WILD MILNOR ATTRACTORS ACCUMULATED BY LOWER DIMENSIONAL DYNAMICS

RAFAEL POTRIE

Abstract. We present new examples of open sets of diffeomorphisms such that a generic diffeomorphisms in those sets have no dynamically indecomposable attractors in the topological sense and have infinitely many chain-recurrence classes. We show that except from one particular class, the other classes are contained in periodic surfaces. This study allows us to obtain existence of Milnor attractors as well as studying ergodic properties of the diffeomorphisms in those open sets by using the ideas and results from [BV] and [BF].

1. Introduction

1.0.1. In 1987, A. Araujo in his thesis ([A]) announced that C^1-generic diffeomorphisms of compact surfaces have hyperbolic attractors. In fact, he claimed to have proved that for a residual subset of diffeomorphisms on a compact surface, either there are infinitely many sinks or there are finitely many hyperbolic attractors whose basin cover a full Lebesgue measure of the manifold. The proof seems to have a gap, but the techniques in [PS] allow to overcome them (and with the recent results of C^1 generic dynamics this can be proven rather easily[1]).

In contrast, an astonishing example was recently constructed in [BLY] where they showed that there exist open sets of diffeomorphisms in any manifold of dimension ≥ 3 such that every C^r-generic diffeomorphism of those open subsets have no attractors and there is an attracting region having infinitely many distinct chain recurrence classes. We recommend reading the introduction of [BLY] for more on the history of this important problem.

The construction in [BLY] relies on some modification of the well known solenoid attractor. Although the construction is rather simple, it is not well understood how is that other chain recurrence classes coexist in this attracting region.

1.0.2. In this paper we propose a new kind of example starting from a non hyperbolic DA attractor (based on an example of [Car], see also [BV]) which allows us to use the

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[1] See [Pot].

properties of semiconjugacy with a linear Anosov diffeomorphisms, and gives a more satisfactory picture of how the infinitely many chain-recurrence classes behave.

Also, this study allows us to obtain some remarkable features of this examples from the ergodic point of view which can be summarized in the following statement (to be stated in a precise form in section 1.2).

Theorem. *There exists a C^1-open set $\mathcal{U}$ of* $\mathrm{Diff}^r(\mathbb{T}^3)$ *($r \geq 1$) and a C^r-generic subset $\mathcal{G}^r \subset \mathcal{U}$ such that every $f \in \mathcal{G}^r$ has no attractors and f has infinitely many chain-recurrence classes. Moreover:*

- *For every $f \in \mathcal{U}$ there exists a chain-recurrence class H such that every chain-recurrence class R different from H is contained in a periodic surface.*
- *For every $f \in \mathcal{U}$ there exists a unique attractor in the sense of Milnor.*
- *For every $f \in \mathcal{U}$ there exists a unique entropy maximizing measure.*
- *If $r \geq 2$ and $f \in \mathcal{U}$ then f admits a unique SRB measure.*

1.0.3. Besides this ergodic point of view, there is another motivation in studying the examples here proposed. Recently, C. Bonatti has proposed a program for studying C^1-generic dynamics ([B]) and in particular, what is known as wild dynamics (see section 1.1.3). He has defined viral homoclinic classes as those classes essentially having a reproductive behavior (until now, the only known mechanism for generating wild dynamics, see [BD2, BCDG]). Even if we are not able to prove that the examples here presented are not viral (we shall not define this notion here, see [B] or [BCDG] for a precise definition), the fact that all chain-recurrence classes except one are contained in periodic surfaces seems to represent a different mechanism for generating wild dynamics.

On the other hand, when studying wild homoclinic classes[2] with a partially hyperbolic structure, it has been announced by C. Bonatti and K. Shinohara that the examples from [BLY] are viral and it seems that the main feature differentiating the behaviors is the topology of the intersections between the homoclinic classes and the center-stable manifolds, this becomes clear in our Proposition 2.1.

1.1. **Some definitions and results which will be used.**

We shall give some definitions and state some results we shall use along the paper, it may be wise to skip this section and return to it when not knowing some definition or when it is referred to by the text.

1.1.1. **Conley's theory and Bonatti-Crovisier's result.**

Given a homeomorphism $f : M \to M$ we can define the following relation on M: we denote $x \dashv y$ whenever for every $\varepsilon > 0$ there exists an $\varepsilon-$pseudo-orbit from x to y, that is, there exists a set of points $x = z_0, \dots, z_n = y$ such that $n \geq 1$ and $d(f(x_i), x_{i+1}) < \varepsilon$.

[2] To be defined in section 1.1.3.

We denote as

$$R(f) = \{x \in M \ : \ x \dashv x\}$$

the *chain-recurrent set* of f. In $R(f)$ the relation $x \vdash\dashv y$ (given by $x \vdash\dashv y$ if and only if $x \dashv y$ and $y \dashv x$) is an equivalence relation, we shall call its equivalence classes *chain-recurrence classes*. An invariant set will be called *chain-transitive* if it is transitive under the relation $\vdash\dashv$.

An open set U is a *filtrating neighborhood* if there exists V_1, V_2 open sets such that $U = V_1 \backslash \overline{V}_2$ and $f(\overline{V_i}) \subset V_i$ for $i = 1, 2$. It is not hard to see that a filtrating neighborhood contains each chain-recurrence class it intersects.

Conley's theorem (see [C2]) implies that given two different chain-recurrence classes $\mathcal{C}_1$ and $\mathcal{C}_2$, there exists a filtrating neighborhood containing $\mathcal{C}_1$ which does not intersect $\mathcal{C}_2$. A chain-recurrence class $\mathcal{C}$ is *isolated* if there exists a filtrating neighborhood U such that $U \cap R(f) = \mathcal{C}$.

We shall pay special attention to certain particular chain-recurrence classes: We say that a compact invariant set Q is a *quasi-attractor* if it is a chain-recurrence class and there exists a decreasing sequence of open neighborhoods $\{U_n\}$ such that $\bigcap U_n = Q$ and $f(\overline{U_n}) \subset U_n$. An important feature is that for quasi-attractors one has that if a point y verifies that there exists $x \in Q$ such that $x \dashv y$, then $y \in Q$, in particular, Q is saturated by unstable sets.

We say that a compact invariant set Q is an *attractor*[3] if it is a quasi-attractor and is isolated as chain-recurrence class.

