $E^\perp$  through  $q$  and the connected component of the (unique) surface of  $\mathcal{F}_{bran}$  intersected with  $B_{2\varepsilon}(p)$  containing  $t$ .

We get a well defined continuous and injective map from  $D \times \gamma \cong \mathbb{R}^3$  to a neighborhood of  $p$  (by the invariance of domain's theorem) such that it sends sets of the form  $D \times \{t\}$  into surfaces of the branching foliation. Since we already know that  $\mathcal{F}_{bran}$  is tangent to a continuous distribution, we get that  $\mathcal{F}_{bran}$  is a true foliation.

□

#### 4.5. Diffeomorphisms isotopic to linear Anosov automorphisms.

4.5.1. Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be a diffeomorphism which is isotopic to a linear Anosov automorphism  $A : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ .



We shall denote as  $A$  to both the diffeomorphism of  $\mathbb{T}^3$  and to the hyperbolic matrix  $A \in GL(3, \mathbb{Z})$  (with determinant of modulus 1) which acts in  $\mathbb{R}^3$  and is the lift of the torus diffeomorphism  $A$  to the universal cover.

As we mentioned in Subsection 2.3 being isotopic to Anosov is equivalent to having  $f_* = A$ .

4.5.2. Classical arguments (see [W]) give that there exists  $K_1$  such that for every  $x \in \mathbb{R}^3$ , there exists a unique  $y \in \mathbb{R}^3$  such that

$$d(\tilde{f}^n(x), A^n y) \leq K_1 \quad \forall n \in \mathbb{Z}$$

We say that the point  $y$  *shadows* the point  $x$ . Notice that uniqueness implies that the point associated with  $x + \gamma$  is  $y + \gamma$  where  $\gamma \in \mathbb{Z}^3$ . We get the following well known result:

**Proposition 4.14.** *There exists  $H : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  continuous and surjective such that  $H \circ \tilde{f} = A \circ H$ . Also, it is verified that  $H(x + \gamma) = H(x) + \gamma$  for every  $x \in \mathbb{R}^3$  and  $\gamma \in \mathbb{Z}^3$  so, there exists also  $h : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  homotopic to the identity such that  $h \circ f = A \circ h$ . Moreover, we have that  $d(H(x), x) < K_1$  for every  $x \in \mathbb{R}^3$ .*

PROOF. Any orbit of  $\tilde{f}$  is a  $K_0$ -pseudo-orbit of the hyperbolic matrix  $A$ . This gives that for every  $x$  we can associate a unique point  $y$  such that

$$d(\tilde{f}^n(x), A^n y) \leq K_1 \quad \forall n \in \mathbb{Z}$$

We define  $H(x) = y$ . It is not hard to show that  $H$  is continuous. Since it is at distance smaller than  $K_1$  from the identity, we deduce that  $H$  is surjective.

Uniqueness of the points that shadows an  $\tilde{f}$ -orbit gives us  $H \circ \tilde{f} = A \circ H$ . □

It is well known and easy to show that  $H(W^\sigma(x, \tilde{f})) \subset W^\sigma(H(x), A)$  with  $\sigma = s, u$ .

4.5.3. We finish this section by reviewing some well known properties of linear Anosov automorphisms on  $\mathbb{T}^3$ .

We say that a matrix  $A \in GL(3, \mathbb{Z})$  (with determinant of modulus 1) is *irreducible* if and only if its characteristic polynomial is irreducible in the field  $\mathbb{Q}$ . It is not hard to prove:

**Proposition 4.15.** *Every hyperbolic matrix  $A \in GL(3, \mathbb{Z})$  with determinant of modulus 1 is irreducible. Moreover, it cannot have an invariant linear two-dimensional torus.*

PROOF. Assume that a matrix  $A$  is not irreducible, this means that  $A$  has one eigenvalue in  $\mathbb{Q}$ .

Notice that the characteristic polynomial of  $A$  has the form  $-\lambda^3 + a\lambda^2 + b\lambda \pm 1$ . By the rational root theorem, if there is a rational root, it must be  $\pm 1$  which is impossible if  $A$  is hyperbolic.

Every linear Anosov automorphism is transitive. Let  $T$  be an linear two-dimensional torus which is invariant under  $A$ . Since the tangent space of  $T$  must also be invariant, we get that it must be everywhere tangent to an eigenspace of  $A$ . Since we have only 3 eigenvalues, this implies that either  $T$  is attracting or repelling, contradicting transitivity.  $\square$

We can obtain further algebraic properties on the eigenvalues of hyperbolic matrices acting in  $\mathbb{T}^3$ :

**Lemma 4.16.** *Let  $A \in GL(3, \mathbb{Z})$  be a hyperbolic matrix with determinant of modulus 1. Then, the eigenvalues are simple and irrational. Moreover, if there is a pair of complex conjugate eigenvalues they must be of irrational angle.*

PROOF. By the previous proposition, we have that the characteristic polynomial of  $A$  is irreducible as a polynomial with rational coefficients.

It is a classic result in Galois' theory that in a field of characteristic zero, irreducible polynomials have simple roots (see [Hun] Definition 3.10 and the remark after the definition).

The matrix  $A^k$  (with  $k \neq 0$ ) will still be a hyperbolic matrix in  $GL(3, \mathbb{Z})$  and its determinant will still be of modulus 1. The argument above implies that for every  $k \neq 0$  the matrix  $A^k$  has simple and algebraic eigenvalues (because it will still be irreducible).

If  $A$  has a complex eigenvalue, then its complex conjugate must also be an eigenvalue of  $A$ . So, the angle of the complex eigenvalue is irrational since otherwise there would exist  $k$  such that  $A^k$  has a double real eigenvalue.  $\square$

4.5.4. Since the eigenspaces of the matrix  $A$  are invariant subspaces for  $A$ , we get that each eigenline or eigenplane has totally irrational direction (by this we mean that its projection to  $\mathbb{T}^3$  is simply connected and dense). Also, these are the only invariant subspaces of  $A$ .

Since  $A$  is hyperbolic and the product of eigenvalues is one, we get that  $A$  must have one or two eigenvalues with modulus smaller than 1. We say that  $A$  has *stable dimension* 1 or 2 depending on how many eigenvalues of modulus smaller than one it has.

We call *stable eigenvalues* (resp. *unstable eigenvalues*) to the eigenvalues of modulus smaller than one (resp. larger than one). The subspace  $E_A^s$  (resp  $E_A^u$ ) corresponds to the eigenspace associated to the stable (resp. unstable) eigenvalues.

*Remark 4.17.* For every  $A \in GL(3, \mathbb{Z})$  hyperbolic with determinant of modulus 1, we know exactly which are the invariant planes of  $A$ . If  $A$  has complex eigenvalues, then, the only invariant plane is the eigenspace associated to that pair of complex conjugate eigenvalues. If  $A$  has 3 different real eigenvalues then there are 3 different invariant planes, one for each pair of eigenvalues. All these planes are totally irrational.

◇

#### 4.6. Diffeomorphisms isotopic to linear partially hyperbolic maps of $\mathbb{T}^3$ .

4.6.1. In the case  $f_*$  is hyperbolic, we saw that each eigendirection of  $f_*$  projects into a immersed line which is dense in  $\mathbb{T}^3$  (the same holds for each plane).

In the non hyperbolic partially hyperbolic case (i.e. when one eigenvalue has modulus 1 and the other two have modulus different from 1), we get that:

**Lemma 4.18.** *Let  $A$  be a matrix in  $GL(3, \mathbb{Z})$  with eigenvalues  $\lambda^s, \lambda^c, \lambda^u$  verifying  $0 < |\lambda^s| < |\lambda^c| = 1 < |\lambda^u| = |\lambda^s|^{-1}$ . Let  $E_A^s, E_A^c, E_A^u$  be the eigenspaces associated to  $\lambda^s, \lambda^c$  and  $\lambda^u$  respectively. We have that:*

- $E_A^c$  projects by  $p$  into a closed circle where  $p: \mathbb{R}^3 \rightarrow \mathbb{T}^3$  is the covering projection.
- The eigenlines  $E_A^s$  and  $E_A^u$  project by  $p$  into immersed lines whose closure coincide with a two dimensional linear torus.

**PROOF.** We can work in the vector space  $\mathbb{Q}^3$  over  $\mathbb{Q}$  where  $A$  is well defined since it has integer entries. As we mentioned in the proof of Lemma 4.16 all rational eigenvalues of  $A$  must be equal to  $+1$  or  $-1$ .

Since  $1$  (or  $-1$ ) is an eigenvalue of  $A$  and is rational, we obtain that there is an eigenvector of  $1$  in  $\mathbb{Q}^3$ . Thus, the  $\mathbb{R}$ -generated subspace (now in  $\mathbb{R}^3$ ) projects under  $p$  into a circle.

Since  $1$  is a simple eigenvalue for  $A$ , there is a rational canonical form for  $A$  which implies the existence of two-dimensional  $\mathbb{Q}$ -subspace of  $\mathbb{Q}^3$  which is invariant by  $A$  (see for example Theorem VII.4.2 of [Hun]).

This plane (as a 2-dimensional  $\mathbb{R}$ -subspace of  $\mathbb{R}^3$ ) must project by  $p$  into a torus since it is generated by two linearly independent rational vectors. Since the eigenvalues  $\lambda^s$  and  $\lambda^u$  are irrational, their eigenlines project into lines which are dense in this torus.

□

*Remark 4.19.* Assume that  $f: \mathbb{T}^3 \rightarrow \mathbb{T}^3$  is a strongly partially hyperbolic diffeomorphism which is not isotopic to Anosov. By Corollary 4.12 we have that  $f_*$  is in the hypothesis of the previous proposition. Let  $P$  be an  $f_*$ -invariant plane, then there are the following 3 possibilities:

- $P$  may project into a torus. In this case,  $P = E_*^s \oplus E_*^u$  (the eigenplane corresponding to the eigenvalues of modulus different from one).
- If  $P = E_*^s \oplus E_*^c$  then  $P$  projects into an immersed cylinder which is dense in  $\mathbb{T}^3$ .
- If  $P = E_*^c \oplus E_*^u$  then  $P$  projects into an immersed cylinder which is dense in  $\mathbb{T}^3$ .

◇

## 5. REEBLESS FOLIATIONS OF $\mathbb{T}^3$

### 5.1. Some preliminaries and statement of results.

5.1.1. We consider a codimension one foliation  $\mathcal{F}$  of  $\mathbb{T}^3$  and  $\mathcal{F}^\perp$  a one dimensional transversal foliation. We shall assume throughout that  $\mathcal{F}$  has no Reeb components and that  $\mathcal{F}^\perp$  is oriented (we can always assume this by considering a double cover).

Let  $p : \mathbb{R}^3 \rightarrow \mathbb{T}^3$  be the canonical covering map whose deck transformations are translations by elements of  $\mathbb{Z}^3$ .

Since  $\mathbb{R}^3$  is simply connected, we can consider an orientation on  $\tilde{\mathcal{F}}^\perp$  (the fact that  $\mathcal{F}^\perp$  is oriented, implies that this orientation is preserved under all deck transformations).

Given  $x \in \mathbb{R}^3$  we get that  $\tilde{\mathcal{F}}^\perp(x) \setminus \{x\}$  has two connected components which we call  $\tilde{\mathcal{F}}_+^\perp(x)$  and  $\tilde{\mathcal{F}}_-^\perp(x)$  according to the chosen orientation of  $\tilde{\mathcal{F}}^\perp$ .

By Corollary 4.7 (ii) we have that for every  $x \in \mathbb{R}^3$  the set  $\tilde{\mathcal{F}}(x)$  separates  $\mathbb{R}^3$  into two connected components<sup>4</sup> whose boundary is  $\tilde{\mathcal{F}}(x)$ . These components will be denoted as  $F_+(x)$  and  $F_-(x)$  depending on whether they contain  $\tilde{\mathcal{F}}_+^\perp(x)$  or  $\tilde{\mathcal{F}}_-^\perp(x)$ .

Since covering transformations preserve the orientation and send  $\tilde{\mathcal{F}}$  into itself, we have that:

$$F_\pm(x) + \gamma = F_\pm(x + \gamma) \quad \forall \gamma \in \mathbb{Z}^3$$

5.1.2. For every  $x \in \mathbb{R}^3$ , we consider the following subsets of  $\mathbb{Z}^3$  seen as deck transformations:

$$\Gamma_+(x) = \{\gamma \in \mathbb{Z}^3 : F_+(x) + \gamma \subset F_+(x)\}$$

$$\Gamma_-(x) = \{\gamma \in \mathbb{Z}^3 : F_-(x) + \gamma \subset F_-(x)\}$$

We also consider  $\Gamma(x) = \Gamma_+(x) \cup \Gamma_-(x)$ .

---

<sup>4</sup>This is an application of the classical Jordan-Brouwer's separation theorem on the 3-dimensional sphere  $S^3$  since by adding one point at infinity we get that  $\tilde{\mathcal{F}}(x)$  is an embedded sphere in  $S^3$ .

*Remark 5.1.* There exists a uniform local product structure between  $\tilde{\mathcal{F}}$  and  $\tilde{\mathcal{F}}^\perp$  (given by the local product structure, c.f. 4.1.2). Since leaves of  $\tilde{\mathcal{F}}$  do not intersect there exists  $\delta > 0$  such that if two points  $x, y$  are at distance smaller than  $\delta$ , then either  $F_+(x) \subset F_+(y)$  or  $F_+(y) \subset F_+(x)$  (the same for  $F_-$ ). In particular, if  $d(\tilde{\mathcal{F}}(x), \tilde{\mathcal{F}}(x) + \gamma) < \delta$ , then  $\gamma \in \Gamma(x)$ . By Corollary 4.7 we also know that two points which are not in the same local unstable manifold (c.f. 4.1.2) cannot be at distance smaller than  $\delta$ .

◇

**Lemma 5.2.** *The following properties hold:*

- (i) *If both  $F_+(x) \cap F_+(y) \neq \emptyset$  and  $F_-(x) \cap F_-(y) \neq \emptyset$  then, either  $F_+(x) \subset F_+(y)$  and  $F_-(y) \subset F_-(x)$  or  $F_+(y) \subset F_+(x)$  and  $F_-(x) \subset F_-(y)$ . In both of this cases we shall say that  $F_+(x)$  and  $F_+(y)$  are nested (similar with  $F_-$ ).*
- (ii) *If  $F_+(x) \cap F_+(y) = \emptyset$  then  $F_+(y) \subset F_-(x)$  and  $F_+(x) \subset F_-(y)$ . A similar property holds if  $F_-(x) \cap F_-(y) = \emptyset$ .*
- (iii) *In particular,  $F_+(x) \subset F_+(y)$  if and only if  $F_-(y) \subset F_-(x)$  (they are nested in both cases).*

PROOF. We will only consider the case where  $\tilde{\mathcal{F}}(x) \neq \tilde{\mathcal{F}}(y)$  since otherwise the Lemma is trivially satisfied (and case (ii) is not possible).

Assume that both  $F_+(x) \cap F_+(y)$  and  $F_-(x) \cap F_-(y)$  are non-empty. Since  $\tilde{\mathcal{F}}(y)$  is connected and does not intersect  $\tilde{\mathcal{F}}(x)$  we have that it is contained in either  $F_+(x)$  or  $F_-(x)$ . We can further assume that  $\tilde{\mathcal{F}}(y) \subset F_+(x)$  the other case being symmetric. In this case, we deduce that  $F_+(y) \subset F_+(x)$ : otherwise, we would have that  $F_-(x) \cap F_-(y) = \emptyset$ . But this implies that  $\tilde{\mathcal{F}}(x) \subset F_-(y)$  and thus that  $F_-(x) \subset F_-(y)$  which concludes the proof of (i).

To prove (ii) notice that if  $F_+(x) \cap F_+(y) = \emptyset$  then we have that  $\tilde{\mathcal{F}}(x) \subset F_-(y)$  and  $\tilde{\mathcal{F}}(y) \subset F_-(x)$ . This gives that both  $F_+(x) \subset F_-(y)$  and  $F_+(y) \subset F_-(x)$  as desired.

Finally, if  $F_+(x) \subset F_+(y)$  we have that  $F_-(x) \cap F_-(y)$  contains at least  $F_-(y)$  so that (i) applies to give (iii).

□

We can prove (see Lemma 3.9 of [BBI<sub>2</sub>]):

**Lemma 5.3.** *For every  $x \in \mathbb{R}^3$  we have that  $\Gamma(x)$  is a subgroup of  $\mathbb{Z}^3$ .*

PROOF. Consider  $\gamma_1, \gamma_2 \in \Gamma_+(x)$ . Since  $\gamma_1 \in \Gamma_+(x)$  we have that  $F_+(x) + \gamma_1 \subset F_+(x)$ . By translating by  $\gamma_2$  we obtain  $F_+(x) + \gamma_1 + \gamma_2 \subset F_+(x) + \gamma_2$ , but since  $\gamma_2 \in \Gamma_+(x)$  we have  $F_+(x) + \gamma_1 + \gamma_2 \subset F_+(x)$ , so  $\gamma_1 + \gamma_2 \in \Gamma_+(x)$ . This shows that  $\Gamma_+(x)$  is a semigroup.

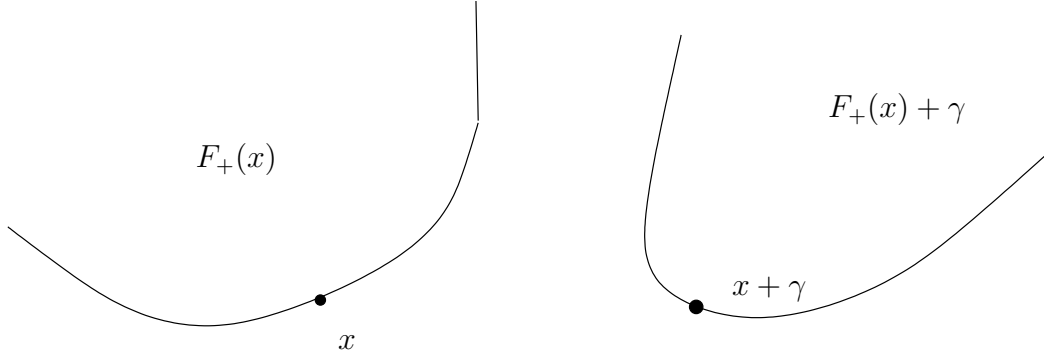


FIGURE 1. When  $F_+(x)$  and  $F_+(x) + \gamma$  are not nested.

