

TAME DYNAMICS AND ROBUST TRANSITIVITY

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ABSTRACT. One main task of smooth dynamical systems consists in finding a good decomposition into elementary pieces of the dynamics. This paper contributes to the study of chain-recurrence classes. It is known that C^1 -generically, each chain-recurrence class containing a periodic orbit is equal to the homoclinic class of this orbit. Our result implies that in general this property is fragile.

We build a C^1 -open set \mathcal{U} of tame diffeomorphisms (their dynamics only splits into finitely many chain-recurrence classes) such that for any diffeomorphism in a C^∞ -dense subset of \mathcal{U} , one of the chain-recurrence classes is not transitive (and has an isolated point). Moreover, these dynamics are obtained among partially hyperbolic systems with one-dimensional center.

1. INTRODUCTION

In the setting of hyperbolic diffeomorphisms, Smale's spectral decomposition theorem organizes the global dynamics by decomposing it into a finite number of pieces. After the first examples of open sets of non hyperbolic systems [AS, New] people focused on non hyperbolic dynamics trying to recover a decomposition in pieces in this new setting. Many such examples have been obtained, and we can distinguish two main different behaviours:

- The dynamics can be globally “undecomposable”: there are open sets of *transitive* systems, i.e. having a half-orbit which is dense in the whole manifold, see [Sh, Ma, BD₁]. In the same spirit, other examples present a decomposition in finitely many disjoint, isolated, robustly transitive compact invariant sets [Ca, BV, BD₁].
- On the opposite, the dynamics of generic systems in some C^r -open sets ($r \geq 1$) can split into infinitely many pieces, see [New, BD₂]. Necessarily in this case, some pieces of the decomposition should be accumulated by other ones.

These two patterns (finite or infinite number of indecomposable pieces) are called *tame* and *wild* systems: in order to formalize these notions we have to explain what we mean by “pieces”. Several definitions have been proposed.

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- A natural way for a piece to be dynamically indecomposable is to be transitive. Among notable transitive sets are the *homoclinic classes* of hyperbolic periodic orbits, that is, the closure of the transversal intersection of their invariant manifolds. These classes contain dense subsets of hyperbolic periodic orbits that are *homoclinically related*: the stable manifold of each of these orbits intersects transversally the unstable manifold of each other.
- Other natural candidates for pieces are the *maximal transitive sets* [BD₂]. They always exist but may fail to be disjoint and to capture a large part of the dynamics.
- By weakening the notion of transitivity, Conley [Co] defined the very general notion of *chain-recurrence classes* for homeomorphisms on a compact manifold M . The *chain-recurrent set* $\mathcal{R}(h)$ is the set of points x contained in closed ε -pseudo-orbits for every $\varepsilon > 0$. This set splits into invariant compact subsets called *chain-recurrence classes*: two points $x, y \in \mathcal{R}(f)$ are in a same class if for every $\varepsilon > 0$ they are contained in a same closed ε -pseudo-orbit. The chain-recurrence classes cover the whole interesting dynamics, as this notion of recurrence is the weakest possible: the chain-recurrence set contains the limit sets and the non-wandering set. Moreover, the different chain-recurrence classes can be separated by filtrations of the dynamics.

Each of these notions has its advantages and drawbacks and in general they do not coincide. When f is a C^1 -generic diffeomorphism, one gets better properties. For instance, [BC] showed that any chain-recurrence class \mathcal{C} containing a hyperbolic periodic orbit O coincides with the homoclinic class $H(O)$ of O , and is the unique maximal transitive set containing O . Of course, these properties are not satisfied by all diffeomorphisms, but once they hold C^1 -generically, one can expect that they persist on larger class of systems, in particular for some close C^r -diffeomorphisms. More precisely one can ask:

Question 1. *Consider a hyperbolic periodic orbit O_f whose continuation exists on an open set \mathcal{U}_0 of C^1 -diffeomorphisms. Under what conditions does there exist a dense open subset $\mathcal{U} \subset \mathcal{U}_0$ such that the chain-recurrence class containing O_f is transitive for any $f \in \mathcal{U}$? coincides with the homoclinic class $H(O_f)$?*

This question could be asked for an important particular case. A chain-recurrence class \mathcal{C} of a diffeomorphism f is *robustly isolated* (for the C^1 -topology) if there exists a neighborhood \mathcal{O} of \mathcal{C} and a C^1 -neighborhood \mathcal{U} of f such that, for every $g \in \mathcal{U}$, the set $\mathcal{R}(g) \cap \mathcal{O}$ of chain-recurrent points in \mathcal{O} coincides precisely with a chain-recurrence class \mathcal{C}_g of g called its *continuation*.

Following [Ab, BD₄], one says that a diffeomorphism f is *tame* if each of its chain-recurrence classes is robustly isolated: in this case, the number of chain-recurrence classes is finite and constant on a C^1 -neighborhood of f . Tame diffeomorphisms are the simplest ones and the dynamics of C^1 -generic such systems is easier to describe: for instance, all its chain-recurrence classes are homoclinic classes [BC]. Until now, the unique known examples of tame dynamics have robustly transitive chain-recurrence classes, i.e. robustly isolated classes whose continuation remains transitive for any C^1 -perturbation. For this reason it was natural to ask:

Question 2 ([BC], problem 1.3). *Is there a dense subset \mathcal{D} in the space $\text{Diff}^1(M)$ of diffeomorphisms such that every robustly isolated chain-recurrence class of any $f \in \mathcal{D}$ is robustly transitive?*

The present paper gives a negative answer to this question. We denote there by $\text{Diff}^r(M)$ the space of C^r -diffeomorphisms of M .

Main Theorem (first version). *Any compact manifold M , with $\dim(M) \geq 3$, admits a smooth diffeomorphism f with a robustly isolated chain-recurrence class \mathcal{C}_f satisfying for any $r > 1$ the following property.*

The diffeomorphisms g such that the continuation \mathcal{C}_g is not transitive form a C^r -dense subset of a C^1 -neighborhood of f in $\text{Diff}^r(M)$.

Actually, the fact for a class to be robustly isolated brings several restriction on the dynamics. On surfaces, it is well-known that such a class needs to be far from homoclinic tangencies [New, PS]. Also it was shown in [BDP] that it presents dominated splittings and volume hyperbolicity. Indeed the dominated splitting is the structure which is antagonist to the homoclinic tangencies [W, G]. Conversely, we expect (see [B], conjecture 11) that, C^1 -generically, any chain-recurrent classes exhibiting enough hyperbolicity is robustly isolated: this should be the case for *partially hyperbolic* classes with a one-dimensional center bundle (see a precise definition in section 2.1).

The aim of this paper is to build an example of a robustly isolated class as close as possible to being hyperbolic, in particular C^1 -far from homoclinic tangencies, which nevertheless is not robustly transitive. There is the precise statement of our result.