For a diffeomorphism f of M and a point $x \in M$ we define

$$W^s(x) = \{y \in M \ : \ d(f^n(x), f^n(y)) \to 0 \ \ n \to +\infty\}$$

and

$$W^u(x) = \{y \in M \ : \ d(f^n(x), f^n(y)) \to 0 \ \ n \to -\infty\}$$

the stable and unstable sets of x.

For hyperbolic periodic points, it is well known that this sets are C^1 injectively immersed manifolds, the *stable index* of a periodic point is the dimension of its stable manifold. We define the *homoclinic class of a periodic point* p as the closure of the transversal intersections between the stable and unstable set of the points in the orbit of p (i.e. $H(p) = \overline{W^s(\mathcal{O}(p)) \pitchfork W^u(\mathcal{O}(p))}$).

[3]It is more usual to find in the literature the following definition: A compact f-invariant set Λ is an *attractor* if it is contains a dense orbit and there is a neigbhorhood U of Λ such that $f(\overline{U}) \subset U$ and $\Lambda = \bigcup_{n\geq 0} f^n(\overline{U})$. Our definition coincides with this one except that we demand the weaker indecomposability hypothesis of being chain-transitive instead of transitive.

If for a point $x \in M$ we have that $W^\sigma(x)$ is a manifold, we shall denote as $W^\sigma_L(x)$ as the disk of radius L centered at x in $W^\sigma(x)$ with the Riemannian metric induced in $W^\sigma(x)$ by the immersion.

It was proved in [BC] that for a residual (G_δ-dense) subset $\mathcal{G}_{BC}$ of $\text{Diff}^1(M)$ one has the following properties:

- Every periodic point of f is hyperbolic (this is the well known Kupka-Smale's theorem).
- $R(f) = \overline{Per(f)}$ where $Per(f)$ denotes the set of periodic points of f.
- If a chain-recurrence class $\mathcal{C}$ of f contains a periodic point p, then $\mathcal{C}$ coincides with its homoclinic class $H(p)$.
- For a residual set of points $G \subset M$ the omega-limit set is a quasi-attractor.

Since homoclinic classes are always transitive, we get from this result that for C^1-generic diffeomorphisms (this will stand for diffeomorphisms in a residual subset of $\text{Diff}^1(M)$) the definition of attractor we gave coincides with the usual one.

1.1.2. **Partial hyperbolicity.** Given a diffeomorphism $f : M \to M$, a compact f-invariant subset $\Lambda \subset M$ and two Df-invariant subbundles E and F of $T_\Lambda M$ we say that F *dominates* E if there exists $N > 0$ such that for every $x \in \Lambda$ and every pair of unit vectors $v \in E$, $w \in F$ we have that:

$$\|Df_x^N v\| < \frac{1}{2}\|Df_x^N w\|$$

Whenever there exists a cone field $\mathcal{E}$ (of constant dimension) such that for every $x \in \Lambda$ one has that $D_x f\mathcal{E}(x) \subset int(\mathcal{E}(f(x)))$ there exists a (unique) Df-invariant sum $T_\Lambda M = E \oplus F$ with $F \subset \mathcal{E}$ and $E \subset \mathcal{E}^c$ and such that F dominates E (see [BDV] appendix B).

We say that Λ is *partially hyperbolic* provided that $T_\Lambda M$ decomposes as a Df-invariant sum $T_\Lambda M = E^{cs} \oplus E^u$ and there exists $N > 0$ such that the following conditions hold:

- E^u dominates E^{cs}.
- For every $x \in \Lambda$ and every unit vector v^u in $E^u(x)$ we have that $\|Df_x^N v^u\| > 2$.

The definitions found in the literature (see particularly [BDV] appendix B or [C2]) require that either f or f^{-1} is partially hyperbolic under our definition. We shall not be concerned with this fact since it will be clear how to adapt the results here to that definition.

In general (see [BDV] appendix B or [HPS]) we have that the bundle E^u integrates into a $f-$invariant lamination $\mathcal{F}^u$ of leaves tangent to E^u which we shall call *strong unstable manifolds*.

When E^{cs} also integrates into a f-invariant lamination $\mathcal{F}^{cs}$ tangent to E^{cs} at Λ we shall say that E^{cs} is *coherent*.

The leaf of $\mathcal{F}^\sigma$ through x shall be denoted as $\mathcal{F}^\sigma(x)$ and $\mathcal{F}^\sigma_L(x)$ will denote the ball of radius L centered at x in $\mathcal{F}^\sigma(x)$ with the induced metric ($\sigma = u, cs$). Notice that for every point $x \in \Lambda$ we have that $\mathcal{F}^u(x) \subset W^u(x)$.

By *lamination* on a set K we mean a collection of disjoint C^1 injectively immersed manifolds of the same dimension (called *leaves*) such that there exists a compact metric space Γ such that for every point $x \in K$ there exists a neighborhood U and a homeomorphism $\varphi : U \cap K \to \Gamma \times \mathbb{R}^d$ such that if L is a leaf of the lamination and $\tilde{L}$ a connected component of $L \cap U$ then $\varphi|_{\tilde{L}}$ is a C^1-diffeomorphism to $\{s\} \times \mathbb{R}^d$ for some $s \in \Gamma$.

When the laminated set K is the whole manifold, we say that the lamination is a *foliation*.

Consider a compact invariant set Λ which is partially hyperbolic with splitting $E^{cs} \oplus E^u$ and such that f is coherent in Λ. Given an open set U of Λ and $x \in \Lambda$, we denote as $\mathcal{F}^{cs}_U(x)$ to the connected component of $\mathcal{F}^{cs}(x) \cap U$ containing x. If $y \in \mathcal{F}^u(x)$ is close to x we can define the *unstable holonomy* $\Pi^{uu}_{x,y}$ from a neighborhood of x in $\mathcal{F}^{cs}_U(x)$ to a neighborhood of y in $\mathcal{F}^{cs}_U(y)$ as projecting the points along the unstable leaves, which is a continuous injective map.

1.1.3. **Non-isolated classes for the C^1-topology.** We say that a diffeomorphism $f \in \text{Diff}^1(M)$ is *tame* if it belongs to the interior of the diffeomorphisms having only finitely many chain-recurrence classes. A diffeomorphism is *wild* if it cannot be accumulated by tame diffeomorphisms in the C^1-topology. We get that in $\text{Diff}^1(M)$ the union of tame and wild diffeomorphisms is open and dense.

