Exponential speed of mixing for skew-products with singularities

R. Markarian[*], M. J. Pacifico[†], J. L. Vieitez[‡]

September 12, 2012

Abstract

Let $f : [0,1] \times [0,1] \backslash \{1/2\} \to [0,1] \times [0,1]$ be the C^∞ endomorphism given by

$$f(x,y) = \left(2x - \lfloor 2x \rfloor,\, y + \frac{c}{|x-1/2|} - \left\lfloor y + \frac{c}{|x-1/2|} \right\rfloor \right),\ c \in I\!R^+$$

We prove that f is topologically mixing and if $c > 1/4$ then f is mixing with respect to Lebesgue measure. Furthermore we prove that the speed of mixing is exponential. This skew-product can be seen as a toy-model related to Lorenz-like attractors.

2000 Mathematics Subject Classification: 37D30, 37C29, 37E30.

Keywords: skew-product with singularities, ergodicity, mixing, rate of mixing.

Contents

1 Introduction **2**

1.1 Toy model of flows with a singularity: slipping effect. 4

1.2 Statement of results. 5

2 Preliminaries **6**

3 The skew-product is topologically mixing **8**

4 Lebesgue measure preserved and mixing. **10**

5 Rate of mixing **16**

[*]partially supported by CSIC (Universidad de la República), SNI-ANII, PEDECIBA, Uruguay

[†]partially supported by CNPq Brazil, Pronex on Dynamical Systems, FAPERJ

[‡]partially supported by CSIC (Universidad de la República), SNI-ANII, PEDECIBA, Uruguay

1 Introduction

A basic problem in dynamics is the understanding of the ergodic behavior of a given dynamical system. Frequently this is translated into the knowledge of mixing properties of the system. Once mixing is established it is natural to ask for the rate or speed of mixing of the system.

For hyperbolic systems and nonuniform hyperbolic ones , without or with singularities, this kind of study is well understood and the techniques to do so have been developed by several authors. See, for instance, [Pe, Ru, BY, Vi, Wa] and the references therein to the interested reader.

When the system T under study has singularities, the phase space is not the whole manifold and in this case one asks zero-Lebesgue measure for the union $\cup_{n=0}^{\infty} T^{-n}S$ of the set of singularities S. This is the case of billiards, studied by Sinai [Si], Chernov [CY], Markarian [CM], Bunimovich [Bu] and others. In these cases we have the additional difficulty that the stable and unstable manifolds of points may be arbitrarily short since their length is conditioned by the distance of the points to S.

In general, the presence of singularities adds complexity into the problem and makes the analysis much more difficult. Nevertheless, in this paper, where we study a certain skew-product with singularities on the fiber, it is the presence of singularities, jointly with the expanding action in the base that enable us to obtain all the chaotic behavior of the system.

In this paper we are interested in the mixing properties of the skew-product[1] given by the C^∞-endomorphism $f : \big([0,1]\backslash\{1/2\}\big) \times \big[0,1\big] \to \big[0,1\big] \times \big[0,1\big]$ defined by

$$f(x,y) = \left(2x - \lfloor 2x \rfloor,\, y + \frac{c}{|x-1/2|} - \left\lfloor y + \frac{c}{|x-1/2|} \right\rfloor \right),\ c \in I\!R^+.$$

Here, given a real number x, $\lfloor x \rfloor$ stands for the greatest integer less or equal to x.

Since the denominator $\dfrac{c}{|x-1/2|}$ vanishes at $x = 1/2$, the line $\{(1/2, y) \,:\, y \in [0,1)\}$ is constituted by singularities of f. Besides that, for $c \neq 0$ we have that the vertical projection of $f(x,y)$ sharply varies when x is close to $1/2$.

[1]Recall that a skew-product T is an automorphism of the measure space $X \times Y$ where X and Y are measure spaces and the action of $T : X \times Y \to X \times Y$ has the form

$$T(x,y) = (A(x), B_x(y))\,; \quad x \in X,\, y \in Y\,,$$

where A is an automorphism of the space X (the "base") and $B_x(y)$, with x fixed, is an automorphism of Y (the "fiber"). The concept of a skew-product extends directly to the case of endomorphisms.

Identifying $[0,1]\times[0,1]$ with the two-dimensional torus $\mathbb{T}^2$, the skew-product may be seen as defined in $\mathbb{T}^2$ where the circle given by $x = 1/2$ is a curve of singularities of f.

The successive iterates by f of a rectangle R of sides parallel to the axes are transformed into a denumerable set of strips accumulating onto the circle $x = 1$ (which is identified with $x = 0$) in the torus. Indeed, when $x \to 1/2$, $2x$ approaches 1. This effect together with the fact that the pre-orbit by $x \mapsto 2x \mod (1)$ of the circle $x = 1/2$ is dense in the torus are responsible of the rich chaotic dynamics observed in this system.

Since the length of vertical segments are preserved under f, the action of f on the vertical borders of R is just a translation depending continously on $x \in [0,1] \setminus \{1/2\}$. Hence, the stretching and accumulation of the iterates of R onto the pre-orbit of the circle $x = 1/2$ in the torus is due to the slipping effect of f in the horizontal borders of R.

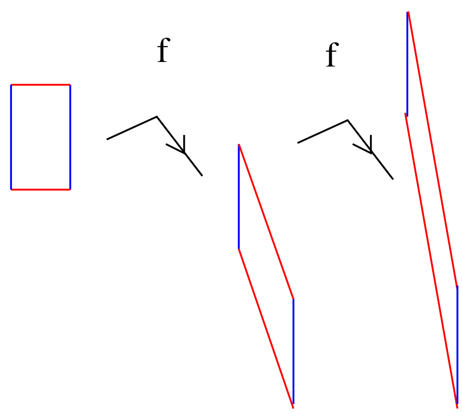

Figure 1: Effect on a rectangle A.

The skew-product f can be also immersed in a one-parameter family of expanding skew-products with the same line of singularities:

$$f_\lambda(x,y) = (2x, \lambda y + \frac{c}{|x-1/2|}),\ \lambda \geq 1\,.$$

Thus, it is interesting to detected the ergodic properties in the limit dynamics given by $\lambda = 1$. For instance, transitivity, mixing and rate of mixing.

In this paper we prove that the skew-product f defined above is topologically mixing (Theorem A), preserves the Lebesgue measure m on the torus, is mixing with respect to m (Theorem B). Finally we prove that the rate of mixing is exponential (Theorem C).

1.1 Toy model of flows with a singularity: slipping effect.

Let M be a 3-dimensional manifold and assume that $\Phi : M \to M$ is a flow containing a transitive attractor $\Lambda \subset M$ with a hyperbolic singularity $p \in \Lambda$. The geometric Lorenz attractor and any Lorenz-like attractor satisfy these conditions, see [GW, Lo, AP].

We consider the case when the singularity has three real eigenvalues λ_i, $1 \leq i \leq 3$, and satisfy $\lambda_2(\sigma) < \lambda_3(\sigma) < 0 < -\lambda_3(\sigma) < \lambda_1(\sigma)$. Via Hartmann-Großman theorem we assume that we have linearized coordinates in a neighborhood $U \supset [-1,1]^3$ of the singularity p in such a way that λ_1 corresponds to x-axis, λ_2 to y-axis and λ_3 to z-axis.

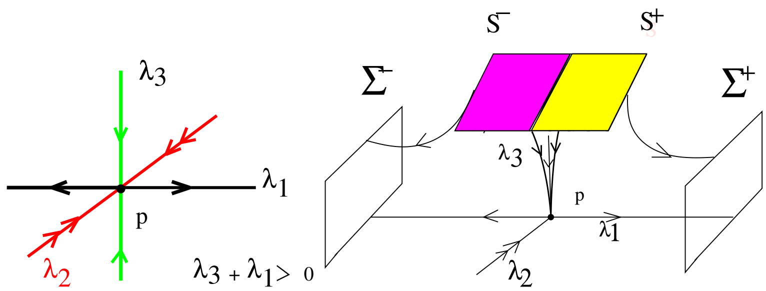

Figure 2: A Lorenz-like equilibrium and a cross-section to the flow.

Let $S = \{(x,y,z) \in U \ : \ z = 1\}$ be a transverse section to the flow so that every trajectory eventually crosses S in the direction of the negative z-axis. Consider also $\Sigma = \{(x,y,z) : |x| = 1\} = \Sigma^+ \cup \Sigma^-$ with $\Sigma^{\pm} = \{(x,y,z) : x = \pm 1\}$.

To each $(x_0, y_0, 1) \in S$ the time τ such that $X^{\tau}(x_0, y_0, 1) \in \Sigma$ is given by $\tau(x_0) = \frac{-1}{\lambda_1} \log |x_0|$, and it is such that $\tau(x_0) \to \infty$ when $x_0 \to 0$. This fact has the effect that different slices parallel to the y-axis of the section S arrives to Σ with a delay. Hence, we cannot see the return of each slice to S at the same time, even when the expecting delay is bounded .

Assume now that we "forget" the effect of the singularity and consider that the return time is the same for points in a same slice. Also "forget" the strong stable direction. Note that the strong stable direction does not interfere in the dynamics of the geometric Lorenz attractor [MPP, Lemma 2.16].