Main Theorem. *When $\dim(M) \geq 3$, there exist a C^1 -open set $\mathcal{U} \subset \text{Diff}^r(M)$, $1 \leq r \leq \infty$, a C^r -dense subset $\mathcal{D} \subset \mathcal{U}$ and an open set $U \subset M$ with the following properties:*

- (I) Isolation: *For every $f \in \mathcal{U}$, the set $\mathcal{C}_f := U \cap \mathcal{R}(f)$ is a chain-recurrence class.*
- (II) Non-robust transitivity: *For every $f \in \mathcal{D}$, the class \mathcal{C}_f is not transitive.*

- (III) Partial hyperbolicity: For any $f \in \mathcal{U}$, the chain-recurrence class \mathcal{C}_f is partially hyperbolic with a one-dimensional central bundle.

More precisely:

- (1) For any $f \in \mathcal{U}$ there exists a subset $H_f \subset \mathcal{C}_f$ that coincides with the homoclinic class of any hyperbolic periodic $x \in \mathcal{C}_f$. Moreover, each pair of hyperbolic periodic points in \mathcal{C}_f with the same stable dimension is homoclinically related.
- (2) For any $f \in \mathcal{U}$ there exist two hyperbolic periodic points $p, q \in \mathcal{C}_f$ satisfying $\dim E_p^s = \dim E_q^s + 1$ and \mathcal{C}_f is the disjoint union of H_f with $W^u(p) \cap W^s(q)$. Moreover the points of $W^u(p) \cap W^s(q)$ are isolated in \mathcal{C}_f .
In particular, if $W^u(p) \cap W^s(q) \neq \emptyset$, the class \mathcal{C}_f is not transitive.
- (3) One has $\mathcal{D} := \{f \in \mathcal{U} : W^u(p) \cap W^s(q) \neq \emptyset\}$. Moreover, this set is a countable union of one-codimensional submanifolds of \mathcal{U} .
- (4) The chain-recurrent set of any $f \in \mathcal{U}$ is the union of \mathcal{C}_f with a finite number of hyperbolic periodic points (which depend continuously on f).

Remark 1. The isolated points $\mathcal{C}_f \setminus H_f$ are nonwandering for f . However, they do not belong to $\Omega(f|_{\Omega(f)})$ (since they are isolated in $\Omega(f)$ and non-periodic).

Let us give now the spirit of our construction. We consider two hyperbolic periodic points p and q with different stable dimension: $\dim(E_p^s) = \dim(E_q^s) + 1$. We assume moreover that they robustly belong to the same chain-recurrence class \mathcal{C}_f , which has a partially hyperbolic structure with 1 dimensional central bundle E^c . Such construction can be obtained using blenders, as in [BD₁].

We can guarantee that a central orientation is preserved so that each point in \mathcal{C}_f has a right and a left central direction. We will choose p to be extremal points of the class \mathcal{C}_f in the following sense (see section 2.2): we assume that the strong stable manifold $W^{ss}(p)$ of p just meets \mathcal{C}_f at p and that the intersection $\mathcal{C}_f \cap W^s(p)$ is contained in a right half submanifold bounded by $W^{ss}(p)$. In this case the class \mathcal{C}_f is contained in a cuspidal prism whose edge is the strong unstable manifold of p , see figure 1. One easily verifies that this hypothesis is robust: a fundamental domain of $W^{ss}(p)$ and of the left component of $W^s(p) \setminus W^{ss}(p)$ goes out of a filtrating neighborhood of \mathcal{C}_f . We call such a point p a *right stable cuspidal point*. Symmetrically one chooses q to be *left unstable cuspidal*: W^{uu} and a right half submanifold bounded by $W^{ss}(p)$ just meet \mathcal{C}_f at q .

As p and q belong to the same class, small perturbations produce heteroclinic intersections $W^u(p) \cap W^s(q)$. In particular, any such intersection belongs to \mathcal{C}_f but the simultaneous left and right cuspidal geometry implies that it is isolated

in \mathcal{C}_f . This formalizes into the following criterion that we use for obtaining our main theorem.

Isolated Point Criterion. *Let \mathcal{C}_f be a partially hyperbolic chain-recurrence class. Assume that its central bundle is one-dimensional and endowed with an invariant continuous orientation. Consider in \mathcal{C}_f a right stable cuspidal point p and a left unstable cuspidal point q , satisfying $\dim(E_p^s) = \dim(E_q^s) + 1$.*

Then, any intersection point $x \in W^u(p) \cap W^s(q)$ is isolated in \mathcal{C}_f .

Our construction uses strongly the existence of strong stable and strong unstable manifolds outside of the chain-recurrence class \mathcal{C}_f . Therefore this class cannot be an *attractor*, that is, a transitive set with a neighborhood U satisfying $\mathcal{C}_f = \bigcap_{n \geq 0} f^n(U)$; nor can it be an attractor for f^{-1} . The following question remains open.

Question 3. *Is there a dense G_δ subset $\mathcal{G} \subset \text{Diff}^1(M)$ such that every attractor of any $f \in \mathcal{G}$ is robustly transitive?*

2. A MECHANISM FOR HAVING ISOLATED POINTS IN A CHAIN-RECURRENCE CLASS

2.1. Preliminaries on invariant bundles. Consider $f \in \text{Diff}^1(M)$ preserving a set Λ .

A Df -invariant subbundle $E \subset T_\Lambda M$ is *uniformly contracted* (resp. *uniformly expanded*) if there exists $N > 0$ such that for every unit vector $v \in E$, we have

$$\|Df^N v\| < \frac{1}{2} \quad (\text{resp. } > 2).$$

A Df -invariant splitting $T_\Lambda M = E^{ss} \oplus E^c \oplus E^{uu}$ is *partially hyperbolic* if E^{ss} is uniformly contracted, E^{uu} is uniformly expanded, both are non trivial, and if there exists $N > 0$ such that for any $x \in \Lambda$ and any unit vectors $v_s \in E_x^{ss}$, $v_c \in E_x^c$ and $v_u \in E_x^{uu}$ we have:

$$\|Df^N v_s\| < \frac{1}{2} \|Df^N v_c\| < \frac{1}{4} \|Df^N v_u\|.$$

E^{ss} , E^c and E^{uu} are called the *strong stable*, *center*, and *strong unstable* bundles.

Remark 2. We will sometimes consider a Df -invariant continuous orientation of E^c . When Λ is the union of two different periodic orbits O_p, O_q and of a heteroclinic orbit $\{f^n(x)\} \subset W^u(O_p) \cap W^s(O_q)$, such an orientation exists if and only if above each orbit O_p, O_q , the tangent map Df preserves an orientation of the central bundle.

On a one-dimensional bundle, an orientation corresponds to a unit vector field tangent.

2.2. Cuspidal periodic points. Let p be a hyperbolic periodic point whose orbit is partially hyperbolic with a one-dimensional central bundle. When the central space is stable, there exists a strong stable manifold $W^{ss}(p)$ tangent to E_p^{ss} that is invariant by the iterates f^τ that fix p . It is contained in and separates the stable manifold $W^s(p)$ in two *half stable manifolds* which contain $W^{ss}(p)$ as a boundary.