When f is wild and C^1-generic, as a consequence of the results of [BC] we have that it has infinitely many chain-recurrence classes, in particular, it will have at least one which is not isolated. When a homoclinic class $H(p)$ is not isolated, we shall say that it is a *wild homoclinic class*.

As a consequence of the main result in [BDP], we obtain the following criterium for partially hyperbolic homoclinic classes to be wild:

Theorem 1.1 ([BDP]). *There exists a residual subset $\mathcal{G}_{BDP}$ of $\text{Diff}^1(M)$ such that if a homoclinic class $H(p)$ verifies that:*

- *The homoclinic class $H(p)$ admits a partially hyperbolic splitting of the form $T_{H(p)}M = E^{cs} \oplus E^u$.*
- *The subbundle E^{cs} admits no decomposition into non-trivial $Df-$invariant subbundles which are dominated.*
- *There is a periodic point $q \in H(p)$ such that $\det(Df_q^{\pi(q)}|_{E^{cs}(q)}) > 1$.*

Then, $H(p)$ is contained in the closure of the set of periodic sources[4] *of f. In particular, $H(p)$ is a wild homoclinic class.*

1.1.4. **Robust tangencies and C^r-generic non-isolation.** As well as in the case of C^1-topology, we can obtain a similar criterium to obtain non-isolation of a homoclinic class for C^r-generic diffeomorphisms combining the main results of [BD3] and [PV]. The only cost will be that we must consider a new open set and that the accumulation by other classes is not as well understood (in Theorem 1.1 we obtained that the class is contained in the closure of the sources, and here we shall only obtain that the class intersects the closure of the sources). We state a consequence of the results in those papers in the following result. We shall only use the result in dimension 3, so we state it in this dimension, it can be modified in order to hold in higher dimension but it would imply defining sectionally dissipative saddles (see [PV]).

Theorem 1.2 ([BD3] and [PV]). *Consider $f \in \mathrm{Diff}^r(M)$ with $\dim M = 3$ and a C^1-open set $\mathcal{U}$ of $\mathrm{Diff}^r(M)$ $(r \geq 1)$ such that there exists a hyperbolic periodic point p of f such that its continuation p_g is well defined for every $g \in \mathcal{U}$ and such that:*

- *The homoclinic class $H(p_g)$ admits a partially hyperbolic splitting of the form $T_{H(p)}M = E^{cs} \oplus E^u$ for every $g \in \mathcal{U}$.*
- *The subbundle E^{cs} admits no decomposition in non-trivial $Dg-$invariant subbundles which are dominated.*
- *There is a periodic point $q \in H(p_f)$ such that $\det(Df_q^{\pi(q)}|_{E^{cs}(q)}) > 1$.*

Then, there exists a C^1-open and dense subset $\mathcal{U}_1 \subset \mathcal{U}$ and a C^r-residual subset $\mathcal{G}_{PV}$ of $\mathcal{U}_1$ such that for every $g \in \mathcal{G}_{PV}$ one has that $H(p_g)$ intersects the closure of the set of periodic sources of g.

The conditions of the Theorem are used in [BD3] in order to create robust tangencies for a hyperbolic set for diffeomorphisms in an C^1-open and dense subset $\mathcal{U}_1$ of $\mathcal{U}$. Then, using similar arguments as in [BLY] (section 3.7) one creates tangencies associated with periodic orbits which are sectionally dissipative for f^{-1} which allows to use the results in [PV] to get the conclusion.

1.1.5. **Milnor attractors, SRB measures and entropy maximizing measures.** Following [Mi], we shall say that a compact invariant chain transitive set Λ is an *attractor in the sense of Milnor* or *Milnor attractor* if and only if the basin $B(\Lambda)$ of Λ has positive Lebesgue measure and for every $\tilde{\Lambda} \subsetneq \Lambda$ compact invariant set, its basin $B(\tilde{\Lambda})$ has strictly smaller measure. Moreover, if for every $\tilde{\Lambda} \subsetneq \Lambda$ compact invariant set, we have that $Leb(B(\tilde{\Lambda})) = 0$, we will say that Λ is a *minimal attractor in the sense of Milnor* or *minimal Milnor attractor*.

[4]Periodic repelling points. Notice that the chain-recurrence class of a periodic source is reduced to the source itself.

Here, *basin* must be understood as the set of points whose forward iterates converge to the compact set (it must not be confused with the statistical basin which is quite more restrictive).

We say that an invariant measure μ is an *SRB-measure* whenever its statistical basin has positive Lebesgue measure, this means that there exists a positive Lebesgue measure set $B(\mu)$ such that for every $x \in B(\mu)$ one has that for every continuous function $\varphi : M \to \mathbb{R}$

$$\frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) \to \int \varphi d\mu$$

We refer the reader to [BDV] chapter 11 (in particular section 11.2) for a nice introduction to SRB measures in this exact context. Notice that the existence of an ergodic SRB measure implies the existence of a minimal Milnor attractor.

Finally, we recall that an ergodic invariant measure μ is called *entropy maximizing measure* for a homeomorphism f whenever its measure-theoretical entropy coincides with the topological entropy of f. In [BF] (see also [BFSV]) some conditions where studied which imply existence and uniqueness of these measures, this conditions will be satisfied on our example.

1.2. **Precise statement of results.** Now we are in conditions to state our main results.

Theorem A. *There exists a* C^1*-open set* $\mathcal{U}$ *of* $\text{Diff}^r(\mathbb{T}^3)$ *such that:*

(a) *For every* $f \in \mathcal{U}$ *we have that* f *is partially hyperbolic with splitting* $TM = E^{cs} \oplus E^u$ *and* E^{cs} *integrates into a* f*-invariant foliation* $\mathcal{F}^{cs}$*.*
(b) *Every* $f \in \mathcal{U}$ *has a unique quasi-attractor* Q_f *which contains a homoclinic class.*
(c) *Every chain recurrence class* $R \neq Q_f$ *is contained in the orbit of a periodic disk in a leaf of the foliation* $\mathcal{F}^{cs}$*.*
(d) *There exists a residual subset* $\mathcal{G}^r$ *of* $\mathcal{U}$ *such that for every* $f \in \mathcal{G}^r$ *the diffeomorphism* f *has no attractors. In particular,* f *has infinitely many chain-recurrence classes accumulating on* Q_f*.*
(e) *For every* $f \in \mathcal{U}$ *there is a unique Milnor attractor* $\tilde{Q} \subset Q_f$*.*
(f) *If* $r \geq 2$ *then every* $f \in \mathcal{U}$ *has a unique SRB measure whose support coincides with a homoclinic class. Consequently,* $\tilde{Q}$ *is a minimal attractor in the sense of Milnor. If* $r = 1$*, then there exists a residual subset* $\mathcal{G}_M$ *of* $\mathcal{U}$ *such that for every* $f \in \mathcal{G}_M$ *we have that* $\tilde{Q}$ *coincides with* Q_f *and is a minimal Milnor attractor.*
(g) *For every* $f \in \mathcal{U}$ *there is a unique entropy maximizing measure.*

By inspection in the proofs, one can easily see that in fact the construction can be made in higher dimensional torus, however, it can only be done in the isotopy classes of Anosov diffeomorphisms. Also, it can be seen that condition (d) can be slightly strengthened in the C^1-topology (see section 3.2.7).