After these identifications, the dynamics in a neighborhood of p occurs in the (x,z) plane, and may be seen as a slipping in the vertical direction in order to annihilate the delay of time. Since the delay goes to infinity as $x \to 0$ the slipping also goes to infinity when $x \to 0$. Thus, the dynamics there is given by $(\phi(x), \psi(x, z))$ with $\psi(x, z) \to \infty$ when $x \to 0$. Moreover, since the ratio $\beta = -\frac{\lambda_2}{\lambda_1}$ is greater than one, the dynamics in the x direction is expanding.

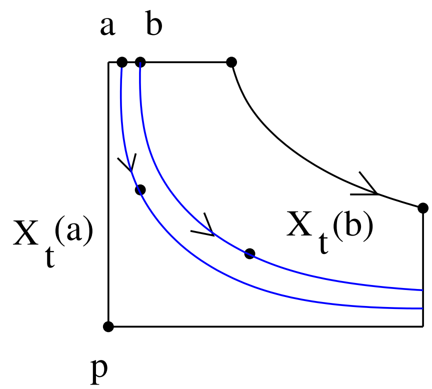

Figure 3: The slipping/shearing effect.

Thus, changing the name of the variable z by y, the skew-product

$$f(x, y) = \left(2x \mod (1), y + \frac{c}{|x - 1/2|} \mod (1)\right)$$

may be seen as a simplified case of the slipping effect in singular hyperbolic attractors, as is the case of a Lorenz-like attractor.

1.2 Statement of results.

To announce in a precise way our results let us introduce some definitions and related facts proved elsewhere.

Definition 1.1. *Let $(X, \mathcal{A}, f, \mu)$ be a dynamical system defined on the space X, $\mathcal{A}$ a σ-algebra of X, and μ an f-invariant probability measure. The map f is mixing if for all pair of sets $A, B \in \mathcal{A}$, we have*

$$\lim_{n \to \infty} \mu(f^{-n}(A) \cap B) = \mu(A)\mu(B) .$$

A form of mixing that can be defined without appealing to measures is the following

Definition 1.2. *Let $f : X \to X$ be a continuous map defined in the topological space X (X is not necessarily compact). We say that the dynamical system defined by f is topologically mixing if for every pair of non-empty open subsets A, B of X there is $N > 0$ such that $\forall n \geq N : \ f^n(A) \cap B \neq \emptyset$.*

There is even a commonly used weaker notion: we say that the system defined by f is topologically transitive if for every pair of non-empty open subsets A, B of X there is $n \in \mathbb{Z}$ such that $f^n(A) \cap B \neq \emptyset$.

It is well known that if a dynamical system is defined on topological space X, $\mathcal{A}$ is the Borel σ-algebra of X and μ is a probability invariant measure such that $\mu(A) > 0$ for every open set A of X, then if the system is mixing it is topological mixing. This and other general results on Ergodic Theory may be found in [Wa], for instance.

The main results in this paper are:

Theorem A. *For all positive c the skew-product $f : \mathbb{T}^2 \to \mathbb{T}^2$ is topologically mixing.*

Theorem B. *The skew-product f preserves the Lebesgue measure m in the torus. For $c > 1/4$, f is mixing with respect to m.*

Theorem C. *The rate of mixing is exponential, that is, there is $0 < \lambda < 1$ such that for all pair of rectangles A and B with sides parallel to the coordinate axes, we have,*

$$|m(f^{-n}(A) \cap B) - m(A)m(B)| < \lambda^n m(A)m(B), \quad \text{for all} \quad n \geq 0\,.$$

2 Preliminaries

In this section we establish some preliminaries properties of f that will be used in the proofs. We identify the set $Q = [0,1] \times [0,1]$ with the 2-torus $\mathbb{T}^2$.

First we list two interesting features of the skew-product f

($\star$) *For all $p = (x,y) \in \mathbb{T}^2$, there is no stable manifold $W^s(p,f)$.* Indeed, given (x,y) and $(x+\Delta x, y+\Delta y)$, assuming that $x < 1/2$, $\Delta x \neq 0$, $x + \Delta x < 1/2$ we obtain

$$\|f(x+\Delta x, y+\Delta y) - f(x,y)\| = \left\| \left(2\Delta x, \Delta y + \frac{c}{1/2 - x} \left[\frac{1}{1 - \frac{\Delta x}{1/2 - x}} - 1 \right] \right) \mod 1 \right\| \geq$$

$$2|\Delta x|, \text{ and similar result holds for } x > 1/2\,.$$

Hence, if $\Delta x \neq 0$, $\mathrm{dist}(f^n(x+\Delta x, y+\Delta y), f^n(x,y)) \geq 2^n \Delta x \mod 1$ which does not converges to 0. On the other hand, if $\Delta x = 0$ then the distance between $f^n(x, y+\Delta y)$ and $f^n(x,y)$ is preserved. Thus, for no $(\Delta x, \Delta y)$ we have $\mathrm{dist}(f^n(x+\Delta x, y+\Delta y), f^n(x,y)) \to 0$ when $n \to +\infty$.

($\star\star$) *The unstable manifolds are not unique.* Indeed, for any itinerary $\{(x_n, y_n)\}_{n \in \mathbb{N}}$ such that $f(x_n, y_n) = (x_{n-1}, y_{n-1})$ it is defined an unstable manifold $W^u((x_0,y_0), f)$ (recall that f is an endomorphism). Thus the unstable manifold of a point is not unique. Moreover, f is not an expanding map since for any $p = (x, y)$ we have $Df_p(0,1) = (0,1)$. Finally, it has no dominated splitting (see Section 2 for the proof of these facts).

Thus, the standard techniques in dynamics using existence of stable and unstable manifolds, for instance, are useless here.

If $0 \leq x < 1/2$ then $f(x,y) = (2x, y + \frac{c}{1/2-x} - \lfloor y + \frac{c}{1/2-x} \rfloor)$ while if $1/2 < x < 1$ then $f(x,y) = (2x-1, y + \frac{c}{x-1/2} - \lfloor y + \frac{c}{x-1/2} \rfloor)$.

The matrix $\left[Df_{(x,y)}\right]$, in the case $0 \leq x < 1/2$, is given by $\begin{pmatrix} 2 & 0 \\ \frac{c}{(1/2-x)^2} & 1 \end{pmatrix}$ and, in the case $1/2 < x \leq 1$, by $\begin{pmatrix} 2 & 0 \\ \frac{-c}{(1/2-x)^2} & 1 \end{pmatrix}$. Therefore it depends only on x. Any vector different of a vertical one is expanded by the action of Df which has two eigenvalues: 1 with eigenvector $(0,1)$, and 2 with eigenvector $(1, \frac{c}{(x-1/2)^2})$ if $0 \leq x < 1/2$ and $(-1, \frac{c}{(x-1/2)^2})$ if $1/2 < x < 1$.

Hence we have no stable manifold at any point of $\mathbb{T}^2$ (see $(\star)$) and points at the left of the line $x = 1/2$ have eigenvectors corresponding to the eigenvalue 2 forming an acute angle with the x-axis such that when $x \to 1/2$ the angle between the eigenvector associated to 2 tends to be vertical. A similar picture is valid at points at the right of $x = 1/2$ taking into account that in that case the eigenvector associated to 2 forms an obtuse angle with the x-axis. From these facts one may see that no non-trivial splitting is preserved. Indeed, given a periodic orbit, no direction different of the vertical one is preserved.

Given a real number $a \in (0,1)$, we write

$$a = \sum_{1}^{\infty} \frac{a_i}{2^i}, \quad a = (0.a_1 \cdots a_n \cdots)_2 \quad a_j \in \{0,1\}$$

for its binary decomposition. Forgetting the leading zero we may identify a point $x \in [0,1)$, $x = (0, b_1 b_2 b_3 \cdots)_2$, written in base 2, with the string $(b_1, b_2, b_3, \cdots)$. We denote this identification as $x \sim (b_1, b_2, b_3, \cdots)$. Hence the dynamics in the x- coordinate given by $x \mapsto 2x \mod (1)$ acts as the shift

$$\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}, \quad \sigma(b_1 b_2 b_3 \cdots) = b_2 b_3 \cdots$$

Each point $x \sim (b_1 b_2 \cdots)$ has two pre-images by this map

$$\sigma^{-1}(b_1 b_2 \cdots) = \begin{cases} x_0 \sim (0\, b_1 b_2 \cdots) \sim x/2 \\ x_1 \sim (1\, b_1 b_2 \cdots) \sim (1+x)/2\,. \end{cases} \tag{1}$$

Since $f(x,y) = (2x, y + c/|x - 1/2|) \mod (1)$ equation (1) implies that any $Z = (x,y) \in \mathbb{T}^2$, with $x = 0.b_1 b_2 \cdots$ has two pre-images Z_0, Z_1 by f given by:

(a) $Z_0 = \left(\frac{x}{2}, y - \frac{2c}{1-x} - \lfloor y - \frac{2c}{1-x} \rfloor\right) = (x_0, y_0)$,

(b) $Z_1 = \left(\frac{1+x}{2}, y - \frac{2c}{x} - \lfloor y - \frac{2c}{x} \rfloor\right) = (x_1, y_1)$.