Let us consider an orientation of E_p^c . The unit vector defining the orientation goes inward on one half stable manifold of p , that we call the *right half stable manifold* $R^s(p)$. The other one is called the *left half stable manifold* $L^s(p)$. These half stable manifolds are invariant by an iterate f^τ which fixes p if and only if the orientation of E_p^c is preserved by Df_p^τ . When the central space is unstable, one defines similarly the right and left half unstable manifolds $R^u(p), L^u(p)$.

Definition 1. A hyperbolic periodic point p is *right stable cuspidal* if:

- its orbit is partially hyperbolic, the central bundle is one-dimensional and stable;
- the left half stable manifold of p intersects the chain-recurrence class of p only at p .

Symmetrically one defines the *left stable cuspidal points*.

When the chain-recurrence class \mathcal{C} containing p is not reduced to the orbit O_p of p , this forces the existence of a Df -invariant orientation on the central bundle of O_p . In this case, the other half stable manifold intersects \mathcal{C} at points different from p . The choice of the name has to do with the geometry it imposes on $\mathcal{C} \cap W^s(p)$ in a neighborhood of p , see figure 1. This notion appears in [BD₄]. It is stronger than the notion of *stable boundary points* in [CP]. We can define in a similar way the *left/right unstable cuspidal points*.

Remark 3. If p is a stable cuspidal point, then the hyperbolic continuation p_g is still stable cuspidal for every g that is C^1 -close to f . Indeed, there exists a compact set $\Delta \subset L^s(p)$ which meets every orbit of $L^s(p) \setminus \{p\}$ and which is disjoint from $\mathcal{R}(f)$. By semi-continuity of the chain-recurrent set, a small neighborhood V of Δ is disjoint from $\mathcal{R}(g)$ for any g close to f and meets every orbit of the continuation of $L^s(p) \setminus \{p\}$.

2.3. Description of the mechanism. Let x be a point in a chain-recurrence class \mathcal{C} . We introduce the following assumptions (see figure 2).

- (H1) \mathcal{C} contains two periodic points p, q such that $\dim(E_p^s) = \dim(E_q^s) + 1$.
- (H2) The point x belongs to $W^u(p) \cap W^s(q)$. The union Λ of the orbits of x, p, q has a partially hyperbolic decomposition with a one-dimensional central bundle. There exists a Df -invariant continuous orientation of the central bundle over Λ

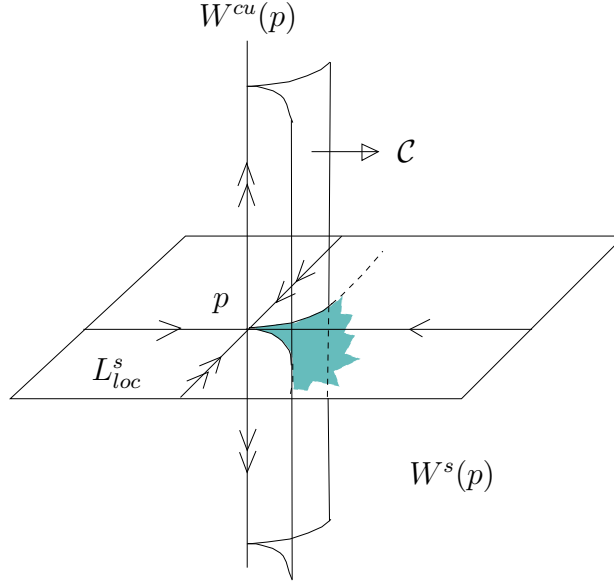


FIGURE 1. Geometry of a chain-recurrence class \mathcal{C} near a stable cuspidal fixed point.

- (H3) For the central orientation on Λ :
- (i) the point p is a right stable cuspidal point;
 - (ii) the point q is a left unstable cuspidal point.

Note that from remark 2 and the fact that a central orientation is preserved for cuspidal points, a Df -invariant continuous orientation of the central bundle over Λ always exists. The following proposition implies the isolation point criterion stated in the introduction.

Proposition 1. *Under (H1)-(H3), the point x is isolated in the chain-recurrence class \mathcal{C} . In particular, \mathcal{C} is not transitive and is not a homoclinic class.*

2.4. Proof of proposition 1. Let q be a periodic point whose orbit is partially hyperbolic and whose central bundle is one-dimensional and unstable. We shall assume that there is an orientation in E_q^c which is preserved by Df . We fix such an orientation of the central bundle E_q^c , so that the left and right half unstable manifolds of q are defined. We denote by $d^u + 1$ the unstable dimension of q .

Any $x \in W^s(q)$ has uniquely defined stable E_x^s and center stable E_x^{cs} directions: the first one is the tangent space $T_x W^s(q)$; a vector $v \in T_x M \setminus \{0\}$ belongs to the second if the direction of its positive iterates $Df^n(v)$ stays away from the directions of E_q^{uu} . If $E' \subset E$ are two vector subspaces of $T_x M$

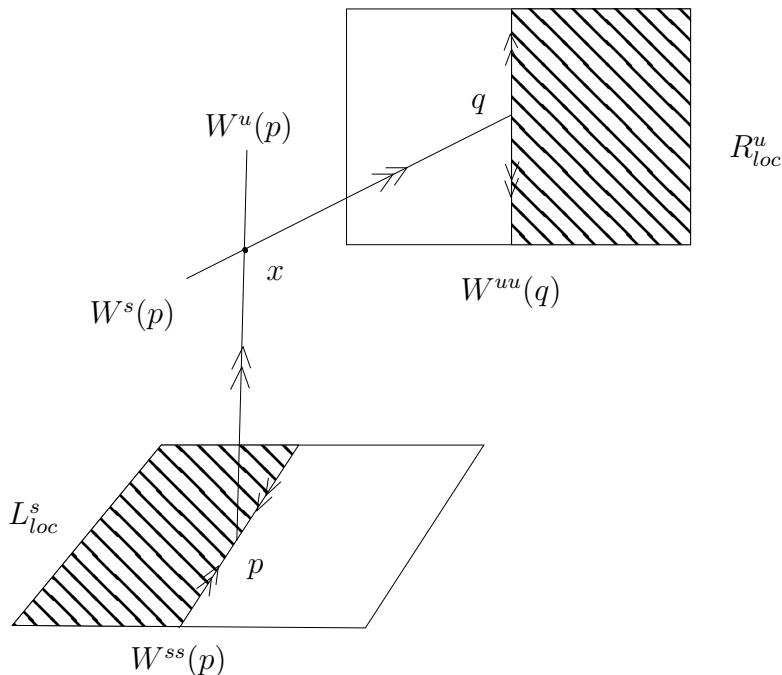


FIGURE 2. Hypothesis (H1)-(H3).

such that E is transverse to E_x^s and E' is transverse to E_x^{cs} (hence E' is one-codimensional in E), then $F = E_x^{cs} \cap E$ is a one-dimensional space whose forward iterates converge to the unstable bundle over the orbit of q . As a consequence, there exists an orientation of F which converges to the orientation of the central bundle by forward iterations.