We are able to construct examples which can be embedded in every isotopy class of diffeomorphisms of manifolds of dimension ≥ 3 on which we were not able to obtain the same ergodic properties:

Theorem B. *For every d-dimensional manifold M and every isotopy class of diffeomorphisms of M there exists a C^1-open set $\mathcal{U}$ of $\text{Diff}^r(M)$ such that for some open neighborhood U in M:*

- (a) *Every $f \in \mathcal{U}$ has a unique quasi-attractor Q_f in U which contains a homoclinic class and has a partially hyperbolic splitting $T_{Q_f}M = E^{cs} \oplus E^u$ which is coherent.*
- (b) *Every chain recurrence class $R \neq Q_f$ is contained in the orbit of a periodic leaf of the lamination $\mathcal{F}^{cs}$ tangent to E^{cs} at Q_f.*
- (c) *There exists a residual subset $\mathcal{G}^r$ of $\mathcal{U}$ such that for every $f \in \mathcal{G}^r$ the diffeomorphism f has no attractors. In particular, f has infinitely many chain-recurrence classes.*
- (d) *For every $f \in \mathcal{U}$ there is a unique Milnor attractor $\tilde{Q} \subset Q_f$.*

The examples here are modifications of the product of a Plykin attractor and the identity on the circle $\mathbb{S}^1$. One can also obtain them in order to provide examples of robustly transitive attractors in dimension 3 with splitting $E^{cs} \oplus E^u$. The author is not aware of other known examples of such attractors other than Carvalho's example which is only possible to be made in certain isotopy classes of diffeomorphisms.

1.3. **Further remarks on the construction and some questions.**

1.3.1. As we mentioned, for a C^1-generic wild diffeomorphism, there are infinitely many chain-recurrence classes. However, it is not known if the cardinal of classes must be necessarily uncountable (this holds in all examples where we know the cardinal of the classes).

We pose the following question:

Question 1. *In the open set of Theorem A, does it hold that for C^1-generic diffeomorphisms in that set there are countably many chain-recurrence classes?*

The motivation for posing this question is the well known Smale's conjecture asserting that for any surface, there is a C^1-open and dense subset of diffeomorphisms of the surface which are hyperbolic (and in particular, they are tame). Since the dynamics in the C^1-open set $\mathcal{U}$ given by Theorem A has all of its chain-recurrence classes except from one contained in periodic normally hyperbolic surfaces, it seems that a positive answer to Smale's conjecture would imply that for C^1-generic diffeomorphisms in $\mathcal{U}$ there are countably many chain-recurrence classes.

1.3.2. We would like to comment that the techniques here are not enough to treat the examples given by [BLY]. In fact, we mentioned that C. Bonatti and K. Shinohara have announced that the quasi-attractor in the example from [BLY] is accumulated by homoclinic classes which are not contained in periodic surfaces so the question about the existence of Milnor attractors in those examples is not settled by now. In more generality one could ask:

Question 2. *Does a C^1-generic diffeomorphism admit a Milnor attractor?*

1.3.3. Finally, we would like to pose yet another question regarding the example given in Theorem B.

Question 3. *Can we say anything about the existence and finiteness of SRB measures and/or entropy maximizing measures for the example of Theorem B?*

In view of recent results ([VY] and [RHRHTU]) one could expect that these measures may exist but not be unique.

1.4. **Organization of the paper.** In section 2 we present a general mechanism for localizing the chain-recurrence classes different from a given one as lower dimensional classes. We apply this mechanism in section 3 to prove Theorem A. In section 4 we indicate the differences of the proof of Theorem B and Theorem A, we also show how to construct in every manifold a robustly transitive attractor with partially hyperbolic splitting $E^{cs} \oplus E^u$ where E^{cs} is not decomposable as two Df-invariant bundles.

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2. A mechanism for localizing chain-recurrence classes

Given a homeomorphism $g : \Gamma \to \Gamma$ where Γ is a compact metric space, we say that g is *expansive* if there exists $\alpha > 0$ such that for any pair of distinct points $x \neq y \in \Gamma$ there exists $n \in \mathbb{Z}$ such that $d(g^n(x), g^n(y)) \geq \alpha$.

Proposition 2.1. *Let f be a C^1-diffeomorphism and U a filtrating set such that its maximal invariant set Λ admits a partially hyperbolic structure $T_\Lambda M = E^{cs} \oplus E^u$ such that E^{cs} is coherent. Assume that there exists a continuous surjective map $h : \Lambda \to \Gamma$ and a homeomorphism $g : \Gamma \to \Gamma$ such that:*

- *h is injective in unstable manifolds.*
- *$h^{-1}(\{h(x)\})$ is contained in $\mathcal{F}^{cs}_U(x)$ and its topological frontier relative to $\mathcal{F}^{cs}_U(x)$ is contained in the same chain-recurrence class Q. In particular $h(Q) = \Gamma$.*
- *The fibers $h^{-1}(\{y\})$ are invariant under unstable holonomy.*
- *g is expansive.*

Then, every chain-recurrence class in U different from Q is contained in the preimage of a periodic orbit by h.

For simplicity, the reader can follow the proof assuming that g is an Anosov diffeomorphism we shall make some footnotes when some differences (which are quite small) appear.

PROOF. Let $R \neq Q$ be a chain recurrence class of f. Then, since $\partial h^{-1}(\{y\}) \subset Q$ for every $y \in \Gamma$, we have that $R \cap int(h^{-1}(\{y\})) \neq \emptyset$ for some $y \in \Gamma$.