Inductively, given a sequence $b = (b_1 b_2 \cdots b_n)$ of *length* $|b| = n$, with $b_j \in \{0,1\}$ $\forall j \le n$, and assuming that $Z_{b_2 b_3 \cdots b_n}$ (one of the 2^{n-1} $(n-1)$-th preimages of Z) is already defined we have that one of the 2^n n-th preimages of Z is $Z_b = (x_b, y_b)$ with

$$(a) \qquad x_b = \frac{b_1 + x_{b_2 b_3 \cdots b_n}}{2} \tag{2}$$

and

$$(b) \quad y_b = \left(y_{b_2 \cdots b_n} - \frac{2c}{(1-b_1) + (2b_1 - 1)x_{b_2 \cdots b_n}}\right) \quad \mod (1)$$

Remark 2.1. *We remark that if* $Z = (x,y)$, $W = (x', y')$ *and* $b = (b_1 b_2 \cdots b_n)$ *then* $|x_b - x'_b| = |x - x'|/2^n$. *We also remark that for any* $x \in [0,1)$ *the set of preimages* $\mathcal{S}_n$ *of* x *for all the different* b*'s of length* n *is almost uniformly distributed in* $[0,1)$*, i.e., for any interval* $I \subset [0,1)$*:*

$$\lim_{n\to\infty} \frac{\#(\mathcal{S}_n \cap I)}{\#\mathcal{S}_n} = \ell(I).$$

Here $\#X$ *means the cardinality of* X ($\#S_n = 2^n$)*, and* $\ell(I)$ *is the length of* I*.*

Convention. We extend this notation to the n-th preimage of an horizontal segment $I = [Z, Y]$: $Z_b(I)$ is the n-th pre-image of I that has Z_b as one of its boundaries. In the same way, if R is a rectangle with borders parallel to the coordinate axes, whose lower bound is I, then $Z_b(R)$ is the n-th pre-image of R with $Z_b(I)$ as one of its "sides".

3 The skew-product is topologically mixing

We start establishing the following result:

Lemma 3.1. *Given a monotone arc* $\gamma(t) = (x(t), y(t))$ *(i.e., an arc such that* $x(t)$ *and* $y(t)$ *are monotone continuous functions when they are seen as defined in* $[0,1) \subset \mathbb{R}$*) there exist* $n_0 \in \mathbb{N}$ *such that the vertical projection* $\Pi_y(f(\gamma))$ *of the image of* $\gamma(t) = (x(t), y(t))$ *covers all* $[0,1)$*.*

Proof. Given a monotone arc $\gamma : [0,1] \to \mathbb{T}^2$, $\gamma(t) = (x(t), y(t))$, the vertical projection of the function $f(x(t), y(t)) = (2x(t) \mod (1), y(t) + \frac{c}{|x(t)-1/2|} \mod (1))$ varies from $y(0) + \frac{c}{|x(0)-1/2|} \mod (1)$ to $y(1) + \frac{c}{|x(1)-1/2|} \mod (1)$.

If the horizontal projection $\Pi_x(\gamma)$ contains $x = 1/2$ then there is $1/8 > \epsilon > 0$ such that $\Pi_x(\gamma) \supset (1/2 - \epsilon, 1/2)$ or $\Pi_x(\gamma) \supset (1/2, 1/2 + \epsilon)$. In both cases the vertical projection of

γ, with $x(t)$ restricted to those segments, is monotone and has infinite length. Hence the result follows. Observe that if for no t it holds that $x(t) = 1/2$ then $\pi_x(\gamma)$ is contained either in $[0, 1/2)$ or in $(1/2, 1]$. In any case $f(\gamma)$ is a monotone arc and we can iterate under the hypothesis of the lemma. Taking into account that if $\ell(\Pi_x(f^{j-1}(\gamma))) < \frac{1}{2}$ then $\ell((\Pi_x(f^j(\gamma))) = 2 \cdot \ell((\Pi_x(f^{j-1}(\gamma)))$ we conclude that there is n_0 such that the horizontal projection $\Pi_x(f^{n_0}(\gamma))$ contains $x = 1/2$. Thus the proof follows. □

Lemma 3.2. *With the same hypothesis as Lemma 3.1 there is a sequence of arcs $\widehat{\gamma}_k$ image of subarcs of γ such that the vertical projection of $\widehat{\gamma}_k$ covers $[0, 1)$ and the horizontal projection is such that $\ell(\Pi_x(\widehat{\gamma}_k)) \to 0$ when $k \to \infty$.*

Proof. Lemma 3.1 implies that there is a (unique) number $t_0 \in [0, 1]$ such that $x(t_0) = 1/2$. There is no loss assume that $t_0 \in (0, 1)$. Let $y(t_0) = y_0$. Since both arcs are monotone we change coordinates and consider $(x, y(x))$, $a \le x \le b$, with $a < 1/2 < b$, as the new one. In these coordinates we have that $y(x) \to y_0$ when $x \to 1/2$. By continuity of the arc $(x, y(x))$ there is $\delta > 0$ such that for $x \in (1/2 - \delta, 1/2)$ it holds that

$$\Pi_y(f(x, y(x))) = y(x) + \frac{c}{1/2 - x} = y_0 + \epsilon(x) + \frac{c}{1/2 - x}, \text{ with } \epsilon(x) \to 0 \text{ when } x \to 1/2.$$

Let $N > 0$ be such that $\frac{1}{Nc} < \delta$. Hence $y(x) = y_0 + \epsilon(x) + N$ for $x = \frac{1}{2} - \frac{1}{cN}$. When x varies from $\frac{1}{2} - \frac{1}{cN}$ to $\frac{1}{2} - \frac{1}{c(N+2)}$ we have that $y(x)$ varies from $y_0 + \epsilon\left(\frac{1}{2} - \frac{1}{cN}\right) + N$ to $y_0 + \epsilon\left(\frac{1}{2} - \frac{1}{c(N+2)}\right) + N + 2$. Since $\epsilon(x) \to 0$ when $x \to 1/2$, we choose δ small enough so that $|\epsilon(x)| < 1/2$. Taking values $\mod (1)$ we obtain that $\Pi_y(f(x, y(x)))$ covers $[0, 1)$ when x varies in a segment of length $\frac{1}{cN} - \frac{1}{c(N+2)}$. We may repeat the arguments to prove that for all $k > 0$ we have that $\Pi_y(f(x, y(x)))$ covers $[0, 1)$ when x varies in the segment $\frac{1}{2} - \frac{1}{c(N+2k)}, \frac{1}{2} - \frac{1}{c(N+2k+2)}$ whose length tends to zero when $k \to +\infty$. This concludes the proof. □

Theorem 3.3. *If $c \in \mathbb{R}^+$ then $f(x, y)$ is topologically mixing.*

Proof. It is enough to prove the statement for open rectangles A and B of sides parallel to the coordinate axes since they form a basis for the standard topology of the plane.

Let $2\delta_A$ be the length of $\Pi_x(A)$ and $2\delta_B$ the length of $\Pi_x(B)$. Let also, in base 2, $x_A = (0.a_1a_2a_3\ldots)_2$ and $x_B = (0.b_1b_2b_3\ldots)_2$ and find N such that for $\delta = \min\{\delta_A, \delta_B\}$ we have $2^N\,\delta > 1$. Now we take

$$r = 0.a_1a_2\ldots a_N0\underbrace{11\ldots1}_{N}0b_1b_2b_3\ldots b_N \quad \text{and} \quad Z = (r, y_A)\,.$$

Clearly r is near x_A and lies in $[x_A - \delta_A, x_A + \delta_A]$ since $|r - x_A| < 1/2^N$.

After N iterates by f we have that $f^N(Z)$ is at a distance less that $1/2^N$ from $\{x = 1/2\}$ (since $\Pi_x(f^N(Z)) = 0.011\dots10b_1b_2b_3\dots$).
Lemmas 3.1 and 3.2 impply that the vertical projection of $f^N(A)$ has length greater than 1 and $\Pi_x(f^{2N+2}(Z)) \in \Pi_x(B)$ and $\ell(\Pi_y(f^{2N+2}(A))) > 1$ too. Therefore $f^{2N+2}(A) \cap B \neq \emptyset$.

Since the length of the horizontal projection doubles under iterations by the action $x \mapsto 2x \mod 1$, there is $N_1 > 0$ such that for $n > N_1$ we have that $\Pi_x(f^n(A))$ covers all $[0, 1]$. It follows that $f^{N_1+2N+2+k}(A) \cap B \neq \emptyset$ for all $k \geq 0$. Thus f is topologically mixing, proving Theorem A □

4 Lebesgue measure preserved and mixing.

In this section we prove Theorem B. We start establishing some auxiliary lemmas. The first says that for $c > 1/4$ we have that the preimages by f expand length in the vertical direction.

In the next lemma ℓ stands for the length in the torus.

