Handel's fixed point theorem revisited

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Abstract

Michael Handel proved in [7] the existence of a fixed point for an orientation preserving homeomorphism of the open unit disk that can be extended to the closed disk, provided that it has points whose orbits form an *oriented cycle of links at infinity*. Later, Patrice Le Calvez gave a different proof of this theorem based only on Brouwer theory and plane topology arguments [9]. These methods permitted to improve the result by proving the existence of a simple closed curve of index 1. We give a new, simpler proof of this improved version of the theorem and generalize it to non-oriented cycles of links at infinity.

1 Introduction

Handel's fixed point theorem [7] has been of great importance for the study of surface homeomorphisms. It guarantees the existence of a fixed point for an orientation preserving homeomorphism f of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ provided that it can be extended to the boundary $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and that it has points whose orbits form an oriented cycle of links at infinity. More precisely, there exist n points $z_i \in \mathbb{D}$ such that

$$\lim_{k\to-\infty} f^k(z_i) = \alpha_i \in S^1, \ \lim_{k\to+\infty} f^k(z_i) = \omega_i \in S^1,$$

$i = 1, \ldots, n$, where the $2n$ points $\{\alpha_i\}$, $\{\omega_i\}$ are different points in S^1 and satisfy the following order property:

(*) α_{i+1} is the only one among these points that lies in the open interval in the oriented circle S^1 from ω_{i-1} to ω_i .

(Although this is not Handel's original statement, it is an equivalent one as already pointed out in [9]).

Le Calvez gave an alternative proof of this theorem [9], relying only in Brouwer theory and plane topology, which allowed him to obtain a sharper result. Namely, he weakened the extension hypothesis by demanding the homeomorphism to be extended just to $\mathbb{D} \cup (\cup_{i\in\mathbb{Z}/n\mathbb{Z}}\{\alpha_i, \omega_i\})$ and he strengthed the conclusion by proving the existence of a simple closed curve of index 1.

We give a new, simpler proof of this improved version of the theorem and we generalize it to non-oriented cycles of links at infinity; that is, we relax the order property (*) as follows.

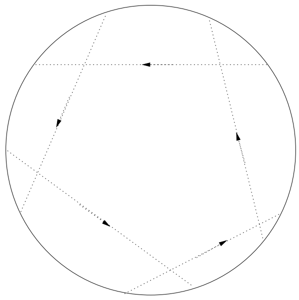
Let $P \subset \mathbb{D}$ be a compact convex n-gon. Let $\{v_i : i \in \mathbb{Z}/n\mathbb{Z}\}$ be its set of vertices and for each $i \in \mathbb{Z}/n\mathbb{Z}$, let e_i be the edge joining v_i and v_{i+1}. We suppose that each e_i is endowed with an orientation, so that we can tell whether P is to the right or to the left of e_i . We say that the orientations of e_i and e_j

coincide if P is to the right (or to the left) of both e_i and e_j, $i, j \in \mathbb{Z}/n\mathbb{Z}$.
We define the *index* of P by

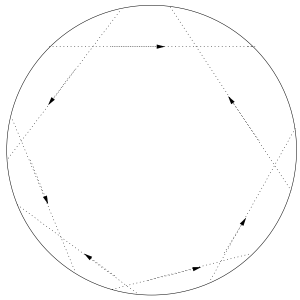
$$i(P) = 1 - \frac{1}{2} \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \delta_i,$$

where $\delta_i = 0$ if the orientations of e_{i-1} and e_i coincide, and $\delta_i = 1$ otherwise.

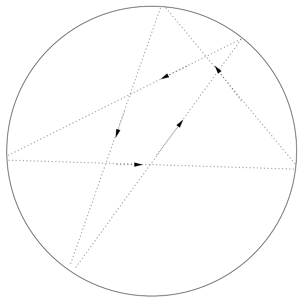
We will note α_i and ω_i the first, and respectively the last, point where the straight line Δ_i containing e_i and inheriting its orientation intersects $\partial\mathbb{D}$.



(a) Handel's index 1 polygon



(b) Index -1 polygon



(c) $\omega_i = \alpha_{i+2}\ \forall i$

Figure 1: The hypothesis of Theorem 1.1.

We say that a homeomorphism $f : \mathbb{D} \to \mathbb{D}$ *realizes* P if there exists a family $(z_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ of points in $\mathbb{D}$ such that for all $i \in \mathbb{Z}/n\mathbb{Z}$,

$$\lim_{k\to-\infty} f^k(z_i) = \alpha_i, \ \lim_{k\to+\infty} f^k(z_i) = \omega_i.$$

We will prove

Theorem 1.1. *Let $f : \mathbb{D} \to \mathbb{D}$ be an orientation preserving homeomorphism which realizes a compact convex polygon $P \subset \mathbb{D}$ where the points $\alpha_i, \omega_i, i \in \mathbb{Z}/n\mathbb{Z}$ are all different. Suppose that f can be extended to a homeomorphism of $\mathbb{D} \cup (\cup_{i\in\mathbb{Z}/n\mathbb{Z}}\{\alpha_i, \omega_i\})$.*
If $i(P) \neq 0$, then f has a fixed point. Furthermore, if $i(P) = 1$, then there exists a simple closed curve $C \subset \mathbb{D}$ of index 1 .

The two polygons appearing in Figure 1 (a) and (b) satisfy the hypothesis of this theorem. However, the polygon illustrated in (c) does not, as there are coincidences among the points $\{\alpha_i\}, \{\omega_i\}$, $i \in \mathbb{Z}/n\mathbb{Z}$.

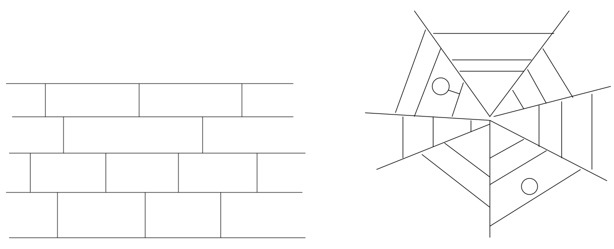
The structure of this article is the following. In Section 2 we will recall the notion of brick decompositions (the main tool of this article), and relate them to the existence of simple closed curves of index 1. We also state the results we use from [9] and give some proofs for the sake of completion. In Section 3 we use brick decompositions to define and study configurations of "repellers and attractors at infinity", with orbits connecting repeller/attractor pairs. We prove that the existence of configurations of this kind guarantees the existence of a fixed point, or even a simple closed curve of index 1. In Section 4 we prove Theorem 1.1: we show that whenever the hypothesis of the theorem are satisfied, either one can construct one of the configurations studied in Section 3, or there exists a simple closed curve of index 1.

I am endebted to Patrice Le Calvez. Not only he suggested me to study possible generalizations of Handel's theorem, but he guided my research through a great number of discussions.

2 Preliminaries

2.1 Brick decompositions

A *brick decomposition* $\mathcal{D}$ of an orientable surface M is a 1- dimensional singular submanifold $\Sigma(\mathcal{D})$ (the *skeleton* of the decomposition), with the property that the set of singularities V is discrete and such that every $\sigma \in V$ has a neighborhood U for which $U \cap (\Sigma(\mathcal{D}) \backslash V)$ has exactly three connected components. We have illustrated two brick decompositions in Figure 2. The *bricks* are the closure of the connected components of $M \backslash \Sigma(\mathcal{D})$ and the *edges* are the closure of the connected components of $\Sigma(\mathcal{D}) \backslash V$. We will write E for the set of edges, B for the set of bricks and finally $\mathcal{D} = (V, E, B)$ for a brick decomposition.



(a) $M = \mathbb{R}^2$ (b) $M = \mathbb{R}^2 \backslash \{0\}$

Figure 2: Brick decompositions

Let $\mathcal{D} = (V, E, B)$ be a brick decomposition of M. We say that $X \subset B$ is *connected* if given two bricks b, $b' \in X$, there exists a sequence $(b_i)_{0 \leq i \leq n}$, where $b_0 = b$, $b_n = b'$ and such that b_i and b_{i+1} have non-empty intersection, $i \in \{0, \ldots, n-1\}$. Whenever two bricks b and b' have non-empty intersection, we say that they are *adjacent*. Moreover, we say that a brick b is *adjacent to a subset* $X \subset B$ if $b \notin X$, but b is adjacent to one of the bricks in X. We say that $X \subset B$ is adjacent to $X' \subset B$ if X and X' have no common bricks but there exists $b \in X$ and $b' \in X'$ which are adjacent.

From now on we will identify a subset X of B with the closed subset of M formed by the union of the bricks in X. By making so, there may be ambiguities (for instance, two adjacent subsets of B have empty intersection in B and nonempty intersection in M), but we will point it out when this happens. We remark that ∂X is a one-dimensional topological manifold and that the connectedness of $X \subset B$ is equivalent to the connectedness of $X \subset M$ and to the connectedness of $\mathrm{Int}(X) \subset M$ as well. We say that the decomposition $\mathcal{D}'$ is a *subdecomposition* of $\mathcal{D}$ if $\Sigma(\mathcal{D}') \subset \Sigma(\mathcal{D})$.

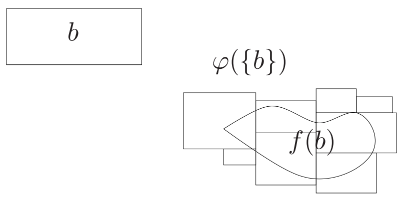
If $f : M \to M$ is a homeomorphism, we define the application $\varphi : \mathcal{P}(B) \to \mathcal{P}(B)$ as follows:

$$\varphi(X) = \{b \in B : f(X) \cap b \neq \emptyset\}.$$

We remark that $\varphi(X)$ is connected whenever X is.

We define analogously an application $\varphi_- : \mathcal{P}(B) \to \mathcal{P}(B)$:

$$\varphi_-(X) = \{b \in B : f^{-1}(X) \cap b \neq \emptyset\}.$$



We define the *future* $[b]_\geq$ and the *past* $[b]_\leq$ of a brick b as follows:

$$[b]_\geq = \bigcup_{k\geq 0} \varphi^k(\{b\}), \; [b]_\leq = \bigcup_{k\geq 0} \varphi_-^k(\{b\}).$$

We also define the *strict future* $[b]_>$ and the *strict past* $[b]_<$ of a brick b :

$$[b]_> = \bigcup_{k>0} \varphi^k(\{b\}), \; [b]_< = \bigcup_{k>0} \varphi_-^k(\{b\}).$$

We say that a set $X \subset B$ is an *attractor* if it verifies $\varphi(X) \subset X$; this is equivalent in M to the inclusion $f(X) \subset \mathrm{Int}(X)$. A *repeller* is any set which verifies $\varphi_-(X) \subset X$. In this way, the future of any brick is an attractor, and the past of any brick is a repeller. We observe that $X \subset B$ is a repeller if and only if $B \backslash X$ is an attractor.