There is thus a connected component of $E \setminus E'$, such that it intersects F in the orientation of F which converges towards the central orientation, its closure is the *right half plane* of $E \setminus E'$. The closure of the other component is the *left half plane* of $E \setminus E'$.

Consider a C^1 -embedding $\varphi: [-1, 1]^{d^u} \rightarrow M$ such that $x := \varphi(0)$ belongs to $W^s(q)$.

Definition 2. The embedding φ is *coherent with the central orientation at q* if

- $E := D_0\varphi(\mathbb{R}^{d^u+1})$ and $E' := D_0\varphi(\{0\} \times \mathbb{R}^{d^u})$ are transverse to E_z^s, E_z^{cs} respectively;
- the half-plaque $\varphi([0, 1] \times [-1, 1]^{d^u})$ is tangent to the right half-plane of $E \setminus E'$.

Let Δ^u be a compact set contained in $R^u(q) \setminus \{q\}$ which meets each orbit of $R^u(q) \setminus \{q\}$.

Lemma 2. *Let $\{\varphi_a\}_{a \in \mathcal{A}}$ be a continuous family of C^1 -embeddings that are coherent with the central orientation at q . Consider some $a_0 \in \mathcal{A}$ and a neighborhood V^u of Δ^u .*

Then, there exist $\delta > 0$ and some neighborhood A of a_0 such that any point $z \in \varphi_a([0, \delta] \times [-\delta, \delta]^{d^u})$ different from $\varphi_a(0)$ has a forward iterate in V^u .

Proof. Let $\tau \geq 1$ be the period of q and $\chi: [-1, 1]^d \rightarrow M$ be some coordinates such that

- $\chi(0) = q$;
- the image $D^u := \chi((-1, 1) \times \{0\}^{d-d^u-1} \times (-1, 1)^{d^u})$ is contained in $W_{loc}^u(q)$;
- the image $D^{uu} := \chi(\{0\}^{d-d^u} \times (-1, 1)^{d^u})$ is contained in $W_{loc}^{uu}(q)$;
- the image $D^{u,+} := \chi([0, 1] \times \{0\}^{d-d^u-1} \times (-1, 1)^{d^u})$ is contained in $R^u(q)$;
- $f^{-\tau}(\overline{D^u})$ is contained in D^u .

One deduces that there exists $n_0 \geq 0$ such that:

- (i) Any point z close to $\overline{D^{u,+}} \setminus f^{-\tau}(D^u)$ has an iterate $f^k(z)$, $|k| \leq n_0$, in V^u .

The graph transform argument (see for instance [KH, section 6.2]) gives the following generalization of the λ -lemma.

Claim. *There exists $N \geq 0$ and, for all a in a neighborhood A of a_0 , there exist some decreasing sequences of disks $(D_{a,n})$ of $[-1, 1]^{d^u+1}$ and $(D'_{a,n})$ of $\{0\} \times [-1, 1]^{d^u}$ which contain 0 and such that for any $n \geq N$ one has, in the coordinates of χ :*

- $f^{n\tau}(D_{a,n})$ is the graph of a function $D^u \rightarrow \mathbb{R}^{d-d^u-1}$ that is C^1 -close to 0;
- $f^{n\tau}(D'_{a,n})$ is the graph of a function $D^{uu} \rightarrow \mathbb{R}^{d-d^u}$ that is C^1 -close to 0.

Let us consider $a \in A$. The image by $f^{n\tau}$ of each component of $D_{a,n} \setminus D'_{a,n}$ is contained in a small neighborhood of a component of $D^u \setminus D^{uu}$. The graph $f^{n\tau}(D'_{a,n})$ which is transverse to a constant cone field around the central direction at q . Since φ is coherent with the central orientation at q , one deduces that

- (ii) $f^{n\tau} \circ \varphi_a([0, 1] \times [-1, 1]^{d^u}) \cap D_{a,n}$ is contained in a small neighborhood of $D^{u,+}$.

For $\delta > 0$ small, any point $z \in \varphi_a([-\delta, \delta] \times [-\delta, \delta]^{d^u})$ different from $\varphi_a(0)$ belongs to some $D_{a,n} \setminus D_{a,n+1}$, with $n \geq N$. Consequently:

- (iii) Any $z \in \varphi_a([- \delta, \delta] \times [- \delta, \delta]^{d^u}) \setminus \{\varphi_a(0)\}$ has a forward iterate in $D^u \setminus f^{-1}(D^u)$.

Putting the properties (i-iii) together, one deduces the announced property. \square

Proof of proposition 1. We denote by $d^s + 1$ (resp. $d^u + 1$) the stable dimension of p (resp. the unstable dimension of q) so that the dimension of M satisfies $d = d^s + d^u + 1$. Consider a C^1 -embedding $\varphi : [-1, 1]^d \rightarrow M$ with $\varphi(0) = x$ such that:

- $\varphi(\{0\} \times [-1, 1]^{d^s} \times \{0\}^{d^u})$ is contained in $W^s(q)$;
- $\varphi(\{0\} \times \{0\}^{d^s} \times [-1, 1]^{d^u})$ is contained in $W^u(p)$;
- $D_0\varphi.(1, 0^{d^s}, 0^{d^u})$ is tangent to E_x^c and has positive orientation.

Note that all the restrictions of φ to $[-1, 1] \times \{a^s\} \times [-1, 1]^{d^u}$ for $a^s \in \mathbb{R}^{d^s}$ close to 0, are coherent with the central orientation at q .

Consider a compact set $\Delta^u \subset R^u(q) \setminus \{q\}$ that meets each orbit of $R^u(q) \setminus \{q\}$. Since \mathcal{C} is closed and q is unstable cuspidal, there is a neighborhood V^u of Δ^u in M that is disjoint from \mathcal{C} . The lemma 2 can be applied: the points in $\varphi([0, \delta] \times \{a^s\} \times [-\delta, \delta]^{d^u})$ distinct from $\varphi(0, a^s, 0^{d^u})$ have an iterate in V^u , hence do not belong to \mathcal{C} . This shows that

$$\mathcal{C} \cap \varphi([0, \delta] \times [-\delta, \delta]^{d-1}) \subset \varphi(\{0\} \times [-\delta, \delta]^{d^s} \times \{0\}^{d^u}).$$

From (H2), if one reverses the central orientation and if one considers the dynamics of f^{-1} , then all the restrictions of φ to $[-1, 1] \times [-1, 1]^{d^s} \times \{a^u\}$ for $a^u \in \mathbb{R}^{d^u}$ close to 0, are coherent with the central orientation at p . One can thus argue analogously and gets:

$$\mathcal{C} \cap \varphi([- \delta, 0] \times [-\delta, \delta]^{d-1}) \subset \varphi(\{0\} \times \{0\}^{d^s} \times [-\delta, \delta]^{d^u}).$$

Both inclusions give that

$$\mathcal{C} \cap \varphi([- \delta, \delta]^d) = \{\varphi(0)\},$$

which says that $x = \varphi(0)$ is isolated in \mathcal{C} . \square

3. CONSTRUCTION OF THE EXAMPLE

In this part we build a collection of diffeomorphisms satisfying the properties (I) and (II) stated in the theorem. The construction will be made only in dimension 3 for notational purposes. The generalization to higher dimensions is straightforward.