Lemma 4.1. *Given $\epsilon > 0$ there is $N(c) = N > 0$ such that for $n \geq N$, every horizontal arc I, $\ell(I) = \Delta x$, every $b = b_1b_2 \cdot b_n$ it results $l(Z_b(I)) > (4 - \epsilon)c\Delta x$.*

Proof. Given a segment $[x, x+\Delta x] \subset [0, 1/2)$, the length of its image by any of its branches: Z_0, Z_1 is given by

$$\int_{x/2}^{x/2+\Delta x/2} \sqrt{1 + \frac{c^2}{(1/2 - s)^4}}\, ds \geq \left.\frac{c}{1/2 - s}\right|_{x/2}^{(x+\Delta x)/2} > 2c\Delta x\,.$$

Analogously for the four second branches Z_{00}, Z_{01}, Z_{10}, Z_{11} we have that, in appropriate coordinates (u, y), $u \in [h, h + \Delta x/4]$, $h < 1/4$, the graph of the pre-images is given by

$$g(u) = y_0 - \frac{2c}{1 - 2u} - \frac{2c}{1 - 4u}.$$

Putting $h' = 4h$, and calculating the length of the graph [2] we get

$$\int_h^{h+\Delta x/4} \sqrt{1 + (g'(u))^2}\, du \geq \int_h^{h+\Delta x/4} |(g'(u))|\, du =$$

$$\frac{2c}{1 - h'}\left(\frac{1}{1 - \frac{\Delta x}{1-h'}} - 1\right) + \frac{2c}{1 - h'/2}\left(\frac{1}{1 - \frac{\Delta x/2}{1-h'/2}} - 1\right) >$$

$$2c\Delta x + c\Delta x = 3c\Delta x\,.$$

[2] note that $1 \geq 1 - h' > 0$

By induction we obtain in the general case ($n \geq 3$) that the length of $Z_b([x_0, x_0+\Delta x], y)$ is bounded from below by

$$\ell(Z_b([x_0, x_0 + \Delta x], y)) \geq (2 + 1 + \sum_{j=1}^{n-2} \frac{1}{2^j})c\Delta x = (3 + (1 - \frac{1}{2^{n-2}})) \cdot c \cdot \Delta x.$$

Choosing $\frac{1}{2^{n-2}} < \epsilon$ the lemma follows if the length of the sequence b is greater or equal to $N = N(c)$. □

Lemma 4.2. *Lebesgue measure m is preserved by the map*

$$f(x, y) = \left(2x - \lfloor 2x \rfloor,\ y + \frac{c}{|x - 1/2|} - \left\lfloor y + \frac{c}{|x - 1/2|} \right\rfloor\right),\ c \in \mathbb{R}^+ .$$

Proof. There is no loss of generality choosing A a rectangle contained in Q, since the family of rectangles generates the σ-algebra associated to the Lebesgue measure [Wa, Theorem 1.17]. Given any small rectangle $A = (a, b) \times (d, e) \subset (0, 1) \times (0, 1)$ it has two pre-images which are the subsets A_0 and A_1 where A_0 is limited by the lines

$$x = \frac{a}{2},\ x = \frac{b}{2},$$

and the graph of the broken hyperbolas

$$y = d - \frac{c}{1/2 - x} - \left\lfloor d - \frac{c}{1/2 - x} \right\rfloor,\ y = e - \frac{c}{1/2 - x} - \left\lfloor e - \frac{c}{1/2 - x} \right\rfloor,\ a/2 \leq x \leq b/2;$$

and A_1 is limited by the lines

$$x = \frac{1 + a}{2},\ x = \frac{1 + b}{2},$$

and the graph of the broken hyperbolas

$$y = d - \frac{c}{x - 1/2} - \left\lfloor d - \frac{c}{x - 1/2} \right\rfloor,\ y = e - \frac{c}{x - 1/2} - \left\lfloor e - \frac{c}{x - 1/2} \right\rfloor,\ (1+a)/2 \leq x \leq (1+b)/2.$$

Calculating the area of A_0 by integration we obtain $(b - a)(e - d)/2$. Similarly for A_1. Summing both areas we obtain $(b-a)(e-d) = \text{Area}(A)$. Since the family of rectangles like A gives a basis for the σ-algebra associated to Lebesgue measure m we have proved that m is f-invariant. □

Figure 4 shows $A = [1/4, 1/3] \times [2/3, 3/4]$ and its pre-images, A_0 and A_1, where we have chosen $c = \pi - 3 \approx 0.1416$. The horizontal sides of A are mapped into the broken graph of the hyperbolas, the top corresponding to the green line and the bottom to the red one.

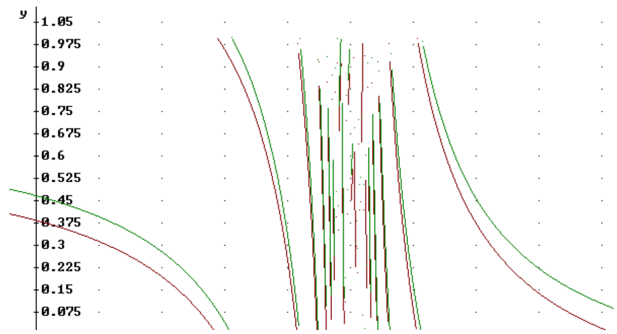

Figure 4: A small rectangle A and A_0, A_1, its pre-images. The almost vertical lines correspond to pre-images of the lines that contain the horizontal boundaries of A.

In the next lemma we refer to the symbols introduced in Section 2 and formula (2).

Lemma 4.3. *Given $b \in [0,1)$, $b \sim b_1 b_2 \cdots b_{N-1} b_N$,*

(a) There areN vertical asymptotes for (the lift to $I\!R^2$ of) $y_b(x_b)$ whose horizontal coordinates are

$$0.b_1 b_2 \cdots b_{N-1} + \frac{1}{2^N}\,, \quad 0.b_1 b_2 \cdots b_{N-2} + \frac{1}{2^{N-1}}\,, \;\ldots\,, \quad 0.b_1 + \frac{1}{2^2}\,, \quad \frac{1}{2}\,. \tag{3}$$

(b) There is at most one point in $(0,1)$ where the derivative $y_b'(x_b)$ can vanish. Moreover, if $y_b'(x_b)$ vanishes, the value of x at which $y_b' = 0$ is between $\Sigma_{i=1}^{j} b_i/2^i$ and $\Sigma_{i=1}^{j} b_i/2^i + 1/2^N$ where b_j is the first digit in $b_1 b_2 \cdots b_{N-1} b_N$ different from b_N (i.e.: $b_N = b_{N-1} = \cdots = b_{j+1} \neq b_j$).

Proof. Let (x,y) be given. For $N=1$ we have that

$$y_{b_1} = y - \frac{c}{|x_{b_1} - 1/2|} \mod (1) = y - \frac{c \cdot (2b_1 - 1)}{x_{b_1} - 1/2} \mod (1)$$

where $x_{b_1} = \frac{x+b_1}{2} \in (\frac{b_1}{2}, \frac{b_1+1}{2})$. Observe that $2b_1 - 1 = -1$ if $b_1 = 0$ and $2b_1 - 1 = 1$ if $b_1 = 1$. Thus

$$y_{b_1}'(x_{b_1}) = \frac{c \cdot (2b_1 - 1)}{(x_{b_1} - 1/2)^2} \quad \text{which does not vanish whenever it exists.}$$

For $N = 2$, on account that

$$x_{b_1 b_2} = \frac{x_{b_2} + b_1}{2} \in (\frac{b_1}{2} + \frac{b_2}{2^2}, \frac{b_1}{2} + \frac{b_2}{2^2} + \frac{1}{2^2})$$

we have $x_{b_2} = 2x_{b_1 b_2} - b_1$ and

$$y_{b_1 b_2}(x_{b_1 b_2}) = y_{b_2} - \frac{c}{|x_{b_1 b_2} - 1/2|} = y_0 - \frac{c}{|x_{b_2} - 1/2|} - \frac{c}{|x_{b_1 b_2} - 1/2|} \mod (1) =$$

$$= y_0 - \frac{c \cdot (2b_2 - 1)}{(2x_{b_1 b_2} - b_1 - 1/2)} - \frac{c \cdot (2b_1 - 1)}{(x_{b_1 b_2} - 1/2)} \mod (1)$$

from which we conclude that

$$y'_{b_1 b_2}(x_{b_1 b_2}) = \frac{2c \cdot (2b_2 - 1)}{(2x_{b_1 b_2} - b_1 - 1/2)^2} + \frac{c \cdot (2b_1 - 1)}{(x_{b_1 b_2} - 1/2)^2},$$

which does not change sign if $b_1 = b_2$ or changes sign only once in its domain if $b_1 \neq b_2$.