Remark 2.1. The following properties can be deduced from the fact that $X \subset B$ is an attractor if and only if $f(X) \subset \mathrm{Int}(X)$:

1. If $X \subset B$ is an attractor and $b \in X$, then $[b]_{\geq} \subset X$; if $X \subset B$ is a repeller and $b \in X$, then $[b]_{\leq} \subset X$,

2. if $X \subset B$ is an attractor and $b \notin X$, then $[b]_{\leq} \cap X = \emptyset$; if $X \subset B$ is a repeller and $b \notin X$, then $[b]_{\geq} \cap X = \emptyset$,

3. if $b \in B$ is adjacent to the attractor $X \subset B$, then $[b]_{>} \cap X \neq \emptyset$; if $b \in B$ is adjacent to the repeller $X \subset B$, then $[b]_{<} \cap X \neq \emptyset$;

4. two attractors are disjoint as subsets of B if and only if they are disjoint as subsets of M; in other words, two disjoint (in B) attractors cannot be adjacent; respectively two disjoint (in B) repellers cannot be adjacent;

The following conditions are equivalent:

$$b \in [b]_{>},\ [b]_{>} = [b]_{\geq},\ b \in [b]_{<},\ [b]_{<} = [b]_{\leq},\ [b]_{<} \cap [b]_{\geq} \neq \emptyset,\ [b]_{\leq} \cap [b]_{>} \neq \emptyset.$$

The existence of a brick $b \in B$ for which any of these conditions is satisfied is equivalent to the existence of a *closed chain of bricks* , i.e a family $(b_i)_{i \in \mathbb{Z}/r\mathbb{Z}}$ of bricks such that for all $i \in \mathbb{Z}/r\mathbb{Z}$, $\cup_{k \geq 1} f^k(b_i) \cap b_{i+1} \neq \emptyset$.

In general, a *chain* for $f \in \text{Homeo}(M)$ is a family $(X_i)_{0 \leq i \leq r}$ of subsets of M such that for all $0 \leq i \leq r-1$, $\cup_{k \geq 1} f^k(X_i) \cap X_{i+1} \neq \emptyset$. We say that the chain is closed if $X_r = X_0$.

We say that a subset $X \subset M$ is *free* if $f(X) \cap X = \emptyset$.

We say that a brick decomposition $\mathcal{D} = (V, E, B)$ is *free* if every $b \in B$ is a free subset of M. If f is fixed point free it is always possible, taking sufficiently small bricks, to construct a free brick decomposition.

We recall the definition of *maximal free decomposition*, which was introduced by Sauzet in his doctoral thesis [11]. Let f be a fixed point free homeomorphism of a surface M. We say that $\mathcal{D}$ is a maximal free decomposition if $\mathcal{D}$ is free and any strict subdecomposition is no longer free. As a consequence of Zorn's lemma, one obtains:

Lemma 2.2. *If $\mathcal{D}$ is a free brick decomposition of M, then there exists a subdecomposition $\mathcal{D}'$ of $\mathcal{D}$ which is free and maximal.*

2.2 Brouwer Theory background.

We say that $\Gamma : [0,1] \to \overline{\mathbb{D}}$ is an *arc*, if it is continuous and injective. We say that an arc Γ joins $x \in \overline{\mathbb{D}}$ to $y \in \overline{\mathbb{D}}$, if $\Gamma(0) = x$ and $\Gamma(1) = y$. We say that an arc Γ joins $X \subset \overline{\mathbb{D}}$ to $Y \subset \overline{\mathbb{D}}$, if Γ joins $x \in X$ to $y \in Y$.

Fix $f \in \text{Homeo}^+(\mathbb{D})$. An arc γ joining $z \notin \text{Fix}(f)$ to $f(z)$ such that $f(\gamma) \cap \gamma = \{z, f(z)\}$ if $f^2(z) = z$ and $f(\gamma) \cap \gamma = \{f(z)\}$ otherwise, is called a *translation arc*.

Proposition 2.3. (Brouwer's translation lemma [1], [2], [4] or [6]) *If any of the two following hypothesis is satisfied, then there exists a simple closed curve of index 1:*

1. *there exists a translation arc γ joining $z \in \text{Fix}(f^2) \setminus \text{Fix}(f)$ to $f(z)$;*

2. *there exists a translation arc γ joining $z \notin \mathrm{Fix}(f^2)$ to $f(z)$ and an integer $k \geq 2$ such that $f^k(\gamma) \cap \gamma \neq \emptyset$.*

If $z \notin \mathrm{Fix}(f)$, there exists a translation arc containing z; this is easy to prove once one has that the connected components of the complement of $\mathrm{Fix}(f)$ are invariant. For a proof of this last fact, see [3] for a general proof in any dimension, or [8] for an easy proof in dimension 2.

We deduce:

Corollary 2.4. *If $\mathrm{Per}(f) \setminus \mathrm{Fix}(f) \neq \emptyset$, then there exists a simple closed curve of index 1.*

Proposition 2.5. (Franks' lemma [5]) *If there exists a closed chain of free, open and pairwise disjoint disks for f, then there exists a simple closed curve of index 1.*

Following Le Calvez [9], we will say that f is *recurrent* if there exists a closed chain of free, open and pairwise disjoint disks for f.

The following proposition is a refinement of Franks' lemma due to Guillou and Le Roux (see [10], page 39).

Proposition 2.6. *Suppose there exists a closed chain $(X_i)_{i \in \mathbb{Z}/r\mathbb{Z}}$ for f of free subsets whose interiors are pairwise disjoint and which verify the following property: given any two points $z, z' \in X_i$ there exists an arc γ joining z and z' such that $\gamma \setminus \{z, z'\} \subset \mathrm{Int}(X_i)$. Then, f is recurrent.*

We deduce:

Proposition 2.7. *Let $\mathcal{D} = (V, E, B)$ be a free brick decomposition of $\mathbb{D} \setminus \mathrm{Fix}(f)$. If there exists $b \in B$ such that $b \in [b]_>$, then f is recurrent.*

2.3 Previous results.

Fix $f \in \mathrm{Homeo}^+(\mathbb{D})$, different from the identity map and *non-recurrent*. We will make use of the following two propositions from [9] (both of them depend on the non-recurrent character of f). The first one (Proposition 2.2 in [9]) is a refinement of a result already appearing in [11]; the second one is Proposition 3.1 in [9].

Proposition 2.8 ([11],[9])**.** *Let $\mathcal{D} = (V, E, B)$ be a free maximal brick decomposition of $\mathbb{D} \setminus \mathrm{Fix}(f)$. Then, the sets $[b]_\geq$, $[b]_>$, $[b]_\leq$ and $[b]_<$ are connected. In particular every connected component of an attractor is an attractor, and every connected component of a repeller is a repeller.*

Proposition 2.9. *[9] If f satisfies the hypothesis of Theorem 1.1, then for all $i \in \mathbb{Z}/n\mathbb{Z}$ we can find a sequence of arcs $(\gamma_i^k)_{k \in \mathbb{Z}}$ such that:*

- *each γ_i^k is a translation arc from $f^k(z_i)$ to $f^{k+1}(z_i)$,*
- *$f(\gamma_i^k) \cap \gamma_i^{k'} = \emptyset$ if $k' < k$,*
- *the sequence $(\gamma_i^k)_{k \leq 0}$ converges to $\{\alpha_i\}$ in the Hausdorff topology,*
- *the sequence $(\gamma_i^k)_{k \geq 0}$ converges to $\{\omega_i\}$ in the Hausdorff topology.*

This last result is a consequence of Brouwer's translation lemma and the hypothesis on the orbits of the points $(z_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$. In particular, the extension hypothesis of Theorem 1.1 is used. Given the centrality of this proposition to our paper, we include a proof due to Le Calvez (already contained in [9]) in what follows.

Fix $f \in \mathrm{Homeo}^+(\mathbb{D})$ satisfying the hypothesis of Theorem 1.1. Let O_i be the orbit of z_i, and $z_i^k = f^k(z_i)$, $k \in \mathbb{Z}$. We will need the three following lemmas, where we will omit the index i for simplicity.

Lemma 2.10. *There exists a sequence of pairwise disjoint arcs $(\gamma'^k)_{k\in\mathbb{Z}}$ such that:*

- $z^k \in \gamma'^k$;
- $\gamma'^k \cap \mathrm{Fix}(f) = \emptyset$:
- $f(\gamma'^k) \cap \gamma'^k \neq \emptyset$;
- *the sequence* $(\gamma'^k)_{k\leq 0}$ *converges to* $\{\alpha\}$;
- *the sequence* $(\gamma'^k)_{k\geq 0}$ *converges to* $\{\omega\}$.

Proof. We can construct a homeomorphism $h : \mathbb{D} \to (-1,1)^2$ such that:

- $\lim_l p_1(x_l) = -1 \Leftrightarrow \lim_l h^{-1}(x_l) = \alpha$;
- $\lim_l p_1(x_l) = 1 \Leftrightarrow \lim_l h^{-1}(x_l) = \omega$, where p_1, p_2 are the projections to the horizontal and vertical coordinates;
- p_1 is injective on $h(O)$, where O is the orbit of z;
- the sequence $p_2(h(z^k))_{k\in\mathbb{Z}}$ is increasing.

Indeed, it is easy to construct a homeomorphism $h' : \mathbb{D} \to (-1,1)^2$ satisfying the two first items. As

$$\lim_{k\to-\infty} z^k = \{\alpha\}, \lim_{k\to\infty} z^k = \{\omega\},$$

for any k there is only a finite number of points in $h'(O)$ in the same vertical line that $h'(z^k)$, and so we can perturb h' in a homeomorphism h'' satisfying the three first items. Once one has injectivity of p_1 on $h''(O)$, we can compose h'' with a homeomorphism fixing each vertical line to have p_2 increasing on $h''(O)$.

For simplicity, we will no longer write h; we will suppose that p_1 and p_2 are defined in $\mathbb{D}$. Let I_k be the open interval of $(-1,1)$ delimited by $p_1(z^k)$ and $p_1(z^{k+1})$, and $U^k = I_k \times (p_2(z^k), p_2(z^{k+1}))$. Then, $(\overline{U^k})_{k\leq 0}$ and $(\overline{U^k})_{k\leq 0}$ are sequences of closed disks in $\mathbb{D}$ converging respectively to $\{\alpha\}$ and $\{\omega\}$.

We can pick a point $z'^0 \in U^0$ such that $f(z'^0)$ does not belong to the same vertical line that any of the points in O. We can also pick a point $z'^1 \in U^1$ such that $f(z'^1)$ does not belong to the same vertical line that any of the points in O, and such that $f(z'^1)$ does not belong either to the same vertical line as z'^0, or the image of this vertical line or its preimage. We can define inductively a sequence $(z'^k)_{k\in\mathbb{Z}}$ such that:

- $z'^k \in U^k$;
- $p_1(f(z'^k)) \neq p_1(z^{k'})$ for all $k, k' \in \mathbb{Z}$;
- $p_1(z'^k) \neq p_1(z'^{k'})$ if $k \neq k'$;
- $p_1(f(z'^k)) \neq p_1(z'^{k'})$ if $k \neq k'$.

So, we can modify h (by composition with a homeomorphism fixing each vertical line, as we did before) so as to have $p_2(z^k) < p_2(f(z'^k)) < p_2(z^{k+1})$.

The arguments that follows depends on the extension hypothesis of Theorem 1.1. As f extends to a homeomorphism of $\mathbb{D} \cup \{\alpha, \omega\}$, and the sequences $(z'^k)_{k \leq 0}$ and $(z'^k)_{k \geq 0}$ converge respectively to $\{\alpha\}$, and $\{\omega\}$, we obtain that the sequences $(f(z'^k))_{k \leq 0}$ and $(f(z'^k))_{k \geq 0}$ also converge to $\{\alpha\}$, and $\{\omega\}$ respectively. It follows that one can construct a sequence $(I'^k)_{k \in \mathbb{Z}}$ of open intervals of $(-1, 1)$ such that:

- $I^k \subset I'^k$;
- $U'^k = I'^k \times (p_2(z^k), p_2(z^{k+1}))$ contains z'^k and $f(z'^k)$;
- the sequences of closed disks $(\overline{U'^k})_{k \leq 0}$ and $(\overline{U'^k})_{k \leq 0}$ converge respectively to $\{\alpha\}$ and $\{\omega\}$.

We will construct our arcs γ'^k to be contained in $U'^k \cup \{z^k\}$. So, these arcs will be pairwise disjoint and the sequences $(\gamma'^k)_{k \leq 0}$ and $(\gamma'^k)_{k \geq 0}$ will converge respectively to $\{\alpha\}$ and $\{\omega\}$.