Notice also that if  $\gamma \in \Gamma_+(x)$  then  $F_+(x) + \gamma \subset F_+(x)$ , by subtracting  $\gamma$  we obtain that  $F_+(x) \subset F_+(x) - \gamma$  which implies that  $F_-(x) - \gamma \subset F_-(x)$  obtaining that  $-\gamma \in \Gamma_-(x)$ . We have proved that  $-\Gamma_+(x) = \Gamma_-(x)$ .

It then remains to prove that if  $\gamma_1, \gamma_2 \in \Gamma_+(x)$ , then  $\gamma_1 - \gamma_2 \in \Gamma(x)$ .

Since  $F_+(x) + \gamma_1 + \gamma_2$  is contained in both  $F_+(x) + \gamma_1$  and  $F_+(x) + \gamma_2$  we have that

$$(F_+(x) + \gamma_1) \cap (F_+(x) + \gamma_2) \neq \emptyset.$$

By Lemma 5.2 (iii) we have that both  $F_-(x) + \gamma_1$  and  $F_-(x) + \gamma_2$  contain  $F_-(x)$  so they also have non-empty intersection.

Using Lemma 5.2 (i), we get that  $F_+(x) + \gamma_1$  and  $F_+(x) + \gamma_2$  are nested and this implies that either  $\gamma_1 - \gamma_2$  or  $\gamma_2 - \gamma_1$  is in  $\Gamma_+(x)$  which concludes. □

5.1.3. We close this subsection by stating the following theorem which provides a kind of classification of Reebless foliations in  $\mathbb{T}^3$  (see also [Pl] where a similar result is obtained for  $C^2$ -foliations of more general manifolds). The proof is deferred to the next subsection.

We say that  $\mathcal{F}$  has a *dead end component* if there exists two (homotopic) torus leaves  $T_1$  and  $T_2$  of  $\mathcal{F}$  such that there is no transversal that intersects both of them. When such a component exists, we have that the leaves of any transversal foliation must remain at bounded distance from some lift of  $T_1$  and  $T_2$ .

**Theorem 5.4.** *Let  $\mathcal{F}$  be a Reebless foliation of  $\mathbb{T}^3$ . Then, there exists a plane  $P \subset \mathbb{R}^3$  and  $R > 0$  such that every leaf of  $\tilde{\mathcal{F}}$  lies in an  $R$ -neighborhood of a translate of  $P$ . Moreover, one of the following conditions hold:*

- (i) *Either for every  $x \in \mathbb{R}^3$  the  $R$ -neighborhood of  $\tilde{\mathcal{F}}(x)$  contains  $P + x$ , or,*
- (ii)  *$P$  projects into a 2-torus and there is a dead-end component of  $\mathcal{F}$ .*

Notice that in condition (ii), since we know that leaves of  $\tilde{\mathcal{F}}$  do not cross we get that every leaf of  $\tilde{\mathcal{F}}$  is at bounded distance from a translate of  $P$ .

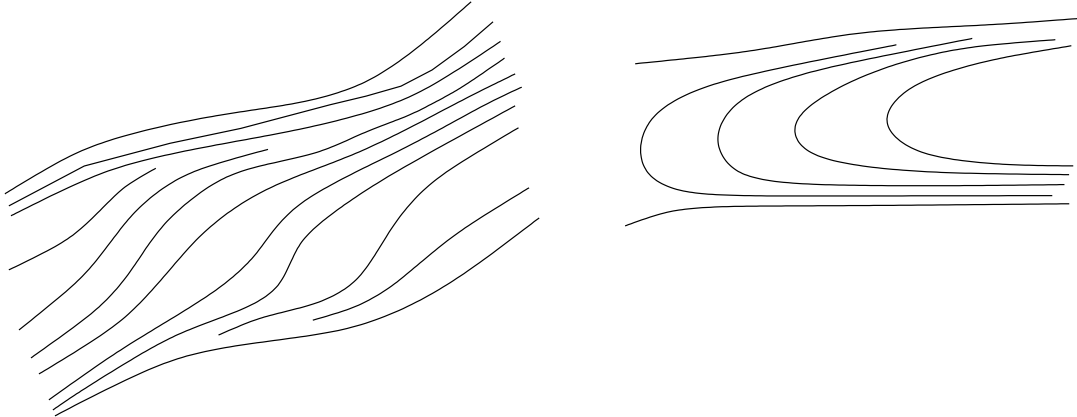


FIGURE 2. How the possibilities on  $\tilde{\mathcal{F}}$  look like.

In case the foliation  $\mathcal{F}^\perp$  is not oriented, we get essentially the same results. In fact, one can lift the foliation to the oriented double cover and apply Theorem 5.4. Notice that the Klein bottle cannot be incompressible in  $\mathbb{T}^3$  (since its fundamental group is not abelian) and thus the boundary component of the projection of the dead end component will still have at least one two-dimensional torus.

## 5.2. Proof of Theorem 5.4.

5.2.1. We essentially follow the proof in [BBI<sub>2</sub>] with some adaptations to get our statement.

We will assume throughout that the foliation is orientable and transversally orientable. This allows us to define as above the sets  $F_\pm(x)$  for every  $x$ .

We define  $G_+(x_0) = \bigcap_{\gamma \in \mathbb{Z}^3} \overline{F_+(x) + \gamma}$  and  $G_-(x_0)$  in a similar way.

First, assume that there exists  $x_0$  such that  $G_+(x_0) = \bigcap_{\gamma \in \mathbb{Z}^3} \overline{F_+(x) + \gamma} \neq \emptyset$  (see Lemma 3.10 of [BBI<sub>2</sub>]). The case where  $G_-(x_0) \neq \emptyset$  is symmetric. The idea is to prove that in this case we will get option (ii) of the theorem.

There exists  $\delta > 0$  such that given a point  $z \in G_+(x_0)$  we can consider a neighborhood  $U_z$  containing  $B_\delta(z)$  given by Corollary 4.7 (ii) such that:

- There is a  $C^1$ -coordinate neighborhood  $\psi_z : U_z \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}$  such that for every  $y \in U_z$  we have that  $\psi(\tilde{\mathcal{F}}(y) \cap U_z)$  consists of the graph of a function  $h_y : \mathbb{R}^2 \rightarrow \mathbb{R}$  (in particular, it is connected).

Since  $\tilde{\mathcal{F}}$  and  $\tilde{\mathcal{F}}^\perp$  are orientable, we get that we can choose the coordinates  $\psi_z$  in order that for every  $y \in U_z$  we have that  $\psi_z(F_+(y) \cap U)$  is the set of points  $(w, t)$  such that  $t \geq h_y(w)$ .

Notice that for every  $\gamma \in \mathbb{Z}^3$  we have that  $(F_+(x) + \gamma) \cap U$  is either the whole  $U$  or the upper part of the graph of a function  $h_{x+\gamma} : \mathbb{R}^2 \rightarrow \mathbb{R}$  in some coordinates in  $U$ .

This implies that the intersection  $G_+(x_0)$  is a 3-dimensional submanifold of  $\mathbb{R}^3$  (modeled in the upper half space) with boundary consisting of leaves of  $\tilde{\mathcal{F}}$  (since the boundary components are always locally limits of local leaves).

The boundary is clearly non trivial since  $G_+(x_0) \subset F_+(x_0) \neq \mathbb{R}^3$ .

**Claim.** *If  $G_+(x_0) \neq \emptyset$  then there exists plane  $P$  and  $R > 0$  such that every leaf of  $\tilde{\mathcal{F}}(x)$  is contained in an  $R$ -neighborhood of a translate of  $P$  and whose projection to  $\mathbb{T}^3$  is a two dimensional torus. Moreover, option (ii) of the proposition holds.*

**PROOF.** Since  $G_+(x_0)$  is invariant under every integer translation, we get that the boundary of  $G_+(x_0)$  descends to a closed surface in  $\mathbb{T}^3$  which is union of leaves of  $\mathcal{F}$ .

By Corollary 4.7 (iii) we get that those leaves are two-dimensional torus whose fundamental group is injected by the inclusion map.

This implies that they are at bounded distance of linear embeddings of  $\mathbb{T}^2$  in  $\mathbb{T}^3$  and so their lifts lie within bounded distance from a plane  $P$  whose projection is a two dimensional torus.

Since leafs of  $\tilde{\mathcal{F}}$  do not cross, the plane  $P$  does not depend on the boundary component. Moreover, every leaf of  $\tilde{\mathcal{F}}(x)$  must be at bounded distance from a translate of  $P$  since every leaf of  $\mathcal{F}$  has a lift which lies within two given lifts of some of the torus leafs.

Consider a point  $x$  in the boundary of  $G_+(x_0)$ . We have that  $\tilde{\mathcal{F}}(x)$  is at bounded distance from  $P$  from the argument above.

Moreover, each boundary component of  $G_+(x_0)$  is positively oriented in the direction which points inward to the interior of  $G_+(x_0)$  (recall that it is a compact 3-manifold with boundary).

We claim that  $\eta_z$  remains between two translates of  $P$  for every  $z \in \tilde{\mathcal{F}}(x)$  and  $\eta_z$  positive transversal to  $\tilde{\mathcal{F}}$ . Indeed, if this is not the case, then  $\tilde{\eta}_z$  would intersect other boundary component of  $G_+(x_0)$  which is impossible since the boundary leafs of  $G_+(x_0)$  point inward to  $G_+(x_0)$  (with the orientation of  $\tilde{\mathcal{F}}^\perp$ ).

Now, consider any point  $z \in \mathbb{R}^3$ , and  $\eta_z$  a positive transversal which we assume does not remain at bounded distance from  $P$ . Then it must intersect some translate of  $\tilde{\mathcal{F}}(x)$ , and the argument above applies. This is a contradiction.

The same argument works for negative transversals since once a leaf enters  $(G_+(x_0))^c$  it cannot reenter any of its translates. We have proved that  $p(G_+(x_0))$  contains a dead end component. This proves the claim.



◇

Now, assume that (ii) does not hold, in particular  $G_{\pm}(x) = \emptyset$  for every  $x$ . Then, for every point  $x$  we have that

$$\bigcup_{\gamma \in \mathbb{Z}^3} (F_+(x) + \gamma) = \bigcup_{\gamma \in \mathbb{Z}^3} (F_-(x) + \gamma) = \mathbb{R}^3$$

As in Lemma 3.11 of [BBI<sub>2</sub>] we can prove:

**Claim.** *We have that  $\Gamma(x) = \mathbb{Z}^3$  for every  $x \in \mathbb{R}^3$ .*

**PROOF.** If for some  $\gamma_0 \notin \Gamma(x)$  one has that  $F_+(x) \cap (F_+(x) + \gamma_0) = \emptyset$  (the other possibility being that  $F_-(x) \cap (F_-(x) + \gamma_0) = \emptyset$ ) then, we claim that for every  $\gamma \notin \Gamma(x)$  we have that  $F_+(x) \cap (F_+(x) + \gamma) = \emptyset$ .

Indeed, by Lemma 5.2 (i) if the claim does not hold, there would exist  $\gamma \notin \Gamma(x)$  such that

$$F_-(x) \cap (F_-(x) + \gamma) = \emptyset \quad \text{and} \quad F_+(x) \cap (F_+(x) + \gamma) \neq \emptyset.$$

By Lemma 5.2 (ii) we have:

- $F_-(x) \subset F_+(x) + \gamma$ .
- $F_+(x) + \gamma_0 \subset F_-(x)$ .

Let  $z \in F_+(x) \cap (F_+(x) + \gamma)$  then  $z + \gamma_0$  must belong both to  $(F_+(x) + \gamma_0) \subset (F_+(x) + \gamma)$  and to  $(F_+(x) + \gamma + \gamma_0)$ . By subtracting  $\gamma$  we get that  $z + \gamma_0 - \gamma$  belongs to  $F_+(x) \cap (F_+(x) + \gamma_0)$  contradicting our initial assumption. So, for every  $\gamma \notin \Gamma(x)$  we have that  $F_+(x) \cap (F_+(x) + \gamma) = \emptyset$ .

Now, consider the set

$$U_+(x) = \bigcup_{\gamma \in \Gamma(x)} (F_+(x) + \gamma).$$

From the above claim, the sets  $U_+(x) + \gamma_1$  and  $U_+(x) + \gamma_2$  are disjoint (if  $\gamma_1 - \gamma_2 \notin \Gamma(x)$ ) or coincide (if  $\gamma_1 - \gamma_2 \in \Gamma(x)$ ).

Since these sets are open, and its translates by  $\mathbb{Z}^3$  should cover the whole  $\mathbb{R}^3$  we get by connectedness that there must be only one. This implies that  $\Gamma(x) = \mathbb{Z}^3$  finishing the proof of the claim.

◇

Consider  $\Gamma_0(x) = \Gamma_+(x) \cap \Gamma_-(x)$ , the set of translates which fix  $\tilde{\mathcal{F}}(x)$ .

If  $\text{rank}(\Gamma_0(x)) = 3$ , then  $p^{-1}(p(\tilde{\mathcal{F}}(x)))$  consists of finitely many translates of  $\tilde{\mathcal{F}}(x)$  which implies that  $p(\tilde{\mathcal{F}}(x))$  is a closed surface of  $\mathcal{F}$ . On the other hand, the fundamental group of this closed surface should be isomorphic to  $\mathbb{Z}^3$  which is impossible since there are

no closed surfaces with such fundamental group ( $[\mathbb{R}]$ ). This implies that  $\text{rank}(\Gamma_0(x)) < 3$  for every  $x \in \mathbb{R}^3$ .

**Claim.** *For every  $x \in \mathbb{R}^3$  there exists a plane  $P(x)$  and translates  $P_+(x)$  and  $P_-(x)$  such that  $F_+(x)$  lies in a half space bounded by  $P_+(x)$  and  $F_-(x)$  lies in a half space bounded by  $P_-(x)$ .*

PROOF. Since  $\text{rank}(\Gamma_0(x)) < 3$  we can prove that  $\Gamma_+(x)$  and  $\Gamma_-(x)$  are half lattices (this means that there exists a plane  $P \subset \mathbb{R}^3$  such that each one is contained in a half space bounded by  $P$ ).

The argument is the same as in Lemma 3.12 of [BBI<sub>2</sub>] (and the argument after that lemma).

Consider the convex hulls of  $\Gamma_+(x)$  and  $\Gamma_-(x)$ . If their interiors intersect one can consider 3 linearly independent points whose coordinates are rational. These points are both positive rational convex combinations of vectors in  $\Gamma_+(x)$  as well as of vectors in  $\Gamma_-(x)$ . One obtains that  $\Gamma_0(x) = \Gamma_+(x) \cap \Gamma_-(x)$  has rank 3 contradicting our assumption.

This implies that there exists a plane  $P(x)$  separating these convex hulls.

Consider  $z \in \mathbb{R}^3$  and let  $\mathcal{O}_+(z) = (z + \mathbb{Z}^3) \cap F_+(x)$ . We have that  $\mathcal{O}_+(z) \neq \emptyset$  (otherwise  $z \in G_-(x)$ ). Moreover,  $\mathcal{O}_+(z) + \Gamma_+(x) \subset \mathcal{O}_+(z)$  because  $\Gamma_+(x)$  preserves  $F_+(x)$ . The symmetric statements hold for  $\mathcal{O}_-(z) = (z + \mathbb{Z}^3) \cap F_-(x)$ .

We get that  $\mathcal{O}_+(z)$  and  $\mathcal{O}_-(z)$  are separated by a plane  $P_z$  parallel to  $P(x)$ . The proof is as follows: we consider the convex hull  $\mathcal{C}\mathcal{O}_+(z)$  of  $\mathcal{O}_+(z)$  and the fact that  $\mathcal{O}_+(z) + \Gamma_+(x) \subset \mathcal{O}_+(z)$  implies that if  $v$  is a vector in the positive half plane bounded by  $P(x)$  we have that  $\mathcal{C}\mathcal{O}_+(z) + v \subset \mathcal{C}\mathcal{O}_+(z)$ . The same holds for the convex hull of  $\mathcal{C}\mathcal{O}_-(z)$  and we get that if the interiors of  $\mathcal{C}\mathcal{O}_+(z)$  and  $\mathcal{C}\mathcal{O}_-(z)$  intersect, then the interiors of the convex hulls of  $\Gamma_+(x)$  and  $\Gamma_-(x)$  intersect contradicting that  $\text{rank}(\Gamma_0(x)) < 3$ .

Consider  $\delta$  given by Corollary 4.7 (ii) such that every point  $z$  has a neighborhood  $U_z$  containing  $B_\delta(z)$  and such that  $\tilde{\mathcal{F}}(y) \cap U_z$  is connected for every  $y \in U_z$ .

Let  $\{z_i\}$  a finite set  $\delta/2$ -dense in a fundamental domain  $D_0$ . We denote as  $P_{z_i}^+$  and  $P_{z_i}^-$  the half spaces defined by the plane  $P_{z_i}$  parallel to  $P(x)$  containing  $\mathcal{O}_+(z_i)$  and  $\mathcal{O}_-(z_i)$  respectively.

We claim that  $F_+(x)$  is contained in the  $\delta$ -neighborhood of  $\bigcup_i P_{z_i}^+$  and the symmetric statement holds for  $F_-(x)$ .

Consider a point  $y \in F_+(x)$ . We get that  $\tilde{\mathcal{F}}(y)$  intersects the neighborhood  $U_y$  containing  $B_\delta(y)$  in a connected component and thus there exists a  $\delta/2$ -ball in  $U_y$  contained in  $F_+(x)$ . Thus, there exists  $z_i$  and  $\gamma \in \mathbb{Z}^3$  such that  $z_i + \gamma$  is contained in  $F_+(x)$  and thus  $z_i + \gamma \in \mathcal{O}_+(z_i) \subset P_{z_i}^+$ . We deduce that  $y$  is contained in the  $\delta$ -neighborhood of  $P_{z_i}^+$  as desired.

The  $\delta$ -neighborhood  $H^+$  of  $\bigcup_i P_{z_i}^+$  is a half space bounded by a plane parallel to  $P(x)$  and the same holds for  $H^-$  defined symmetrically. We have proved that  $F_+(x) \subset H^+$  and  $F_-(x) \subset H^-$ . This implies that  $\tilde{\mathcal{F}}(x)$  is contained in  $H^+ \cap H^-$ , a strip bounded by planes  $P_+(x)$  and  $P_-(x)$  parallel to  $P(x)$  concluding the claim.