3.1. Construction of a diffeomorphism. Let us consider an orientation-preserving C^∞ diffeomorphism H of the plane \mathbb{R}^2 and a closed subset $D = D^- \cup C \cup D^+$ such that:

- $H(\overline{D}) \subset \text{Int}(D)$ and $H(\overline{D^- \cup D^+}) \subset \text{Int}(D^-)$;
- the forward orbit of any point in D^- converges towards a sink $S \in D^-$;
- C is the cube $[0, 5]^2$ whose maximal invariant set is a hyperbolic horseshoe.

On $C \cap H^{-1}(C)$ the map H is piecewise linear, it preserves and contracts by $1/5$ the horizontal direction and it preserves and expands by 5 the vertical direction (see figure 3):

- The set $C \cap H(C)$ is the union of 4 disjoint vertical bands I_1, I_2, I_3, I_4 of width 1. We will assume that $I_1 \cup I_2 \subset (0, 2 + \frac{1}{3}) \times [0, 5]$ and $I_3 \cup I_4 \subset (2 + \frac{2}{3}, 5) \times [0, 5]$.
- The preimage $H^{-1}(C) \cap C$ is the union of 4 horizontal bands $H^{-1}(I_i)$. We will assume that $H^{-1}(I_1 \cup I_2) \subset [0, 5] \times (0, 2 + \frac{1}{3})$ and $H^{-1}(I_3 \cup I_4) \subset [0, 5] \times (2 + \frac{2}{3}, 5)$.

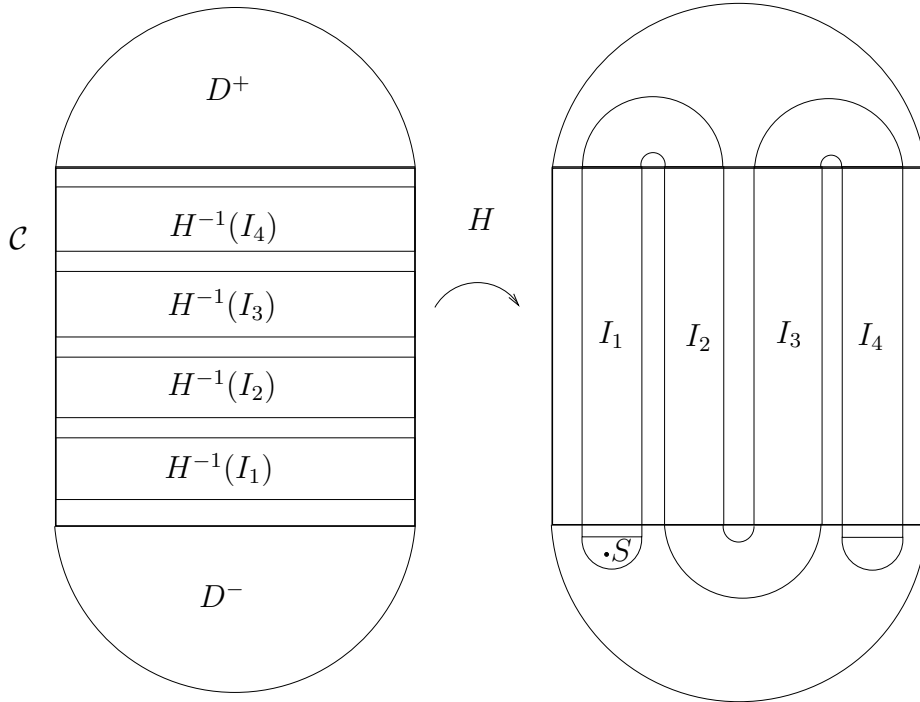


FIGURE 3. The map H .

We define a C^∞ diffeomorphism F of \mathbb{R}^3 whose restriction to a neighborhood of $D \times [-1, 6]$ it is a skew product of the form

$$F: (x, t) \mapsto (H(x), g_x(t)),$$

where the diffeomorphisms g_x are orientation-preserving and satisfy (see figure 4):

- (P1) g_x does not depend on x in the sets $H^{-1}(I_i)$ for every $i = 1, 2, 3, 4$.
- (P2) For every $(x, t) \in D \times [-1, 6]$ one has $4/5 < g'_x(t) < 6/5$.
- (P3) g_x has exactly two fixed points inside $[-1, 6]$, which are $\{0, 4\}$, $\{3, 4\}$, $\{1, 2\}$ and $\{1, 5\}$, when x belongs to $H^{-1}(I_i)$ for i respectively equal to 1, 2, 3 and 4. All fixed points are hyperbolic, moreover,
 - $g'_x(t) < 1$ for $t \in [-1, 3 + 1/2]$ and $x \in H^{-1}(I_1) \cup H^{-1}(I_2)$.
 - $g'_x(t) > 1$ for $t \in [1 + 1/2, 6]$ and $x \in H^{-1}(I_3) \cup H^{-1}(I_4)$.
- (P4) For every $(x, t) \in (D^- \cup D^+) \times [-1, 6]$ one has $g_x(t) > t$.

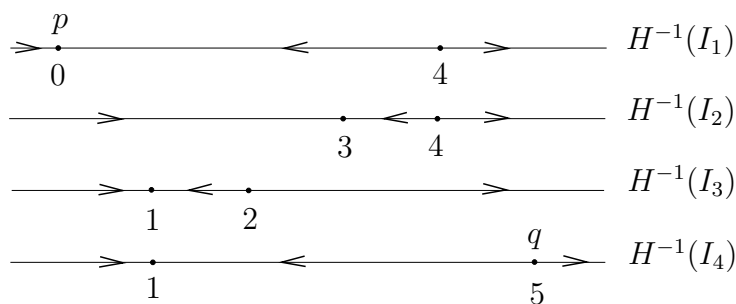


FIGURE 4. The map g_x above each rectangle $H^{-1}(I_i)$.

We assume furthermore that the following properties are satisfied:

- (P5) $F(D \times [6, 8]) \subset \text{Int}(D \times [6, 8])$;
- (P6) there exists a sink which attracts the orbit of any point of $D \times [6, 8]$;
- (P7) F coincides with a linear homothety outside a compact domain;
- (P8) any forward orbit meets $D \times [-1, 8]$.

One can build a diffeomorphism which coincides with the identity on a neighborhood of the boundary of $D_0 \times (-2, 9)$ and coincides with F in $D \times (-1, 8)$ (D_0 denotes a small neighborhood of D in \mathbb{R}^2). This implies that, on any 3-dimensional manifold, every isotopy class of diffeomorphisms contains an element whose restriction to an invariant set is C^∞ -conjugated to F .