If there is only a finite number of fixed points in U'^k, we can suppose that z'^k is not fixed and take an arc $\gamma'^k \subset U'^k \cup \{z^k\}$ disjoint from $\mathrm{Fix}(f)$, with an endpoint in z^k, and containing both z'^k and $f(z'^k)$.

If there are infinitely many fixed points in U'^k, we can construct three arcs contained in $U'^k \cup \{z^k\}$, each one of them with an endpoint in z^k and the other one in a fixed point, such that these arcs meet only in z^k. We can also suppose that the only fixed point of these arcs is their other endpoint. If one of these arcs meets its image outside its fixed extremity, we can find a subarc γ'^k disjoint from the fixed point set and meeting its image as we want. Otherwise, as f is orientation preserving, necessarily the union of two of these three segments must meet its image outside the fixed point set. If we delete a neighbourhood of the fixed extremity for both ot these arcs, we obtain our arc γ'^k.

□

By thickening the arcs given by the preceeding lemma, and then taking the "smallest" disk which is no longer free, we obtain:

Lemma 2.11. *There exists a sequence of pairwise disjoint closed disks $(D'^k)_{k \in \mathbb{Z}}$ such that:*

- $z^k \in \partial D'^k$*;*
- $D'^k \cap \mathrm{Fix}(f) = \emptyset$*:*
- $f(D'^k) \cap D'^k \neq \emptyset$*;*
- $f(\mathrm{Int}((D'^k))) \cap \mathrm{Int}(D'^k) = \emptyset$*;*

- *the sequence* $(D'^k)_{k\leq 0}$ *converges to* $\{\alpha\}$*;*
- *the sequence* $(D'^k)_{k\geq 0}$ *converges to* $\{\omega\}$*.*

This last lemma allows us to construct the desired translation arcs.

Lemma 2.12. *Suppose that f is not recurrent. Then, there exists a sequence of pairwise disjoint closed disks $(D^k)_{k\in\mathbb{Z}}$ such that:*

- $z^k \in \mathrm{Int}(D^k)$*;*
- $D^k \cap \mathrm{Fix}(f) = \emptyset$*:*
- $f(D^k) \cap D^k \neq \emptyset$*;*
- $f(D^k) \cap D^{k'} = f^2(D^k) \cap D^{k'} = \emptyset$ *if* $k' < k$*;*
- *the sequence* $(D^k)_{k\leq 0}$ *converges to* $\{\alpha\}$*;*
- *the sequence* $(D^k)_{k\geq 0}$ *converges to* $\{\omega\}$*.*

Proof. Let $(D'^k)_{k\in\mathbb{Z}}$ be the sequence of pairwise disjoint closed disks given by Lemma 2.11. If γ is an arc joining z^k and a point $z' \in \partial D'^k$ which is contained in $\mathrm{Int}(D'^k)$ except for its endpoints, then γ is free. Indeed as $\mathrm{Int}(D'^k)$ is free, $f(\gamma) \cap \gamma \neq \emptyset$ implies either $z^k \in f(\gamma) \cap \gamma$ or $z'^k \in f(\gamma) \cap \gamma$. The first case is impossible because z^{k-1}, the preimage of z^k, is contained in D'^{k-1} which is disjoint from D'^k. The second case implies (as $D'^k \cap \mathrm{Fix}(f) \neq \emptyset$) that $z'^k = f(z^k)$, which is also imposible as z^{k+1} is contained in D'^{k+1} which is disjoint from D'^k.

Take a point $x_k \in \partial D'^k \cap f^{-1}(\partial D'^k)$, and two arcs γ_-^k, γ_+^k contained in $\mathrm{Int}(D'^k)$ except for its endpoints, the former joining x^k and z^k, and the latter joining z^k and $f(x^k)$, and such that $\gamma_-^k \cap \gamma_+^k = \{z_k\}$. If $k' < k$, then the positive orbit of $\gamma_-^{k'}$ and $\gamma_+^{k'}$ meets γ_-^k and γ_+^k. As these arcs are all free, and we are supposing that f is not recurrent, we obtain that the positive orbit of γ_-^k and γ_+^k never meets $\gamma_-^{k'}$ or $\gamma_+^{k'}$. Besides, as

$$\lim_{k\to-\infty} \gamma_-^k\gamma_+^k = \{\alpha\}, \lim_{k\to-\infty} \gamma_-^k\gamma_+^k = \{\alpha\},$$

we can find a closed disk D^0 neighbourhood of $\gamma_-^0\gamma_+^0$ such that:

- $D^0 \cap \mathrm{Fix}(f) = \emptyset$;
- $D^0 \cap \gamma_-^k\gamma_+^k = f(D^0) \cap \gamma_-^k\gamma_+^k = f^2(D^0) \cap \gamma_-^k\gamma_+^k = \emptyset$, if $k < 0$;
- $D^0 \cap \gamma_-^k\gamma_+^k = f^{-1}(D^0) \cap \gamma_-^k\gamma_+^k = f^{-2}(D^0) \cap \gamma_-^k\gamma_+^k = \emptyset$, if $k > 0$.

We obtain:

- $z^0 \in \mathrm{Int}(D^0)$;
- $f(D^0 \cap D^0) \neq \emptyset$.

Now we can choose a closed disk D^1 neighbourhood of $\gamma_-^1\gamma_+^1$ such that:

- $D^1 \cap \mathrm{Fix}(f) = \emptyset$;

- $D^1 \cap \gamma_-^k \gamma_+^k = f(D^1) \cap \gamma_-^k \gamma_+^k = f^2(D^1) \cap \gamma_-^k \gamma_+^k = \emptyset$, if $k < 1$;
- $D^1 \cap D^0 = f(D^1) \cap D^0 = f^2(D^1) \cap D^0 = \emptyset$;
- $D^1 \cap \gamma_-^k \gamma_+^k = f^{-1}(D^1) \cap \gamma_-^k \gamma_+^k = f^{-2}(D^0) \cap \gamma_-^k \gamma_+^k = \emptyset$, if $k > 1$.

So,

- $z^1 \in \mathrm{Int}(D^1)$;
- $f(D^1 \cap D^1) \neq \emptyset$.

We proceed inductively to construct our sequence $(D^k)_{k \in \mathbb{Z}}$.

□

Now we are ready to prove Proposition 2.9:

Proof. Suppose that f is non-recurrent, and take a sequence of closed disks $(D^k)_{k \in \mathbb{Z}}$ as in the preceding lemma. By taking a smaller disk if necessary, we can suppose that the interior of each D^k is free. Take a point $x_k \in \partial D'^k \cap f^{-1}(\partial D'^k)$, and two arcs γ_-^k, γ_+^k contained in $\mathrm{Int}(D'^k)$ except for one endpoint, the former joining x^k and z^k, and the latter joining z^k and $f(x^k)$, and such that $\gamma_-^k \cap \gamma_+^k = \{z_k\}$. Then, $\gamma_-^k \gamma_+^k$ is a translation arc. As f is not recurrent, $\gamma^k = \gamma_-^k \gamma_+^k$ is a translation arc as well. Besides, γ^k joins z^k and z^{k+1}. The other required properties of γ^k are verified because:

- $\gamma^k \subset D^k \cup f(D^k)$ and f extends to a homeomorphism of $\mathbb{D} \cup \{\alpha, \omega\}$;
- $D^{k'} \cup f(D^{k'})$ is disjoint from $f(D^k) \cup f^2(D^k)$ if $k' < k$.

□

Proposition 2.9 allows us to construct a particular brick decomposition suitable for our purposes:

Lemma 2.13. *Let f satisfy the hypothesis of Theorem 1.1. For every $i \in \mathbb{Z}/n\mathbb{Z}$, take U_i^- a neighbourhood of α_i in $\overline{\mathbb{D}}$ and U_i^+ a neighbourhood of ω_i in $\overline{\mathbb{D}}$ such that the sets $U_i^-, U_i^+, i \in \mathbb{Z}/n\mathbb{Z}$ are pairwise disjoint. There exists two families $(b_i'^l)_{i \in \mathbb{Z}/n\mathbb{Z}, l \geq 1}$ and $(b_i'^l)_{i \in \mathbb{Z}/n\mathbb{Z}, l \leq -1}$ of closed disks in $\mathbb{D}$, and $K > 0$ such that:*

1. *each $b_i'^l$ is free and contained in U_i^- $(l \leq -1)$ or in U_i^+ $(l \geq 1)$,*
2. $\mathrm{Int}(b_i'^l) \cap \mathrm{Int}(b_i'^{l'}) = \emptyset$, *if $l \neq l'$,*
3. *for all $i \in \mathbb{Z}/n\mathbb{Z}$ $f^{K+l}(z_i) \in \mathrm{Int}(b_i'^{l+1})$ for all $l \geq 0$, and $f^{-K-l}(z_i) \in \mathrm{Int}(b_i'^{-l-1})$ for all $l \geq 0$,*
4. *$f^k(z_j) \in b_i'^l$ if and only if $j = i$ and $k = K + l - 1$,*
5. *for every $k > 1$ the sets $(b_i'^l)_{1 \leq l \leq k}$ and $(b_i'^l)_{-k \leq l \leq -1}$ are connected,*
6. *for all $i \in \mathbb{Z}/n\mathbb{Z}$, $\partial \cup_{l \in \mathbb{Z} \setminus \{0\}} b_i'^l$ is a one dimensional submanifold,*
7. *if $x \in \mathbb{D}$, then x belongs to at most two different disks in the family $(b_i'^l)_{l \in \mathbb{Z} \setminus \{0\}}$, $i \in \mathbb{Z}/n\mathbb{Z}$,*

8. *the sequence* $(b_i'^l)_{l\geq 1}$ *converges to* $\{\omega_i\}$ *in the Hausdorff topology and the sequence* $(b_i'^l)_{l\leq -1}$ *converges to* $\{\alpha_i\}$ *in the Hausdorff topology.*

We have illustrated these families in Figure 3. We remark that this consequence of Proposition 2.9 is also contained in [9].

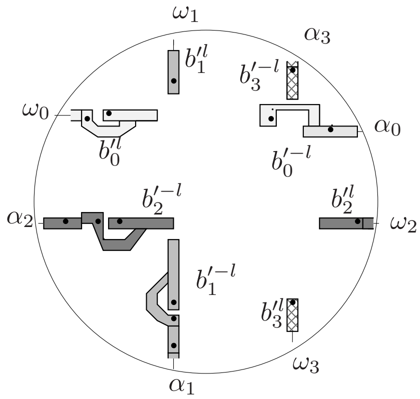


Figure 3: The families $b_i'^l$

Proof. We first note that we may supppose that $\gamma_i^k \cap O_i = \{z_i^k, z_i^{k+1}\}$ and $\gamma_i^k \cap O_{i'} = \emptyset$ if $i' \neq i$. Indeed, if $\gamma' \subset \gamma_i^k \backslash \{z_i^k, z_i^{k+1}\}$ is an arc, there exists a neighbourhood U of γ' such that any arc joining z_i^k and z_i^{k+1} contained in $\gamma_i^k \cup U$ is a translation arc: just note that

$$f(\gamma') \cap \gamma' = f(\gamma') \cap \gamma_i^k = \gamma' \cap f(\gamma_i^k) = \emptyset.$$

We take $K > 0$ large enough such that for all $i \in \mathbb{Z}/n\mathbb{Z}$ and $k \geq K$, $\gamma_i^k \subset U_i^+$ and $\gamma_i^{-k} \subset U_i^-$. We define arcs $\beta_i^l, l \in \mathbb{Z}\backslash\{0\}$, by deleting the loops of the curves $\prod_{k\leq -K} \gamma_i^k$ and $\prod_{k\geq K} \gamma_i^k$ respectively. More precisely: Let $\beta_i^1 = \gamma_i^K$ and define inductively $\beta_i^{l+1} = \gamma_i^{K+l}([t_i^l, 1])$, $l \geq 1$, where t_i^l is the last point where γ_i^{K+l} intersects $\cup_{j=1}^l \beta_i^j$.