In general the expression of $y_b = y_{b_1 b_2 \cdots b_N}$ as a function of $x_b = x_{b_1 b_2 \cdots b_N}$ is given by

$$y_b(x_b) = y_0 - \frac{c \cdot (2b_N - 1)}{(2^{N-1}x_b - (2^{N-2}b_1 + 2^{N-3}b_2 + \cdots + 2b_{N-2} + b_{N-1}) - 1/2)} -$$

$$\frac{c \cdot (2b_{N-1} - 1)}{(2^{N-2}x_b - (2_{N-3}b_1 + 2^{N-4}b_2 + \cdots + 2b_{N-3} + b_{N-2}) - 1/2)} - \cdots - \frac{c \cdot (2b_1 - 1)}{(x_b - 1/2)} \mod (1)$$

from which the derivative of y_b with respect to x_b, whenever it exists, is given by

$$y'_b(x_b) = \frac{2^{N-1}c \cdot (2b_N - 1)}{(2^{N-1}x_b - (2_{N-2}b_1 + 2^{N-3}b_2 + \cdots + 2b_{N-2} + b_{N-1}) - 1/2)^2} +$$

$$\frac{2^{N-2}c \cdot (2b_{N-1} - 1)}{(2^{N-2}x_b - (2_{N-3}b_1 + 2^{N-4}b_2 + \cdots + 2b_{N-3} + b_{N-2}) - 1/2)^2} + \cdots + \frac{c \cdot (2b_1 - 1)}{(x_b - 1/2)^2}.$$

The last expression can be written as

$$y'_b(x_b) = \frac{c \cdot (2b_N - 1)}{2^{N-1}\left(x_b - (\frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_{N-2}}{2^{N-2}} + \frac{b_{N-1}}{2^{N-1}}) - \frac{1}{2^N}\right)^2} +$$

$$\frac{c \cdot (2b_{N-1} - 1)}{2^{N-2}\left(x_b - (\frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_{N-3}}{2^{N-3}} + \frac{b_{N-2}}{2^{N-2}}) - \frac{1}{2^{N-1}}\right)^2} + \cdots + \frac{c \cdot (2b_1 - 1)}{\left(x_b - \frac{1}{2}\right)^2}.$$

Hence $$y'_b(x_b) = \frac{c \cdot (2b_N - 1)}{2^{N-1}\left(x_b - 0.b_1 b_2 \cdots b_{N-2} b_{N-1} - \frac{1}{2^N}\right)^2} +$$

$$\frac{c \cdot (2b_{N-1} - 1)}{2^{N-2}\left(x_b - 0.b_1 b_2 \cdots b_{N-3} b_{N-2} - \frac{1}{2^{N-1}}\right)^2} + \cdots + \frac{c \cdot (2b_1 - 1)}{\left(x_b - \frac{1}{2}\right)^2}$$

where we have written $\frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_{N-2}}{2^{N-2}} + \frac{b_{N-1}}{2^{N-1}} = 0.b_1 b_2 \cdots b_{N-1}$, and similarly for the other terms.

Observe that each term in the expression or $y_b'(x_b)$ is positive or negative depending on the values of b_i, and that there are N vertical asymptotes for (the lift to $\mathbb{R}^2$ of) $y_b(x_b)$ with horizontal coordinates as at (3).

Claim 4.1. *All the terms with positive sign in $y_b'(x_b)$ have their asymptotes at points of coordinates less than those which have negative sign. Moreover $y_b'(x_b)$ will vanish only once at x_ξ located between the closest asymptotes of different sign.*

Proof. The proof goes by induction. For $N = 1$ there is nothing to prove. For $N = 2$, if $b_2 = 0$ and $b_1 = 1$ then we have an asymptote $x = \frac{1}{2}$ and the other $x = \frac{3}{4}$ with derivative

$$y'_{b_1 b_2}(x_{b_1 b_2}) = y'_{10}(x_{10}) = \frac{-c}{2(x_{10} - 3/4)^2} + \frac{c}{(x_{10} - 1/2)^2}\,.$$

If $b_2 = 1$ and $b_1 = 0$ then we have an asymptote $x = 1/4$ and the other $x = 1/2$ and the derivative is

$$y'_{b_1 b_2}(x_{b_1 b_2}) = y'_{01}(x_{01}) = \frac{c}{2(x_{01} - 1/4)^2} + \frac{-c}{(x_{01} - 1/2)^2}\,.$$

Hence the claim is true for $N = 1, 2$.

Assume that the claim is true for $b_2 b_3 \cdots b_N$ and let us prove it for $b = b_1 b_2 b_3 \cdots b_N$.

If $b_1 = 0$ then all the values of the asymptotes are divided by 2 and the corresponding asymptotes of the positive terms in $y_b'(x_b)$ rest to the left of the smallest asymptote corresponding to a negative term (if there is any one). By induction the difference between the smallest asymptote of a negative term and the largest asymptote of a positive term is $1/2^{N-1}$ for $b_2 b_3 \cdots b_N$ and when we divide by 2 the difference becomes $1/2^N$. Moreover, all terms are less than $1/2$ and we add a negative term corresponding to the asymptote $x = 1/2$. Thus the claim is true for $b_1 = 0$.

If $b_1 = 1$ then all the values of the asymptotes are divided by 2 and to these values we add $1/2$. Therefore the corresponding asymptotes of the positive terms in $y_b'(x_b)$ rest to the left of the smallest asymptote corresponding to a negative term (if there is any). The difference between the smallest asymptote of a negative term and the largest asymptote of a positive term becomes $1/2^N$ as above. All terms are greater than $1/2$ and we add a positive term corresponding to the asymptote $x = 1/2$. Thus the claim is proved.

□

The proof of the lemma follows readily from Claim 4.1, doing standard computations similar to those for the case $N = 2$.

□

Remark 4.4. *Let $A = [a,b] \times [d,e]$ be a small rectangle with sides parallel to the coordinate axes, and A_0 and A_1 its preimages. As a consequence of the proof of Lemma 4.2 we have that $m(A_0) = m(A_1) = m(A)/2$. The measure of each of the n-preimages is $m(A)/2^n$. Moreover, as a consequence of the previous lemma the shape of each preimage is an "almost" parallelogram bounded by two vertical lines of length $(e-d)$ and two almost vertical parallel lines, preimages of $[a,b] \times \{d\}$ and $[a,b] \times \{e\}$ respectively.*

Remark 4.5. *Since $x \in [0;1]$, although the number of asymptotes is N, we have that x_b belongs to an interval of length $1/2^N$, determined by two of these asymptotes .*

Next, we introduce the following convention that will be used in the sequel

Convention Given a positive constant C and real numbers A and B, we write $A \asymp B$ if $C^{-1} < A/B < C$. Given two sequences of real numbers $\{A_k\}_{k\in \mathbb{N}}$ and $\{B_k\}_{k\in \mathbb{N}}$ we write $A_k \simeq B_k$ if $A_k \asymp B_k$ for all $k \in \mathbb{N}$ and the constant $C_k > 0$ such that $C_k^{-1} < A_k/B_k < C_k$ converges to 1 when $k \to \infty$. When using the symbol $\simeq$ we usually will not indicate the parameter k.

Theorem 4.6. *If $c > 1/4$ then the map f is mixing with respect to Lebesgue measure.*

Proof. As mentioned in Lemma 4.2, there is no loss of generality choosing A and R as rectangles with borders parallel to the coordinate axes contained in Q. We have to show

$$\lim_{n\to\infty} |m(f^{-n}(A) \cap R) - m(A) \cdot m(R)| = 0,$$

where $f^{-n}(A)$ include all the preimages of A: $\cup\{Z_a(A), |a| = n\}$. To do so we proceed as follows. Let $W = (x^R, y^R) \in R$, be the center point of R and $Z = (x^A, y^A)$ the center point of A. Write $x^R \sim b_1 b_2 \cdots$ for the binary decomposition of x^R and let $\hat{x} \sim \overline{b_1 b_2 \cdots b_N} = b$, that is, $\hat{x}$ is the N-periodic point of the map $x \mapsto 2x \bmod 1$.

Taking N sufficiently large, we have that the vertical line $x = \hat{x}$ crosses the rectangle R nearby its central point W. Indeed, $|\hat{x} - x^R| < 2^{-N}$. If $x_b = \Pi_x(Z_b)$, as a consequence of the Remark 2.1, we obtain $|x_b - \hat{x}| < 2^{-N}$, and also $|x_b - x^R| < 2^{-N}$.

Moreover Lemma 4.1 implies that for $n \geq N$, with N large enough, the n-th pre-image of any horizontal segment of length Δx contained in $\pi_x(A)$, is almost vertical and its length is greater than $((4-\epsilon)c)^{\lfloor n/N \rfloor} \Delta x$. For $c > 1/4$ we can choose ϵ such that $d = (4-\epsilon)c > 1$.

Claim 4.2. *Let $Z_a(A)$, $a = a_1 a_2 \cdots a_n$, be one of the 2^n preimages of A, recall (2). The distance h_a between the pre-images of the horizontal sides (top and bottom) of A is*

$$h_a \simeq \frac{m(A)}{2^n \cdot \Delta x_A \cdot L_a}, \tag{4}$$

where Δx_A is the length of the bottom of A and $L_a > d^{[n/N]}$, $d > 1$.