On any 3-dimensional manifold, one can consider an orientation-preserving Morse-Smale diffeomorphism and by surgery replace the dynamics on a neighborhood of a sink by the dynamics of F . We denote by f_0 the obtained diffeomorphism.

3.2. First robust properties. We list some properties satisfied by f_0 , which are also satisfied by any diffeomorphism f in a small C^1 -neighborhood \mathcal{U}_0 of f_0 .

Fixed points: By (P3), in each rectangle $\text{Int}(I_i) \times (-1, 6)$, there exists two hyperbolic fixed points p_i, q_i . Their stable dimensions are respectively equal to 2 and 1. Since p_1 and q_4 will play special roles, we shall denote them as $p = p_1$ and $q = q_4$.

Isolation: The two open sets $V_0 = \text{Int}(D) \times (-1, 8)$ and $V_1 = V_0 \setminus (C \times [-1, 6])$ are isolating blocks, i.e. satisfy $f(\overline{V_0}) \subset V_0$ and $f(\overline{V_1}) \subset V_1$.

For V_0 , the property follows immediately from the construction. The closure of the second set V_1 can be decomposed as the union of:

- $D^+ \times [-1, 6]$, which is mapped into $(D^- \times [-1, 6]) \cup (D \times [6, 8])$,
- $D^- \times [-1, 6]$ which is also mapped into $(D^- \times [-1, 6]) \cup (D \times [6, 8])$ and moreover has a forward iterate in $D \times [6, 8]$ by (P4),
- $D \times [6, 8]$ which is mapped into itself and whose limit set is a sink.

Hence, any chain-recurrence class which meets the rectangle $C \times [-1, 6]$ is contained inside. The maximal invariant set of f in $C \times [-1, 6]$ will be denoted by \mathcal{C}_f .

Any chain-recurrence class which meets V_1 coincides with the sink of $D \times [6, 9]$.

Partial hyperbolicity (property (III) of the theorem): On $C \times [-1, 6] \subset \mathbb{R}^3$, there exists some narrow cone fields $\mathcal{E}^s, \mathcal{E}^{cs}$ around the coordinate direction $(1, 0, 0)$ and the plane $(x, 0, z)$ which are invariant by Df^{-1} . The vectors tangent to \mathcal{E}^s are uniformly expanded by Df^{-1} . Similarly there exists some forward invariant cone fields $\mathcal{E}^u, \mathcal{E}^{cu}$ close to the direction $(0, 1, 0)$ and the plane $(0, y, z)$.

In particular \mathcal{C}_f is partially hyperbolic. Moreover the tangent map Df preserves the orientation of the central direction such that any positive unitary central vector is close to the vector $(0, 0, 1)$.

Central expansion: Property (P2) holds for f when one replaces the derivative $g'_x(t)$ by the tangent map $\|Df|_{E^c}(x, t)\|$ along the central bundle.

Properties (H2) and (H3): The point p is stable cuspidal and the point q is unstable cuspidal. More precisely the left half plaque of $W^s(p)$ and the right half plaque of $W^u(q)$ are disjoint from \mathcal{C}_f : since

the chain-recurrence classes of p and q are contained in \mathcal{C}_f this implies property (H3). Moreover if there exists an intersection point $x \in W^u(p) \cap W^s(q)$ for f , then by the isolating property it is contained in \mathcal{C}_f . By preservation of the central orientation, (H2) holds also.

Let us explain how to prove these properties: it is enough to discuss the case of the left half-plaque of $W^s(p)$ and (arguing as in remark 3) to assume that $f = f_0$. From (P2) and (P3), we have:

- every point in $C \times [-1, 0)$ has a backward iterate outside $C \times [-1, 6]$;
- the same holds for every point in $(C \setminus I_1) \times \{0\}$;
- any point in $I_1 \times \{0\}$ has some backward image in $(C \setminus I_1) \times \{0\}$, unless it belongs to $W^u(p)$.

Combining these properties, one deduces that the connected component of $W^s(p) \cap (C \times [-1, 0])$ containing p intersects \mathcal{C}_f only at p . Note that this is a left half plaque of $W^s(p)$, giving the required property.

Hyperbolic regions: By (P3), the maximal invariant set in $Q_p := [0, 5] \times [0, 2 + \frac{1}{3}] \times [-1, 3 + \frac{1}{2}]$ and $Q_q := [0, 5] \times [2 + \frac{2}{3}, 5] \times [1 + \frac{1}{2}, 6]$ are two locally maximal transitive hyperbolic sets, denoted by K_p and K_q . Their stable dimensions are 2 and 1 respectively. The first one contains p, p_2 , the second one contains q, q_3 .

Tameness (property (4) of the theorem): Since f_0 has been obtained by surgery of a Morse-Smale diffeomorphism, the chain-recurrent set in $M \setminus \mathcal{C}_f$ is a finite union of hyperbolic periodic orbits.

Any $x \in \mathcal{C}_f$ has a strong stable manifold $W^{ss}(x)$. Its *local* strong stable manifold $W_{loc}^{ss}(x)$ is the connected component containing x of the intersection $W^{ss}(x) \cap C \times [-1, 6]$. It is a curve bounded by $\{0, 5\} \times [0, 5] \times [-1, 6]$. Symmetrically, we define $W^{uu}(x)$ and $W_{loc}^{uu}(x)$.

3.3. Central behaviours of the dynamics. We analyze the local strong stable and strong unstable manifolds of points of \mathcal{C}_f depending on their central position.

Lemma 3. *There exists an open set $\mathcal{U}_1 \subset \mathcal{U}_0$ such that for every $f \in \mathcal{U}_1$ and $x \in \mathcal{C}_f$:*

- (R1) *If $x \in R_1 := C \times [-1, 4 + \frac{1}{2}]$, then $W_{loc}^{uu}(x) \cap W^s(p) \neq \emptyset$.*
- (R2) *If $x \in R_2 := C \times [\frac{1}{2}, 6]$, then $W_{loc}^{ss}(x) \cap W^u(q) \neq \emptyset$.*
- (R3) *If $x \in R_3 := C \times [\frac{1}{2}, 2 + \frac{1}{2}]$, then $W_{loc}^{ss}(x) \cap W_{loc}^{uu}(y) \neq \emptyset$ for some $y \in K_p$.*
- (R4) *If $x \in R_4 := C \times [2 + \frac{1}{2}, 4 + \frac{1}{2}]$, then $W_{loc}^{uu}(x) \cap W_{loc}^{ss}(y) \neq \emptyset$ for some $y \in K_q$.*

Moreover p_2 belongs to R_2 and q_3 belongs to R_1 .

Proof. Properties (R1) and (R2) follow directly from the continuous variation of the stable and unstable manifolds. Similarly $p_2 \in R_2$ and $q_3 \in R_1$ by continuity.