We can now thicken these arcs $(\beta_i^l)_{l\geq 1}$ to obtain closed disks $(b_i'^l)_{l\geq 1}$ in such a way that they satisfy all the conditions of the lemma. We proceed analogously to obtain the family of closed disks $(b_i'^l)_{l\leq -1}$. ☐

Remark 2.14. The fact that the sequence $(b_i'^l)_{l\geq 1}$ converges in the Hausdorff topology to ω_i, implies that we can find an arc $\Gamma_i^+ : [0,1] \to \mathrm{Int}(\cup_{l\geq 0} b_i'^l) \cup \{\omega_i\}$ such that $\Gamma_i^+(1) = \omega_i$, $i \in \mathbb{Z}/n\mathbb{Z}$. Similarly, we can find an arc $\Gamma_i^- : [0,1] \to \mathrm{Int}(\cup_{l\geq 0} b_i'^{-l}) \cup \{\alpha_i\}$ such that $\Gamma_i^-(1) = \alpha_i$, $i \in \mathbb{Z}/n\mathbb{Z}$.

Corollary 2.15. *There exists a free brick decomposition* (V, E, B) *of* $\mathbb{D}\backslash \mathrm{Fix}(f)$ *such that for all* $i \in \mathbb{Z}/n\mathbb{Z}$ *and all* $l \in \mathbb{Z}\backslash\{0\}$*, there exists* $b_i^l \in B$ *such that* $b_i'^l \subset b_i^l$*. Moreover, one can suppose that this decomposition is maximal (see Lemma 2.2).*

We will make use of proposition 2.8 in the next section. Propositions 2.9 and Lemma 2.13 will not be used until section 4.

3 Repeller/Attractor configurations at infinity

3.1 Cyclic order at infinity.

Let $(a_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ be a family of non-empty, pairwise disjoint, closed, connected subsets of $\mathbb{D}$, such that $\overline{a}_i \cap \partial\mathbb{D} \neq \emptyset$ and $U = \mathbb{D}\backslash(\cup_{i\in\mathbb{Z}/n\mathbb{Z}} a_i)$ is a connected open set. As U is connected, and its complementary set in $\mathbb{C}$

$$\{z \in \mathbb{C} : |z| \geq 1\} \cup \cup_{i\in\mathbb{Z}/n\mathbb{Z}} a_i$$

is connected, U is simply connected.

With these hypotheses, there is a natural cyclic order on the sets $\{a_i\}$. Indeed, U is conformally isomorphic to the unit disc via the Riemann map $\varphi : U \to \mathbb{D}$, and one can consider the Carathéodory's extension of φ,

$$\hat{\varphi} : \hat{U} \to \overline{\mathbb{D}},$$

which is a homeomorphism between the prime ends completion $\hat{U}$ of U and the closed unit disk $\overline{\mathbb{D}}$. The set $\hat{J}_i$ of prime ends whose impression is contained in a_i is open and connected. It follows that the images $J_i = \hat{\varphi}(\hat{J}_i)$ are pairwise disjoint open intervals in S^1, and are therefore cyclically ordered following the positive orientation in the circle.

3.2 Repeller/Attractor configurations.

We fix $f \in \mathrm{Homeo}^+(\mathbb{D})$ together with a free maximal decomposition in bricks $\mathcal{D}= (V, E, B)$ of $\mathbb{D}\backslash \mathrm{Fix}(f)$.

Let $(R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ and $(A_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ be two families of connected, pairwise disjoint subsets of B such that :

1. For all $i \in \mathbb{Z}/n\mathbb{Z}$:

   (a) R_i is a repeller and A_i is an attractor;

   (b) there exists non-empty, closed, connected subsets of $\mathbb{D}$, $r_i \subset \mathrm{Int}(R_i)$, $a_i \subset \mathrm{Int}(A_i)$ such that $\overline{r_i} \cap \partial\mathbb{D} \neq \emptyset$ and $\overline{a_i} \cap \partial\mathbb{D} \neq \emptyset$,

2. $\mathbb{D}\backslash(\cup_{i\in\mathbb{Z}/n\mathbb{Z}}(a_i \cup r_i))$ is a connected open set.

We say that the pair $((R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/n\mathbb{Z}})$ is a *Repeller/Attractor configuration of order* n .
We will note

$$\mathcal{E} = \{R_i, A_i : i \in \mathbb{Z}/n\mathbb{Z}\}.$$

Property 2 in the previous definition allows us to give a cyclic order to the sets $r_i, a_i, i \in \mathbb{Z}/n\mathbb{Z}$ (see the beginning of this section).

We say that a Repeller/Attractor configuration of order $n \geq 3$ is an *elliptic configuration* if :

1. the cyclic order of the sets $r_i, a_i,\ i \in \mathbb{Z}/n\mathbb{Z}$, satisfies the *elliptic order property*:

$$a_0 \to r_2 \to a_1 \to \ldots \to a_i \to r_{i+2} \to a_{i+1} \to \ldots \to a_{n-1} \to r_1 \to a_0.$$

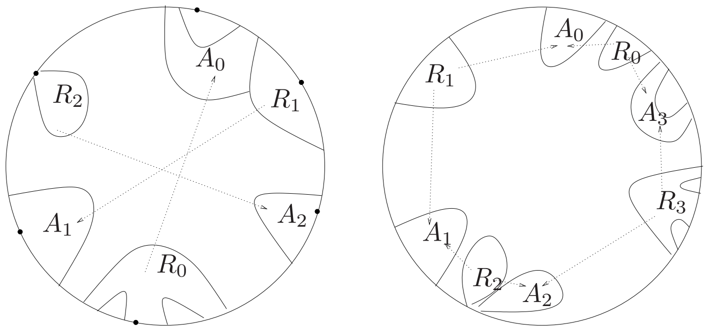
2. for all $i \in \mathbb{Z}/n\mathbb{Z}$ there exists a brick $b_i \in R_i$ such that $[b_i]_\geq \cap A_i \neq \emptyset$;

We say that a Repeller/Attractor configuration is a *hyperbolic configuration* if:

1. the cyclic order of the sets r_i, a_i, $i \in \mathbb{Z}/n\mathbb{Z}$, satisfies the *hyperbolic order property*:

   $r_0 \to a_0 \to r_1 \to a_1 \to \ldots \to r_i \to a_i \to r_{i+1} \to a_{i+1} \to \ldots \to r_{n-1} \to a_{n-1} \to r_0.$

2. for all $i \in \mathbb{Z}/n\mathbb{Z}$ there exists two bricks $b_i^i, b_i^{i-1} \in R_i$ such that $[b_i^i]_> \cap A_i \neq \emptyset$, and $[b_i^{i-1}]_> \cap A_{i-1} \neq \emptyset$;



(a) An elliptic configuration (b) A hyperbolic configuration

We will show:

Proposition 3.1. *If there exists an elliptic configuration of order $n \geq 3$, then f is recurrent.*

Proposition 3.2. *If there exists a hyperbolic configuration of order $n \geq 2$, then* $\mathrm{Fix}(f) \neq \emptyset$.

One could think that Proposition 3.2 should give a negative-index fixed point, as the example that comes to mind is that of a saddle point (see Figure 4 below)

However, this is not the case, as the following example shows.

Example 1. Let f_1 be the time-one map of the flow whose orbits are drawn in Figure 5:

One can perturb f_1 in a homeomorphism f such that:

1. $\mathrm{Fix}(f) = \mathrm{Fix}(f_1) = \{x\}$,
2. $f = f_1$ in a neighbourhood of x,
3. $f = f_1$ in a neighbourhood of S^1 (and so f preserves the repellers and attractors drawn in dotted lines),
4. there is an f-orbit from R_0 to A_1,
5. there is an f-orbit from R_1 to A_0.

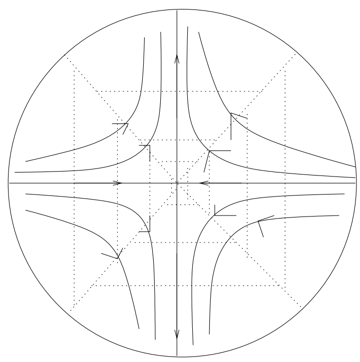


Figure 4: A hyperbolic configuration arising from a saddle point.

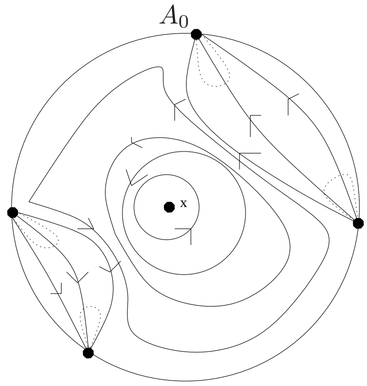


Figure 5: A hyperbolic configuration without a fixed point of negative index.

So, $((R_i)_{i\in\mathbb{Z}/2\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/2\mathbb{Z}})$ is a hyperbolic configuration for f, but the only fixed point f has is an index-one fixed point.

We define an order relationship in the set of Repeller/Attractor configurations of order n :

$$((R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/n\mathbb{Z}}) \leq ((R'_i)_{i\in\mathbb{Z}/n\mathbb{Z}}, (A'_i)_{i\in\mathbb{Z}/n\mathbb{Z}})$$

if and only if for all $i \in \mathbb{Z}/n\mathbb{Z}$

$$A_i \subseteq A'_i \text{ and } R_i \subseteq R'_i.$$

As the union of attractors (resp. repellers) is an attractor (resp. repeller), the existence of an elliptic (resp. hyperbolic) Repeller/Attractor configuration implies the existence of a maximal elliptic (resp.hyperbolic) Repeller/Attractor configuration by Zorn's lemma.

Example 2. The hyperbolic configuration in Figure 4 is maximal.

We will assume for the rest of this section that f is non-recurrent. In particular, for any brick $b \in B$, the sets $[b]_{\geq}$, $[b]_{>}$, $[b]_{\leq}$ and $[b]_{<}$ are connected (see Proposition 2.8).

The following lemma is an immediate consequence of the maximality of configurations:

Lemma 3.3. *Let $((R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/n\mathbb{Z}})$ be a maximal configuration (either elliptic or hyperbolic), and consider a brick $b \in B\setminus \cup_{i\in\mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$. If b is adjacent to R_i, then there exists, $j \neq i$, such that $[b]_< \cap R_j \neq \emptyset$ in B. If b is adjacent to A_i, then there exists, $j \neq i$, such that $[b]_> \cap A_j \neq \emptyset$ in B.*

Proof. Let $b \in B\setminus \cup_{i\in\mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$ be adjacent to R_i. As both R_i and $[b]_{\leq}$ are connected and they intersect, it follows that the repeller $R = [b]_{\leq} \cup R_i$ is connected. As our configuration is maximal and $R_i \subsetneq R$, there exists $X \in \mathcal{E}\setminus\{R_i\}$, such that $R \cap X \neq \emptyset$ (in B). As the sets in $\mathcal{E}$ are pairwise disjoint, and b does not belong to X, this implies that $[b]_< \cap X \neq \emptyset$ (in B). So, $X = R_j$ for some $j \neq i$, because $[b]_{\leq}$ cannot intersect any attractor (see Remark 2.1, item 2). The second statement in the lemma is proved analogously.

□

We say that a brick $b \in B$ is a *connexion brick* from R_j to A_j if:

1. $b \in B\setminus \cup_{i\in\mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$,
2. b is adjacent to R_j and
3. $[b]_>$ contains a brick $b' \in B\setminus \cup_{i\in\mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$ which is adjacent to A_j.