◇

We have proved that for every  $x \in \mathbb{R}^3$  there exists a plane  $P(x)$  and translates  $P_+(x)$  and  $P_-(x)$  such that  $F_{\pm}(x)$  lies in a half space bounded by  $P_{\pm}(x)$ . Let  $R(x)$  be the distance between  $P_+(x)$  and  $P_-(x)$ , we have that  $\tilde{\mathcal{F}}(x)$  lies at distance smaller than  $R(x)$  from  $P_+(x)$ .

Now, we must prove that the  $R(x)$ -neighborhood of  $\tilde{\mathcal{F}}(x)$  contains  $P_+(x)$ . To do this, it is enough to show that the projection from  $\tilde{\mathcal{F}}(x)$  to  $P_+(x)$  by an orthogonal vector to  $P(x)$  is surjective. If this is not the case, then there exists a segment joining  $P_+(x)$  to  $P_-(x)$  which does not intersect  $\tilde{\mathcal{F}}(x)$ . This contradicts the fact that every curve from  $F_-(x)$  to  $F_+(x)$  must intersect  $\tilde{\mathcal{F}}(x)$ .

Since the leaves of  $\tilde{\mathcal{F}}$  do not intersect,  $P(x)$  cannot depend on  $x$ . Since the foliation is invariant under integer translations, we get (by compactness) that  $R(x)$  can be chosen uniformly bounded. This concludes the proof of Theorem 5.4.

□

*Remark 5.5.* It is direct to show that for a given Reebless foliation  $\mathcal{F}$  of  $\mathbb{T}^3$ , the plane  $P$  given by Theorem 5.4 is unique. Indeed, the intersection of the  $R$ -neighborhoods of two different planes is contained in a  $2R$ -neighborhood of their intersection line  $L$ . If two planes would satisfy the thesis of Theorem 5.4 then we would obtain that the complement of every leaf contains a connected component which is contained in the  $2R$ -neighborhood of  $L$ . This is a contradiction since as a consequence of Theorem 5.4 we get that there is always a leaf of  $\tilde{\mathcal{F}}$  whose complement contains two connected components each of which contains a half space of a plane<sup>5</sup>.

◇

We have used strongly the fact that  $\tilde{\mathcal{F}}$  is the lift of a foliation in  $\mathbb{T}^3$  so that the foliation is invariant under integer translations, this is why there is more rigidity in the possible foliations of  $\mathbb{R}^3$  which are lifts of foliations on  $\mathbb{T}^3$ .

### 5.3. Further properties of the foliations.

5.3.1. Given a codimension one foliation  $\tilde{\mathcal{F}}$  of  $\mathbb{R}^3$  whose leaves are homeomorphic to two dimensional planes, one defines the *leaf space*  $\mathcal{L}$  of  $\tilde{\mathcal{F}}$  by the quotient space of  $\mathbb{R}^3$  with

<sup>5</sup>Notice that if case (ii) holds this is direct from the existence of a torus leaf and in case (i) this follows from the statement of the last claim in the proof.

the equivalence relationship of being in the same leaf of  $\tilde{\mathcal{F}}$ . It is well known that in this case<sup>6</sup>, we have that  $\mathcal{L} = \mathbb{R}^3/\tilde{\mathcal{F}}$  is a (possibly non-Hausdorff) one-dimensional manifold (see [CC] Proposition II.D.1.1).

It is not hard to see that:

**Proposition 5.6.** *Let  $\mathcal{F}$  be a Reebless foliation of  $\mathbb{T}^3$ , if option (i) of Theorem 5.4 holds then the leaf space  $\mathcal{L} = \mathbb{R}^3/\tilde{\mathcal{F}}$  is homeomorphic to  $\mathbb{R}$ .*

PROOF. The space of leafs  $\mathcal{L}$  with the quotient topology has the structure of a (possibly non-Hausdorff) one-dimensional manifold. In fact, this follows directly from Corollary 4.7 as well as the fact that it is simply connected as a one-dimensional manifold (see Corollary 4.7 (i)). To prove the proposition is thus enough to show that it is Hausdorff.

We define an ordering in  $\mathcal{L}$  as follows

$$\tilde{\mathcal{F}}(x) \geq \tilde{\mathcal{F}}(y) \quad \text{if} \quad F_+(x) \subset F_+(y).$$

If option (i) of Theorem 5.4 holds, given  $x, y$  we have that  $F_+(x) \cap F_+(y) \neq \emptyset$  and  $F_-(x) \cap F_-(y) \neq \emptyset$ .

Then, Lemma 5.2 (i) implies that  $F_+(x)$  and  $F_+(y)$  are nested. In conclusion, we obtain that the relationship we have defined is a total order.

Let  $\tilde{\mathcal{F}}(x)$  and  $\tilde{\mathcal{F}}(y)$  two different leaves of  $\tilde{\mathcal{F}}$ . We must show that they belong to disjoint open sets.

Without loss of generality, since it is a total order, we can assume that  $\tilde{\mathcal{F}}(x) < \tilde{\mathcal{F}}(y)$ . This implies that  $F_+(y)$  is strictly contained in  $F_+(x)$ . On the other hand, this implies that  $F_-(y) \cap F_+(x) \neq \emptyset$ , in particular, there exists  $z$  such that  $\tilde{\mathcal{F}}(x) < \tilde{\mathcal{F}}(z) < \tilde{\mathcal{F}}(y)$ .

Since the sets  $F_+(z)$  and  $F_-(z)$  are open and disjoint and we have that  $\tilde{\mathcal{F}}(x) \subset F_-(z)$  and  $\tilde{\mathcal{F}}(y) \in F_+(z)$  we deduce that  $\mathcal{L}$  is Hausdorff as desired.

□

Since  $\tilde{\mathcal{F}}$  is invariant under deck transformations, we obtain that we can consider the quotient action of  $\mathbb{Z}^3 = \pi_1(\mathbb{T}^3)$  in  $\mathcal{L}$ . For  $[x] = \tilde{\mathcal{F}}(x) \in \mathcal{L}$  we get that  $\gamma \cdot [x] = [x + \gamma]$  for every  $\gamma \in \mathbb{Z}^3$ .

5.3.2. Notice that all leaves of  $\mathcal{F}$  in  $\mathbb{T}^3$  are simply connected if and only if  $\pi_1(\mathbb{T}^3)$  acts without fixed point in  $\mathcal{L}$ . In a similar fashion, existence of fixed points, or common fixed points allows one to determine the topology of leaves of  $\mathcal{F}$  in  $\mathbb{T}^3$ .

In fact, we can prove:

---

<sup>6</sup>When all leaves are properly embedded copies of  $\mathbb{R}^2$ .

**Proposition 5.7.** *Let  $\mathcal{F}$  be a Reebless foliation of  $\mathbb{T}^3$ . If the plane  $P$  given by Theorem 5.4 projects into a two dimensional torus by  $p$ , then there is a leaf of  $\mathcal{F}$  homeomorphic to a two-dimensional torus.*

PROOF. Notice first that if option (ii) of Theorem 5.4 holds, the existence of a torus leaf is contained in the statement of the theorem.

So, we can assume that option (i) holds. By considering a finite index subgroup, we can further assume that the plane  $P$  is invariant under two of the generators of  $\pi_1(\mathbb{T}^3) \cong \mathbb{Z}^3$  which we denote as  $\gamma_1$  and  $\gamma_2$ .

Since leaves of  $\tilde{\mathcal{F}}$  remain close in the Hausdorff topology to the plane  $P$  we deduce that the orbit of every point  $[x] \in \mathcal{L}$  by the action of the elements  $\gamma_1$  and  $\gamma_2$  is bounded.

Let  $\gamma_3$  be the third generator: its orbit cannot be bounded, otherwise translation by  $\gamma_3$  would fix the plane  $P$ . So, the quotient of  $\mathcal{L}$  by the action of  $\gamma_3$  is a circle. We can make the group generated by  $\gamma_1$  and  $\gamma_2$  act on this circle and we obtain two commuting circle homeomorphisms with zero rotation number. This implies they have a common fixed point which in turn gives us the desired two-torus leaf of  $\mathcal{F}$ .

□

5.3.3. Also, depending on the topology of the projection of the plane  $P$  given by Theorem 5.4 we can obtain some properties on the topology of the leaves of  $\mathcal{F}$ .

**Lemma 5.8.** *Let  $\mathcal{F}$  be a Reebless foliation of  $\mathbb{T}^3$  and  $P$  be the plane given by Theorem 5.4.*

- (i) *Every closed curve in a leaf of  $\mathcal{F}$  is homotopic in  $\mathbb{T}^3$  to a closed curve contained in  $p(P)$ . This implies in particular that if  $p(P)$  is simply connected, then all the leaves of  $\mathcal{F}$  are also simply connected.*
- (ii) *If a leaf of  $\mathcal{F}$  is homeomorphic to a two dimensional torus, then, it is homotopic to  $p(P)$  (in particular,  $p(P)$  is also a two dimensional torus).*

PROOF. To see (i), first notice that leaves are incompressible. Given a closed curve  $\gamma$  in a leaf of  $\mathcal{F}$  which is not null-homotopic, we know that when lifted to the universal cover it remains at bounded distance from a linear one-dimensional subspace  $L$ . Since  $\gamma$  is a circle, we get that  $p(L)$  is a circle in  $\mathbb{T}^3$ . If the subspace  $L$  is not contained in  $P$  then it must be transverse to it. This contradicts the fact that leaves of  $\mathcal{F}$  remain at bounded distance from  $P$ .

To prove (ii), notice that a torus leaf  $T$  which is incompressible must remain close in the universal cover to a plane  $P_T$  which projects to a linear embedding of a 2-dimensional torus. From the proof of Theorem 5.4 and the fact that  $\mathcal{F}$  is a foliation we get that  $P_T = P$ . See also the proof of Lemma 3.10 of [BBI<sub>2</sub>].

□

5.3.4. We end this section by obtaining some results about branching foliations we will use in only in Section 8.

**Proposition 5.9.** *Let  $\mathcal{F}_{bran}$  be a branching foliation of  $\mathbb{T}^3$  and consider a sequence of points  $x_k$  such that there are leaves  $L_k \in \mathcal{F}_{bran}(x_k)$  which are closed, incompressible and homotopic to each other. If  $x_k \rightarrow x$ , then there is a leaf  $L \in \mathcal{F}_{bran}(x)$  which is incompressible and homotopic to the leaves  $\mathcal{F}_{bran}(x_k)$ .*

PROOF. Recall that if  $x_k \rightarrow x$  and we consider a sequence of leaves through  $x_k$  we get that the leaves converge to a leaf through  $x$ .

Consider the lifts of the leaves  $\mathcal{F}_{bran}(x_k)$  which are homeomorphic to a plane since they are incompressible. Moreover, the fundamental group of each of the leaves must be  $\mathbb{Z}^2$  and the leaves must be homeomorphic to 2-torus, since it is the only possibly incompressible surface in  $\mathbb{T}^3$ .

Since all the leaves  $\mathcal{F}_{bran}(x_k)$  are homotopic, their lifts are invariant under the same elements of  $\pi_1(\mathbb{T}^3)$ . The limit leaf must thus be also invariant under those elements. Notice that it cannot be invariant under further elements of  $\pi_1(\mathbb{T}^3)$  since no surface has such fundamental group.

□

## 6. GLOBAL PRODUCT STRUCTURE: QUANTITATIVE RESULTS

6.1. **Statement of results.** Recall that given two transverse foliations (this in particular implies that their dimensions are complementary)  $\mathcal{F}_1$  and  $\mathcal{F}_2$  of a manifold  $M$  we say they admit a *global product structure* if given two points  $x, y \in \tilde{M}$  the universal cover of  $M$  we have that  $\tilde{\mathcal{F}}_1(x)$  and  $\tilde{\mathcal{F}}_2(y)$  intersect in a unique point.

Notice that by continuity of the foliations and Invariance of Domain's Theorem we have that if a manifold has two transverse foliations with a global product structure, then, the universal cover of the manifold must be homeomorphic to the product of  $\tilde{\mathcal{F}}_1(x) \times \tilde{\mathcal{F}}_2(x)$  for any  $x \in \tilde{M}$ . In particular, leaves of  $\tilde{\mathcal{F}}_i$  must be simply connected and all homeomorphic between them.

We are interested in giving conditions under which global product structure is guaranteed. We start by noticing that as a consequence of Reeb's stability Theorem (see Proposition 4.3) we have that if a codimension 1 foliation  $\mathcal{F}$  has one compact leaf  $L$  and the foliation is without holonomy then for any transverse foliation  $\mathcal{F}^\perp$  there will be a global product structure between  $\mathcal{F}$  and  $\mathcal{F}^\perp$ . This can be generalized to the following theorem proved in [HH] (where it says that the  $C^2$ -version is due to Novikov):

**Theorem 6.1** (Theorem VIII.2.2.1 of [HH]). *Consider a codimension one foliation  $\mathcal{F}$  without holonomy of a compact manifold  $M$ . Then, for every  $\mathcal{F}^\perp$  foliation transverse to  $\mathcal{F}$  we have that  $\mathcal{F}$  and  $\mathcal{F}^\perp$  have global product structure.*

Other than the case where there is a compact leaf without holonomy, the other important case in which this result applies is when every leaf of the foliation is simply connected. Unfortunately, there will be some situations where we will be needing to obtain global product structure but not having neither all leaves of  $\mathcal{F}$  simply connected nor that the foliation lacks of holonomy in all its leaves.

We will use instead the following quantitative version of the previous result which does not imply it other than it the situations we will be needing it. We hope this general result on the existence of global product structure may find other applications.

**Theorem 6.2.** *Let  $M$  be a compact manifold and  $\delta > 0$ . Consider a set of generators of  $\pi_1(M)$  and endow  $\pi_1(M)$  with the word length for generators. Then, there exists  $K > 0$  such that if  $\mathcal{F}$  is a codimension one foliation and  $\mathcal{F}^\perp$  a transverse foliation such that:*

- *There is a local product structure of size  $\delta$  between  $\mathcal{F}$  and  $\mathcal{F}^\perp$ .*
- *The leaves of  $\tilde{\mathcal{F}}$  are simply connected and no element of  $\pi_1(M)$  of size less than  $K$  fixes a leaf of  $\tilde{\mathcal{F}}$ .*
- *The leaf space  $\mathcal{L} = \tilde{M}/\tilde{\mathcal{F}}$  is homeomorphic to  $\mathbb{R}$ .*
- *$\pi_1(M)$  is abelian.*

*Then,  $\mathcal{F}$  and  $\mathcal{F}^\perp$  admit a global product structure.*

The hypothesis of  $\pi_1(M)$  being abelian is only used in the last step of the proof. This hypothesis guarantees that deck transformations are in one to one correspondence with free homotopy classes of loops. We believe that the hypothesis of commutativity of  $\pi_1(M)$  can be removed, but it would need a different proof: in fact, although we will not make it explicit, one can see that our proof gives *polynomial global product structure* (see [H<sub>3</sub>]).

**6.2. Proof of Theorem 6.2.** The first step is to show that leaves of  $\tilde{\mathcal{F}}$  and  $\tilde{\mathcal{F}}^\perp$  intersect in at most one point:

**Lemma 6.3.** *We have that for every  $x \in \tilde{M}$  one has that  $\tilde{\mathcal{F}}(x) \cap \tilde{\mathcal{F}}^\perp(x) = \{x\}$ .*

PROOF. Assume otherwise, then, by Proposition 4.2 (Haefliger's argument) one would obtain that there is a non-simply connected leaf of  $\tilde{\mathcal{F}}$  a contradiction. □

In  $\mathcal{L} = \tilde{M}/\tilde{\mathcal{F}}$  we can consider an ordering of leaves (by using the ordering from  $\mathbb{R}$ ). We denote as  $[x]$  to the equivalence class in  $\tilde{M}$  of the point  $x$ , which coincides with  $\tilde{\mathcal{F}}(x)$ .

The following condition will be the main ingredient for obtaining a global product structure:

- (\*) For every  $z_0 \in \tilde{M}$  there exists  $y^-$  and  $y^+ \in \tilde{M}$  verifying that  $[y^-] < [z_0] < [y^+]$  and such that for every  $z_1, z_2 \in \tilde{M}$  satisfying  $[y^-] \leq [z_i] \leq [y^+]$  ( $i = 1, 2$ ) we have that  $\tilde{\mathcal{F}}^\perp(z_1) \cap \tilde{\mathcal{F}}(z_2) \neq \emptyset$ .

We get

**Lemma 6.4.** *If property (\*) is satisfied, then  $\tilde{\mathcal{F}}$  and  $\tilde{\mathcal{F}}^\perp$  have a global product structure.*

PROOF. Consider any point  $x_0 \in \tilde{M}$  and consider the set  $G = \{z \in \tilde{M} : \tilde{\mathcal{F}}^\perp(x_0) \cap \tilde{\mathcal{F}}(z) \neq \emptyset\}$ . We have that  $G$  is open from the local product structure (Remark 5.1) and by definition it is saturated by  $\tilde{\mathcal{F}}$ . We must show that  $G$  is closed and since  $\tilde{M}$  is connected this would conclude.

Now, consider  $z_0 \in \overline{G}$ , using assumption (\*) we obtain that there exists  $[y^-] < [z_0] < [y^+]$  such that every point  $z$  such that  $[y^-] < [z] < [y^+]$  verifies that its unstable leaf intersects both  $\tilde{\mathcal{F}}(y^-)$  and  $\tilde{\mathcal{F}}(y^+)$ .

Since  $z_0 \in \overline{G}$  we have that there are points  $z_n \in G$  such that  $z_n \rightarrow \tilde{\mathcal{F}}(z_0)$ .

We get that eventually,  $[y^-] < [z_k] < [y^+]$  and thus we obtain that there is a point  $y \in \tilde{\mathcal{F}}^\perp(x_0)$  verifying that  $[y^-] < [y] < [y^+]$ . We get that every leaf between  $\tilde{\mathcal{F}}(y^-)$  and  $\tilde{\mathcal{F}}(y^+)$  is contained in  $G$  from assumption (\*). In particular,  $z_0 \in G$  as desired.  $\square$

We must now show that property (\*) is verified. To this end, we will need the following lemma:

**Lemma 6.5.** *If  $K$  is well chosen, there exists  $\ell > 0$  such that every segment of  $\mathcal{F}^\perp(x)$  of length  $\ell$  intersects every leaf of  $\mathcal{F}$ .*

We postpone the proof of this lemma to the next subsection 6.3.