Conley's theory gives us an open neighborhood V of R whose closure is disjoint with Q and such that every two points $x, z \in R$ are joined by arbitrarily small pseudo-orbits contained in V.

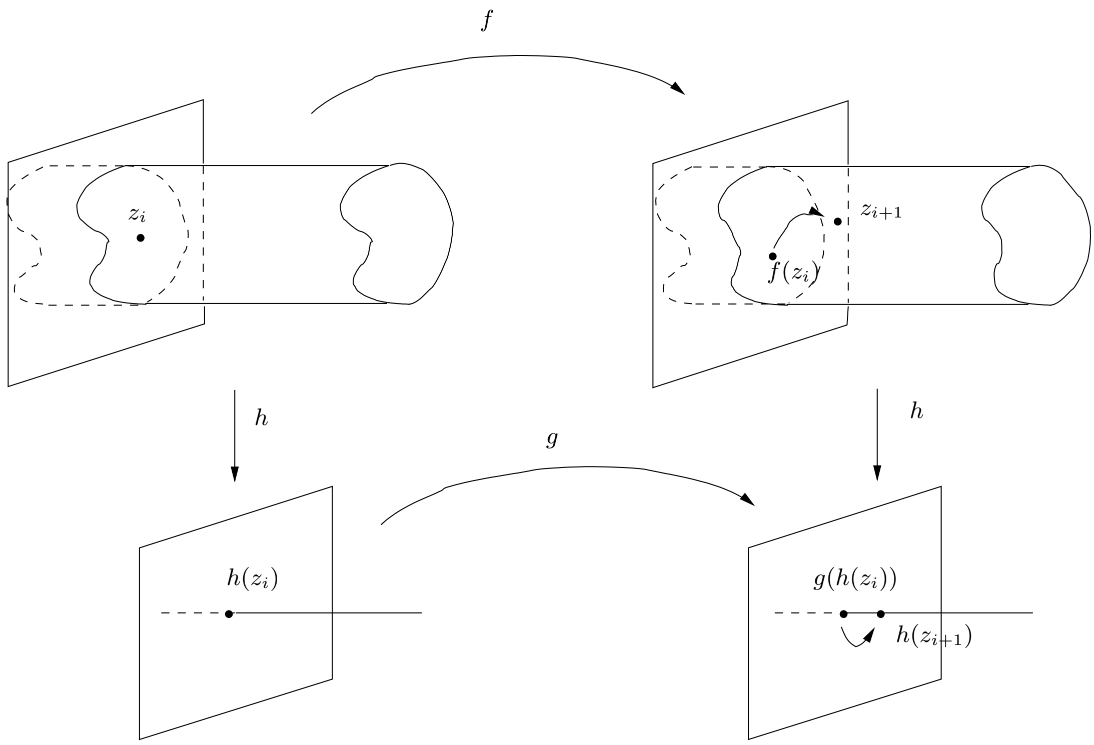

FIGURE 1. Pseudo-orbits for f are sent to pseudo-orbits of g with jumps in the unstable sets.

Since $\overline{V}$ does not intersect Q, using the invariance under unstable holonomy of the fibers, we get that there exists η_0 such that if $d(x,z) < \eta_0$ and $x \in V$, then $h(x)$ and $h(z)$ lie in the same local unstable manifold[5] (it suffices to choose $\eta_0 < d(\overline{V}, Q)$).

Given $\zeta > 0$ we choose $\eta > 0$ such that $d(x,z) < \eta$ implies $d(h(x),h(z)) < \zeta$. The semiconjugacy implies then that if $z_0, \dots z_n$ is a $\eta-$pseudo orbit for f, then $h(z_0), \dots, h(z_n)$ is a ζ-pseudo orbit for g (that is, $d(g(h(z_i)), h(z_{i+1})) < \zeta$). Also, if $\eta < \eta_0$ and $z_0, \dots z_n$ is contained in U, then we get that the the pseudo-orbit $h(z_0), \dots, h(z_n)$ has jumps inside local unstable sets (i.e. $h(z_{i+1}) \in W^u_\zeta(g(h(z_i)))$).

Take $x \in R$. Then, for every $\eta < \eta_0$ we take $x = z_0, z_1, \dots, z_n = x$ ($n \geq 1$) a $\eta-$pseudo orbit contained in V joining x to itself. Thus, we have that

$$g^n(W^u(h(x))) = W^u(h(x))$$

so, $W^u(h(x))$ is the unstable manifold for g of a periodic orbit $\mathcal{O}$. Since R is g-invariant and since the semiconjugacy implies that $g^{-n}(x)$ accumulates on $h^{-1}(\mathcal{O})$, we get that R intersects the fiber $h^{-1}(\mathcal{O})$.

We must now prove that $R \subset h^{-1}(\mathcal{O})$ which concludes.

Given $\varepsilon > 0$ there exists $\delta > 0$ such that if $z_0, \dots z_n$ is a $\delta-$pseudo orbit for g with jumps in the unstable manifold, then $z_n \in W^u_\varepsilon(\mathcal{O})$ implies that $z_0 \in \mathcal{O}$ (notice that a pseudo orbit with jumps in the unstable manifold of a periodic orbits can be regarded as a pseudo orbit for a homothety[6] in $\mathbb{R}^k$).

Assume that there is a point $z \in R$ such that $h(z) \in W^u(\mathcal{O}) \backslash \mathcal{O}$. So, there are arbitrarily small pseudo orbits contained in V joining z with a point in $h^{-1}(\mathcal{O})$. This implies that after sending the pseudo orbit by h we would get arbitrarily small pseudo orbits for g, with jumps in the unstable manifold, joining $h(z)$ with $\mathcal{O}$. This contradicts the remark made in the last paragraph.

So, we get that R is contained in $h^{-1}(\mathcal{O})$ where $\mathcal{O}$ is a periodic orbit of g.

□

3. Examples in $\mathbb{T}^3$. Proof of Theorem A

[5]The $\zeta-$local unstable set of a point x for an expansive homeomorphism g is the set of points whose orbit remains at distance smaller than ζ for every past iterate. For an expansive homeomorphism, this set is contained in the unstable set.

[6]In the general case of g being an expansive homeomorphism, it is very similar since one has that restricted to the unstable set of a periodic orbit, one can obtain a metric inducing the same topology where g^{-1} is an uniform contraction. This follows from [F] and can also be deduced using the uniform expansion of f in unstable leaves and the injectivity of the semi-conjugacy along unstable leaves.