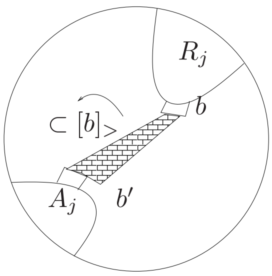


Figure 6: A connexion brick.

Lemma 3.4. *Let $((R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/n\mathbb{Z}})$ be a maximal elliptic or hyperbolic configuration. The following two conditions guarantee the existence of a connexion brick from R_i to A_i:*

1. *There exists a brick $b \notin \cup_{i\in\mathbb{Z}/n\mathbb{Z}}(R_i \cup A_i)$ which is adjacent to both R_i and A_i,*

2. *R_i is not adjacent to A_i.*

Proof. 1. Let $b' \notin \cup_{i\in\mathbb{Z}/n\mathbb{Z}}(R_i \cup A_i)$ be adjacent to both R_i and A_i. As a subset of B, the repeller $[b']_<$ meets a repeller R_j different from R_i (Lemma 3.3), meets R_i because b' is adjacent to R_i (Remark 2.1, item 3), and does not meet any A_j, $j \in \mathbb{Z}/n\mathbb{Z}$ (Remark 2.1, item 2). As it is connected, $[b']_<$ contains a brick b which is adjacent to R_i, which implies that $b \notin \cup_{i\in\mathbb{Z}/n\mathbb{Z}}(R_i \cup A_i)$ (Remark 2.1, item 4). As $b' \in [b]_>$, and b' is adjacent to A_i, b is a connexion brick from R_i to A_i.

2. Assume that R_i is not adjacent to A_i. We know there exists $b_i \in R_i$ such that $[b_i]_\geq \cap A_i \neq \emptyset$. As $[b_i]_\geq$ is connected, it contains a brick b' adjacent to A_i. This brick b' is not contained in R_i; otherwise, R_i would be adjacent to A_i. Neither it is contained in any attractor or in any repeller other that R_i (Remark 2.1, items 2 and 4). Therefore, $b' \notin \cup_{i\in\mathbb{Z}/n\mathbb{Z}}(R_i \cup A_i)$.

As $b_i \in [b']_\leq$ and $[b']_\leq$ is connected, $[b']_\leq$ contains a brick b adjacent to R_i. If $b \in [b']_<$, then b is a connexion brick from R_i to A_i (again, $b \notin \cup_{i\in\mathbb{Z}/n\mathbb{Z}}(R_i \cup A_i)$ by Remark 2.1, items 2 and 4). If $b = b'$, then b is adjacent to both R_i and A_i and we are done by the previous item.

□

Remark 3.5. Connexion bricks do not always exist; Figure 4 exhibits an example. Of course, none of the conditions of Lemma 3.4 is satisfied. Indeed, in this example $\cup_{i\in\mathbb{Z}/2\mathbb{Z}}(R_i \cup A_i) = B$ and R_i is adjacent to A_i for all $i \in \mathbb{Z}/2\mathbb{Z}$.

3.3 The elliptic case.

The following consequences of the elliptic order property will be used in the proof of Proposition 3.1:

Lemma 3.6. *Let $((R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/n\mathbb{Z}})$ be an elliptic configuration.*

1. *If $C \subset B$ is a connected set containing both R_i and A_i, and $C \cap (R_{i+1} \cup A_{i+1}) = \emptyset$ in B, then R_{i+1} and A_{i+1} belong to different connected components of $\mathbb{D}\setminus \mathrm{Int}(C)$; in particular $R_{i+1} \cap A_{i+1} = \emptyset$ in $\mathbb{D}$.*

2. *If $C \subset B$ is a connected set containing both R_i and R_{i+1}, and $C \cap (R_{i-1} \cup A_{i-1}) = \emptyset$ in B, then R_{i-1} and A_{i-1} belong to different connected components of $\mathbb{D}\setminus \mathrm{Int}(C)$; in particular $R_{i-1} \cap A_{i-1} = \emptyset$ in $\mathbb{D}$.*

3. *If $C \subset B$ is a connected set containing every repeller R_i, and disjoint (in B) from every attractor A_i, then the n attractors $\{A_i\}$ belong to n different connected components of $\mathbb{D}\setminus \mathrm{Int}(C)$.*

Proof. 1. First we remark that $C \cap (R_{i+1} \cup A_{i+1}) = \emptyset$ in B implies $\mathrm{Int}(R_{i+1}) \cap \mathrm{Int}(C) = \emptyset$ and $\mathrm{Int}(A_{i+1}) \cap \mathrm{Int}(C) = \emptyset$. Besides, $\mathrm{Int}(C)$ is a connected set containing both r_i and a_i. So, the elliptic order property implies that r_{i+1} and a_{i+1} belong to different connected components of $\mathbb{D}\setminus \mathrm{Int}(C)$. Now, $\mathrm{Int}(R_{i+1})$ and $\mathrm{Int}(A_{i+1})$ belong to different connected components of $\mathbb{D}\setminus \mathrm{Int}(C)$. As each connected component of $\mathbb{D}\setminus \mathrm{Int}(C)$ is closed (in $\mathbb{D}$), we obtain that R_{i+1} and A_{i+1} belong to different connected components of $\mathbb{D}\setminus \mathrm{Int}(C)$; in particular $R_{i+1} \cap A_{i+1} = \emptyset$ in $\mathbb{D}$.

2. As before, we know that $\text{Int}(R_{i-1}) \cap \text{Int}(C) = \emptyset$ and $\text{Int}(A_{i-1}) \cap \text{Int}(C) = \emptyset$. Besides, $\text{Int}(C)$ is a connected set containing both r_i and r_{i+1}. So, the elliptic order property implies that r_{i-1} and a_{i-1} belong to different connected components of $\mathbb{D} \setminus \text{Int}(C)$. It follows that $\text{Int}(R_{i-1})$ and $\text{Int}(A_{i-1})$ belong to different connected components of $\mathbb{D} \setminus \text{Int}(C)$, and we conclude as in the preceding item.

3. As before, we know that $\text{Int}(A_i) \cap \text{Int}(C) = \emptyset$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. Furthermore, $\text{Int}(C)$ is a connected set containing r_i for all $i \in \mathbb{Z}/n\mathbb{Z}$. So, the elliptic order property implies that each a_i, $i \in \mathbb{Z}/n\mathbb{Z}$ belong to a different connected component of $\mathbb{D} \setminus \text{Int}(C)$. It follows that each $\text{Int}(A_i)$, $i \in \mathbb{Z}/n\mathbb{Z}$, belong to a different connected component of $\mathbb{D} \setminus \text{Int}(C)$, and we conclude as in the preceding item.

□

Lemma 3.7. *Let $((R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/n\mathbb{Z}})$ be a maximal elliptic configuration. Then, for some $i \in \mathbb{Z}/n\mathbb{Z}$ there exists a connexion brick from R_i to A_i.*

Proof. Because of lemma 3.4, it is enough to show that for some $i \in \mathbb{Z}/n\mathbb{Z}$, R_i is not adjacent to A_i.

If R_i is adjacent to A_i, then $C = R_i \cup A_i$ is a connected set containing R_i and A_i. Besides, $C \cap (R_{i+1} \cup A_{i+1}) = \emptyset$ in B, because the sets in $\mathcal{E}$ are pairwise disjoint. So, item 1 of the preceeding lemma tells us that $R_{i+1} \cap A_{i+1} = \emptyset$ in $\mathbb{D}$. In particular, R_{i+1} cannot be adjacent to A_{i+1}. □

The following lemma tells us that it is enough to prove Proposition 3.1 for configurations of order $n = 3$:

Lemma 3.8. *Let $((R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/n\mathbb{Z}})$ be an elliptic configuration of order $n > 3$. Then, there exists an elliptic configuration $((R'_i)_{i\in\mathbb{Z}/(n-1)\mathbb{Z}}, (A'_i)_{i\in\mathbb{Z}/(n-1)\mathbb{Z}})$ of order $n-1$.*

Proof. We claim that there exists a brick $b \in R_0$ such that $[b]_\geq \cap A_1 \neq \emptyset$. Indeed, we recall that for all $i \in \mathbb{Z}/n\mathbb{Z}$ there exists $b_i \in R_i$ such that $[b_i]_\geq \cap A_i \neq \emptyset$, and that the future of any brick is connected. We have,

$$(R_0 \cup [b_0]_\geq \cup A_0) \cap R_1 = \emptyset \text{ in } B,$$

by Remark 2.1, item 2. So, Lemma 3.6, item 1 implies that either

$$(R_0 \cup [b_0]_\geq \cup A_0) \cap A_1 \neq \emptyset \text{ in } \ B,$$

or $\text{Int}(R_0 \cup [b_0]_\geq \cup A_0)$ separates R_1 from A_1. In the first case, necessarily

$$[b_0]_\geq \cap A_1 \neq \emptyset \text{ in } \ B,$$

and we take $b = b_0$. In the second case, we obtain

$$(R_0 \cup [b_0^-]_\geq \cup A_0) \cap (R_1 \cup [b_1^+]_\leq \cup A_1) \neq \emptyset \text{ in } \ B,$$

where $b_1^+ \in [b_1]_\geq \cap A_1$. By Remark 2.1, item 2, we know that $[b_0]_\geq \cap R_1 = \emptyset$ and $[b_1^+]_\leq \cap A_0 = \emptyset$. So, in fact

$$(R_0 \cup [b_0]_\geq) \cap ([b_1^+]_\leq \cup A_1) \neq \emptyset \text{ in } \ B.$$

If $R_0 \cap [b_1^+]_\leq \neq \emptyset$ in B, we take any brick $b \in R_0 \cap [b_1^+]_\leq$; if $[b_0]_\geq \cap ([b_1^+]_\leq \cup A_1) \neq \emptyset$ in B, we take $b = b_0$. (Note that $b \in [b_1^+]_\leq$ implies $b_1^+ \in [b]_\geq \cap A_1$). This finishes the proof of our claim.

Now, by defining

$$R'_0 = R_0,\ R'_i = R_{i+1} \text{ for } 1 \leq i \leq n-2,$$

$$A'_i = A_{i+1} \text{ for } 0 \leq i \leq n-2,$$

we are done.

□

We are now ready to prove Proposition 3.1 :

Proof. Because of the previous lemma, we can suppose that there exists an elliptic configuration of order $n = 3$ and take a maximal one

$$((R_i)_{i \in \mathbb{Z}/3\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/3\mathbb{Z}}).$$

We will show that our assumption that f is not recurrent contradicts the maximality of this configuration. Lemma 3.7 allows us to consider a connexion brick b from R_i to A_i, for some $i \in \mathbb{Z}/3\mathbb{Z}$, and there is no loss of generality in supposing $i = 0$. Let $b' \in B \setminus \cup_{i \in \mathbb{Z}/3\mathbb{Z}} (R_i \cup A_i)$ be adjacent to A_0 and such that $b' \in [b]_>$. We will first show that $[b]_<$ meets every repeller and no attractor in the configuration. Then, by defining A'_i as to be the connected component of $B \setminus (\cup_{i \in \mathbb{Z}/3\mathbb{Z}} R_i \cup [b]_<)$ containing A_i, we will be able to show that $((R_i)_{i \in \mathbb{Z}/3\mathbb{Z}}, (A'_i)_{i \in \mathbb{Z}/3\mathbb{Z}})$ is an elliptic configuration strictly bigger than the initial configuration, due to the fact that $b' \in A'_0 \setminus A_0$.