PROOF OF THEOREM 6.2. We must prove that condition (\*) is verified. We consider  $\delta$  given by the size of local product structure boxes (see Remark 5.1) and by Lemma 6.5 we get a value of  $\ell > 0$  such that every segment of  $\mathcal{F}^\perp$  of length  $\ell$  intersects every leaf of  $\mathcal{F}$ .

There exists  $k > 0$  such that every curve of length  $k\ell$  will verify that it has a subarc whose endpoints are  $\delta$ -close and joined by a curve in  $\mathcal{F}^\perp$  of length larger than  $\ell$  (so, intersecting every leaf of  $\mathcal{F}$ ).

Consider a point  $z_0 \in \tilde{M}$  and a point  $z \in \tilde{\mathcal{F}}(z_0)$ . Let  $\tilde{\eta}_z$  be the segment in  $\tilde{\mathcal{F}}^\perp(z)$  of length  $k\ell$  with one extreme in  $z$ . We can project  $\tilde{\eta}_z$  to  $M$  and we obtain a segment  $\eta_z$  transverse to  $\mathcal{F}$  which contains two points  $z_1$  and  $z_2$  at distance smaller than  $\delta$  and such



that the segment from  $z_1$  to  $z_2$  in  $\eta_z$  intersects every leaf of  $\mathcal{F}$ . We denote  $\tilde{z}_1$  and  $\tilde{z}_2$  to the lift of those points to  $\tilde{\eta}_z$ .

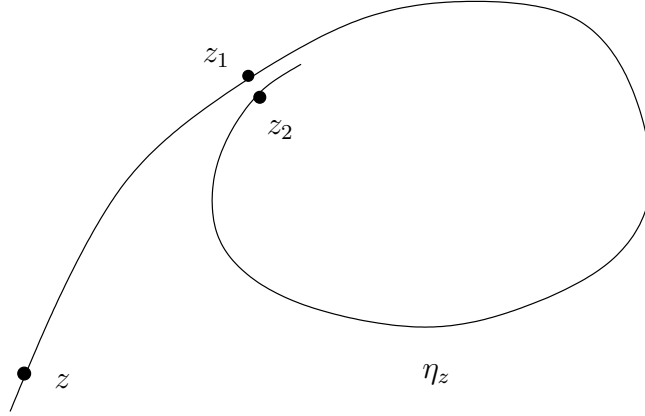


FIGURE 3. The curve  $\eta_z$ .

Using the local product structure, we can modify slightly  $\eta_z$  in order to create a closed curve  $\eta'_z$  through  $z_1$  which is contained in  $\eta_z$  outside  $B_\delta(z_1)$ , intersects every leaf of  $\mathcal{F}$  and has length smaller than  $k\ell + \delta$ .

We can define  $\Gamma_+$  as the set of elements in  $\pi_1(M)$  which send the half space bounded by  $\mathcal{F}(x)$  in the positive orientation into itself.

Since  $\eta_z$  essentially contains a loop of length smaller than  $k\ell + \delta$  we have that  $\tilde{\eta}_z$  connects  $[z_0]$  with  $[\tilde{z}_1 + \gamma]$  where  $\gamma$  belongs to  $\Gamma_+$  and can be represented by a loop of length smaller than  $k\ell + \delta$ . Moreover, since from  $z$  to  $\tilde{z}_1$  there is a positively oriented arc of  $\tilde{\mathcal{F}}^\perp$  we get that  $[z_0] = [z] \leq [\tilde{z}_1]$  (notice that it is possible that  $z = \tilde{z}_1$ ).

This implies that  $[\tilde{z}_1 + \gamma] \geq [z_0 + \gamma] > [z_0]$ , where the last inequality follows from the fact that the loop is positively oriented and non-trivial (recall that by Lemma 6.3 a curve transversal to  $\tilde{\mathcal{F}}$  cannot intersect the same leaf twice).

Notice that there are finitely many elements in  $\Gamma_+$  which are represented by loops of length smaller than  $k\ell + \delta$ . This is because the fundamental group is abelian so that deck transformations are in one to one correspondence with free homotopy classes of loops.

The fact that there are finitely many such elements in  $\Gamma_+$  implies the following: There exists  $\gamma_0 \in \Gamma_+$  such that for every  $\gamma \in \Gamma_+$  which can be represented by a positively oriented loop transverse to  $\mathcal{F}$  of length smaller than  $k\ell + \delta$ , we have

$$[z_0] < [z_0 + \gamma_0] \leq [z_0 + \gamma]$$

We have obtained that for  $y^+ = z_0 + \gamma_0$  there exists  $L = k\ell > 0$  such that for every point  $z \in \tilde{\mathcal{F}}(z_0)$  the segment of  $\tilde{\mathcal{F}}_+^\perp(z)$  of length  $L$  intersects  $\tilde{\mathcal{F}}(y^+)$ .

This defines a continuous injective map from  $\tilde{\mathcal{F}}(z_0)$  to  $\tilde{\mathcal{F}}(y^+)$  (injectivity follows from Lemma 6.3). Since the length of the curves defining the map is uniformly bounded, this map is proper and thus, a homeomorphism. The same argument applies to any leaf  $\tilde{\mathcal{F}}(z_1)$  such that  $[z_0] \leq [z_1] \leq [y^+]$ .

For any  $z_1$  such that  $[z_0] \leq [z_1] \leq [y^+]$  we get that  $\tilde{\mathcal{F}}^\perp(z_1)$  intersects  $\tilde{\mathcal{F}}(z_0)$ . Since the map defined above is a homeomorphism, we get that also  $\tilde{\mathcal{F}}^\perp(z_0) \cap \tilde{\mathcal{F}}(z_1) \neq \emptyset$ .

A symmetric argument allows us to find  $y^-$  with similar characteristics. Using the fact that intersecting with leaves of  $\tilde{\mathcal{F}}^\perp$  is a homeomorphism between any pair of leaves of  $\tilde{\mathcal{F}}$  between  $[y^-]$  and  $[y^+]$  we obtain (\*) as desired.

Lemma 6.4 finishes the proof. □

**6.3. Proof of Lemma 6.5.** We first prove the following Lemma which allows us to bound the topology of  $M$  in terms of coverings of size  $\delta$ . Notice that we are implicitly using that  $\pi_1(M)$  as before to be able to define a correspondence between (free) homotopy classes of loops with elements of  $\pi_1(M)$ .

**Lemma 6.6.** *Given a covering  $\{V_1, \dots, V_k\}$  of  $M$  by contractible open subsets there exists  $K > 0$  such that if  $\eta$  is a loop in  $M$  intersecting each  $V_i$  at most once<sup>7</sup>, then  $[\eta] \in \pi_1(M)$  has norm less than  $K$ .*

**PROOF.** We can consider the lift  $p^{-1}(V_i)$  to the universal cover of each  $V_i$  and we have that each connected component of  $p^{-1}(V_i)$  has bounded diameter since they are simply connected in  $M$ . Let  $C_V > 0$  be a uniform bound on those diameters.

Let  $K$  be such that every loop of length smaller than  $2kC_V$  has norm less than  $K$  in  $\pi_1(M)$ .

Now, consider a loop  $\eta$  which intersects each of the open sets  $V_i$  at most once. Consider  $\eta$  as a function  $\eta : [0, 1] \rightarrow M$  such that  $\eta(0) = \eta(1)$ . Consider a lift  $\tilde{\eta} : [0, 1] \rightarrow M$  such that  $p(\tilde{\eta}(t)) = \eta(t)$  for every  $t$ .

We claim that the diameter of the image of  $\tilde{\eta}$  cannot exceed  $kC_V$ . Otherwise, this would imply that  $\eta$  intersects some  $V_i$  more than once. Now, we can homotope  $\tilde{\eta}$  fixing the extremes in order to have length smaller than  $2kC_V$ . This implies the Lemma. □

Given  $\delta$  of the uniform local product structure, we say that a loop  $\eta$  is a  $\delta$ -loop if it is transverse to  $\mathcal{F}$  and consists of a segment of a leaf of  $\mathcal{F}^\perp$  together with a curve of length smaller than  $\delta$ .

<sup>7</sup>More precisely, if  $\eta$  is  $\eta : [0, 1] \rightarrow M$  with  $\eta(0) = \eta(1)$  this means that  $\eta^{-1}(V_i)$  is connected for every  $i$ .

**Lemma 6.7.** *There exists  $K \geq 0$  such that if  $O \subset M$  is an open  $\mathcal{F}$ -saturated set such that  $O \neq M$ . Then, there is no  $\delta$ -loop contained in  $O$ .*

PROOF. For every point  $x$  consider  $N_x = B_\delta(x)$  with  $\delta$  the size of the local product structure boxes. We can consider a finite subcover  $\{N_{x_1}, \dots, N_{x_n}\}$  for which Lemma 6.6 applies giving  $K > 0$ .

Consider, an open set  $O \neq M$  which is  $\mathcal{F}$ -saturated. We must prove that  $O$  cannot contain a  $\delta$ -loop.

Let  $\tilde{O}_0$  a connected component of the lift  $\tilde{O}$  of  $O$  to the universal cover  $\tilde{M}$ . We have that the boundary of  $\tilde{O}_0$  consists of leaves of  $\tilde{\mathcal{F}}$  and if a translation  $\gamma \in \pi_1(M)$  verifies that

$$\tilde{O}_0 \cap \gamma \tilde{O}_0 \neq \emptyset$$

then we must have that  $\tilde{O}_0 = \gamma + \tilde{O}_0$ . This implies that  $\gamma$  fixes the boundary leaves of  $\tilde{O}_0$ : This is because the leaf space  $\mathcal{L} = \tilde{M}/\tilde{\mathcal{F}}$  is homeomorphic to  $\mathbb{R}$  so that  $\tilde{O}_0$  being connected and  $\tilde{\mathcal{F}}$  saturated is an open interval of  $\mathcal{L}$ . Since deck transformations preserve orientation, if they fix an open interval then they must fix the boundaries.

The definition of  $K$  then guaranties that if an element  $\gamma$  of  $\pi_1(M)$  makes  $\tilde{O}_0$  intersect with itself, then  $\gamma$  must be larger than  $K$ . In particular, any  $\delta$ -loop contained in  $O$  must represent an element of  $\pi_1(M)$  of length larger than  $K$ .

Now consider a  $\delta$ -loop  $\eta$ . Corollary 4.7 (i) implies that  $\eta$  is in the hypothesis of Lemma 6.6. We deduce that  $\eta$  cannot be entirely contained in  $O$  since otherwise its lift would be contained in  $\tilde{O}_0$  giving a deck transformation  $\gamma$  of norm less than  $K$  fixing  $\tilde{O}_0$  a contradiction.

□

**Corollary 6.8.** *For the  $K \geq 0$  obtained in the previous Lemma, if  $\eta$  is a  $\delta$ -loop then it intersects every leaf of  $\mathcal{F}$ .*

PROOF. The saturation by  $\mathcal{F}$  of  $\eta$  is an open set which is  $\mathcal{F}$ -saturated by definition. Lemma 6.7 implies that it must be the whole  $M$  and this implies that every leaf of  $\mathcal{F}$  intersects  $\eta$ .

□

PROOF OF LEMMA 6.5. Choose  $K$  as in Lemma 6.7. Considering a covering  $\{V_1, \dots, V_k\}$  of  $M$  by neighborhoods with local product structure between  $\mathcal{F}$  and  $\mathcal{F}^\perp$  and of diameter less than  $\delta$ .

There exists  $\ell_0 > 0$  such that every oriented unstable curve of length larger than  $\ell_0$  traverses at least one of the  $V_i$ 's. Choose  $\ell > (k + 1)\ell_0$  and we get that every curve of

length larger than  $\ell$  must intersect some  $V_i$  twice in points say  $x_1$  and  $x_2$ . By changing the curve only in  $V_i$  we obtain a  $\delta$ -loop which will intersect the same leaves as the initial arc joining  $x_1$  and  $x_2$ .

Corollary 6.8 implies that the mentioned arc must intersect all leaves of  $\mathcal{F}$ .

□

#### 6.4. Consequences of the global product structure in $\mathbb{T}^3$ .

6.4.1. We say that a foliation  $\mathcal{F}$  in a Riemannian manifold  $M$  is *quasi-isometric* if there exists  $a, b \in \mathbb{R}$  such that for every  $x, y$  in the same leaf of  $\mathcal{F}$  we have:

$$d_{\mathcal{F}}(x, y) \leq ad(x, y) + b$$

where  $d$  denotes the distance in  $M$  induced by the Riemannian metric and  $d_{\mathcal{F}}$  the distance induced in the leaves of  $\mathcal{F}$  by restricting the metric of  $M$  to the leaves of  $\mathcal{F}$ .

**Proposition 6.9.** *Let  $\mathcal{F}$  be a codimension one foliation of  $\mathbb{T}^3$  and  $\mathcal{F}^{\perp}$  a transverse foliation. Assume the foliations  $\tilde{\mathcal{F}}$  and  $\tilde{\mathcal{F}}^{\perp}$  lifted to the universal cover have global product structure. Then, the foliation  $\tilde{\mathcal{F}}^{\perp}$  is quasi-isometric. Moreover, if  $P$  is the plane given by Theorem 5.4, there exists a cone  $\mathcal{E}$  transverse to  $P$  in  $\mathbb{R}^3$  and  $K > 0$  such that for every  $x \in \mathbb{R}^3$  and  $y \in \tilde{\mathcal{F}}^{\perp}(x)$  at distance larger than  $K$  from  $x$  we have that  $y - x$  is contained in the cone  $\mathcal{E}$ .*

PROOF. Notice that the global product structure implies that  $\mathcal{F}$  is Reebless. Let  $P$  be the plane given by Theorem 5.4.

Consider  $v$  a unit vector perpendicular to  $P$  in  $\mathbb{R}^3$ .

**Claim.** *For every  $N > 0$  there exists  $L$  such that every segment of  $\tilde{\mathcal{F}}^{\perp}$  of length  $L$  centered at a point  $x$  intersects both  $P + x + Nv$  and  $P + x - Nv$ .*

PROOF. If this was not the case, we could find arbitrarily large segments  $\gamma_k$  of leaves of  $\tilde{\mathcal{F}}^{\perp}$  centered at a point  $x_k$  with length larger than  $k$  and such that they do not intersect either  $P + x_k + Nv$  or  $P + x_k - Nv$ . Without loss of generality, and by taking a subsequence we can assume that they do not intersect  $P + x_k + Nv$ .

Since the foliations are invariant under translations, we can assume that the sequence  $x_k$  is bounded and by further considering a subsequence, that  $x_k \rightarrow x$ .

We deduce that  $\tilde{\mathcal{F}}^{\perp}(x)$  cannot intersect  $P + x + (N + 1)v$  which in turn implies (by Theorem 5.4) that  $\tilde{\mathcal{F}}^{\perp}(x)$  cannot intersect the leaf of  $\tilde{\mathcal{F}}$  through the point  $x + (N + 1 + 2R)v$  contradicting global product structure.

◇

This implies quasi-isometry since having length larger than  $kL$  implies that the endpoints are at distance at least  $kN$ .

It also implies the second statement since assuming that it does not hold, we get a sequence of points  $x_n, y_n$  such that the distance is larger than  $n$  and such that the angle between  $y_n - x_n$  with  $P$  is smaller than  $1/n$ . This implies that the length of the arc of  $\tilde{\mathcal{F}}^\perp$  joining  $x_n$  and  $y_n$  is larger than  $n$  and that it does not intersect  $P + x_n + (n \sin(\frac{1}{n}) + 2R)v$  contradicting the claim.

□

## 7. PARTIALLY HYPERBOLIC DIFFEOMORPHISMS ISOTOPIC TO ANOSOV

7.0.2. In this section we give a proof of Theorem A.

We shall assume that  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  is an almost dynamical coherent partially hyperbolic diffeomorphism isotopic to a linear Anosov automorphism  $A : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  with splitting of the form  $T\mathbb{T}^3 = E^{cs} \oplus E^u$  with  $\dim E^u = 1$ .

This means that  $f_*$  coincides with the lift of  $A$  to  $\mathbb{R}^3$  and is a hyperbolic matrix. We will abuse notation and use  $A$  to denote both the hyperbolic matrix in  $\mathbb{R}^3$  and the diffeomorphism of  $\mathbb{T}^3$ .

It is important to remark that we are not assuming that the stable dimension of  $A = f_*$  coincides with the one of  $E^{cs}$ . In fact, many of the arguments below become much easier in the case  $A$  has stable dimension 2. We believe that the fact of being able to treat the case when  $A$  has stable dimension 1 is one of the main contributions of this paper.

7.0.3. We will denote as  $\mathcal{F}$  to the foliation given by the definition of almost dynamical coherent which we know is Reebless and it thus verifies the hypothesis of Theorem 5.4. Being Reebless, Corollary 4.7 will play an important role.

We denote as  $p : \mathbb{R}^3 \rightarrow \mathbb{T}^3$  the covering projection and we denote as  $\tilde{f}$ ,  $\tilde{\mathcal{F}}$  and  $\tilde{\mathcal{F}}^u$  the lifts of  $f$ ,  $\mathcal{F}$  and  $\mathcal{F}^u$  to the universal cover. Recall that from the properties of  $\tilde{\mathcal{F}}$  we have that  $\mathbb{R}^3 \setminus \tilde{\mathcal{F}}(x)$  has exactly two connected components  $F_+(x)$  and  $F_-(x)$  whose boundary coincides with  $\tilde{\mathcal{F}}(x)$ . The choice of the  $+$  or  $-$  depends on the chosen orientation of  $E^u$  for which we know that  $F_+(x)$  is the connected component of  $\tilde{\mathcal{F}}_+^u(x)$ , the positively oriented connected component of  $\tilde{\mathcal{F}}^u(x) \setminus \{x\}$ .

### 7.1. Consequences of the semiconjugacy.

7.1.1. Proposition 4.14 implies the existence of a continuous  $\mathbb{Z}^3$ -periodic surjective function  $H : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  which verifies

$$H \circ \tilde{f} = A \circ H$$

and such that  $d(H(x), x) < K_1$  for every  $x \in \mathbb{R}^3$ .