3.1. **Construction of the example.** In this section we shall construct an open set $\mathcal{U}$ of $\text{Diff}^r(\mathbb{T}^3)$ for $r \geq 1$ verifying Theorem A.

The construction is very similar to the one of Carvalho's example ([Car]) following [BV] with the difference that instead of creating a source, we create an expanding saddle.

3.1.1. We start with a linear Anosov diffeomorphism $A : \mathbb{T}^3 \to \mathbb{T}^3$ admitting a splitting $E^s \oplus E^u$ where $\dim E^s = 2$.

We assume that A has complex eigenvalues on the E^s direction so that E^s cannot split as a dominated sum of other two subspaces. For example, the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

which has characteristic polynomial $1 + \lambda^2 - \lambda^3$ works since it has only one real root, and it is larger than one.

Considering an iterate, we may assume that there exists $\lambda < 1/3$ satisfying:

$$\|(DA)_{/E^s}\| < \lambda \ \ ; \ \ \|(DA)^{-1}_{/E^u}\| < \lambda$$

3.1.2. Let q and r be different fixed points of A.

Consider δ small enough such that $B(q, 6\delta)$ and $B(r, 6\delta)$ are pairwise disjoint and at distance larger than 400δ (this implies in particular that the diameter of $\mathbb{T}^3$ is larger than 400δ).

Let $\mathcal{E}^u$ be a family of closed cones around the subspace E^u of A which is preserved by DA (that is $D_xA(\mathcal{E}^u(x)) \subset int(\mathcal{E}^u(Ax))$). We shall consider the cones are narrow enough so that any curve tangent to $\mathcal{E}^u$ of length bigger than L intersects any stable disk of radius δ. Let $\mathcal{E}^{cs}$ be a family of closed cones around E^s preserved by DA.

From now on, δ remains fixed. Given $\varepsilon > 0$ such that $\varepsilon \ll \delta$,[7] we can choose ν sufficiently small such that every diffeomorphism g which is ν-C^0-close to A is semiconjugated to A with a continuous surjection h which is ε-C^0-close to the identity (this is a classical result on topological stability of Anosov diffeomorphisms, see [W]).

3.1.3. We shall modify A inside $B(q, \delta)$ such that we get a new diffeomorphism $F : \mathbb{T}^3 \to \mathbb{T}^3$ that verifies the following properties:

- F coincides with A outside $B(q, \delta)$ and lies at C^0-distance smaller than ν from A.

[7] If K bounds $\|A\|$ and $\|A^{-1}\|^{-1}$ then $\frac{\delta}{10K}$ is enough.

- The point q is a hyperbolic saddle fixed point of stable index 1 and such that the product of its two eigenvalues with smaller modulus is larger than 1. We also assume that the length of the stable manifold of q is larger than δ.
- $D_xF(\mathcal{E}^u(x)) \subset int(\mathcal{E}^u(F(x)))$. Also, for every $w \in \mathcal{E}^u(x)\backslash\{0\}$ we have $\|DF_x^{-1}w\| < \lambda\|w\|$.
- F preserves the stable foliation of A. Notice that the foliation will no longer be stable.
- For some small $\beta > 0$ we have that $\|D_xFv\| < (1+\beta)\|v\|$ for every v tangent to the stable foliation of A preserved by F and every x.

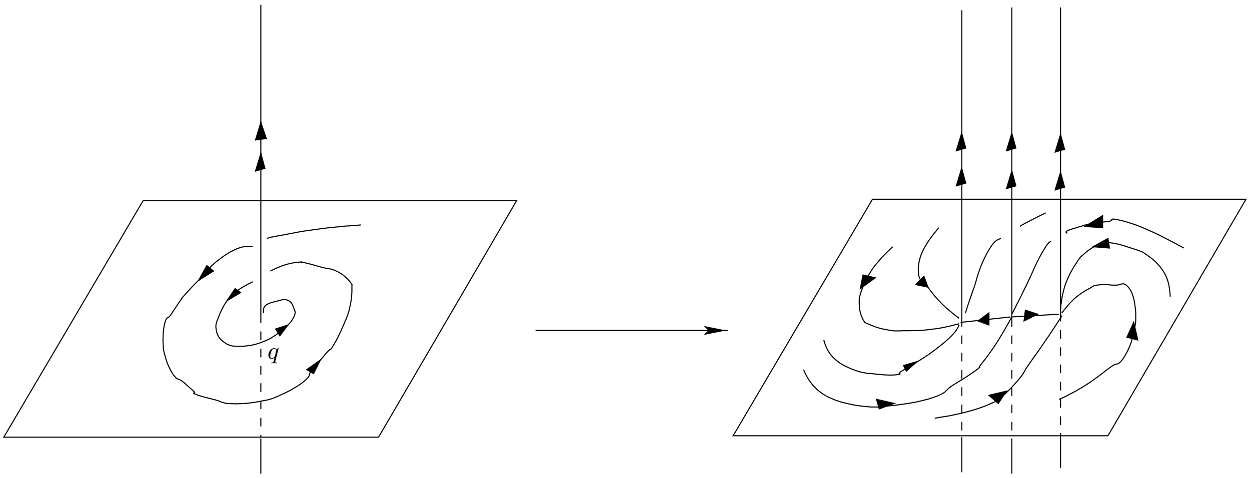

FIGURE 2. Modification of A in a neighborhood of q.

This construction can be made using classical methods (see [BV] section 6). Indeed, consider a small neighborhood U of q such that $U \subset B(q, \nu/2)$ such that U admits a chart $\varphi : U \to \mathbb{D}^2 \times [-1, 1]$ which sends q to $(0, 0)$ and sends stable manifolds of A in sets of the form $\mathbb{D}^2 \times \{t\}$ and unstable ones into sets of the form $\{s\} \times [-1, 1]$. We can modify A by isotopy inside U in such a way that the sets $\mathbb{D}^2 \times \{t\}$ remain an invariant foliation but such that the derivative of q becomes the identity in the tangent space to $\varphi^{-1}(\mathbb{D}^2 \times \{0\})$ which is invariant and such that the dynamics remains conjugated to the initial one. At this point, the norm of the images of unit vectors tangent to the stable foliation of A are not expanded by the derivative.