Indeed, we know by Lemma 3.3 that $[b]_\leq \cap R_j \neq \emptyset$ for some $j \in \{1, 2\}$. We will suppose $[b]_\leq \cap R_1 \neq \emptyset$; the proof is analogous in the other case. We claim that this implies $[b]_\leq \cap R_2 \neq \emptyset$. To see this, note that item 2 of Lemma 3.6 implies

$$R \cap (R_2 \cup [b_2]_\geq \cup A_2) \neq \emptyset,$$

where

$$R = R_0 \cup [b]_\leq \cup R_1.$$

So, actually

$$[b]_\leq \cap (R_2 \cup [b_2]_\geq) \neq \emptyset,$$

which implies $[b]_\leq \cap R_2 \neq \emptyset$.

We have obtained that $R' = \cup_{i \in \mathbb{Z}/3\mathbb{Z}} R_i \cup [b]_\leq$ is a connected repeller disjoint (in B) from every attractor A_i, $i \in \mathbb{Z}/3\mathbb{Z}$ (Remark 2.1, item 2). Let A'_j be the connected component of $B \setminus R'$ containing A_j for all $j \in \mathbb{Z}/3\mathbb{Z}$. Then, the sets A'_j $j \in \mathbb{Z}/3\mathbb{Z}$ are pairwise disjoint (in $\mathbb{D}$) by the elliptic order property. We know that $b' \in B \setminus R'$; otherwise, we would have $b' \in [b]_\leq$ as $b' \notin \cup_{i \in \mathbb{Z}/3\mathbb{Z}} (R_i \cup A_i)$, which is impossible because $b' \in [b]_>$ and we are supposing that f is non-recurrent. So, A_0 is strictly contained in A'_0 and we deduce that $((R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A'_i)_{i \in \mathbb{Z}/3\mathbb{Z}})$ is an elliptic configuration strictly greater than $((R_i)_{i \in \mathbb{Z}/3\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/3\mathbb{Z}})$, contradicting the maximality of the configuration.

□

3.4 The hyperbolic case.

In what follows, we deal with the hyperbolic case. The proof of the following lemma is analogous to that of Lemma 3.6, substituting of course the elliptic order property by the hyperbolic order property.

Lemma 3.9. *Let* $((R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/n\mathbb{Z}})$ *be a hyperbolic configuration. If* $C \subset B$ *is a connected set containing* R_i *and* R_{i+1}*, and* $C \cap A_m = \emptyset$ *in* B *for all* $m \in \mathbb{Z}/n\mathbb{Z}$*, then* $\mathrm{Int}(C)$ *separates (in* $\mathbb{D}$*)* A_i *from any* A_j*,* $j \neq i$*.*

Lemma 3.10. *Let* $((R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/n\mathbb{Z}})$ *be a hyperbolic configuration. If* $X \in \mathcal{E}$*, then there is only one connected component of* $B\backslash X$ *containing sets in* $\mathcal{E}$*.*

Proof. We will suppose that $X = R_j, j \in \mathbb{Z}/n\mathbb{Z}$; the proof is analogous for any $X \in \mathcal{E}$. We will show that the connected component C of $B\backslash R_j$ containing A_j contains every $X \in \mathcal{E}, X \neq R_j$. As $B\backslash R_j$ is an attractor, and there is a brick in R_{j+1} whose (connected) future intersects A_j, we have that $R_{j+1} \subset C$ (we recall that every connected component of an attractor is an attractor, see Proposition 2.8). As there is also a brick in R_{j+1} whose future intersects A_{j+1}, the same argument shows that $A_{j+1} \in C$. By induction, we get that every $X \in \mathcal{E}\backslash\{R_j\}$ belongs to C. □

Lemma 3.11. *Let* $((R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/n\mathbb{Z}})$ *be a maximal hyperbolic configuration. One of the following is true:*

1. $\mathrm{Fix}(f) \neq \emptyset$,
2. *there exists a connexion brick from* R_j *to* A_j *for some* $j \in \mathbb{Z}/n\mathbb{Z}$*.*

Proof. We will show that if $\mathrm{Fix}(f) = \emptyset$, then there exists a connexion brick from R_j to A_j for some $j \in \mathbb{Z}/n\mathbb{Z}$. By Lemma 3.4, we can suppose that R_i is adjacent to A_i for all $i \in \mathbb{Z}/n\mathbb{Z}$. If R_i is adjacent to A_i, either there is one connected component γ of ∂R_i which is also a connected component of ∂A_i or there is a point $x \in R_i \cap A_i \cap \partial(R_i \cup A_i)$. If $\mathrm{Fix}(f) = \emptyset$, then every connected component of ∂X is an embedded line in $\mathbb{D}$, for any $X \in \mathcal{E}$. So, if there were one connected component γ of ∂R_i which is also a connected component of ∂A_i, γ would separate $\mathbb{D}$ into two connected components C_1 and C_2, containing $\mathrm{Int}(A_i)$ and $\mathrm{Int}(R_i)$ respectively. Then, Lemma 3.10 would imply that every set in $\mathcal{E}\backslash R_i$ belongs to C_1, and that every set in $\mathcal{E}\backslash A_i$ belongs to C_2, which is clearly impossible.

We are left with the case where there is a point $x \in R_i \cap A_i \cap \partial(R_i \cup A_i)$. This point x is necessarily a vertex of $\Sigma(\mathcal{D})$. It belongs to three bricks: one that belongs to R_i, another one which belongs to A_i, and a third one which is adjacent to both R_i and A_i. This third brick does not belong to any repeller or attractor, as it is adjacent to both R_i and A_i (see Remark 2.1, item 4). So, by Lemma 3.4, item 1, there exists a connexion brick from R_i to A_i.

□

We will prove Proposition 3.2 by induction on the order of the configuration. We begin by the case $n = 2$:

Proposition 3.12. *If there exists a hyperbolic configuration of order* 2*, then* $\mathrm{Fix}(f) \neq \emptyset$*.*

Proof. Suppose there exists such a configuration and take a maximal one

$$((R_i)_{i\in\mathbb{Z}/2\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/2\mathbb{Z}}).$$

Because of Lemma 3.11, we can suppose that there exists a connexion brick b from R_j to A_j for some $j \in \mathbb{Z}/2\mathbb{Z}$, and there is no loss of generality in supposing $j = 0$. We take a brick b' such that $b' \in [b]_>$, $b' \in B\backslash \cup_{i\in\mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$ and b' is adjacent to A_0. Here again, we will first show that $[b]_<$, the strict past of b, meets every repeller and no attractor in the configuration. Then, by defining A'_i as the connected component of $B\backslash(\cup_{i\in\mathbb{Z}/2\mathbb{Z}} R_i \cup [b]_<)$ containing A_i, we will be able to show that $((R_i)_{i\in\mathbb{Z}/2\mathbb{Z}}, (A'_i)_{i\in\mathbb{Z}/2\mathbb{Z}})$ is a hyperbolic configuration strictly greater than the original one, due to the fact that $b' \in A'_0\backslash A_0$.

Because of Lemma 3.3 we know that $[b]_< \cap R_1 \neq \emptyset$ in B. So,

$$R = R_0 \cup b_\leq \cup R_1$$

is connected and disjoint from every attractor in the configuration (see Remark 2.1, item 2). It follows that $\mathrm{Int}(R)$ separates A_0 from A_1, this being the content of Lemma 3.9. Let A'_i be the connected component of $B\backslash R$ containing A_i, $i \in \mathbb{Z}/2\mathbb{Z}$. Then, $A'_0 \cap A'_1 = \emptyset$. We know that $b' \notin R$, because $b' \in [b]_>$, and otherwise f would be recurrent. So, b' belongs to $A'_0\backslash A_0$, contradicting the maximality of $((R_i)_{i\in\mathbb{Z}/2\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/2\mathbb{Z}})$.

□

Now we are ready to prove Proposition 3.2:

Proof. We will show that given a maximal hyperbolic configuration of order $n > 2$

$$((R_i)_{i\in\mathbb{Z}/n\mathbb{Z}}, (A_i)_{i\in\mathbb{Z}/n\mathbb{Z}}),$$

we can construct a new hyperbolic configuration whose order is strictly smaller than n (and yet greater or equal to 2). We can suppose there exists a connexion brick b from R_0 to A_0. We take a brick $b' \in [b]_>$ such that $b' \in B\backslash \cup_{i\in\mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$ and b' is adjacent to A_0. By Lemma 3.3,

$$[b]_\leq \cap R_i \neq \emptyset \text{ for some } \ i \neq 0.$$

We can suppose that $i \neq 1$; otherwise, we could use the same argument we used for the case $n = 2$. Indeed, Lemma 3.9 would imply that $R_0 \cup R_1 \cup [b]_\leq$ is a connected repeller which separates A_0 from any other A_j, $j \neq 0$. So, by replacing A_0 by A'_0, the connected component of $B\backslash(R_0 \cup R_1 \cup [b]_\leq)$ containing A_0, we would have a hyperbolic configuration strictly bigger than the original one.

So, we may suppose that

$$i = \min\{j \in \{1, \dots, n-1\} : [b]_\leq \cap R_j \neq \emptyset\} \neq 1.$$

We define

$$R = R_0 \cup [b]_\leq \cup R_i,$$

which is a connected repeller.

If we set $R'_0 = R$, $R'_j = R_j$ for all $1 \leq j \leq i-1$, and $A'_j = A_j$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, $0 \leq j \leq i-1$. Then, $((R'_j)_{j\in\mathbb{Z}/i\mathbb{Z}}, (A'_j)_{j\in\mathbb{Z}/i\mathbb{Z}})$ is a hyperbolic configuration of order i, $2 \leq i < n$.

□

4 Proof of the Theorem

In this section we prove Theorem 1.1. We fix an orientation preserving homeomorphism $f : \mathbb{D} \to \mathbb{D}$ which realizes a compact convex polygon $P \subset \mathbb{D}$, and can be extended to a homeomorphism of $\mathbb{D} \cup (\cup_{i\in\mathbb{Z}/n\mathbb{Z}}\{\alpha_i, \omega_i\})$. We suppose that $i(P) \neq 0$, and we will show that either f is recurrent, or we can construct an elliptic or hyperbolic Repeller/Attractor configuration. We recall that recurrence of f implies the existence of a simple closed curve of index one by Proposition 2.5

Some polygons can be simplified, due to the fact that they may have "extra" edges. More precisely, we will say that the polygon P is minimal if for every $i \in \mathbb{Z}/n\mathbb{Z}$, the lines $\{\Delta_j : j \neq i\}$ do not bound a compact convex polygon. The following lemma tells us that it is enough to deal with minimal polygons.

Lemma 4.1. *The map f realizes a minimal polygon P' such that $i(P') = i(P)$, or a triangle T such that $i(T) = 1$.*

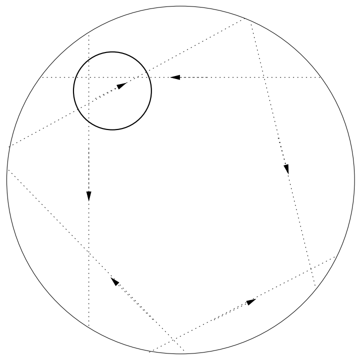


Figure 7: A non-minimal hexagon of index -2 presenting an index 1 subtriangle.

Proof. If P is not minimal, then there exists $i \in \mathbb{Z}/n\mathbb{Z}$ such that the straight lines $\{\Delta_j : j \neq i\}$ bound a compact polygon $P' \subset \mathbb{D}$. The line Δ_i intersects in $\mathbb{D}$ both Δ_{i-1} and Δ_{i+1}; it follows that necessarily

$$\Delta_{i-1} \cap \Delta_{i+1} \cap \mathbb{D} \neq \emptyset.$$

So, the lines Δ_{i-1}, Δ_i and Δ_{i+1} bound a triangle $T \subset \mathbb{D}$. Moreover,

$$i(P') = i(P) + i(T),$$

and the only possibilities for the index of a triangle are 0 or 1.