Proof. Note that $\mathrm{m}(Z_a(A)) = \frac{\mathrm{m}(A)}{2^n}$. Moreover, as a consequence of Lemma 4.1 the length of $\Pi_x(Z_a(A))$ is given by $\Delta x_A \cdot L_a$ and so the area of $Z_a(A) \simeq (\Delta x_A \cdot L_a) h_a$. Thus the height h_a of the "almost" parallelogram given by $Z_a(A)$ is

$$h_a \simeq \frac{m(A)}{2^n \cdot \Delta x_A \cdot L_a}$$

proving the claim. □

Now note that if n is large enough, as a consequence of Remark 4.4, $Z_a(A)$ is a long "vertical" strip. Remark 2.1 implies that this happens to each of the n-th pre-images of A that are almost uniformly distributed in the torus. To each one of these strips there are $[\Delta x_A L_a]$ small strips cutting R (the number of connected components of $Z_a(A) \pmod 1$ in $[0,1]^2$). By claim 4.2 the total area of the intersection of a single $Z_a(A)$ with R is

$$\Delta y_R \cdot h_a \simeq \frac{\Delta y_R \cdot m(A)}{\Delta x_A \cdot 2^n \cdot L_a} = \frac{\Delta y_R \cdot m(A) \cdot [\Delta x_A \cdot L_a]}{2^n \cdot \Delta x_A \cdot L_a} \simeq \frac{\Delta y_R \cdot m(A)}{2^n}.$$

Since the number of pre-images $Z_a(A)$ satisfying the previous computations is $\simeq 2^n \cdot \Delta x_R$ we obtain that

$$m(f^{-n}(A) \cap R) \simeq \frac{\Delta y_R \cdot m(A)}{2^n} \cdot 2^n \cdot \Delta x_R \to m(A) \cdot m(R) \quad \text{when} \quad n \to \infty,$$

finishing the proof. □

Lemma 4.2 together with Theorem 4.6 prove Theorem B.

5 Rate of mixing

In this section we prove that the rate of mixing for $c > 1/4$ is exponential. We start with the folowing observation

Remark 5.1. *Given a rectangle* $A = [x_A - \Delta x/2, x_A + \Delta x/2] \times [y_A, y_A + \Delta y]$*, Lemma 4.1 implies that there is a first* $N_0 = N_0(\Delta x)$ *such that* $\ell(Z_b([x_A - \Delta x/2, x_A + \Delta x/2])) \geq 1$ *if* $|b| = N_0$*. Recall* $|b| = n$ *if* $b = (b_1 \cdots b_n)$*.*

The following lemma says that far from the asymptotes the growth of lengths of the pre-images is bounded from above.

Lemma 5.2. *Let* $c > 0$*. Given* $K > 0$ *and* $\epsilon > 0$ *there is* $\delta > 0$ *such that if* $0 < \Delta x \leq \delta$ *then* $N_0 > K$ *for a subset of* Σ *of measure greater or equal than* $1 - K\epsilon$*.*

Proof. Let us choose a vertical strip $S_\epsilon = [1/2 - \epsilon/2, 1/2 + \epsilon/2] \times [0, 1)$ and assume that $I = [x - \Delta x/2, x + \Delta x/2] \times \{y\}$ does not intersect S_ϵ. Let us bound from above the length of the pre-images of I. Recall, see equation (2), that these pre-images are given by

$$Z_0 = \left(\frac{x}{2}, y - \frac{2c}{1-x} - \lfloor y - \frac{2c}{1-x} \rfloor\right), \text{ and } Z_1 = \left(\frac{1+x}{2}, y - \frac{2c}{x} - \lfloor y - \frac{2c}{x} \rfloor\right).$$

We set $Z_0 = (x_0, y_0)$ and $Z_1 = (x_1, y_1)$. Let us assume that $x + \Delta x/2 < 1/2 - \epsilon/2$, the other cases are similar. This implies in particular that $1 - x > \epsilon$. We obtain:

$$\ell(Z_0(I)) = \int_{x/2-\Delta x/2}^{x/2+\Delta x/2} \sqrt{1 + \frac{c^2}{(1/2 - s)^4}} ds \leq \int_{x/2-\Delta x/2}^{x/2+\Delta x/2} \frac{(1/16 + c^2)^{1/2}}{(1/2 - s)^2} ds <$$

$$\left.\frac{c + 1/4}{1/2 - s}\right|_{(x-\Delta x)/2}^{(x+\Delta x)/2} < \frac{2c + 1/2}{\epsilon^2} \left(\frac{1}{(1 - \delta^2/\epsilon^2)}\right) 2\delta$$

where we have put $\Delta x < \delta < \epsilon$. This gives

$$\ell(Z_0(I)) < \frac{4c + 1}{\epsilon^2 - \delta^2} \delta.$$

The same bound is valid for the case $x > 1/2$ and for the other pre-image given by Z_1.

Let us denote by $b^{(K)} = b_1 b_2 b_3 \cdots b_K$ the finite subsequence given by the first K terms of $b = \{b_n\}_{n \in I\!N} \in \Sigma$ and

$$X_K(x, y) = \{(x_{b^{(K)}}, y_{b^{(K)}}) : f^K(x_{b^{(K)}}, y_{b^{(K)}}) = (x, y)\}.$$

There is $\epsilon > 0$ such that if $f^j([u - \Delta x/2, u + \Delta x/2], v) \cap [1/2 - \epsilon/2, 1/2 + \epsilon/2] \times [0, 1) = \emptyset$ for all $j = 0, 1, 2, \ldots, K - 1$ then

$$\ell(Z_{b^{(K)}}(([u - \Delta x/2, u + \Delta x/2], v))) < \left(\frac{4c + 1}{\epsilon^2 - \delta^2} \delta\right)^K$$

from which the thesis follows shrinking δ if necessary .

□

By Remark 5.1 after N_0 iterations the length of $Z_{b^{(N_0)}}$ is at least 1. Thus if k_0 denotes the time needed for a monotone arc γ to duplicate its length when computing $Z_{(b)^{(k_0)}}$ (see Lemma 4.1) we obtain the following

Corollary 5.3. *If $N = N_0 + k_0 h$, $h \geq 0$, then for b such that $|b| = N$ we have that*

$$\ell(Z_b([x_A - \Delta x/2, x_A + \Delta x/2] \times \{y\})) \geq 2^h.$$

Corollary 5.3 implies that the pre-image $Z_b(A)$ has 2^h connected components in $[0,1]\times[0,1]$ which are almost vertical strips. The value of N_0 is bounded but depends on the length of Δx and the position of x_A.

The next lemma estimates the width of each of these strips. Before we state it, let us sort out the intersections between $Z_b(A)$ and $[0,1]\times[0,1]$ in the following way:

($\star$) The image in $\mathbb{R}^2$ of the top side $T = [x_A - \Delta x/2, x_A + \Delta x/2] \times \{y + \Delta y\}$ of A is an arc almost parallel to the vertical axis Oy with reverse orientation. We assign the label n to the connected component of this arc whose projection covers the interval $[-n+1, -n]$ in Oy (see Figure 5). Similarly for the bottom segment $B = [x_A - \Delta x/2, x_A + \Delta x/2] \times \{y\}$.

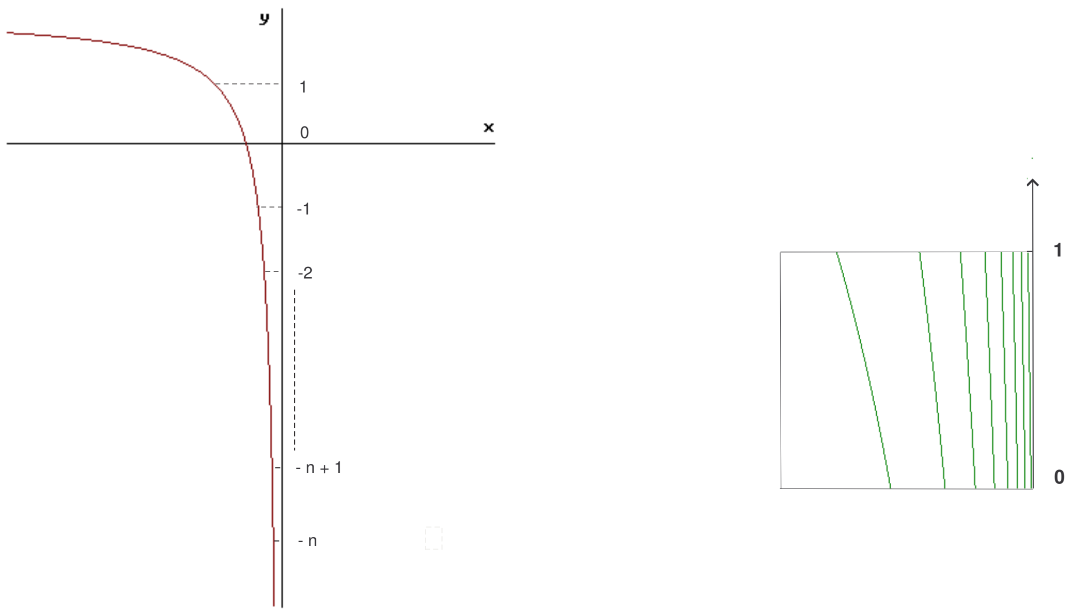

Figure 5: The image of T in $\mathbb{R}^2$ and in $[0,1]\times[0,1]$.