We can prove:

**Lemma 7.1.** *For every  $x \in \mathbb{R}^3$  we have that  $H(\tilde{\mathcal{F}}_+^u(x))$  is unbounded.*

PROOF. Otherwise, for some  $x \in \mathbb{R}^3$ , the unstable leaf  $\tilde{\mathcal{F}}_+^u(x)$  would be bounded. Since its length is infinite one can find two points in  $\tilde{\mathcal{F}}_+^u(x)$  in different local unstable leaves at arbitrarily small distance. This contradicts Corollary 4.7 (i). □

*Remark 7.2.* Notice that for every  $x \in \mathbb{R}^3$  the set  $F_+(x)$  is unbounded and contains a half unstable leaf of  $\tilde{\mathcal{F}}^u$ .

- In the case the automorphism  $A$  has stable dimension 2, this implies that  $H(F_+(x))$  contains a half-line of irrational slope. Indeed, by Lemma 7.1 we have that  $H(\tilde{\mathcal{F}}_+^u(x))$  is non bounded and since we know that  $H(\tilde{\mathcal{F}}^u(x)) \subset W^u(H(x), A)$  we conclude.
- When  $A$  has stable dimension 1, we only obtain that  $H(\tilde{\mathcal{F}}_+^u(x))$  contains an unbounded connected set in  $W^u(H(x), A)$  which is two dimensional plane parallel to  $E_A^u$ .

◇

7.1.2. One can push forward Lemma 7.1 in order to show that  $H$  is almost injective in each unstable leaf of  $\tilde{\mathcal{F}}^u$ , in particular, a simple argument gives that at most finitely many points of an unstable leaf can have the same image under  $H$ . Later, we shall obtain that in fact,  $H$  is injective on unstable leaves (see Remark 7.10) so we do not give the details here.

## 7.2. A planar direction for the foliation transverse to $E^u$ .

7.2.1. Since  $\mathcal{F}$  is transverse to the unstable direction, we get by Corollary 4.7 that it is a Reebless foliation so that we can apply Theorem 5.4. We intend to prove in this section that option (ii) of this Theorem 5.4 is not possible when  $f$  is isotopic to Anosov (see the example in [RHRHU<sub>2</sub>] where that possibility occurs).

*Remark 7.3.* Notice that if we apply  $\tilde{f}^{-1}$  to the foliation  $\tilde{\mathcal{F}}$ , then the new foliation  $f^{-1}(\tilde{\mathcal{F}})$  is still transverse to  $E^u$  so that Theorem 5.4 still applies. In fact, if  $P$  is the plane obtained for  $\tilde{\mathcal{F}}$ , then the plane which the proposition will give for  $f^{-1}(\tilde{\mathcal{F}})$  will be  $A^{-1}(P)$ : This is immediate by recalling that  $\tilde{f}$  and  $A$  are at bounded distance while two planes which are not parallel have points at arbitrarily large distance. ◇

7.2.2. The result that follows can be deduced more easily if one assumes that  $A$  has stable dimension 2.

We say that a subspace  $P$  is *almost parallel* to a foliation  $\tilde{\mathcal{F}}$  if there exists  $R > 0$  such that for every  $x \in \mathbb{R}^3$  we have that  $P + x$  lies in an  $R$ -neighborhood of  $\tilde{\mathcal{F}}(x)$  and  $\tilde{\mathcal{F}}(x)$  lies in a  $R$ -neighborhood of  $P + x$ .

**Proposition 7.4.** *Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be a partially hyperbolic diffeomorphism of the form  $T\mathbb{T}^3 = E^{cs} \oplus E^u$  (with  $\dim E^{cs} = 2$ ) isotopic to a linear Anosov automorphism and  $\mathcal{F}$  a foliation transverse to  $E^u$ . Then, there exists a two dimensional subspace  $P \subset \mathbb{R}^3$  which is almost parallel to  $\tilde{\mathcal{F}}$ .*

PROOF. It is enough to show that option (i) of Theorem 5.4 holds, since it implies the existence of a plane  $P$  almost parallel to  $\tilde{\mathcal{F}}$ .

Assume by contradiction that option (ii) of Proposition 5.4 holds. Then, there exists a plane  $P \subset \mathbb{R}^3$  whose projection to  $\mathbb{T}^3$  is a two dimensional torus and such that every leaf of  $\tilde{\mathcal{F}}^u$ , being transverse to  $\tilde{\mathcal{F}}$ , remains at bounded distance<sup>8</sup> from  $P$ .

Since  $f$  is isotopic to a linear Anosov automorphism  $A$  we know that  $P$  cannot be invariant under  $A$  (see Proposition 4.15). So, we have that  $P$  and  $A^{-1}(P)$  intersect in a one dimensional subspace  $L$  which projects into a circle in  $\mathbb{T}^3$  (notice that a linear curve in  $\mathbb{T}^2$  is either dense or a circle, so, if a line belongs to the intersection of two linear two dimensional torus in  $\mathbb{T}^3$  which do not coincide, it must be a circle).

We get that for every point  $x$  we have that  $\tilde{\mathcal{F}}^u(x)$  must lie within bounded distance from  $P$  as well as from  $A^{-1}(P)$  (since when we apply  $\tilde{f}^{-1}$  to  $\tilde{\mathcal{F}}$  the leaf close to  $P$  becomes close to  $A(P)$ , see Remark 7.3). This implies that in fact  $\tilde{\mathcal{F}}^u(x)$  lies within bounded distance from  $L$ .

On the other hand, we have that  $H(\tilde{\mathcal{F}}^u_+(x))$  is contained in  $W^u(H(x), A) = E^u_A + H(x)$  for every  $x \in \mathbb{R}^3$ . We get that  $\tilde{\mathcal{F}}^u(x)$  lies within bounded distance from  $E^u_A$ , the eigenspace corresponding to the unstable eigenvalues of  $A$ .

Since  $E^u_A$  must be totally irrational (4.5.3) and both  $P$  and  $A^{-1}(P)$  project to two-dimensional tori, we get that the intersection between the three subspaces must be trivial. This contradicts the fact that  $\tilde{\mathcal{F}}^u_+(x)$  is unbounded (Lemma 7.1).

□

### 7.3. Global Product Structure.

<sup>8</sup>Notice that if  $A$  has stable dimension 2, this already gives us a contradiction since  $H(\tilde{\mathcal{F}}^u(x)) = E^u_A + H(x)$  which is totally irrational and cannot accumulate in a plane which projects into a two-torus.

7.3.1. When the plane  $P$  almost parallel to  $\tilde{\mathcal{F}}$  is totally irrational, one can see that the foliation  $\mathcal{F}$  in  $\mathbb{T}^3$  is without holonomy, and thus there is a global product structure between  $\tilde{\mathcal{F}}$  and  $\tilde{\mathcal{F}}^u$  which follows directly from Theorem 6.1.

This would be the case if we knew that the plane  $P$  given by Theorem 5.4 is  $A$ -invariant (see Appendix A). To obtain the global product structure in our case we will use the fact that iterating the plane  $P$  backwards by  $A$  it will converge to an irrational plane and use instead Theorem 6.2.

7.3.2. Proposition 7.4 implies that the foliation  $\tilde{\mathcal{F}}$  is quite well behaved. In this section we shall show that the properties we have showed for the foliations and the fact that  $\tilde{\mathcal{F}}^u$  is  $\tilde{f}$ -invariant while the foliation  $\tilde{\mathcal{F}}$  remains with a uniform local product structure with  $\tilde{\mathcal{F}}^u$  when iterated backwards (see Lemma 7.6) implies that there is a global product structure. Some of the arguments become simpler if one assumes that  $A$  has stable dimension 2.

The main result of this section is thus the following:

**Proposition 7.5.** *Given  $x, y \in \mathbb{R}^3$  we have that  $\tilde{\mathcal{F}}(x) \cap \tilde{\mathcal{F}}^u(y) \neq \emptyset$ . This intersection consists of exactly one point.*

Notice that uniqueness of the intersection point follows directly from Corollary 4.7 (i). The proof consists in showing that for sufficiently large  $n$  we have that  $f^{-n}(\mathcal{F})$  and  $\mathcal{F}^u$  are in the conditions of Theorem 6.2.

7.3.3. We shall proceed with the proof of Proposition 7.5.

We start by proving a result which gives that the size of local product structure boxes between  $f^{-n}(\mathcal{F})$  and  $\mathcal{F}^u$  can be chosen independent of  $n$ . We shall denote as  $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| \leq 1\}$ .

**Lemma 7.6.** *There exists  $\delta > 0$  such that for every  $x \in \mathbb{R}^3$  and  $n \geq 0$  there exists a closed neighborhood  $V_x^n$  containing  $B_\delta(x)$  such that it admits  $C^0$ -coordinates  $\varphi_x^n : \mathbb{D}^2 \times [-1, 1] \rightarrow \mathbb{R}^3$  such that:*

- $\varphi_x^n(\mathbb{D}^2 \times [-1, 1]) = V_x^n$  and  $\varphi_x^n(0, 0) = x$ .
- $\varphi_x^n(\mathbb{D}^2 \times \{t\}) = \tilde{f}^{-n}(\tilde{\mathcal{F}}(\tilde{f}^n(\varphi_x^n(0, t)))) \cap V_x^n$  for every  $t \in [-1, 1]$ .
- $\varphi_x^n(\{s\} \times [-1, 1]) = \tilde{\mathcal{F}}^u(\varphi_x^n(s, 0)) \cap V_x^n$  for every  $s \in \mathbb{D}^2$ .

PROOF. Notice first that  $f^{-n}(\mathcal{F})$  is tangent to a cone transverse to  $E^u$  and independent of  $n$ . Let us call this cone  $\mathcal{E}^{cs}$ .

Given  $\epsilon > 0$  we can choose a neighborhood  $V_\epsilon$  of  $x$  contained in  $B_\epsilon(x)$  such that the following is verified:



- There exists a two dimensional disk  $D$  containing  $x$  such that  $V_\epsilon$  is the union of segments of  $\mathcal{F}^u(x)$  of length  $2\epsilon$  centered at points in  $D$ . This defines two boundary disks  $D^+$  and  $D^-$  contained in the boundary of  $V_\epsilon$ .
- By choosing  $D$  small enough, we get that there exists  $\epsilon_1 > 0$  such that every curve of length  $\epsilon_1$  starting at a point  $y \in B_{\epsilon_1}(x)$  tangent to  $\mathcal{E}^{cs}$  must leave  $V_\epsilon$  and intersects  $\partial V_\epsilon$  in  $\partial V_\epsilon \setminus (D^+ \cup D^-)$ .

Notice that both  $\epsilon$  and  $\epsilon_1$  can be chosen uniformly in  $\mathbb{R}^3$  because of compactness of  $\mathbb{T}^3$  and uniform transversality of the foliations.

This implies that every disk of radius  $\epsilon$  tangent to  $\mathcal{E}^{cs}$  centered at a point  $z \in B_{\epsilon_1}(x)$  must intersect the unstable leaf of every point in  $D$ , in particular, there is a local product structure of uniform size around each point in  $\mathbb{R}^3$ .

Now, we can choose a continuous chart (recall that the foliations are only continuous, but with  $C^1$ -leaves) around each point which sends horizontal disks into disks transverse to  $E^u$  and vertical lines into leaves of  $\tilde{\mathcal{F}}^u$  containing a fixed ball around each point  $x$  independent of  $n \geq 0$  giving the desired statement. □

*Remark 7.7.* We obtain that there exists  $\epsilon > 0$  such that for every  $x \in \mathbb{R}^3$  there exists  $V_x \subset \bigcap_{n \geq 0} V_x^n$  containing  $B_\epsilon(x)$  admitting  $C^1$ -coordinates  $\psi_x : \mathbb{D}^2 \times [-1, 1] \rightarrow \mathbb{R}^3$  such that:

- $\psi_x(\mathbb{D}^2 \times [-1, 1]) = V_x$  and  $\psi_x(0, 0) = x$ .
- If we consider  $V_x^\epsilon = \psi_x^{-1}(B_\epsilon(x))$  then one has that for every  $y \in V_x^\epsilon$  and  $n \geq 0$  we have that:

$$\psi_x^{-1}(\tilde{f}^{-n}(\tilde{\mathcal{F}}(\tilde{f}^n(y)))) \cap V_x$$

is the graph of a function  $h_y^n : \mathbb{D}^2 \rightarrow [-1, 1]$  which has uniformly bounded derivative in  $y$  and  $n$ .

Indeed, this is given by considering a  $C^1$ -chart  $\psi_x$  around every point such that its image covers the  $\epsilon$ -neighborhood of  $x$  and sends the  $E$ -direction to an almost horizontal direction and the  $E^u$ -direction to an almost vertical direction (see Proposition 4.13). See for example [BuW<sub>2</sub>] section 3 for more details on this kind of constructions. ◇

7.3.4. The next lemma shows that after iterating the foliation backwards, one gets that it becomes nearly irrational so that we can apply Theorem 6.2.

**Lemma 7.8.** *Given  $K > 0$  there exists  $n_0 > 0$  such that for every  $x \in \mathbb{R}^3$  and for every  $\gamma \in \mathbb{Z}^3$  with norm less than  $K$  we have that*

$$\tilde{f}^{-n_0}(\tilde{\mathcal{F}}(x)) + \gamma \neq \tilde{f}^{-n_0}(\tilde{\mathcal{F}}(x)) \quad \forall x \in \mathbb{R}^3.$$

PROOF. Notice that  $\tilde{f}^{-n}(\tilde{\mathcal{F}})$  is almost parallel to  $A^{-n}(P)$ . Notice that  $A^{-n}(P)$  has a converging subsequence towards a totally irrational plane  $\tilde{P}$  (see Remark 4.17 and Lemma 4.16).

We can choose  $n_0$  large enough such that no element of  $\mathbb{Z}^3$  of norm smaller than  $K$  fixes  $A^{-n_0}(P)$ .

Notice first that  $\tilde{f}^{-n_0}(\tilde{\mathcal{F}})$  is almost parallel to  $A^{-n_0}(P)$  (see Remark 7.3). Now, assuming that there is a translation  $\gamma$  which fixes a leaf of  $\tilde{f}^{-n_0}(\tilde{\mathcal{F}}(x))$  we get that the leaf  $p(\tilde{f}^{-n_0}(\tilde{\mathcal{F}}(x)))$  contains a loop homotopic to  $\gamma$ .

This implies that it is at bounded distance from the line which is the lift of the canonical (linear) representative of  $\gamma$ . This implies that  $\gamma$  fixes  $A^{-n_0}(P)$  and thus has norm larger than  $K$  as desired (see also Lemma 5.8). □

We can now complete the proof of Proposition 7.5.

PROOF OF PROPOSITION 7.5. By Corollary 4.7 we know that all the leaves of  $\tilde{\mathcal{F}}$  are simply connected. Proposition 5.6 implies that the leaf space of  $\tilde{\mathcal{F}}$  is homeomorphic to  $\mathbb{R}$ . All this properties remain true for the foliations  $\tilde{f}^{-n}(\tilde{\mathcal{F}})$  since they are diffeomorphisms at bounded distance from linear transformations.

Lemma 7.6 gives that the size of the local product structure between  $\tilde{f}^{-n}(\tilde{\mathcal{F}})$  and  $\tilde{\mathcal{F}}^u$  does not depend on  $n$ .

Using Lemma 7.8 we get that for some sufficiently large  $n$  the foliations  $\tilde{f}^{-n}(\tilde{\mathcal{F}})$  and  $\tilde{\mathcal{F}}^u$  are in the hypothesis of Theorem 6.2 which gives global product structure between  $\tilde{f}^{-n}(\tilde{\mathcal{F}})$  and  $\tilde{\mathcal{F}}^u$ . Since  $\tilde{\mathcal{F}}^u$  is  $\tilde{f}$ -invariant and  $f$  is a diffeomorphism we get that there is a global product structure between  $\tilde{\mathcal{F}}$  and  $\tilde{\mathcal{F}}^u$  as desired. □

Using Proposition 6.9 we deduce the following (see figure 4) :

**Corollary 7.9.** *The foliation  $\tilde{\mathcal{F}}^u$  is quasi-isometric. Moreover, there exist one dimensional subspaces  $L_1$  and  $L_2$  of  $E_A^u$  transverse to  $P$  and  $K > 0$  such that for every  $x \in \mathbb{R}^3$  and  $y \in \tilde{\mathcal{F}}^u(x)$  at distance larger than  $K$  from  $x$  we have that  $H(y) - H(x)$  is contained in the cone of  $E_A^u$  with boundaries  $L_1$  and  $L_2$  and transverse to  $P$ .*

Notice that if  $A$  has stable dimension 2 then  $L_1 = L_2 = E_A^u$ .

PROOF. This is a direct consequence of Proposition 6.9 and the fact that the image of  $\tilde{\mathcal{F}}^u(x)$  by  $H$  is contained in  $E_A^u + H(x)$ . □

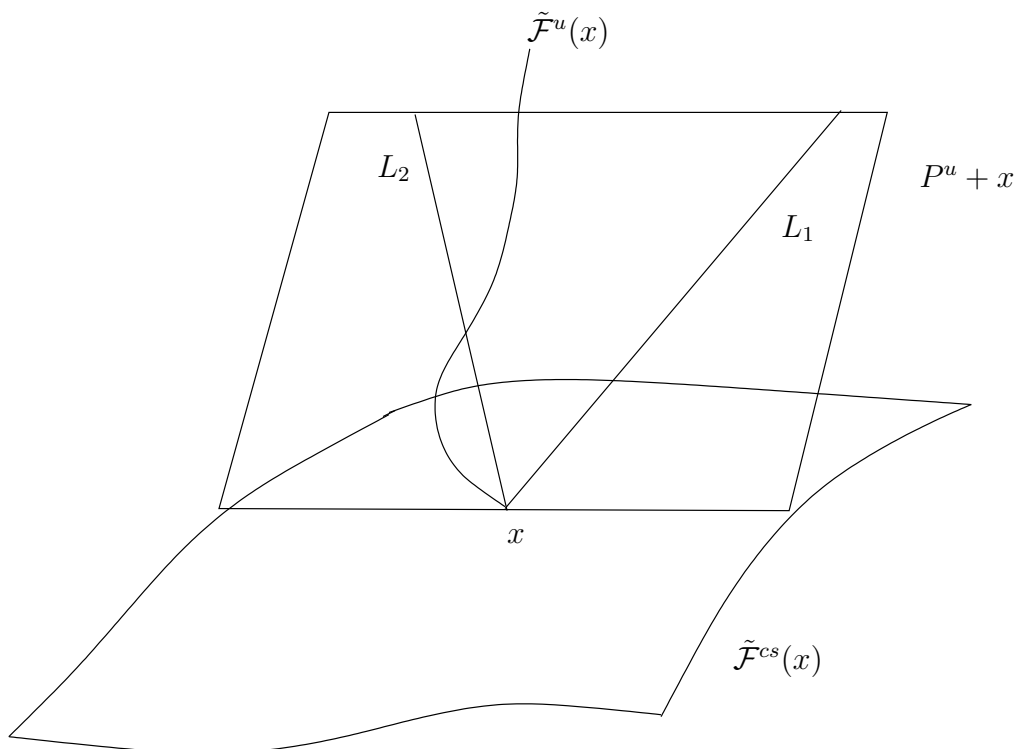


FIGURE 4. The unstable leaf of  $x$  remains close to the cone bounded by  $L_1$  and  $L_2$ .