We prove (R3) with classical blender arguments (see [BD₁] and [BDV, chapter 6] for more details). The set K_p is called *blender-horseshoes* in [BD₃, section 3.2].

A *cs-strip* \mathcal{S} is the image by a diffeomorphism $\phi : [-1, 1]^2 \rightarrow Q_p = [0, 5] \times [0, 2 + \frac{1}{3}] \times [-1, 3 + \frac{1}{2}]$ such that:

- The surface \mathcal{S} is tangent to the center-stable cone field and meets $C \times [\frac{1}{2}, 2 + \frac{1}{2}]$.
- The curves $\phi(t, [-1, 1])$, $t \in [-1, 1]$, are tangent to the strong stable cone field and crosses Q_p , i.e. $\phi(t, [-1, 1]) \subset \{0, 5\} \times [0, 2 + \frac{1}{3}] \times [-1, 3 + \frac{1}{2}]$.
- \mathcal{S} does not intersect $W_{loc}^u(p) \cup W_{loc}^u(p_2)$.

The *width* of \mathcal{S} is the minimal length of the curves contained in \mathcal{S} , tangent to the center cone, and that joins $\phi(-1, [-1, 1])$ and $\phi(1, [-1, 1])$.

Condition (P2) is important to get the following (see [BDV, lemma 6.6] for more details):

Claim. *There exists $\lambda > 1$ such that if \mathcal{S} is a cs-strip of width ε , then, either $f^{-1}(\mathcal{S})$ intersects $W_{loc}^u(p) \cup W_{loc}^u(p_2)$ or it contains at least one cs-strip with width $\lambda\varepsilon$.*

Proof. Using (P2), the set $f^{-1}(\mathcal{S}) \cap C \times [-1, 6]$ is the union of two bands crossing $C \times [-1, 6]$: the first has its two first coordinates near $H^{-1}(I_1)$, the second near $H^{-1}(I_2)$. Their width is larger than $\lambda\varepsilon$ where $\lambda > 1$ is a lower bound of the expansion of Df^{-1} in the central direction inside Q . We assume by contradiction that none of them intersects $W_{loc}^u(p) \cup W_{loc}^u(p_2)$, nor $C \times [\frac{1}{2}, 2 + \frac{1}{2}]$.

Since \mathcal{S} intersects $C \times [\frac{1}{2}, 2 + \frac{1}{2}]$, from conditions (P2) and (P3) the first band intersects $C \times [\frac{1}{2}, 4]$. By our assumption it is thus contained in $C \times (2 + \frac{1}{2}, 4]$. Using (P2) and (P3) again, this shows that \mathcal{S} is contained in $C \times (2, 4]$. The same argument with the second band shows that \mathcal{S} is contained in $C \times [-1, 2)$, a contradiction. \square

Repeating this procedure, we get an intersection point between $W_{loc}^u(p) \cup W_{loc}^u(p_2)$ and a backward iterate of the *cs-strip*. It gives in turn a transverse intersection point z between the initial *cs-strip* and $W^u(p) \cup W^u(p_2)$. By construction, all the past iterates of z belong to Q_p . Hence z has a well defined local strong unstable manifold. In particular, the intersection y between $W_{loc}^{uu}(z)$ and $W_{loc}^s(p)$ (which exists by (R1)) remains in Q_p both for future and past iterates, thus, it belongs to K_p .

For any point $x \in \mathcal{C}_f \cap R_3$, one builds a cs -strip by thickening in the central direction the local strong stable manifold. We have proved that this cs -strip intersects $W_{loc}^{uu}(y)$ for some $y \in K_p$. One can consider a sequence of thinner strips. Since K_p is closed and the local strong unstable manifolds vary continuously, we get at the limit an intersection between $W_{loc}^{ss}(x)$ and $W_{loc}^{uu}(y')$ for some $y' \in K_p$ as desired.

This gives (R3). Property (R4) can be obtained similarly. \square

We have controled the local strong unstable manifold of points in $R_1 \cup R_4$ and the local strong stable manifold of points in $R_2 \cup R_3$. Since neither $R_1 \cup R_4$ nor $R_2 \cup R_3$ cover completely $C \times [-1, 6]$ we shall also make use of the following result:

Lemma 4. *For every diffeomorphism in a small C^1 -neighborhood $\mathcal{U}_2 \subset \mathcal{U}_0$ of f_0 , the only point whose complete orbit is contained in $C \times [-1, \frac{1}{2}]$ is p ; symmetrically, the only point whose complete orbit is contained in $C \times [4 + \frac{1}{2}, 6]$ is q .*

Proof. We argue as for property (H3) in section 3.2: the set of points whose past iterates stay in $C \times [-1, \frac{1}{2}]$ is the local strong unstable manifold of p . Since p is the only point in its local unstable manifold whose future iterates stay in $C \times [-1, \frac{1}{2}]$ is p we conclude. \square

3.4. Properties (I) and (II) of the theorem. We now check that (I) and (II) hold for the region $U = \text{Int}(C \times [-1, 6])$ and the neighborhood $\mathcal{U} := \mathcal{U}_1 \cap \mathcal{U}_2$.

Proposition 5. *For any $f \in \mathcal{U}$, $x \in \mathcal{C}_f$, there are arbitrarily large $n_q, n_p \geq 0$ such that $W_{loc}^{uu}(f^{n_q}(x)) \cap W^{ss}(y_q) \neq \emptyset$ and $W_{loc}^{ss}(f^{-n_p}(x)) \cap W^{uu}(y_p) \neq \emptyset$ for some $y_q \in K_q$, $y_p \in K_p$.*

Proof. If $\{f^n(x), n \geq n_0\} \subset C \times [4 + \frac{1}{2}, 6]$, for some $n_0 \geq 0$, then $x \in W^{ss}(q)$ by lemma 4.

In the remaining case, there exist some arbitrarily large forward iterates $f^n(x)$ in R_1 , so that $W_{loc}^{uu}(f^n(x))$ meets $W^s(p)$ by lemma 3. Since p is homoclinically related with p_2 , by the λ -lemma there exists $k \geq 0$ such that $f^k(W_{loc}^{uu}(f^n(x)))$ contains $W_{loc}^{uu}(x')$ for some $x' \in W^s(p_2) \cap R_4$ because $p_2 \in R_4$. By lemma 3, $f^k(W_{loc}^{uu}(f^n(x)))$ intersects $W_{loc}^{ss}(y'_q)$ for some $y'_q \in K_q$ showing that $W_{loc}^{uu}(f^n(x)) \cap W^{ss}(y_q) \neq \emptyset$ with $y_q = f^{-k}(y'_q)$ in K_q .

We have obtained the first property in all the cases. The second property is similar. \square

The following corollary (together with the isolation property of section 3.2) implies that for every $f \in \mathcal{U}$, the properties (I) and (H1) are verified.