To simplify notations, let us introduce the following convention:

Convention Given K, L depending on $\varepsilon > 0$, we set $K \asymp L$ if given $\delta > 0$ there exists ε such that $|K - L| < \delta$.

Lemma 5.4. *Let $A = [x_A - \Delta x/2, x_A + \Delta x/2] \times [y_A, y_A + \Delta y]$ and $N = N_0 + k_0\, h$, $h \geq 0$, be as above. Denote by T the top and B the bottom sides of the rectangle A. Then, for b such that $|b| = N$ there is a positive constant $C = C(A, b, n) < K(c)$ such that*

$$\mathrm{dist}(Z_b(T)_n, Z_b(B)_n) = C\frac{\Delta y}{2^N n^2}\,,$$

where $Z_b(T)_n$, and $Z_b(B)_n$ are the n^{th}-connected component of $Z_b(T)$ and $Z_b(B)$ respectively.

Proof. For the bottom side B the expression of $y_b = y_{b_1 b_2 \cdots b_N}$ as a function of $x_b = x_{b_1 b_2 \cdots b_N}$ is given by (see the proof of Lemma 4.3)

$$y_b(x_b) = y_A - \frac{c \cdot (2b_N - 1)}{(2^{N-1}x_b - (2^{N-2}b_1 + 2^{N-3}b_2 + \cdots + 2b_{N-2} + b_{N-1}) - 1/2)} -$$
$$\frac{c \cdot (2b^{N-1} - 1)}{(2^{N-2}x_b - (2N - 3b_1 + 2^{N-4}b_2 + \cdots + 2b_{N-3} + b_{N-2}) - 1/2)} - \cdots - \frac{c \cdot (2b_1 - 1)}{(x_b - 1/2)} .$$

This curve has N asymptotes $x_1, x_2, \ldots, x_N$, see equation (3). As a consequence of the previous formula and (2), close to one of each asymptotes with horizontal coordinate x_j, $y_b(x_b)$ can be written as

$$y_b(x_b) = y_A + H(x_b) + \frac{c\,(2b_j - 1)}{2^{j-1}(x_b - x_j)} ,$$

where $H(x_b)$ has a finite limit H_j when $x_b \to x_j$. Similarly for the top side T we have

$$y_b(x_b) = y_A + \Delta y + H(x_b) + \frac{c\,(2b_j - 1)}{2^{j-1}(x_b - x_j)} .$$

The only values that give asymptotes in the domain of x_b correspond to the case $j = N$ that give

$$y_b(x_b) = y_A + H(x_b) + \frac{c\,(2b_N - 1)}{2^{N-1}(x_b - x_N)} ,$$

and

$$y_b(x_b) = y_A + \Delta y + H(x_b) + \frac{c\,(2b_N - 1)}{2^{N-1}(x_b - x_N)} .$$

For a fixed y_b, varying from $-n + 1$ to $-n$, there exist $\tilde{x}_b, \widehat{x}_b$ such that

$$(\star_1)\ y_b = y_A + H(\tilde{x}_b) + \frac{c\,(2b_N - 1)}{2^{N-1}(\tilde{x}_b - x_N)} \quad \text{and} \quad (\star_2)\ y_b = y_A + \Delta y + H(\widehat{x}_b) + \frac{c\,(2b_N - 1)}{2^{N-1}(\widehat{x}_b - x_N)} .$$

For a given small $\varepsilon > 0$, there is n_0 such that for $n > n_0$ it holds that $|H(x_b) - H_N| < \varepsilon$. Thus, from the first equation $(\star_1)$ we have that, for $n > n_0$, it holds

$$y_b \asymp y_A + H_N + \frac{c\,(2b_N - 1)}{2^{N-1}(\tilde{x}_b - x_N)} \Longrightarrow \tilde{x}_b \asymp x_N + \frac{c(2b_N - 1)}{(y_b - y_A - H_N)2^{N-1}} .$$

From the second one $(\star_2)$ we obtain that

$$y_b \asymp y_A + H_N + \Delta y + \frac{c\,(2b_N - 1)}{2^{N-1}(\widehat{x}_b - x_N)} \Longrightarrow \widehat{x}_b \asymp x_N + \frac{c(2b_N - 1)}{(y_b - y_A - H_N - \Delta y)2^{N-1}} .$$

This implies that

$$|\widehat{x}_b - \tilde{x}_b| \asymp \left| \frac{c(2b_N - 1)}{(y_b - y_A - H_N - \Delta y)2^{N-1}} - \frac{c(2b_N - 1)}{(y_b - y_A - H_N)2^{N-1}} \right| .$$

Taking into account that $-n+1 > y_b > -n$ and $2b_N - 1 = \pm 1$ we conclude that

$$|\widehat{x}_b - \tilde{x}_b| \asymp \frac{c}{2^{N-1}} \left(\frac{1}{(-n - y_A - H_N - \Delta y)} - \frac{1}{(-n - y_A - H_N)} \right) =$$

$$\frac{c}{2^{N-1}} \frac{\Delta y}{(n + y_A + H_N)(n + y_A + H_N + \Delta y)} = E \frac{\Delta y}{2^N n^2} .$$

The constant E depends, in particular, on ε. Now we choose the constant $C > 0$ such that the relation of the statement holds for all $n \geq 1$ and not only for $n > n_0$. □

Theorem 5.5. *There is $0 < \theta < 1$ such that after $N = N_0 + k_0\, h$ iterates by any branch Z_b of f^{-1}, the Lebesgue measure of the set of points that has returned to A is greater or equal to $m(A) \cdot \theta$.*

Proof. Let $\ell(Z_b([x_A - \Delta x/2, x_A + \Delta x/2] \times \{y\})) \geq 2^h$ with $|b| = N$ (Corollary 5.3), and assume that $\Pi_x(Z_b(x_A, y) \in [x_A - \Delta x/2 + \frac{1}{2^N}, x_A + \Delta x/2 - \frac{1}{2^N}]$. Then $Z_b(A)$ cuts A in at least 2^h strips which are almost vertical except perhaps for one where the derivative $y_b'(x_b)$ can vanish. We don't take into account this strip so that we either have 2^h almost vertical strips or $(2^h - 1)$ of them. By Lemma 5.4 the (almost) vertical sides of the strips (which are at a distance $\frac{C\Delta y}{2^N n^2}$ between them) intersected with A are mapped by f^N, $N = |b|$, in part of the horizontal sides of length $C\Delta y/n^2$ with $N_0 \leq n \leq 2^h$; recall that C is uniformly bounded although it depends on A, b and n.

Thus the area of A covered by the f^N-image of one of the strips is equal to a constant D multiplied by the length $\Delta y/n^2$ of the horizontal sub-intervals, by the height Δy which gives

$$\text{Area}_n = D \cdot \frac{\Delta y}{n^2} \cdot \Delta y .$$

It follows that the area of the f^N-image of the 2^hstrips is $\simeq D \cdot (\Delta y)^2 \sum_{n=N_0}^{2^h} \frac{1}{n^2}$. Since any point in A has 2^N preimages by the different Z_b, $|b| = N$, we have to divide this number by 2^N in order not to multiple count. This gives us

$$D \cdot (\Delta y)^2 \frac{\sum_{n=N_0}^{2^h} \frac{1}{n^2}}{2^N} .$$

Since the number of preimages from N_0 to N that cut A is given by the action of $x \mapsto 2x$ mod (1) in $[0,1)$ we have that this number is $\simeq 2^{k_0 h}\Delta x$. Hence we have that the coverded area of A by the set of points that have returned after N pre-images is

$$\simeq D \cdot (\Delta y)^2 \frac{\sum_{n=N_0}^{2^h} \frac{1}{n^2}}{2^N} \Delta x \cdot 2^{k_0 h} = \Delta x \Delta y \left(D \cdot \Delta y \cdot \frac{2^{k_0 h}}{2^{N_0 + k_0 h}} \cdot \sum_{n=N_0}^{2^h} \frac{1}{n^2} \right) =$$

$$= m(A) \left(D \cdot \Delta y \cdot 2^{-N_0} \cdot \sum_{n=N_0}^{2^h - 1} \frac{1}{n^2} \right) = m(A) \cdot \theta,$$

$$\text{where} \quad \theta = D \cdot \Delta y \cdot 2^{-N_0} \cdot \sum_{n=N_0}^{2^h} \frac{1}{n^2} < 1\,.$$

Therefore the measure of the set of points that have not returned yet is at most

$$m(A)(1 - \theta).$$

After taking 2^N new pre-images (i.e.: by backward iteration N times following all the possible 2^N branches Z_b, $|b| = N$, from the new starting point) we cover $m(A)(1-\theta)\theta$ which implies that it rests at most $m(A)(1-\theta)^2$ points that have not returned to A yet. We conclude by induction that for $|b| = nN$ the measure of points not covered after taking all 2^{nN} pre-images is less than $m(A)(1-\theta)^n \to 0$ when $n \to \infty$. ☐

Corollary 5.6. *We have that after $n\,N$ iterates, the Lebesgue measure of points that have not yet returned is less than $m(A)(1-\theta)^n$.*

The next corollary gives that the rate of recurrence of f is exponential.