*Remark 7.10.* Since points which are sent to the same point by  $H$  must have orbits remaining at bounded distance, the quasi-isometry of  $\tilde{\mathcal{F}}^u$  implies that  $H$  must be injective on leaves of  $\tilde{\mathcal{F}}^u$ .

◇

**7.4. Complex eigenvalues.** The following proposition has interest only in the case  $A$  has stable dimension one. It establishes the last statement of Theorem A.

**Proposition 7.11.** *The matrix  $A$  cannot have complex unstable eigenvalues.*

**PROOF.** Assume that  $A$  has complex unstable eigenvalues, in particular  $E_A^u$  is two-dimensional. Consider a fixed point  $x_0$  of  $\tilde{f}$ .

Recall that by Lemma 7.1 the set  $\eta = H(\tilde{\mathcal{F}}_+^u(x_0))$  is an unbounded continuous curve in  $E_A^u$ . Since  $x_0$  is fixed and since  $H$  is a semiconjugacy, we have that  $\eta$  is  $A$ -invariant.

On the other hand, by Corollary 7.9 we have that  $\eta$  is eventually contained in a cone between two lines  $L_1$  and  $L_2$ .

This implies that  $A$  cannot have complex unstable eigenvalues (recall that they should have irrational angle by Lemma 4.16) since a matrix which preserves an unbounded connected subset of a cone cannot have complex eigenvalues with irrational angle.

□

## 7.5. Dynamical Coherence.

7.5.1. In this section we shall show dynamical coherence of almost dynamically coherent partially hyperbolic diffeomorphisms isotopic to linear Anosov automorphisms (and uniqueness of the  $f$ -invariant foliation tangent to  $E^{cs}$ ). This will complete the proof of Theorem A.

The proof of the following theorem becomes much simpler if one assumes that the plane  $P$  almost parallel to  $\tilde{\mathcal{F}}$  is  $A$ -invariant which as we mentioned before represents an important case (see Appendix A).

**Theorem 7.12.** *Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be an almost dynamically coherent partially hyperbolic diffeomorphism of the form  $T\mathbb{T}^3 = E^{cs} \oplus E^u$  isotopic to a linear Anosov automorphism. Then, there exists an  $f$ -invariant foliation  $\mathcal{F}^{cs}$  tangent to  $E^{cs}$ . If  $\tilde{\mathcal{F}}^{cs}$  denotes the lift to  $\mathbb{R}^3$  of this foliation, then  $H(\tilde{\mathcal{F}}^{cs}(x)) = P^{cs} + H(x)$  where  $P^{cs}$  is an  $A$ -invariant subspace and  $E_A^u$  is not contained in  $P^{cs}$ .*

PROOF. Consider the foliation  $\tilde{\mathcal{F}}$ , by Proposition 7.4 we have a plane  $P$  which is almost parallel to  $\tilde{\mathcal{F}}$ .

Let  $P^{cs}$  be the limit of  $A^{-n}(P)$  which is an  $A$ -invariant subspace. Since we have proved that  $A$  has no complex unstable eigenvalues (Proposition 7.11) and since  $P$  is transverse to  $E_A^u$  (Proposition 6.9), this plane is well defined (see Remark 4.17).

Notice that the transversality of  $P$  with  $E_A^u$  implies that  $P^{cs}$  contains  $E_A^s$ , the eigenspace associated with stable eigenvalues (in the case where  $A$  has stable dimension 2 we thus have  $P^{cs} = E_A^s$ ).

Since  $P^{cs}$  is  $A$ -invariant, we get that it is totally irrational so that no deck transformation fixes  $P^{cs}$ .

Using Remark 7.7 we obtain  $\varepsilon > 0$  such that for every  $x \in \mathbb{R}^3$  there are neighborhoods  $V_x$  containing  $B_\varepsilon(x)$  admitting  $C^1$ -coordinates  $\psi_x : \mathbb{D}^2 \times [-1, 1] \rightarrow V_x$  such that:

- For every  $y \in B_\varepsilon(x)$  we have that if we denote as  $W_n^x(y)$  to the connected component containing  $y$  of  $V_x \cap \tilde{f}^{-n}(\tilde{\mathcal{F}}(f^n(y)))$  then the set  $\psi_x^{-1}(W_n^x(y))$  is the graph of a  $C^1$ -function  $h_{x,y}^n : \mathbb{D}^2 \rightarrow [-1, 1]$  with bounded derivatives.

By a standard graph transform argument (see [HPS] or [BuW<sub>2</sub>] section 3) using the fact that these graphs have bounded derivative we get that  $\{h_n^{x,y}\}$  is pre-compact in the space of functions from  $\mathbb{D}^2$  to  $[-1, 1]$ .

For every  $y \in B_\varepsilon(x)$  there exists  $\mathcal{J}_y^x$  a set of indices such that for every  $\alpha \in \mathcal{J}_y^x$  we have a  $C^1$ -function  $h_{\infty,\alpha}^{x,y} : \mathbb{D}^2 \rightarrow [-1, 1]$  and  $n_j \rightarrow +\infty$  such that:

$$h_{\infty,\alpha}^{x,y} = \lim_{j \rightarrow +\infty} h_{n_j}^{x,y}$$

Every  $h_{\infty,\alpha}^{x,y}$  gives rise to a graph whose image by  $\psi_x$  we denote as  $W_{\infty,\alpha}^x(y)$ . This manifold verifies that it contains  $y$  and is everywhere tangent to  $E^{cs}$ .

**Claim.** *We have that  $H(W_{\infty,\alpha}^x(z)) \subset P^{cs} + H(z)$  for every  $z \in B_\varepsilon(x)$  and every  $\alpha \in \mathcal{J}_z^x$ .*

PROOF. Consider  $y \in W_{\infty,\alpha}^x(z)$  for some  $\alpha \in \mathcal{J}_z^x$ . One can find  $n_j \rightarrow \infty$  such that  $W_{n_j}^x(z) \rightarrow W_{\infty,\alpha}^x(z)$ .

In the coordinates  $\psi_x$  of  $V_x$ , we can find a sequence  $z_{n_j} \in W_{n_j}^x(z) \cap \tilde{\mathcal{F}}^u(y)$  such that  $z_{n_j} \rightarrow y$ . Moreover, we have that  $\tilde{f}^{n_j}(z_{n_j}) \in \tilde{\mathcal{F}}(f^{n_j}(z))$ . Assume that  $H(y) \neq H(z)$  (otherwise there is nothing to prove).

We have, by continuity of  $H$  that  $H(z_{n_j}) \rightarrow H(y) \neq H(z)$ .

We choose a metric in  $\mathbb{R}^3$  so that  $(P^{cs})^\perp$  with this metric is  $A$ -invariant. We denote as  $\lambda$  to the eigenvalue of  $A$  in the direction  $(P^{cs})^\perp$ .

By Proposition 7.4 and the fact that  $H$  is at bounded distance from the identity, there exists  $R > 0$  such that for every  $n_j \geq 0$  we have that  $A^{n_j}(H(z_{n_j}))$  is at distance smaller than  $R$  from  $P + A^{n_j}(H(z))$ .

Suppose that  $H(z_{n_j})$  does not converge to  $P^{cs} + H(z)$ . We must reach a contradiction.

Consider then  $\alpha > 0$  such that the angle between  $P^{cs}$  and the vector  $H(y) - H(z)$  is larger than  $\alpha > 0$ . This  $\alpha$  can be chosen positive under the assumption that  $H(z_{n_j})$  does not converge to  $P^{cs} + H(z)$ .

Let  $n_j > 0$  be large enough such that:

- The angle between  $A^{-n_j}(P)$  and  $P^{cs}$  is smaller than  $\alpha/4$ ,
- $\|H(z_{n_j}) - H(z)\| > \frac{3}{4}\|H(y) - H(z)\|$ ,
- $\lambda^{n_j} \gg 2R(\sin(\frac{\alpha}{2})\cos(\beta)\|H(y) - H(z)\|)^{-1}$ .

Let  $v_{n_j}$  be the vector which realizes  $d(H(z_{n_j}) - H(z), A^{-n_j}(P))$  and as  $v_{n_j}^\perp$  the projection of  $v_{n_j}$  to  $(P^{cs})^\perp$ . We have that

$$\|v_{n_j}^\perp\| > \frac{1}{2} \sin\left(\frac{\alpha}{2}\right) \|H(y) - H(z)\|$$

Notice that the distance between  $A^{n_j}(H(z_{n_j}))$  and  $P + A^{n_j}(H(z))$  is larger than  $\|A^{n_j}v_{n_j}^\perp\| \cos(\beta)$ .

This is a contradiction since this implies that  $A^{n_j}(H(z_{n_j}))$  is at distance larger than

$$\lambda^{n_j} \|v_{n_j}^\perp\| \cos(\beta) \gg R$$

from  $P + A^{n_j}(H(z))$ . This concludes the claim.

◇

Assuming that  $P^{cs}$  does not intersect the cone bounded by  $L_1$  and  $L_2$  this finishes the proof since one sees that each leaf of  $\tilde{\mathcal{F}}^u$  can intersect the pre-image by  $H$  of  $P^{cs} + y$  in a unique point, thus showing that the partition of  $\mathbb{R}^3$  by the pre-images of the translates of  $P^{cs}$  defines a  $\tilde{f}$ -invariant foliation (and also invariant under deck transformations). We leave to the interested reader the task of filling the details of the proof in this particular case, since we will continue by giving a proof which works in all cases.

We will prove that  $H$  cannot send unstable intervals into the same plane parallel to  $P^{cs}$ .

**Claim.** *Given  $\gamma : [0, 1] \rightarrow \mathbb{R}$  a non-trivial curve contained in  $\tilde{\mathcal{F}}^u(x)$  we have that  $H(\gamma([0, 1]))$  is not contained in  $P^{cs} + H(\gamma(0))$ .*

PROOF. Consider  $C_\varepsilon$  given by Corollary 4.7 (iv) for  $\varepsilon$ . Moreover, consider  $L$  large enough such that  $C_\varepsilon L > \text{Vol}(\mathbb{T}^3)$ .

Since  $\tilde{\mathcal{F}}^u$  is  $\tilde{f}$ -invariant and  $P^{cs}$  is  $A$ -invariant we deduce that we can assume that the length of  $\gamma$  is arbitrarily large, in particular larger than  $2L$ .

We will show that  $H(B_\varepsilon(\gamma([a, b]))) \subset P^{cs} + H(\gamma(0))$  where  $0 < a < b < 1$  and the length of  $\gamma([a, b])$  is larger than  $L$ .

Having volume larger than  $\text{Vol}(\mathbb{T}^3)$  there must be a deck transformation  $\gamma \in \mathbb{Z}^3$  such that  $\gamma + B_\varepsilon(\gamma([a, b])) \cap B_\varepsilon(\gamma([a, b])) \neq \emptyset$ . This in turn gives that  $\gamma + H(B_\varepsilon(\gamma([a, b]))) \cap H(B_\varepsilon(\gamma([a, b]))) \neq \emptyset$  and thus  $\gamma + P^{cs} \cap P^{cs} \neq \emptyset$ . Since  $P^{cs}$  is totally irrational this is a contradiction.

It remains to show that  $H(B_\varepsilon(\gamma([a, b]))) \subset P^{cs} + H(\gamma(0))$ . By the previous claim, we know that if  $z, w \in W_{\infty, \alpha}^x(y)$  for some  $\alpha \in \mathcal{J}_y$ , then  $H(z) - H(w) \in P^{cs}$ .

Consider  $a, b \in [0, 1]$  such that  $\tilde{\mathcal{F}}^u(x) \cap B_\varepsilon(\gamma([a, b])) \subset \gamma([0, 1])$ . By Corollary 4.7 we have that such  $a, b$  exist and we can choose them in order that the length of  $\gamma([a, b])$  is larger than  $L$ .

Let  $z \in B_\varepsilon(\gamma([a, b]))$  and choose  $w \in \gamma([a, b])$  such that  $z \in B_\varepsilon(w)$ . We get that for every  $\alpha \in \mathcal{J}_z^w$  we have that  $W_{\infty, \alpha}^w(z) \cap \gamma([0, 1]) \neq \emptyset$ . Since  $H(\gamma([0, 1])) \subset P^{cs} + H(\gamma(0))$  and by the previous claim, we deduce that  $H(w) \subset P^{cs} + H(\gamma(0))$  finishing the proof.

◇

Now we are in conditions of showing that for every point  $x$  and for every point  $y \in B_\varepsilon(x)$  there is a unique manifold  $W_\infty^x(y)$  tangent to  $E^{cs}$  which is a limit of the manifolds  $W_n^x(y)$ . Using the same argument as in Proposition 4.13 we get that the foliations  $\tilde{f}^{-n}(\tilde{\mathcal{F}})$  converge to a  $f$ -invariant foliation  $\tilde{\mathcal{F}}^{cs}$  tangent to  $E^{cs}$  concluding the proof of the Theorem.

Indeed, assume that the manifolds  $W_n^x(y)$  have a unique limit for every  $x \in \mathbb{R}^3$  and  $y \in B_\varepsilon(x)$  and that for any pair points  $y, z \in B_\varepsilon(x)$  these limits are either disjoint or equal (see the claim below). One has that the set of manifolds  $W_\infty^x(y)$  forms an  $f$ -invariant plaque family in the following sense:

- $\tilde{f}(W_\infty^x(y)) \cap W_\infty^{\tilde{f}(x)}(\tilde{f}(y))$  is relatively open whenever  $\tilde{f}(y) \in B_\varepsilon(\tilde{f}(x))$ .

We must thus show that these plaque families form a foliation. For this, we use the same argument as in Proposition 4.13. Consider  $z, w \in B_\varepsilon(x)$  we have that  $W_\infty^x(z) \cap \tilde{\mathcal{F}}^u(w) \neq \emptyset$  and in fact consists of a unique point (see Corollary 4.7 (i)). Since the intersection point varies continuously and using that plaques are either disjoint or equal we obtain a continuous map from  $\mathbb{D}^2 \times [-1, 1]$  to a neighborhood of  $x$  sending horizontal disks into plaques. This implies that the plaques form an  $f$ -invariant foliation as desired.

It thus remains to show the following:

**Claim.** *Given  $x \in \mathbb{R}^3$  and  $y, z \in B_\varepsilon(x)$  we have that there is a unique limit of  $W_\infty^x(y)$  and  $W_\infty^x(z)$  and they are either disjoint or coincide. More precisely, for every  $\alpha \in \mathcal{J}_y^x$  and  $\beta \in \mathcal{J}_z^x$  ( $z$  could coincide with  $y$ ) we have that  $h_{\infty, \alpha}^{x, y} = h_{\infty, \beta}^{x, z}$  or the graphs are disjoint.*

PROOF. Assuming the claim does not hold, one obtains  $y, z \in B_\varepsilon(x)$  such that  $h_{\infty, \alpha}^{x, y}$  and  $h_{\infty, \beta}^{x, z}$  coincide at some point but whose graphs are different for some  $\alpha \in \mathcal{J}_y^x$  and  $\beta \in \mathcal{J}_z^x$ . In particular, there exists a point  $t \in \mathbb{D}^2$  which is in the boundary of where both functions coincide. We assume for simplicity<sup>9</sup> that  $\psi_x(t)$  belongs to  $B_\varepsilon(x)$ .

Let  $\gamma : [0, 1] \rightarrow B_\varepsilon(x)$  be a non-trivial arc of  $\tilde{\mathcal{F}}^u$  joining the graphs of  $h_{\infty, \alpha}^{x, y}$  and  $h_{\infty, \beta}^{x, z}$ . Since the graphs of both  $h_{\infty, \alpha}^{x, y}$  and  $h_{\infty, \beta}^{x, z}$  separate  $V_x$  we have that every point  $w \in \gamma((0, 1))$  verifies that for every  $\delta \in \mathcal{J}_w^x$  one has that  $W_{\infty, \delta}^x(w)$  intersects at least one of  $W_{\infty, \alpha}^x(y)$  or  $W_{\infty, \beta}^x(z)$ . By the first claim we get that  $H(w) \in P^{cs} + H(y) = P^{cs} + H(z)$  a contradiction with the second claim.

◇

□

7.5.2. We can in fact obtain a stronger property since our results allows us to show that in fact  $E^{cs}$  is uniquely integrable into a  $f$ -invariant foliation. Notice that there are stronger forms of unique integrability (see [BuW<sub>1</sub>] and [BF]).

**Proposition 7.13.** *There is a unique  $f$ -invariant foliation  $\mathcal{F}^{cs}$  tangent to  $E^{cs}$ . Moreover, the plane  $P^{cs}$  given by Theorem 5.4 for this foliation is  $A$ -invariant and contains the stable eigenspace of  $A$ .*

<sup>9</sup>If it were not the case we would need to change the coordinates and perform the same proof, but to avoid charging the notation we choose to make this (unnecessary) assumption.

PROOF. Assume there are two different  $f$ -invariant foliations  $\mathcal{F}_1^{cs}$  and  $\mathcal{F}_2^{cs}$  tangent to  $E^{cs}$ .

Since they are transverse to  $E^u$  they must be Reebless (see Corollary 4.7) so that Theorem 5.4 applies.

By Remark 7.3 we know that since the foliations are  $f$ -invariant, the planes  $P_1^{cs}$  and  $P_2^{cs}$  given by Theorem 5.4 are  $A$ -invariant. The fact that  $P^{cs}$  contains the stable direction of  $A$  is given by Remark 4.17 and Corollary 7.9 since it implies that  $P^{cs}$  cannot be contained in  $E_A^u$ .