If $i(T) = 1$, we are done. Otherwise, $i(P') = i(P)$. If P' is minimal, we are done. If not, we apply the same procedure as before. We continue like this until we obtain an index 1 triangle, or a minimal polygon with the same index as P. □

Let us state our first proposition:

Proposition 4.2. *If $i(P) = 1$, then f is recurrent.*

We observe that lemma 4.1 allows us to suppose that P is minimal; we will also suppose that the boundary of P is positively oriented. With these assumptions, the order of the points $\alpha_i, \omega_i, i \in \mathbb{Z}/n\mathbb{Z}$ at the circle at infinity satisfies:

$\omega_0 \to \alpha_2 \to \omega_1 \to \ldots \to \omega_i \to \alpha_{i+2} \to \omega_{i+1} \to \ldots \to \omega_{n-1} \to \alpha_1 \to \omega_0.$

From now on, we suppose that f is not recurrent. We apply Lemma 2.13 and obtain a family of closed disks $(b_i'^l)_{l\in\mathbb{Z}\setminus\{0\}, i\in\mathbb{Z}/n\mathbb{Z}}$.

Remark 4.3. The sets $\Gamma_i^- \cap \mathbb{D}$, $\Gamma_i^+ \cap \mathbb{D}$ defined in Remark 2.14 satisfy the elliptic order property.

By Corollary 2.15, we can construct a maximal free brick decomposition (V, E, B) such that for all $i \in \mathbb{Z}/n\mathbb{Z}$ and for all $l \in \mathbb{Z}\setminus\{0\}$, there exists $b_i^l \in B$ such that $b_i'^l \subset b_i^l$.

Remark 4.4. As $\cup_{l>0}[b_i^l]_\leq$ is a connected set whose closure contains both α_i and ω_i, if $\Gamma : [0,1] \to \overline{\mathbb{D}}$ is an arc that separates α_i from ω_i, then $\Gamma \cap \cup_{l>0}[b_i^l]_\leq \neq \emptyset$.

Lemma 4.5. *If for some $k > 0$, $m > 0$ and $j \in \mathbb{Z}/n\mathbb{Z}$, both b_j^k and b_{j+1}^k are contained in $[b_i^{-m}]_>$, then there exists $l > 0$ such that $b_{j+2}^l \in [b_i^{-m}]_>$.*

Proof. If b_j^k and b_{j+1}^k are contained in $[b_i^{-m}]_>$, then b_j^p and b_{j+1}^p are contained in $[b_i^{-m}]_>$ for all $p \geq k$ (note that $[b_i^{-m}]_>$ is an attractor, and that Lemma 2.13, item 6. implies that $b_j^p \subset [b_j^k]_\geq$ for all $p \geq k$). So, as $[b_i^{-m}]_>$ is connected, we can find an arc

$$\Gamma : [0,1] \to [b_i^{-m}]_> \cup \{\omega_j, \omega_{j+1}\}$$

joining ω_j and ω_{j+1} (see Remark 2.14). Then, Γ separates α_{j+2} from ω_{j+2} in $\overline{\mathbb{D}}$. By Remark 4.4, we obtain

$$\Gamma \cap (\cup_{l>0}[b_{j+2}]_<^l) \neq \emptyset.$$

So,

$$[b_i^{-m}]_> \cap (\cup_{l>0}[b_{j+2}]_<^l) \neq \emptyset,$$

from which one gets (as the future of any brick is an attractor) that there exists $l > 0$ such that $b_{j+2}^l \in [b_i^{-m}]_>$.

□

Lemma 4.6. (Domino effect) *There exists $k > 0$ such that for all $i, j \in \mathbb{Z}/n\mathbb{Z}$, $[b_i^{-k}]_>$ contains b_j^k.*

Proof. Fix $i \in \mathbb{Z}/n\mathbb{Z}$. There exists an arc

$$\Gamma : [0,1] \to \cup_{l>0}[b_i^{-l}]_> \cup \{\alpha_i, \omega_i\}$$

joining α_i and ω_i (see Remark 2.14). Then, Γ separates α_{i+1} from ω_{i+1} in $\overline{\mathbb{D}}$. So, Remark 4.4 gives us

$$\Gamma \cap (\cup_{l>0}[b_{i+1}^l]_<) \neq \emptyset.$$

So,

$$(\cup_{l>0}[b_i^{-l}]_>) \cap (\cup_{l>0}[b_{i+1}^l]_<) \neq \emptyset,$$

from which one immediately gets that there exists $l_i, m_i > 0$ such that $b_{i+1}^{l_i} \in [b_i^{-m_i}]_>$. As $b_i^{l_i} \in [b_i^{-m_i}]_>$ as well, the previous lemma tells us that there exists $l > 0$ such that $b_{i+2}^l \in [b_i^{-m_i}]_>$. We finish the proof of the lemma by induction, and then taking $k > 0$ large enough. □

We are now ready to prove Proposition 4.2:

Proof. We will show that $(([b_i^{-k}]_<)_{i\in\mathbb{Z}/n\mathbb{Z}}, ([b_i^k]_>)_{i\in\mathbb{Z}/n\mathbb{Z}})$ is an elliptic configuration, where $k > 0$ is given by the preceding lemma. This contradicts our assumption that f is not recurrent, by Proposition 3.1.

We define $r_i = \Gamma_i^- \cap \cup_{m\geq k} b_i^{-k}$, and $a_i = \Gamma_i^+ \cap \cup_{m\geq k} b_i^k$, $i \in \mathbb{Z}/n\mathbb{Z}$; we may suppose that the sets r_i, a_i, $i \in \mathbb{Z}/n\mathbb{Z}$ are arcs (the sets $\Gamma_i^- \cap \mathbb{D}$, $\Gamma_i^+ \cap \mathbb{D}$ were defined in Remark 2.14). These arcs a_i, r_i, $i \in \mathbb{Z}/n\mathbb{Z}$ satisfy the elliptic order property (see Remark 4.3). Besides, for all $i \in \mathbb{Z}/n\mathbb{Z}$,

- $r_i \subset [b_i^{-k}]_<$,
- $a_i \subset [b_i^k]_>$, and
- $b_i^k \in [b_i^{-k}]_>$.

So, we only have to show that the sets $\{[b_i^{-k}]_<\}, \{[b_j^k]_>\}$, are pairwise disjoint. As we are supposing that f is not recurrent, the preceding lemma gives us that for any pair of indices i, j in $\mathbb{Z}/n\mathbb{Z}$:

$$[b_i^{-k}]_< \cap [b_j^k]_> = \emptyset.$$

Let us show that for for any pair of different indices i, j in $\mathbb{Z}/n\mathbb{Z}$ one has

$$[b_i^{-k}]_< \cap [b_j^{-k}]_< = \emptyset.$$

Otherwise, there would exist $i \neq j$ such that $[b_i^{-k}]_< \cup [b_j^{-k}]_<$ is a connected set containing r_i and r_j. As $[b_i^{-k}]_>$ is a connected set containing a_p for all $p \in \mathbb{Z}/n\mathbb{Z}$ (again by the preceding lemma), the elliptic order property tells us:

$$([b_i^{-k}]_< \cup [b_j^{-k}]_<) \cap [b_i^{-k}]_> \neq \emptyset.$$

We deduce (as f is not recurrent) that

$$[b_j^{-k}]_< \cap [b_i^{-k}]_> \neq \emptyset,$$

but then $[b_j^{-k}]_<$ is a connected set containing both r_j and r_i, and once again the preceding lemma and the elliptic order property imply

$$[b_j^{-k}]_< \cap [b_j^{-k}]_> \neq \emptyset,$$

a contradiction. To prove that for any pair of different indices i, j in $\mathbb{Z}/n\mathbb{Z}$ one also has

$$[b_i^k]_> \cap [b_j^k]_> = \emptyset,$$

it is enough to interchange the roles of $<$ and $>$, k and $-k$ in the proof we just did. □

Our next proposition finishes the proof of Theorem 1.1:

Proposition 4.7. *If* $i(P) < 0$*, then* $\mathrm{Fix}(f) \neq \emptyset$.

By Lemma 4.1 and Proposition 4.2, we can suppose that P is minimal. We would also like to suppose that $\delta_i = 1$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, so as to fix the cyclic order of the points $\{\alpha_i\}, \{\omega_i\}$, at the circle at infinity. For this reason, we introduce the following lemma.

Lemma 4.8. *If* $\delta_i = 0$ *for some* $i \in \mathbb{Z}/n\mathbb{Z}$*, then there exists* $g \in \mathrm{Homeo}^+(\mathbb{D})$ *such that :*

1. $\mathrm{Fix}(g) = \mathrm{Fix}(f)$*;*
2. $g = f$ *on the orbits of the points* z_j*,* $j \notin \{i-1, i\}$*,*
3. *there exists* $z \in \mathbb{D}$ *such that* $\lim_{k\to-\infty} g^k(z) = \alpha_{i-1}$ *and* $\lim_{k\to+\infty} g^k(z) = \omega_i$*.*

We will need the following lemma, which is nothing but an adaptation of Franks' Lemma (see 2.2).

Lemma 4.9. *Let* $(D_i)_{0\leq i\leq p}$ *be a chain of free, open and pairwise disjoint disks for* f*, and take two points* $x \in D_0$ *and* $y \in D_p$*.*

Then, there exists $g \in \mathrm{Homeo}^+(\mathbb{D})$ *and an integer* $q \geq p$ *such that:*

- $\mathrm{Fix}(g) = \mathrm{Fix}(f)$*,*
- $g = f$ *outside* $\cup_{i=0}^{p} D_i$*,*
- $g^q(x) = f(y)$*.*

Proof. Take $z_i \in D_i$ and $k_i > 0$ the smallest positive integer such that $f^{k_i}(z_i) \in D_{i+1}$, $i \in \{0, \ldots, p-1\}$. We may suppose that the chain $(D_i)_{0\leq i\leq p}$ is of minimal length; that is, every $f^k(z_i), 0 < k < k_i$ is outside $\cup_{j=0}^{p} D_j$. We construct a homeomorphism h_0 which is the identity outside D_0 and such that $h_0(x) = z_0$, and a homeomorphism h_p which is the identity outside D_p and such that $h_p(f^{k_{p-1}}(z_{p-1})) = y$. For $i \in \{1, \ldots, p-1\}$, we construct homeomorphisms h_i such that:

- h_i is the identity outside D_i,
- $h_i(f^{k_{i-1}}(z_{i-1})) = z_i$

Finally, we construct a homeomorphism h which is the identity outside $\cup_{j=0}^{p} D_j$ and identical to h_i in D_i, $i \in \{0, \ldots, p\}$.

So, as the disks $\{D_i\}$ are free, $g = f \circ h$ satisfy all the conditions of the lemma.

□

The proof of Lemma 4.8 follows.

Proof. We will first construct a brick decomposition that suits our purposes. As the points $\alpha_{i-1}, \alpha_i, \omega_{i-1}, \omega_i$ are all different and f is not recurrent, we can construct families of closed disks $(b_i'^k)_{k\in\mathbb{Z}\setminus\{0\}}$, $(b_{i-1}'^k)_{k\in\mathbb{Z}\setminus\{0\}}$ as in Lemma 2.13 with the property that the interiors of the bricks in these families are pairwise disjoint.

Let $O = \cup_{i\in\mathbb{Z}/n\mathbb{Z}, k\in\mathbb{Z}} f^k(z_i)$. Here again we construct a maximal free brick decomposition such that for all $l \in \mathbb{Z}\setminus\{0\}$, there exists $b_i^l, b_{i-1}^l \in B$ such that $b_i'^l \subset b_i^l$ and $b_{i-1}'^l \subset b_{i-1}^l$. Furthermore, we may suppose that for all $x \in O$ there exists $b_x \in B$ such that $x \in \mathrm{Int}(b_x)$.