Corollary 6. *For every $f \in \mathcal{U}$ the set \mathcal{C}_f is contained in a chain-transitive class.*

Proof. For any $\varepsilon > 0$ and $x \in \mathcal{C}_f$, there exists a ε -pseudo-orbit $p = x_0, x_1, \dots, x_n = p$, $n \geq 1$, which contains x . Indeed by proposition 5, and using that K_p, K_q are transitive and contain respectively p and q_3 , there exists a ε -pseudo-orbit from p to q_3 which contains x . By lemma 3, the unstable manifold of q_3 intersects the stable manifold of p , hence there exists a ε -pseudo-orbit from q_3 to p . We take the concatenation of these pseudo-orbits. \square

Now, we show that (H2) holds for a C^r dense set \mathcal{D} of \mathcal{U} . Since (H1) and (H3) are satisfied, proposition 1 implies that the property (II) of the theorem holds with the set $\mathcal{D} \subset \mathcal{U}$. In fact, as we noticed in section 3.2 it is enough to get the following.

Corollary 7. *For every $r \geq 1$, the set*

$$\mathcal{D} = \{f \in \mathcal{U}, W^u(p) \cap W^s(q) \neq \emptyset\}$$

is dense in $\mathcal{U} \cap \text{Diff}^r(M)$. It is a countable union of one-codimensional submanifolds.

In the C^1 topology, this result is direct consequence of the connecting lemma (together with proposition 5). The additional structure of our specific example allows to make these perturbations in any C^r -topology.

Proof. Fix any $f \in \mathcal{U}$. By proposition 5, there exists $x \in K_q$ such that $W^u(p)$ intersects $W^{ss}(x)$ at a point y (notice that $y \notin K_q \cup \{p\}$). Let U be a neighborhood of y such that:

- U is disjoint from the iterates of y , i.e. $\{f^n(y) : n \in \mathbb{Z}\} \cap U = \{y\}$;
- U is disjoint from $K_q \cup \{p\}$.

Given a C^r neighborhood \mathcal{V} of the identity, there exists a neighborhood $V \subset U$ of y such that, for every $z \in V$, the set \mathcal{V} contains a diffeomorphism g_z which coincides with the identity in the complement of U and maps y at z .

Since K_q is locally maximal, there exists $\bar{x} \in K_q \cap W^s(q)$ near x . In particular $W_{loc}^{ss}(\bar{x})$ intersects V in a point z whose backward orbit is disjoint from U .

For the diffeomorphism $h = g_z \circ f$ (which is C^r -close to f) the manifolds $W^s(q)$ and $W^u(p)$ intersect. Indeed both f and h satisfy $f^{-1}(y) \in W^u(p)$ and $z \in W_{loc}^{ss}(\bar{x})$. Since $W_{loc}^{ss}(\bar{x}) \subset W^{ss}(q)$ and $h(f^{-1}(y)) = z$ we get the conclusion.

For each integer $n \geq 1$, the manifolds $f^n(W_{loc}^{uu}(p))$ and $W_{loc}^{ss}(q)$ have disjoint boundary and intersect in at most finitely many points. One deduces that the set \mathcal{D}_n of diffeomorphisms such that they intersect is a finite union of one-codimensional submanifold of \mathcal{U} . The set \mathcal{D} is the countable union of the \mathcal{D}_n . \square

3.5. Other properties. We here show properties (1), (2) and (3) of the theorem.

Proposition 8. *For every $f \in \mathcal{U}$ and $x \in \mathcal{C}_f$ we have:*

- *If $x \notin W^s(q)$, there exist large $n \geq 0$ such that $W_{loc}^{uu}(f^n(x)) \cap W^s(p) \neq \emptyset$.*
- *If $x \notin W^u(p)$, there exists large $n \geq 0$ such that $W_{loc}^{ss}(f^{-n}(x)) \cap W^u(q) \neq \emptyset$.*

Moreover, in the first case x belongs to the homoclinic class of p and in the second it belongs to the homoclinic class of q .

Proof. By lemma 4, any point $x \in \mathcal{C}_f \setminus W^s(q)$ has arbitrarily large iterates $f^n(x)$ in R_1 , proving that $W_{loc}^{uu}(f^n(x)) \cap W^s(p) \neq \emptyset$.

In particular, $W^s(p)$ intersects transversally $W_{loc}^{uu}(x)$ at points arbitrarily close to x . On the other hand by proposition 5, there exists a sequence z_n converging to x and points $y_n \in K_p$ such that $z_n \in W^u(y_n)$ for each n , proving that $W_{loc}^{uu}(z_n)$ intersects $W^u(p)$ transversally at a point close to x when n is large. By the λ -lemma, $W_{loc}^{uu}(y_n)$ is the C^1 -limit of a sequence of discs contained in $W^u(p)$. This proves that $W^u(p)$ and $W^s(p)$ have a transverse intersection point close to x , hence x belongs to the homoclinic class of p .

The other properties are obtained analogously. \square

Let H_f denotes the homoclinic class of p . The next gives property (1) of the theorem.

Corollary 9. *For every $f \in \mathcal{U}$, the homoclinic class of any hyperbolic periodic point of \mathcal{C}_f coincides with H_f . Moreover, the periodic points in \mathcal{C}_f of the same stable index are homoclinically related.*

Proof. Let $z \in \mathcal{C}_f$ be a hyperbolic periodic point whose stable index is 2. By proposition 5 $W^{ss}(z)$ intersects $W_{loc}^{uu}(y)$ for some $y \in K_p$, this implies that $W^s(z)$ intersects $W_{loc}^{uu}(y)$ and since $W_{loc}^{uu}(y)$ is accumulated by $W^u(p)$ we get that $W^s(z)$ intersects $W^u(p)$. Now, by proposition 8, $W^u(z)$ intersects $W^s(p)$. Moreover the partial hyperbolicity implies that the intersections are transversal, proving that z and p are homoclinically related. One shows in the same way that any hyperbolic periodic point whose stable index is 1 is homoclinically related to q .

It remains to prove that the homoclinic classes of p and q coincide. The homoclinic class of q contains a dense set of points x that are homoclinic to q_3 . In particular, x does not belong to $W^u(q)$, hence belongs to the homoclinic class of p by proposition 8. This gives one inclusion. The other one is similar. \square

Properties (2) and (3) of the theorem follow from corollary 7 and the following.

Corollary 10. *For every $f \in \mathcal{U}$ we have $\mathcal{C}_f \setminus H_f = W^s(q) \cap W^u(p)$.*

Proof. By corollary 9, a point $x \in \mathcal{C}_f \setminus H_f$ does not belong to the homoclinic class of q (nor to the homoclinic class of p by definition of H_f). Proposition 8 gives $\mathcal{C}_f \setminus H_f \subset W^s(q) \cap W^u(p)$. Proposition 1 proves that the points of $W^s(q) \cap W^u(p)$ are isolated in \mathcal{C}_f . Since any point in a non-trivial homoclinic class is limit of a sequence of distinct periodic points of the class we conclude that $W^s(q) \cap W^u(p)$ and H_f are disjoint. \square

The proof of the theorem is now complete.

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