Now, one can modify slightly the dynamics in $\varphi^{-1}(\mathbb{D}^2 \times \{0\})$ in order to obtain the desired conditions on the eigenvalues of q for F. It is not hard to see that for backward iterates there will be points outside $\varphi^{-1}(\mathbb{D}^2 \times \{0\})$ which will approach q so one can obtain the desired length of the stable manifold of q by maybe performing yet another small modification. All this can be made in order that the vectors tangent to the stable foliation of A are expanded by DF by a factor of at most $(1+\beta)$ with β as small as we desire.

The fact that we can keep narrow cones invariant under DF seems difficult to obtain in view that we made all this modifications. However, the argument of [BV] (page 190) allows to obtain it: This is achieved by conjugating the modification with appropriate homotheties in the stable direction.

The last condition on the norm of DF in the tangent space to the stable foliation of A seems quite restrictive, more indeed in view of the condition on the eigenvalues of q. This condition (as well as property (P7) below) shall be only used (and will be essential) to obtain the ergodic properties of the diffeomorphisms in the open set we shall construct. Nevertheless, one can construct such a diffeomorphism as explained above.

3.1.4. There exists a C^1-open neighborhood $\mathcal{U}_1$ of F such that for every $f \in \mathcal{U}_1$ we have that:

(P1) There exists a continuation q_f of q and r_f of r. The point r_f has stable index 2 and complex eigenvalues. The point q_f is a saddle fixed point of stable index 1, such that the product of its two eigenvalues with smaller modulus is larger than 1 and such that the length of the stable manifold is larger than δ.

(P2) $D_x f(\mathcal{E}^u(x)) \subset int(\mathcal{E}^u(f(x)))$. Also, for every $w \in \mathcal{E}^u(x)$ we have
$$\|Df_x w\| \geq \lambda \|w\|.$$

(P3) f preserves a foliation $\mathcal{F}^{cs}$ which is C^0-close to the stable foliation of A. Also, each leaf of $\mathcal{F}^{cs}$ is C^1-close to a leaf of the stable foliation of A.

(P4) For every $x \notin B(q,\delta)$ we have that if $v \in \mathcal{E}^{cs}(x)$ then
$$\|D_x f v\| \leq \lambda \|v\|.$$
This is satisfied for F since $F = A$ outside $B(q,\delta)$.

(P5) There exists a continuous and surjective map $h_f : \mathbb{T}^3 \to \mathbb{T}^3$ such that
$$h_f \circ f = A \circ h_f$$
and $d(h(x), x) < \varepsilon$ for every $x \in \mathbb{T}^3$.

The fact that properties (P1), (P2) and (P4) are C^1-robust is immediate, robustness of (P5) follows from the choice of ν.

Property (P3) holds in a neighborhood of F since F preserves the stable foliation of A which is a C^1-foliation (see [HPS] chapter 7). The foliation $\mathcal{F}^{cs}$ will be tangent to E^{cs} a bidimensional bundle which is f-invariant and contained in $\mathcal{E}^{cs}$. Other way to proceed in order to obtain an invariant foliation is to use Theorem 3.1 of [BF] of which all hypothesis are verified here but we shall not state it.

Since the cones $\mathcal{E}^u$ are narrow and from (P3) one has that:

(P6) Every curve of length L tangent to $\mathcal{E}^u$ will intersect any disc of radius 2δ in $\mathcal{F}^{cs}$.

Finally, there exists an open set $\mathcal{U}_2 \subset \mathcal{U}_1$ such that for $f \in \mathcal{U}_2$ we have:

(P7) $\|D_x f v\| \leq (1+\beta)\|v\|$ for every $v \in \mathcal{E}^{cs}(x)$ and every x.

3.1.5. We shall close this section by proving that for these examples there exists a unique quasi-attractor for the dynamics.

Lemma 3.1. *For every $f \in \mathcal{U}_1$ there exists an unique quasi-attractor Q_f. This quasi attractor contains the homoclinic class of r_f, the continuation of r.*

Proof. We use the same argument as in [BLY].

There is a center stable disc of radius bigger than 2δ contained in the stable manifold of r_f ((P3) and (P4)). So, every unstable manifold of length bigger than L will intersect the stable manifold of r_f ((P6)).

Let Q be a quasi attractor, so, there exists a sequence U_n, of neighborhoods of Q such that $f(\overline{U_n}) \subset U_n$ and $Q = \bigcap_n \overline{U_n}$.

Since U_n is open, there is a small unstable curve γ contained in U_n. Since Df expands vectors in $\mathcal{E}^u$ we have that the length of $f^k(\gamma)$ tends to $+\infty$ as $n \to +\infty$. So, there exists k_0 such that $f^{k_0}(\gamma) \cap W^s(r_f) \neq \emptyset$. So, since $f(\overline{U_n}) \subset U_n$ we get that $U_n \cap W^s(r_f) \neq \emptyset$, using again the forward invariance of U_n we get that $r_f \in \overline{U_n}$.

This holds for every n so $r_f \in Q$. Since the homoclinic class of r_f is chain transitive, we also get that $H(r_f) \subset Q$.

From Conley's theory (cf. 1.1.1), every homeomorphism of a compact metric space there is at least one chain recurrent class which is a quasi attractor. This concludes.

□

3.2. **The example verifies the mechanism.** We shall consider $f \in \mathcal{U}_1$ so that it verifies (P1)-(P6).

3.2.1. Let $\mathcal{A}^s$ and $\mathcal{A}^u$ be, respectively, the stable and unstable foliations of A, which are linear foliations. Since A is a linear Anosov diffeomorphism, the distances inside the leaves of the foliations and the distances in the manifold are equal in small neighborhoods of the points if we choose a convenient metric.

Let $\mathcal{A}^s_\eta(x)$ denote the ball of radius η around x inside the leaf of x of $\mathcal{A}^s$. For any $\eta > 0$, it is satisfied that $A(\mathcal{A}^s_\eta(x)) \subset \mathcal{A}^s_{\eta/3}(Ax)$ (an analogous property is satisfied by $\mathcal{A}^u_\eta(x)$ and backward iterates).

3.2.2. The distance inside the leaves of $\mathcal{F}^{cs}$ is similar to the ones in the ambient manifold since each leaf of $\mathcal{F}^{cs}$ is C^1-close to a leaf of $\mathcal{A}^s$. That is, there exists $\rho \approx 1$ such that if x, y belong to a connected component of $\mathcal{F}^{cs}(z) \cap B(z, 10\delta)$ then $\rho^{-1} d_{cs}(x,y) < d(x,y) < \rho d_{cs}(x,y)$ where $\mathcal{F}^{cs}(z)$ denotes the leaf of the foliation passing through z and d_{cs} the distance restricted to the leaf.