Assume first that the planes  $P_1^{cs}$  and  $P_2^{cs}$  coincide. The foliations remain at distance  $R$  from translates of the planes. By Corollary 7.9 we know that two points in the same unstable leaf must separate in a direction transverse to  $P_1^{cs} = P_2^{cs}$ . If  $\mathcal{F}_1^{cs}$  is different from  $\mathcal{F}_2^{cs}$  we have a point  $x$  such that  $\mathcal{F}_1^{cs}(x) \neq \mathcal{F}_2^{cs}(x)$ . By the global product structure we get a point  $y \in \mathcal{F}_1^{cs}(x)$  such that  $\tilde{\mathcal{F}}^u(y) \cap \mathcal{F}_2^{cs}(x) \neq \{y\}$ . Iterating forward and using Corollary 7.9 we contradict the fact that leaves of  $\mathcal{F}_1^{cs}$  and  $\mathcal{F}_2^{cs}$  remain at distance  $R$  from translates of  $P_1^{cs} = P_2^{cs}$ .

Now, if  $P_1^{cs} \neq P_2^{cs}$  we know that  $A$  has stable dimension 1 since we know that  $E_A^s$  is contained in both. Using Corollary 7.9 and the fact that the unstable foliation is  $\tilde{f}$  invariant we see that this cannot happen. □

Notice also that from the proof of Theorem 7.12 we deduce that given a foliation  $\mathcal{F}$  transverse to  $E^u$  we have that the backward iterates of this foliation must converge to this unique  $f$ -invariant foliation. This implies that:

**Corollary 7.14.** *Given a dynamically coherent partially hyperbolic diffeomorphism  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  isotopic to Anosov we know that it is  $C^1$ -robustly dynamically coherent and that the  $f_*$ -invariant plane  $P$  given by Theorem 5.4 for the unique  $f$ -invariant foliation  $\mathcal{F}^{cs}$  tangent to  $E^{cs}$  does not change for diffeomorphisms  $C^1$ -close to  $f$ .*

PROOF. The robustness of dynamical coherence follows from the fact that being dynamically coherent it is robustly almost dynamically coherent so that Theorem A applies.

From uniqueness and the fact that for a perturbation  $g$  of  $f$  we can use the foliation  $\mathcal{F}_f^{cs}$  tangent to  $E_f^{cs}$  as a foliation transverse to  $E_g^u$ . We get that the plane  $P^{cs}$  almost parallel to  $\mathcal{F}_f^{cs}$  which is invariant under  $f_*$  is also invariant under  $g_* = f_*$ . This implies that the plane which is almost parallel to the unique  $g$ -invariant foliation is again  $P^{cs}$  and proves the corollary. □



8. STRONG PARTIALLY HYPERBOLIC DIFFEOMORPHISMS OF  $\mathbb{T}^3$ 

8.0.3. In this section we prove Theorem B.

The idea of the proof is to obtain a global product structure between the foliations involved in order to then get dynamical coherence. In a certain sense, this is a similar idea to the one used for the proof of Theorem A. However, the fact that global product structure implies dynamical coherence is much easier in this case due to the existence of  $f$ -invariant branching foliations tangent to the center-stable direction (see subsection 4.4).

8.0.4. This approach goes in the inverse direction to the one made in [BBI<sub>2</sub>] (and continued in [H]). In [BBI<sub>2</sub>] the proof proceeds by showing that the planes close to the foliations are different (by using absolute domination) for then showing (again by using absolute domination) that leaves of  $\tilde{\mathcal{F}}^u$  are quasi-isometric so that Brin's criterium for absolutely dominated partially hyperbolic systems ([Br]) can be applied.

Then, in [H] it is proved that in fact, the planes  $P^{cs}$  and  $P^{cu}$  close to the  $f$ -invariant foliations are the expected ones in order to obtain global product structure and then leaf conjugacy to linear models.

Another difference with their proof there is that in our case it will be important to discuss depending on the isotopy class of  $f$ . In a certain sense, the reason why in each case there is a global product structure can be regarded as different: In the isotopic to Anosov case (see Appendix A) the reason is that we deduce that the foliations are without holonomy and use Theorem 6.1. In the case which is isotopic to a non-hyperbolic matrix we must find out which are the planes close to each foliation first in order to then get the global product structure using this fact.

## 8.1. Preliminary discussions.

8.1.1. Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be a strong partially hyperbolic diffeomorphism with splitting  $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$ .

We denote as  $\mathcal{F}^s$  and  $\mathcal{F}^u$  to the stable and unstable foliations given by Theorem 2.3 which are one dimensional and  $f$ -invariant.

As in the previous sections, we will denote as  $p : \mathbb{R}^3 \rightarrow \mathbb{T}^3$  to the covering projection and  $\tilde{f}$  will denote a lift of  $f$  to the universal cover. Recall that  $f_* : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  which denotes the linear part of  $f$  is at bounded distance ( $K_0 > 0$ ) from  $\tilde{f}$ .

8.1.2. We have proved:

**Theorem 8.1.** *Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be a strong partially hyperbolic diffeomorphism isotopic to Anosov, then  $f$  is dynamically coherent.*

This follows from Theorem A and the fact that strongly partially hyperbolic diffeomorphisms are almost dynamical coherent (Corollary 4.11). We will give an independent proof in Appendix A since in the context of strong partial hyperbolicity the proof becomes simpler.

8.1.3. The starting point of our proof of Theorem B is the existence of  $f$ -invariant branching foliations  $\mathcal{F}_{bran}^{cs}$  and  $\mathcal{F}_{bran}^{cu}$  tangent to  $E^s \oplus E^c$  and  $E^c \oplus E^u$  respectively. By using Theorem 4.9 and Theorem 5.4 we can deduce the following:

**Proposition 8.2.** *There exist an  $f_*$ -invariant plane  $P^{cs}$  and  $R > 0$  such that every leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$  (the lift of  $\mathcal{F}_{bran}^{cs}$ ) lies in the  $R$ -neighborhood of a plane parallel to  $P^{cs}$ .*

*Moreover, one can choose  $R$  such that one of the following conditions holds:*

- (i) *The projection of the plane  $P^{cs}$  is dense in  $\mathbb{T}^3$  and the  $R$ -neighborhood of every leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$  contains a plane parallel to  $P^{cs}$ , or,*
- (ii) *The projection of  $P^{cs}$  is a linear two-dimensional torus and there is a leaf of  $\mathcal{F}_{bran}^{cs}$  which is a two-dimensional torus homotopic to  $p(P^{cs})$ .*

*An analogous dichotomy holds for  $\mathcal{F}_{bran}^{cu}$ .*

PROOF. We consider sufficiently small  $\varepsilon > 0$  and the foliation  $\mathcal{S}_\varepsilon$  given by Theorem 4.9.

Let  $h_\varepsilon^{cs}$  be the continuous and surjective map which is  $\varepsilon$ -close to the identity sending leaves of  $\mathcal{S}_\varepsilon$  into leaves of  $\mathcal{F}_{bran}^{cs}$ .

This implies that given a leaf  $L$  of  $\mathcal{F}_{bran}^{cs}$  there exists a leaf  $S$  of  $\mathcal{S}_\varepsilon$  such that  $L$  is at distance smaller than  $L$  from  $S$  and viceversa.

Since the foliation  $\mathcal{S}_\varepsilon$  is transverse to  $E^u$  we can apply Theorem 5.4 and we obtain that there exists a plane  $P^{cs}$  and  $R > 0$  such that every leaf of the lift  $\tilde{\mathcal{S}}_\varepsilon$  of  $\mathcal{S}_\varepsilon$  to  $\mathbb{R}^3$  lies in an  $R$ -neighborhood of a translate of  $P^{cs}$ .

From the previous remark, we get that every leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$ , the lift of  $\mathcal{F}_{bran}^{cs}$  to  $\mathbb{R}^3$  lies in an  $(R + \varepsilon)$ -neighborhood of a translate of  $P^{cs}$ . Since  $\mathcal{F}_{bran}^{cs}$  is  $f$ -invariant, we deduce that the plane  $P^{cs}$  is  $f_*$ -invariant (see also Remark 7.3).

By Proposition 5.7 we know that if  $P^{cs}$  projects into a two-dimensional torus, we obtain that the foliation  $\mathcal{S}_\varepsilon$  must have a torus leaf. The image of this leaf by  $h_\varepsilon^{cs}$  is a torus leaf of  $\mathcal{F}_{bran}^{cs}$ .

Since a plane whose projection is not a two-dimensional torus must be dense we get that if option (ii) does not hold, we have that the image of  $P^{cs}$  must be dense. Moreover, option (i) of Theorem 5.4 must hold for  $\mathcal{S}_\varepsilon$  and this concludes the proof of this proposition.

□

**8.2. Global product structure implies dynamical coherence.** Assume that  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  is a strong partially hyperbolic diffeomorphism. Let  $\mathcal{F}_{bran}^{cs}$  be the  $f$ -invariant branching foliation tangent to  $E^s \oplus E^c$  given by Theorem 4.9 and let  $\mathcal{S}_\varepsilon$  be a foliation tangent to an  $\varepsilon$ -cone around  $E^s \oplus E^c$  which remains  $\varepsilon$ -close to the lift of  $\mathcal{F}_{bran}^{cs}$  to the universal cover for small  $\varepsilon$ .

When the lifts of  $\mathcal{S}_\varepsilon$  and  $\mathcal{F}^u$  to the universal cover have a global product structure, we deduce from Proposition 6.9 the following:

**Corollary 8.3.** *The foliation  $\tilde{\mathcal{F}}^u$  is quasi-isometric. Indeed, if  $v \in (P^{cs})^\perp$  is a unit vector, there exists  $L > 0$  such that every unstable curve starting at a point  $x$  of length larger than  $nL$  intersects  $P^{cs} + nv + x$  or  $P^{cs} - nv + x$ .*

Before we show that global product structure implies dynamical coherence, we must show that global product structure is equivalent to a similar property with the branching foliation:

**Lemma 8.4.** *There exists  $\varepsilon > 0$  such that  $\tilde{\mathcal{F}}^u$  and  $\tilde{\mathcal{S}}_\varepsilon$  have global product structure if and only if:*

- For every  $x, y \in \mathbb{R}^3$  and for every  $L \in \tilde{\mathcal{F}}_{bran}^{cs}(y)$  we have that  $\tilde{\mathcal{F}}^u(x) \cap L \neq \emptyset$ .

**PROOF.** First notice that any of the hypothesis implies that  $\tilde{\mathcal{S}}_\varepsilon$  cannot have dead-end components. In particular, there exists  $R > 0$  and a plane  $P^{cs}$  such every leaf of  $\tilde{\mathcal{S}}_\varepsilon$  and every leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$  verifies that it is contained in an  $R$ -neighborhood of a translate of  $P^{cs}$  and the  $R$ -neighborhood of the leafs contains a translate of  $P^{cs}$  too (see Proposition 8.2).

We prove the direct implication first. Consider  $x, y \in \mathbb{R}^3$  and  $L$  a leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}(y)$ . Now, we know that  $L$  separates in  $\mathbb{R}^3$  the planes  $P^{cs} + y + 2R$  and  $P^{cs} + y - 2R$ . One of them must be in the connected component of  $\mathbb{R}^3 \setminus L$  which not contains  $x$ , without loss of generality we assume that it is  $P^{cs} + y + 2R$ . Now, we know that there is a leaf  $S$  of  $\tilde{\mathcal{S}}_\varepsilon$  which is contained in the half space bounded by  $P^{cs} + y + R$  not containing  $L$  (notice that  $L$  does not intersect  $P^{cs} + y + R$ ). Global product structure implies that  $\tilde{\mathcal{F}}^u(x)$  intersects  $S$  and thus, it also intersects  $L$ .

The converse direction has an analogous proof. □

We can prove the following result which does not make use of the isotopy class of  $f$ .

**Proposition 8.5.** *Assume that there is a global product structure between the lift of  $\mathcal{S}_\varepsilon$  and the lift of  $\mathcal{F}^u$  to the universal cover. Then there exists an  $f$ -invariant foliation  $\mathcal{F}^{cs}$  everywhere tangent to  $E^s \oplus E^u$ .*

PROOF. We will show that the branched foliation  $\tilde{\mathcal{F}}_{bran}^{cs}$  must be a true foliation. Using Proposition 4.13 this is reduced to showing that each point in  $\mathbb{R}^3$  belongs to a unique leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$ .

Assume otherwise, i.e. there exists  $x \in \mathbb{R}^3$  such that  $\tilde{\mathcal{F}}_{bran}^{cs}(x)$  has more than one complete surface. We call  $L_1$  and  $L_2$  different leaves in  $\tilde{\mathcal{F}}_{bran}^{cs}(x)$ . There exists  $y$  such that  $y \in L_1 \setminus L_2$ . Using global product structure and Lemma 8.4 we get  $z \in L_2$  such that:

$$- y \in \tilde{\mathcal{F}}^u(z).$$

Consider  $\gamma$  the arc in  $\tilde{\mathcal{F}}^u(z)$  whose endpoints are  $y$  and  $z$ . Let  $R$  be the value given by Proposition 8.2 and  $\ell > 0$  given by Corollary 8.3. We consider  $N$  large enough so that  $\tilde{f}^N(\gamma)$  has length larger than  $n\ell$  with  $n \gg R$ .

By Corollary 8.3 we get that the distance between  $P^{cs} + \tilde{f}^N(z)$  and  $\tilde{f}^N(y)$  is much larger than  $R$ . However, we have that, by  $\tilde{f}$ -invariance of  $\tilde{\mathcal{F}}_{bran}^{cs}$  there is a leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$  containing both  $\tilde{f}^N(z)$  and  $\tilde{f}^N(x)$  and another one containing both  $\tilde{f}^N(y)$  and  $\tilde{f}^N(x)$ . This contradicts Proposition 8.2 showing that  $\tilde{\mathcal{F}}_{bran}^{cs}$  must be a true foliation.  $\square$

### 8.3. Torus leaves.

8.3.1. This subsection is devoted to prove the following:

**Lemma 8.6.** *If  $\mathcal{F}_{bran}^{cs}$  contains a leaf which is a two-dimensional torus, then there is a leaf of  $\mathcal{F}_{bran}^{cs}$  which is a torus and it is fixed by  $f^k$  for some  $k$ . Moreover, this leaf is repelling.*

PROOF. Let  $T \subset \mathbb{T}^3$  be a leaf of  $\mathcal{F}^{cs}$  homeomorphic to a two-torus. Since  $\mathcal{F}^{cs}$  is  $f$ -invariant and  $P^{cs}$  is invariant under  $f_*$  we get that the image of  $T$  by  $f$  is homotopic to  $T$  and a leaf of  $\mathcal{F}_{bran}^{cs}$ .

Notice that having an  $f_*$ -invariant plane which projects into a torus already implies that  $f_*$ -cannot be hyperbolic (see Proposition 4.15).

We have that by Corollary 4.12 and Remark 4.19 that the plane  $P^{cs}$  coincides with  $E_*^s \oplus E_*^u$  (the eigenspaces corresponding to the eigenvalues of modulus different from 1 of  $f_*$ )

Since the eigenvalue of  $f_*$  in  $E_*^c$  is of modulus 1, this implies that if we consider two different lifts of  $T$ , then they remain at bounded distance when iterated by  $\tilde{f}$ . Indeed, if we consider two different lifts  $\tilde{T}_1$  and  $\tilde{T}_2$  of  $T$  we have that  $\tilde{T}_2 = \tilde{T}_1 + \gamma$  with  $\gamma \in E_*^c \cap \mathbb{Z}^3$ . Now, we have that  $\tilde{f}(\tilde{T}_2) = \tilde{f}(\tilde{T}_1) + f_*(\gamma) = \tilde{f}(\tilde{T}_1) \pm \gamma$ .

We shall separate the proof depending on how the orbit of  $T$  is.

**Case 1:** Assume the torus  $T$  is fixed by some iterate  $f^n$  of  $f$ . Then, since it is tangent to the center stable distribution, we obtain that it must be repelling as desired.

**Case 2:** If the orbit of  $T$  is dense, we get that  $\mathcal{F}_{bran}^{cs}$  is a true foliation by two-dimensional torus which we call  $\mathcal{F}^{cs}$  from now on. This is obtained by the fact that one can extend the foliation to the closure using the fact that there are no topological crossings between the torus leaves (see Proposition 4.4).

Since all leaves must be two-dimensional torus which are homotopic we get that the foliation  $\mathcal{F}^{cs}$  has no holonomy see Proposition 4.3 and Proposition 5.9).

Using Theorem 6.1, we get that the unstable direction  $\tilde{\mathcal{F}}^u$  in the universal cover must have a global product structure with  $\tilde{\mathcal{F}}^{cs}$ .

Let  $S$  be a leaf of  $\mathcal{F}^{cs}$  and consider  $\tilde{S}_1$  and  $\tilde{S}_2$  two different lifts of  $S$  to  $\mathbb{R}^3$ .

Consider an arc  $J$  of  $\tilde{\mathcal{F}}^u$  joining  $\tilde{S}_1$  to  $\tilde{S}_2$ . Iterating the arc  $J$  by  $\tilde{f}^n$  we get that its length grows exponentially, while the extremes remain the the forward iterates of  $\tilde{S}_1$  and  $\tilde{S}_2$  which remain at bounded distance by the argument above.

By considering translations of one end of  $\tilde{f}^n(J)$  to a fundamental domain and taking a convergent subsequence we obtain a leaf of  $\tilde{\mathcal{F}}^u$  which does not intersect every leaf of  $\tilde{\mathcal{F}}^{cs}$ . This contradicts global product structure.

**Case 3:** Let  $T_1, T_2 \in \mathcal{F}_{bran}^{cs}$  two different torus leaves. Since there are no topological crossings, we can regard  $T_2$  as embedded in  $\mathbb{T}^2 \times [-1, 1]$  where both boundary components are identified with  $T_1$  and such that the embedding is homotopic to the boundary components (recall that any pair of torus leaves must be homotopic). In particular, we get that  $\mathbb{T}^3 \setminus (T_1 \cup T_2)$  has at least two different connected components and each of the components has its boundary contained in  $T_1 \cup T_2$ .

If the orbit of  $T$  is not dense, we consider  $\mathcal{O} = \overline{\bigcup_n f^n(T)}$  the closure of the orbit of  $T$  which is an invariant set.