If $\delta_i = 0$ for some $i \in \mathbb{Z}/n\mathbb{Z}$, then P is either to the right of both Δ_i and Δ_{i-1} or either to the left of both Δ_i and Δ_{i-1}. We will suppose that P is to the left of both lines, as the other case is analogous. By Remark 2.14, we can find an arc

$$\Gamma : [0,1] \to \cup_{l>0}[b_i^l]_<$$

joining α_i and ω_i. So, Γ separates in $\overline{\mathbb{D}}$ α_{i-1} from ω_{i-1}. This implies that there exist two positive integers j, k such that

$$[b_{i-1}^{-j}]_> \cap [b_i^k]_< \neq \emptyset$$

(note that $\cup_{j>0}[b_{i-1}^{-j}]_>$ is a connected set whose closure contains α_{i-1} and ω_{i-1}). So, we can find a sequence of bricks $(b_m)_{0\leq m\leq p}$ such that $b_0 = b_{i-1}^{-j}$, $b_p = b_i^k$ and $f(b_m) \cap b_{m+1} \neq \emptyset$ if $m \in \{0, \dots, p-1\}$. We will suppose that this sequence is of minimal length, that is:

$$f(b_m) \cap b_{m'} \neq \emptyset \Rightarrow m' = m + 1 (*).$$

We define for all $1 \leq m \leq p-1$

$$X_m = b_m \setminus O.$$

We also define

$$X_0 = b_0 \setminus (O - \{f^{-k_{i-1}-j+1}(z_{i-1})\})$$

and

$$X_p = b_p \setminus (O - \{f^{k_i+k-1}(z_i)\})$$

(we recall from Lemma 2.13 that $f^{-l_{i-1}-j+1}(z_{i-1})$ is the only point of the orbit of z_{i-1} which lies in b_0, and that $f^{l_i+k-1}(z_i)$ is the only point of the orbit of z_i which lies in b_p). As every $x \in O$ belongs to the interior of a brick, we know that

$$f(X_m) \cap X_{m+1} \neq \emptyset$$

if $m \in \{0, \dots, p-1\}$.

For each $m \in \{0, \dots, p-1\}$, we take $x_m \in X_m$ such that $f(x_m) \in X_{m+1}$. We take an arc $\gamma_0 \subset X_0$ from $f^{-k_{i-1}-j+1}(z_{i-1})$ to x_0, and an arc $\gamma_p \subset X_p$ from $f(x_{p-1})$ to $f^{k_i+k-1}(z_i)$. For each $m \in \{1, \dots, p-1\}$ we take an arc $\gamma_m \subset X_m$ joining $f(x_{m-1})$ and x_m. As the interiors of the sets $\{X_m\}$ are pairwise disjoint, the arcs $\{\gamma_m\}$ can only meet in their extremities. However, condition $(*)$ implies that the points $\{x_m\}$ (and thus the points $\{f(x_m)\}$) are all different. Indeed, if $x_m = x_{m'}$, then $f(x_m) \in X_{m'+1}$, and so $f(b_m) \cap b_{m'+1} \neq \emptyset$. It follows by $(*)$ that $m = m'$. On the other hand, if $f(x_m) = x_{m'}$, we obtain that $f(b_m) \cap b_{m'} \neq \emptyset$, and so $m' = m + 1$. This means that the arcs $\{\gamma_m\}$ are pairwise disjoint (some of them maybe reduced to a point).

It follows that we can thicken this arcs $\{\gamma_m\}$ into free, open and pairwise disjoint disks $\{D_m\}$, such that $\gamma_m \subset D_m$, and such that $D_m \cap O = \emptyset$.

We are done by Lemma 4.9.

□

Lemma 4.10. *Let f realize a minimal n-gon P such that $i(P) < 0$. If $\delta_i = 0$ for some $i \in \mathbb{Z}/n\mathbb{Z}$, then here exists $g \in Homeo^+(\mathbb{D})$ realizing an n-1-gone P' such that $i(P') = i(P)$ and $\mathrm{Fix}(g) = \mathrm{Fix}(f)$.*

Proof. By Lemma 4.8, there exists $g \in \mathrm{Homeo}^+(\mathbb{D})$ such that :

1. $\mathrm{Fix}(g) = \mathrm{Fix}(f)$;
2. $g = f$ on the orbits of the points z_j, $j \in \mathbb{Z}/n\mathbb{Z}$, $j \notin \{i-1, i\}$,
3. there exists $z \in \mathbb{D}$ such that $\lim_{k\to-\infty} g^k(z) = \alpha_{i-1}$ and $\lim_{k\to+\infty} g^k(z) = \omega_i$.

The lines $(\Delta_j)_{j\in\mathbb{Z}/n\mathbb{Z}\setminus\{i,i-1\}}$ and the straight (oriented) line Δ_* from α_{i-1} to ω_i bound an $n-1$- gon P' such that $i(P') = i(P)$, and g realizes P'.

□

By applying the previous lemma inductively, there exists $g \in \mathrm{Homeo}^+(\mathbb{D})$ such that $\mathrm{Fix}(g) = \mathrm{Fix}(f)$ and g realizes a minimal n-gon P such that $i(P) < 0$, and $\delta_i = 1$ for all $i \in \mathbb{Z}/n\mathbb{Z}$.

This next lemma finishes the proof of Proposition 4.7:

Lemma 4.11. *If f realizes a minimal n-gon P such that $i(P) < 0$, and $\delta_i = 1$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, then $\mathrm{Fix}(f) \neq \emptyset$.*

Remark 4.12. With these assumptions, the cyclic order of the points $\{\alpha_i\}$, $\{\omega_i\}$, at the circle at infinity satisfies:

$$\alpha_i \to \alpha_{i-1} \to \omega_{i+1} \to \omega_i \to \alpha_{i+2}$$

for all even values of $i \in \mathbb{Z}/2m\mathbb{Z}$.

We apply Lemma 2.13 and obtain a family of closed disks $(b_i'^l)_{l\in\mathbb{Z}\setminus\{0\},i\in\mathbb{Z}/2m\mathbb{Z}}$. We construct a maximal free brick decomposition (V, E, B) such that for all $i \in \mathbb{Z}/2m\mathbb{Z}$ and for all $l \in \mathbb{Z}\setminus\{0\}$, there exists $b_i^l \in B$ such that $b_i'^l \subset b_i^l$ (see Corollary 2.15).

We will suppose that f is not recurrent, and we will show that we can construct a hyperbolic configuration.

Lemma 4.13. (Hyperbolic domino effect) *There exists $k > 0$ such that for all even values of $i \in \mathbb{Z}/2m\mathbb{Z}$, both attractors $[b_i^{-k}]_>$ and $[b_{i-1}^{-k}]_>$ contain b_l^k for all $l \in \{i-2, i-1, i, i+1\}$.*

Remark 4.14. Note that for all $i = 0 \mod 2$:

$$\omega_{i-1} \to \omega_{i-2} \to \alpha_i \to \alpha_{i-1} \to \omega_{i+1} \to \omega_i.$$

So, the "future indices" $\{i-2, i-1, i, i+1\}$ are those coming immediately before and immediately after the "past indices" $\{i, i-1\}$ in the cyclic order.

Proof. By Remark 2.14, we can find an arc

$$\Gamma : [0,1] \to \cup_{l\geq 1}[b_i^{-l}]_> \cup \{\alpha_i, \omega_i\}$$

joining α_i and ω_i. So, Γ separates α_{i-1} from ω_{i-1} and α_{i+1} from ω_{i+1} (in $\overline{\mathbb{D}}$). So, there exists $l > 0$ such that $[b_i^{-l}]_> \cap [b_{i-1}^l]_< \neq \emptyset$ and $[b_i^{-l}]_> \cap [b_{i+1}^l]_< \neq \emptyset$. So,

$$(\cup_{k\geq l} b_{i-1}^k) \cap (\cup_{k\geq l} b_{i+1}^k) \subset [b_i^{-l}]_>.$$

Using Remark 2.14 again, we can find an arc

$$\Gamma' : [0,1] \to [b_i^{-l}]_> \cup \{\omega_{i+1}, \omega_{i-1}\}$$

joining ω_{i+1} and ω_{i-1}. The cyclic order at S^1 of the points $\{\alpha_i\}, \{\omega_i\}$, implies that Γ' separates ω_{i-2} from α_{i-2} in $\overline{\mathbb{D}}$. So,

$$\Gamma' \cap \cup_{k\geq 1}[b_{i-2}^k]_< \neq \emptyset,$$

which implies that there exists $j > 0$ such that $b_{i-2}^j \in [b_i^{-l}]_>$. By taking $m > 0$ large enough, we obtain that for all $l \in \{i-2, i-1, i, i+1\}$, $b_l^m \in [b_i^{-m}]_>$. Analogously we obtain $b_l^p \in [b_{i-1}^{-p}]_>$ for all $l \in \{i-2, i-1, i, i+1\}$, for a suitable $p > 0$. We finish by taking $k \geq \max\{m, p\}$ □

We are now ready to prove Lemma 4.11:

Proof. We will show that $(([b_i^{-k}]_<)_{i=0 \mod 2}, ([b_i^k]_>)_{i=0 \mod 2})$ is a hyperbolic configuration, where $k > 0$ is given by Lemma 4.13 (the choice of even indices is arbitrary; we may as well have chosen the odd indices).

By Remark 4.12 and Lemma 4.13, we just have to show that the sets $[b_i^{-k}]_<$, $[b_i^k]_>$, for i even, are pairwise disjoint. Lemma 4.13 also gives us,

$$[b_i^{-k}]_< \cap [b_{i-2}^k]_> = \emptyset,$$

for i even. If $[b_i^{-k}]_< \cap [b_j^k]_> \neq \emptyset$ for an even j other than $i-2$, then we can find an arc $\Gamma : [0,1] \to [b_i^{-k}]_< \cup \{\alpha_i, \alpha_j\}$ joining α_i and α_j. The cylic order at S^1 of the points $\{\alpha_i\}, \{\omega_i\}$ implies that Γ separates ω_i from ω_{i-2} in $\overline{\mathbb{D}}$. As $[b_i^{-k}]_>$ is a connected set whose closure contains both ω_i and ω_{i-2} (by the previous lemma), one gets

$$[b_i^{-k}]_> \cap \Gamma \neq \emptyset$$

and so

$$[b_i^{-k}]_> \cap [b_i^{-k}]_< \neq \emptyset,$$

which implies that f is recurrent. So, we have:

$$[b_i^{-k}]_< \cap [b_j^k]_> = \emptyset,$$

for any pair of even indices i, j. We will show that

$$[b_i^{-k}]_< \cap [b_j^{-k}]_< = \emptyset$$

for any two different even indices i, j. Otherwise, we could find an arc

$$\Gamma : [0,1] \to [b_i^{-k}]_< \cup [b_j^{-k}]_< \cup \{\alpha_i, \alpha_j\}$$

joining α_i and α_j, from which we deduce again using the preceding lemma that

$$([b_i^{-k}]_< \cup [b_j^{-k}]_<) \cap [b_i^{-k}]_> \neq \emptyset.$$

So, as f is not recurrent, we have

$$[b_j^{-k}]_< \cap [b_i^{-k}]_> \neq \emptyset.$$

But now we can find an arc $\Gamma : [0,1] \to [b_j^{-k}]_< \cup \{\alpha_i, \alpha_j\}$ joining α_i and α_j, which implies

$$[b_j^{-k}]_< \cap [b_j^{-k}]_> \neq \emptyset,$$

contradicting that f is not recurrent. The proof of the fact that $[b_i^k]_> \cap [b_j^k]_> = \emptyset$ for any two different even indices i, j, is completely analogous.

□

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