Corollary 5.7. *There exists $\tau < 1$ such that $|m(f^{-n}(A) \cap A) - m^2(A)| \leq \tau^n m^2(A)$.*

Proof. Note that by Theorem 5.5 we have that the measure of points that have returned to A after N iterations is

$$m(A) \left(D \cdot \Delta y \cdot 2^{-N_0} \cdot \sum_{n=N_0}^{2^h} \frac{1}{n^2} \right).$$

We write this expression as

$$m(A) \left(D \cdot \Delta x \cdot \Delta y \cdot \frac{2^{-N_0}}{\Delta x} \cdot \sum_{n=N_0}^{2^h \Delta x} \frac{1}{n^2} \right) = m^2(A) \left(D \cdot \frac{2^{-N_0}}{\Delta x} \cdot \sum_{n=N_0}^{2^h \Delta x} \frac{1}{n^2} \right).$$

For N_0 sufficiently large we have that $\lambda = \left(D \cdot \frac{2^{-N_0}}{\Delta x} \cdot \sum_{n=N_0}^{n=2^h} \frac{1}{n^2} \right) < 1$ and therefore we obtain that after $n = hN$ iterations

$$|m(f^{-n}(A) \cap A) - m^2(A)| \leq |(m(A))^2 \big(1 - \lambda^{[\frac{n_i}{N}]}\big) - (m(A))^2| = (m(A))^2 \lambda^{[\frac{n}{N}]}.$$

Putting $\lambda^{1/N} = \tau < 1$ we have

$$|m(f^{-n}(A) \cap A) - m^2(A)| \leq m^2(A) \tau^n,$$

proving the thesis. □

The following theorem is similar to Theorem 5.5.

Theorem 5.8. *Given small rectangles* $A = [x_A - \frac{\Delta x_A}{2}, x_A + \frac{\Delta x_A}{2}] \times [y_A, y_A + \Delta y_A]$ *and* $B = [x_B - \frac{\Delta x_B}{2}, x_B + \frac{\Delta x_B}{2}] \times [y_B, y_B + \Delta y_B]$ *there is* $0 < \theta < 1$ *such that the set of points of* A *that has visited* B *after* N *iterates is greater or equal than* $m(B) \cdot \theta$.

Proof. By Corollary 5.3 we have that $\ell(Z_b([x_A - \frac{\Delta x_A}{2}, x_A + \frac{\Delta x_A}{2}] \times \{y\})) = 2^h$ with $|b| = N$. Assume that $\Pi_x(Z_b(x_a, y)) \in [x_B - \frac{\Delta x_B}{2} + \frac{1}{2^N}, x_B + \frac{\Delta x_B}{2} - \frac{1}{2^N}]$. Then $Z_b(A)$ cuts B in 2^h strips which are almost vertical except perhaps for one of them corresponding to that strip where the derivative $y_b'(x_b)$ vanishes. We don't take into account this strip so that we either have $2^h \Delta x_A$ or $(2^h - 1)\Delta x_A$ almost vertical strips. The area of $Z_b(A)$ is $m(A)/2^N$.

By Lemma 5.4 and taking into account the sorting given at ($\star$), the intersection of the (almost) vertical sides of the strips with B are mapped by f^N, $N = |b|$, in a subsegment of the horizontal sides of A with length $\asymp C \Delta y_B / n^2$, $N_0 \leq n \leq 2^h$, recall Corollary 5.3. Thus, the area covered by the f^N-image of the n^{th}-strip is (Area$_n \simeq D \cdot \frac{\Delta y_B}{n^2} \cdot \Delta y_A$, where D is a constant, Δy_B is the length of the vertical side of B, and Δy_A is the length of the vertical side of A.

Therefore, the area of the f^N-image of all the $(2^h - N_0)$ strips is

$$\sum_{n=N_0}^{2^h} \text{Area}_n \simeq D \cdot (\Delta y_A)(\Delta y_B) \sum_{n=N_0}^{2^h} \frac{1}{n^2} \,.$$

Since any point in A has 2^N pre-images by the different Z_b, $|b| = N$, we have to divide this number by 2^N in order not to multiple count. This gives us

$$D \cdot (\Delta y_A)(\Delta y_B) \frac{\sum_{n=N_0}^{2^h} \frac{1}{n^2}}{2^N} \,.$$

Since the number of pre-images from N_0 to $N = N_0 + 2^{k_0 h}$ that cut B is given by the action of $x \mapsto 2x \mod (1)$ in $[0, 1)$, which is Bernoulli, we have that this number is $\approx 2^{k_0 h} \Delta x_B$.

Hence we have that the area of the set of points that have cut B after N pre-images is

$$\simeq D\cdot(\Delta y_A)(\Delta y_B)\frac{\sum_{n=N_0}^{2^h\Delta x_A}\frac{1}{n^2}}{2^N}\Delta x_B\cdot 2^{k_0h}=\Delta x_B\Delta y_B\left(D\cdot\Delta y_A\cdot\frac{2^{k_0h}}{2^{N_0+k_0h}}\cdot\sum_{n=N_0}^{2^h}\frac{1}{n^2}\right)=$$

$$=m(B)\left(D\cdot\Delta y_A\cdot 2^{-N_0}\cdot\sum_{n=N_0}^{2^h}\frac{1}{n^2}\right)=m(B)\cdot\theta,$$

$$\text{where}\quad \theta=D\cdot\Delta y_A\cdot 2^{-N_0}\cdot\sum_{n=N_0}^{2^h\Delta x}\frac{1}{n^2}<1\,.$$

Therefore the measure of the set of points of A that have not visited yet the set B is at most $m(B)(1-\theta)$. After taking 2^N new pre-images (i.e.: by backward iteration N times following all the possible 2^N branches Z_b, $|b|=N$, from the new starting point) we cover $m(B)(1-\theta)\theta$ which implies that it rests at most $m(B)(1-\theta)^2$ points that have not visited B yet.

By induction we conclude that for $|b|=nN$ the measure of points not covered after taking all 2^{nN} pre-images is less than $m(B)(1-\theta)^n\to 0$ when $n\to\infty$. □

The proof of the following corollary is similar to that of Corollary 5.7. It finishes the proof of Theorem C.

Corollary 5.9. *The exist $\tau<1$ such that For any two rectangles A,B , it holds that $|m(f^{-n}(A)\cap B)-m(A)m(B)|\leq\tau^n m(A)m(B)$.*

References

[AP] V. Araújo and M. J. Pacifico. *Three-dimensional flows*, volume 53 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer, Heidelberg, 2010.

[BY] M. Benedicks, L.-S. Young, *Markov extensions and decay of correlations for certain Hénon maps*, Astérisque 261 (2000), 13-56.

[Bu] Bunimovich, L.A.: *On the ergodic properties of nowhere dispersing billiards.* Commun. Math. Phys. 65, 295-312 (1979)

[CM] Chernov, N. Markarian R. *Chaotic billiards* Mathematical Surveys and Monographs 127 AMS, Providence, 320 pp (2006)

[CY] N. Chernov & L.-S. Young. *Decay of correlations for Lorentz gases and hard balls.* Encyclopaedia of Mathematical Sciences 101, Springer-Verlag, (2000) 89 - 120

[GW] J. Guckenheimer and R. F. Williams. Structural stability of Lorenz attractors. *Publ. Math. IHES*, 50:59–72, 1979.

[Lo] E. N. Lorenz. Deterministic nonperiodic flow. *J. Atmosph. Sci.*, 20:130–141, 1963.

[MPP] C. A. Morales, M. J. Pacifico and E. R. Pujals. Robust transitive sets for 3-flows are partially hyperbolic attractors or repellers. *Annal. of Math.*, 160:375–432, 2004.

[Pe] Pesin, Ya.B.: *Characteristic Lyapunov exponents and smooth ergodic theory.* Russ. Math. Surv. 32:4, 55-112 (1977)

[Ru] Ruelle D. *Ergodic Theory of Dynamical Systems*, Publ Math IHES, 50, 27 - 58, (1979).

[Si] Sinai, Ya.G.: *Dynamical systems with elastic reflections.* Russ. Math. Surv.

[Vi] Viana, M. *Stochastic dynamics of deterministic systems*, Lecture Notes 21^{st} Braz. Math Colloq, IMPA, Rio de Janeiro, (1997).

[Wa] P. Walters. *An Introduction to Ergodic Theory.* Graduate Texts in Mathematics 79, Springer-Verlag, New York-Berlin, 1982. ix+250 pp.

R. Markarian: Instituto de Matemática y Estadística (IMERL), Facultad de Ingeniería, Universidad de la República, CC30, CP 11300, Montevideo, Uruguay. E-mail: roma@fing.edu.uy.

M. J. Pacifico: Instituto de Matemática, Universidade Federal do Rio de Janeiro, C. P. 68.530, CEP 21.945-970, Rio de Janeiro, RJ, Brazil. E-mail: pacifico@im.ufrj.br .

J. Vieitez: Regional Norte, Universidad de la Republica, Rivera 1350, CP 50000, Salto, Uruguay. E-mail: jvieitez@unorte.edu.uy.