Recall that we can assume completeness of  $\mathcal{F}_{bran}^{cs}$  (i.e. for every  $x_n \rightarrow x$  and  $L_n \in \mathcal{F}_{bran}^{cs}(x_n)$  we have that  $L_n$  converges in the  $C^1$ -topology to  $L_\infty \in \mathcal{F}_{bran}^{cs}(x)$ ). We get that  $\mathcal{O}$  is saturated by leaves of  $\mathcal{F}_{bran}^{cs}$  all of which are homotopic torus leaves (see Proposition 5.9).

Let  $U$  be a connected component of the complement of  $\mathcal{O}$ . By the previous remarks we know that its boundary  $\partial U$  is contained in the union of two torus leaves of  $\mathcal{F}_{bran}^{cs}$ .

If some component  $U$  of  $\mathcal{O}^c$  verifies that there exists  $n \geq 1$  such that  $f^n(U) \cap U \neq \emptyset$ , by invariance of  $\mathcal{O}^c$  we get that  $f^{2n}$  fixes both torus leaves whose union contains  $\partial U$ . This implies the existence of a periodic normally repelling torus as in Case 1.

We claim that if every connected component of  $\mathcal{O}^c$  is wandering, then we can show that every leaf of  $\tilde{\mathcal{F}}^u$  intersects every leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$  which allows to conclude exactly as in Case 2.

To prove the claim, consider  $\delta$  given by the local product structure between these two transverse foliations (one of them branched). This means that given  $x, y$  such that  $d(x, y) < \delta$  we have that  $\tilde{\mathcal{F}}^u(x)$  intersects every leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$  passing through  $y$ .

Assume there is a point  $x \in \mathbb{R}^3$  such that  $\tilde{\mathcal{F}}^u(x)$  does not intersect every leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$ . We know that each leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$  separates  $\mathbb{R}^3$  into two connected components so we can choose among the lifts of torus leaves, the leaf  $\tilde{T}_0$  which is the lowest (or highest depending on the orientation of the semi-unstable leaf of  $x$  not intersecting every leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$ ) not intersecting  $\tilde{\mathcal{F}}^u(x)$ . We claim that  $\tilde{T}_0$  must project by the covering projection into a torus leaf which intersects the boundary of a connected component of  $\mathcal{O}^c$ . Indeed, there are only finitely many connected components  $U_1, \dots, U_N$  of  $\mathcal{O}^c$  having volume smaller than the volume of a  $\delta$ -ball, so if a point is not in  $U_i$  for some  $i$ , we know that it must be covered by local product structure boxes forcing its unstable leaf to advance until one of those components.

On the other hand, using  $f$ -invariance of  $\mathcal{F}^u$  and the fact that every connected component of  $\mathcal{O}^c$  is wandering, we get that every point in  $U_i$  must eventually fall out of  $\bigcup_i U_i$  and then its unstable manifold must advance to other component. This concludes the claim, and as we explained, allows to use the same argument as in Case 2 to finish the proof in Case 3. □

M.A. Rodriguez Hertz and R. Ures were kind to comunicate an alternative proof of this lemma by using an adaptation of an argument due to Haefliger for branched foliations (it should appear in [RHRHU<sub>2</sub>]).

## 8.4. Existence of a Global Product Structure.

8.4.1. In this section we will prove the following result which will allow us to conclude in the case where  $f_*$  is not isotopic to Anosov.

**Proposition 8.7.** *Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be a strongly partially hyperbolic diffeomorphism which is not isotopic to Anosov and does not have any two-dimensional torus tangent to  $E^s \oplus E^c$ . Then, the plane  $P^{cs}$  given by Proposition 8.2 corresponds to the eigenplane corresponding to the eigenvalues of modulus smaller or equal to 1. Moreover, there is a global product structure between  $\tilde{\mathcal{F}}_{bran}^{cs}$  and  $\tilde{\mathcal{F}}^u$ . A symmetric statement holds for  $\tilde{\mathcal{F}}_{bran}^{cu}$  and  $\tilde{\mathcal{F}}^s$ .*

8.4.2. Corollary 4.12 says that even if a strongly partially hyperbolic diffeomorphism is not isotopic to Anosov, then,  $f_*$  still must have one eigenvalue of modulus larger than one and one smaller than one.

Recall from subsection 4.6 that there are exactly three  $f_*$ -invariant lines  $E_*^s$ ,  $E_*^c$  and  $E_*^u$  corresponding to the eigenvalues of  $f_*$  of modulus smaller, equal and larger than one respectively.

Before we prove Proposition 8.7, we will prove the following:

**Lemma 8.8.** *For every  $R > 0$  and  $x \in \mathbb{R}^3$  we have that  $\tilde{\mathcal{F}}^u(x)$  is not contained in an  $R$ -neighborhood of  $(E_*^s \oplus E_*^c) + x$ . Symmetrically, for every  $R > 0$  and  $x \in \mathbb{R}^3$  the leaf  $\tilde{\mathcal{F}}^s(x)$  is not contained in an  $R$ -neighborhood of  $(E_*^c \oplus E_*^u) + x$ .*

PROOF. Let  $C$  be a connected set contained in an  $R$ -neighborhood of a translate of  $E_*^s \oplus E_*^c$ , we will estimate the diameter of  $\tilde{f}(C)$  in terms of the diameter of  $C$ .

**Claim.** *There exists  $K_R$  which depends only on  $\tilde{f}$ ,  $f_*$  and  $R$  such that:*

$$\text{diam}(\tilde{f}(C)) \leq \text{diam}(C) + K_R$$

PROOF. Let  $K_0$  be the  $C^0$ -distance between  $\tilde{f}$  and  $f_*$  and consider  $x, y \in C$  we get that:

$$\begin{aligned} d(\tilde{f}(x), \tilde{f}(y)) &\leq d(f_*(x), f_*(y)) + d(f_*(x), \tilde{f}(x)) + d(f_*(y), \tilde{f}(y)) \leq \\ &\leq d(f_*(x), f_*(y)) + 2K_0 \end{aligned}$$

We have that the difference between  $x$  and  $y$  in the unstable direction of  $f_*$  is bounded by  $2R$  given by the distance to the plane  $E_*^s \oplus E_*^u$  which is transverse to  $E_*^u$ .

Since the eigenvalues of  $f_*$  along  $E_*^s \oplus E_*^c$  we have that  $f_*$  does not increase distances in this direction: we thus have that  $d(f_*(x), f_*(y)) \leq d(x, y) + 2|\lambda^u|R$  where  $\lambda^u$  is the eigenvalue of modulus larger than 1. We have obtained:

$$d(\tilde{f}(x), \tilde{f}(y)) \leq d(x, y) + 2K_0 + 2|\lambda^u|R = d(x, y) + K_R$$

which concludes the proof of the claim. ◇

Now, this implies that if we consider an arc  $\gamma$  of  $\tilde{\mathcal{F}}^u$  of length 1 and assume that its future iterates remain in a slice parallel to  $E_*^s \oplus E_*^c$  of width  $2R$  we have that

$$\text{diam}(\tilde{f}^n(\gamma)) < \text{diam}(\gamma) + nK_R \leq 1 + nK_R$$

So that the diameter grows linearly with  $n$ .

The volume of balls in the universal cover of  $\mathbb{T}^3$  grows polynomially with the radius (see Step 2 of [BBI] or page 545 of [BI], notice that the universal) so that we have that  $B_\delta(\tilde{f}^{-n}(\gamma))$  has volume which is polynomial  $P(n)$  in  $n$ .

On the other hand, we know from the partial hyperbolicity that there exists  $C > 0$  and  $\lambda > 1$  such that the length of  $\tilde{f}^n(\gamma)$  is larger than  $C\lambda^n$ .

Using Corollary 4.7 (iv), we obtain that there exists  $n_0$  uniform such that every arc of length 1 verifies that  $\tilde{f}^{n_0}(\gamma)$  is not contained in the  $R$ -neighborhood of a translate of  $E_*^s \oplus E_*^c$ . This implies that no unstable leaf can be contained in the  $R$ -neighborhood of a translate of  $E_*^s \oplus E_*^c$  concluding the proof of the lemma. □

8.4.3. We are now ready to prove Proposition 8.7

PROOF OF PROPOSITION 8.7. Consider the plane  $P^{cs}$  given by Proposition 8.2 for the branching foliation  $\mathcal{F}_{bran}^{cs}$ .

If option (ii) of Proposition 8.2 holds, we get that there must be a torus leaf in  $\mathcal{F}_{bran}^{cs}$  which we assume there is not.

By Lemma 4.18 and Remark 4.19 the plane  $P^{cs}$  must be either  $E_*^s \oplus E_*^c$  or  $E_*^c \oplus E_*^u$ .

Lemma 8.8 implies that  $P^{cs}$  cannot be  $E_*^c \oplus E_*^u$  since  $\mathcal{F}^s$  is contained in  $\mathcal{F}_{bran}^{cs}$ . This implies that  $P^{cs} = E_*^s \oplus E_*^c$  as desired.

Now, using Lemma 8.8 for  $\tilde{\mathcal{F}}^u$  we see that the unstable foliation cannot remain close to a translate of  $P^{cs}$  and must intersect every leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$  obtaining the desired global product structure. □

8.5. **Proof of Theorem B.** To prove Theorem B, we first assume that  $f_*$  is not isotopic to Anosov.

Consider the branching foliation  $\mathcal{F}_{bran}^{cs}$  given by Theorem 4.9. We can apply Proposition 8.2 to  $\mathcal{F}_{bran}^{cs}$  and obtain a plane  $P^{cs}$  which is close to the lift of leaves of  $\mathcal{F}_{bran}^{cs}$ .

If the plane  $P^{cs}$  projects into a torus, there must be a two-dimensional torus as a leaf of  $\mathcal{F}_{bran}^{cs}$  then, by Lemma 8.6 we obtain a repelling periodic two-dimensional torus.

If  $P^{cs}$  is not a torus, then Proposition 8.7 applies giving a global product structure between the lift of the unstable foliation and the lift of  $\mathcal{F}_{bran}^{cs}$ .

By Proposition 8.5 we get that there exists an  $f$ -invariant foliation  $\mathcal{F}^{cs}$  tangent to  $E^s \oplus E^c$ .

The proof shows that there must be a unique  $f$ -invariant foliation tangent to  $E^{cs}$  (and to  $E^{cu}$ ).



Indeed, we get that every foliation tangent to  $E^{cs}$  must verify option (i) of Proposition 8.2 when lifted to the universal cover and that the plane which is close to the foliation must correspond to the eigenspace of  $f_*$  corresponding to the smallest eigenvalues (Proposition 8.7).

Using quasi-isometry of the strong foliations, this implies that if there is another surface tangent to  $E^{cs}$  through a point  $x$ , then this surface will not extend to an  $f$ -invariant foliation since we get that forward iterates will get arbitrarily far from this plane.

This concludes the proof of Theorem B in case  $f$  is not isotopic to Anosov, Theorem 8.1 concludes. □

It may be that there are other foliations tangent to  $E^{cs}$  (see [BF]) or, even if there are no such foliations there may be complete surfaces tangent to  $E^{cs}$  which do not extend to foliations. The techniques here presented do not seem to be enough to discard such situations.

#### APPENDIX A. A SIMPLER PROOF OF THEOREM 8.1. THE ISOTOPY CLASS OF ANOSOV.

In Section 7 we proved a general result which implies Theorem 8.1. We present here a simpler proof of this result which is independent of Section 7. We also prove in this appendix a result in the vein of Proposition 8.7 for the isotopy class of Anosov (the difference is in this case it is an a fortiori result while in the other case it is needed to get dynamical coherence), the proof of this last result is based on what is proved in Section 7.

**PROOF OF THEOREM 8.1.** Notice that if  $f_*$  is hyperbolic, then, every invariant plane must be totally irrational (see Remark 4.17), so that it projects into a plane in  $\mathbb{T}^3$ .

Let  $\mathcal{F}_{bran}^{cs}$  be the branched foliation tangent to  $E^{cs}$  given by Theorem 4.9. By Proposition 8.2 we get a  $f_*$ -invariant plane  $P^{cs}$  in  $\mathbb{R}^3$  which we know cannot project into a two-dimensional torus since  $f_*$  has no invariant planes projecting into a torus, this implies that option (i) of Proposition 8.2 is verified.

Since for every  $\varepsilon > 0$ , Theorem 4.9 gives us a foliation  $\mathcal{S}_\varepsilon$  whose lift is close to  $\tilde{\mathcal{F}}^{cs}$ , we get that the foliation  $\tilde{\mathcal{S}}_\varepsilon$  remains close to  $P^{cs}$  which must be totally irrational. By Lemma 5.8 (i) we get that all leaves of  $\mathcal{S}_\varepsilon$  are simply connected, thus, we get that the foliation  $\mathcal{S}_\varepsilon$  is without holonomy.

We can apply Theorem 6.1 and we obtain that for every  $\varepsilon > 0$  there is a global product structure between  $\tilde{\mathcal{S}}_\varepsilon$  and  $\tilde{\mathcal{F}}^u$  which is transverse to  $\mathcal{S}_\varepsilon$  if  $\varepsilon$  is small enough.

The rest of the proof follows from Proposition 8.5.

□

In fact, using the same argument as in Proposition 7.13 we get uniqueness of the foliation tangent to  $E^s \oplus E^c$ .

We are also able to prove the following proposition which is similar to Proposition 8.7 in the context of partially hyperbolic diffeomorphisms isotopic to Anosov, this will be used in [HP] to obtain leaf conjugacy to the linear model.

Notice first that the eigenvalues of  $f_*$  verify that they are all different (see Lemma 4.16 and Proposition 7.11).

We shall name them  $\lambda_1, \lambda_2, \lambda_3$  and assume they verify:

$$|\lambda_1| < |\lambda_2| < |\lambda_3| \quad ; \quad |\lambda_1| < 1, \quad |\lambda_2| \neq 1, \quad |\lambda_3| > 1$$

we shall denote as  $E_*^i$  to the eigenline of  $f_*$  corresponding to  $\lambda_i$ .

**Proposition A.1.** *The plane close to the branched foliation  $\tilde{\mathcal{F}}^{cs}$  corresponds to the eigenplane corresponding to the eigenvalues of smaller modulus (i.e. the eigenspace  $E_*^1 \oplus E_*^2$  corresponding to  $\lambda_1$  and  $\lambda_2$ ). Moreover, there is a global product structure between  $\tilde{\mathcal{F}}^{cs}$  and  $\tilde{\mathcal{F}}^u$ . A symmetric statement holds for  $\tilde{\mathcal{F}}^{cu}$  and  $\tilde{\mathcal{F}}^s$ .*

PROOF. This proposition follows from the existence of a semiconjugacy  $H$  between  $\tilde{f}$  and its linear part  $f_*$  which is at bounded distance from the identity.

The existence of a global product structure was proven above. Assume first that  $|\lambda_2| < 1$ , in this case, we know that  $\tilde{\mathcal{F}}^u$  is sent by the semiconjugacy into lines parallel to the eigenspace of  $\lambda_3$  for  $f_*$ . This readily implies that  $P^{cs}$  must coincide with the eigenspace of  $f_*$  corresponding to  $\lambda_1$  and  $\lambda_2$  otherwise we would contradict the global product structure.

The case were  $|\lambda_2| > 1$  is more difficult. First, it is not hard to show that the eigenspace corresponding to  $\lambda_1$  must be contained in  $P^{cs}$  (otherwise we can repeat the argument in Lemma 8.8 to reach a contradiction).

Assume by contradiction that  $P^{cs}$  is the eigenspace corresponding to  $\lambda_1$  and  $\lambda_3$ .

First, notice that by the basic properties of the semiconjugacy  $H$ , for every  $x \in \mathbb{R}^3$  we have that  $\tilde{\mathcal{F}}^u(x)$  is sent by  $H$  into  $E_*^u + H(x)$  (where  $E_*^u = E_*^2 \oplus E_*^3$  is the eigenspace corresponding to  $\lambda_2$  and  $\lambda_3$  of  $f_*$ ).

We claim that this implies that in fact  $H(\tilde{\mathcal{F}}^u(x)) = E_*^2 + H(x)$  for every  $x \in \mathbb{R}^3$ . In fact, we know from Corollary 7.9 that points of  $H(\tilde{\mathcal{F}}(x))$  which are sufficiently far apart are contained in a cone of  $(E_*^2 \oplus E_*^3) + H(x)$  bounded by two lines  $L_1$  and  $L_2$  which are transverse to  $P^{cs}$ . If  $P^{cs}$  contains  $E_*^3$  this implies that if one considers points in the same unstable leaf which are sufficiently far apart, then their image by  $H$  makes an angle with

$E_*^3$  which is uniformly bounded from below. If there is a point  $y \in \tilde{\mathcal{F}}^u(x)$  such that  $H(y)$  not contained in  $E_*^2$  then we have that  $d(\tilde{f}^n(y), \tilde{f}^n(x))$  goes to  $\infty$  with  $n$  while the angle of  $H(y) - H(x)$  with  $E_*^3$  converges to 0 exponentially contradicting Corollary 7.9.

Consider now a point  $x \in \mathbb{R}^3$  and let  $y$  be a point which can be joined to  $x$  by a finite set of segments  $\gamma_1, \dots, \gamma_k$  tangent either to  $E^s$  or to  $E^u$  (an  $su$ -path, see [DW]). We know that each  $\gamma_i$  verifies that  $H(\gamma_i)$  is contained either in a translate of  $E_*^1$  (when  $\gamma_i$  is tangent to  $E^s$ , i.e. it is an arc of the strong stable foliation  $\tilde{\mathcal{F}}^s$ ) or in a translate of  $E_*^2$  (when  $\gamma_i$  is tangent to  $E^u$  from what we have shown in the previous paragraph). This implies that the *accessibility* class of  $x$  (see [DW, BRHRHTU] for a definition and properties) verifies that its image by  $H$  is contained in  $(E_*^1 \oplus E_*^2) + H(x)$ . The projection of  $E_*^1 \oplus E_*^2$  to the torus is not the whole  $\mathbb{T}^3$  so in particular, we get that  $f$  cannot be accessible. From Corollary 7.14 this situation should be robust under  $C^1$ -perturbations since those perturbations cannot change the direction of  $P^{cs}$ .

On the other hand, in [DW, BRHRHTU] it is proved that by an arbitrarily small ( $C^1$  or  $C^r$ ) perturbation of  $f$  one can make it accessible. This gives a contradiction and concludes the proof. □

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