For $z \in \mathbb{T}^3$ we define $W^{cs}_{loc}(z)$ (the *local center stable manifold* of z) as the 2δ-neighborhood of z in $\mathcal{F}^{cs}(z)$ with the distance d_{cs}.

Also, we can assume that for some $\gamma < \min\{\|A\|^{-1}, \|A^{-1}\|^{-1}, \delta/10\}$ the plaque $W^{cs}_{loc}(x)$ is contained in a $\gamma/2$ neighborhood of $\mathcal{A}^s_{2\delta}(x)$, the disc of radius 2δ of the stable foliation of A around x.

Lemma 3.2. *We have that* $f(\overline{W^{cs}_{loc}(x)}) \subset W^{cs}_{loc}(f(x))$.

PROOF. Consider around each $x \in \mathbb{T}^3$ a continuous map $b_x : \mathbb{D}^2 \times [-1,1] \to \mathbb{T}^3$ such that $b_x(\{0\} \times [-1,1]) = \mathcal{A}^u_{3\delta}(x)$ and $b_x(\mathbb{D}^2 \times \{t\}) = \mathcal{A}^s_{3\delta}(b_x(\{0\} \times \{t\}))$. For example, one can choose b_x to be affine in each coordinate to the covering of $\mathbb{T}^3$.

Thus, it is not hard to see that one can assume also that $b_x(\frac{1}{3}\mathbb{D}^2 \times \{t\}) = \mathcal{A}^s_\delta(b_x(\{0\} \times \{t\}))$ and that $b_x(\{y\} \times [-1/3,1/3]) = \mathcal{A}^u_\delta(b_x(\{y\} \times \{0\}))$. Let

$$B_x = b_x(\mathbb{D}^2 \times [-\gamma/2, \gamma/2]).$$

We have that $A(B_x)$ is contained in $b_{Ax}(\frac{1}{3}\mathbb{D}^2 \times [-1/2,1/2])$. Since f is ε-C^0-near A, we get that $f(B_x) \subset b_{f(x)}(\frac{1}{2}\mathbb{D}^2 \times [-1,1])$.

Let $\pi_1 : \mathbb{D}^2 \times [-1,1] \to \mathbb{D}^2$ such that $\pi_1(x,t) = x$. We have that $\pi_1(b^{-1}_{f(x)}(W^{cs}_{loc}(f(x))))$ contains $\frac{1}{2}\mathbb{D}^2$ from how we chose γ and from how we have defined the local center stable manifolds[8].

Since $f(\mathcal{F}^{cs}(x)) \subset \mathcal{F}^{cs}(f(x))$ and $f(W^{cs}_{loc}(x)) \subset b_{f(x)}(\frac{1}{2}\mathbb{D}^2 \times [-1,1])$ we get the desired property.

□

3.2.3. The fact that $f \in \mathcal{U}_1$ is semiconjugated with A together with the fact that the semiconjugacy is ε-C^0-close to the identity gives us the following easy properties about the fibers (preimages under h_f) of the points.

As in 1.1.2, we denote

$$\Pi^{uu}_{x,z} : U \subset W^{cs}_{loc}(x) \to W^{cs}_{loc}(z)$$

the unstable holonomy where $z \in \mathcal{F}^u(x)$ and U is a neighborhood of x in $W^{cs}_{loc}(x)$ which can be considered large if z is close to x in $\mathcal{F}^u(x)$. In particular, let $\gamma > 0$ be such that if $z \in \mathcal{F}^u_\gamma(x)$ then the holonomy is defined in a neighborhood of radius ε of x.

Proposition 3.3. *Consider* $y = h_f(x)$ *for* $x \in \mathbb{T}^3$*:*

[8]In fact, $b^{-1}_{f(x)}(W^{cs}_{loc}(g(x))) \cap \frac{1}{2}\mathbb{D}^2 \times [-1,1]$ is the graph of a C^1 function from $\frac{1}{2}\mathbb{D}^2$ to $[-\gamma/2, \gamma/2]$ if b_x is well chosen.

(1) $h_f^{-1}(\{y\})$ *is a compact connected set contained in* $W^{cs}_{loc}(x)$.
(2) *If* $z \in \mathcal{F}^u_\gamma(x)$, *then* $h_f(\Pi^{uu}_{x,z}(h_f^{-1}(\{y\})))$ *is exactly one point.*

Proof. (1) Since h_f is ε-C^0-close the identity, we get that for every point $y \in \mathbb{T}^3$, $h_f^{-1}(\{y\})$ has diameter smaller than ε. Since ε is small compared to δ, it is enough to prove that $h_f^{-1}(\{y\}) \subset W^{cs}_{loc}(x)$ for some $x \in h_f^{-1}(\{y\})$.

Assume that for some $y \in \mathbb{T}^3$, $h_f^{-1}(\{y\})$ intersects two different center stable leaves of $\mathcal{F}^{cs}$ in points x_1 and x_2.

Since the points are near, we have that $\mathcal{F}^u_\gamma(x_1) \cap W^{cs}_{loc}(x_2) = \{z\}$. Thus, by forward iteration, we get that for some $n_0 > 0$ we have $d(f^{n_0}(x_1), f^{n_0}(z)) > 3\delta$.

Lemma 3.2 gives us that $d(f^{n_0}(x_2), f^{n_0}(z)) < 2\delta$ and so, we get that $d(f^{n_0}(x_1), f^{n_0}(x_2)) > \delta$ which is a contradiction since $\{f^{n_0}(x_1), f^{n_0}(x_2)\} \subset h_f^{-1}(\{A^{n_0}(y)\})$ which has diameter smaller than $\varepsilon \ll \delta$.

Also, since the dynamics is trapped in center stable manifolds, we get that the fibers must be connected since one can write them as

$$h^{-1}(\{h(x)\}) = \bigcap_{n\geq 0} f^n(W^{cs}_{loc}(f^{-n}(x))).$$

(2) Since $f^{-n}(h_f^{-1}(\{y\})) = h_f^{-1}(\{A^{-n}(y)\})$ we get that $\operatorname{diam}(f^{-n}(h_f^{-1}(\{y\}))) < \varepsilon$ for every $n > 0$.

This implies that there exists n_0 such that if $n > n_0$ then $f^{-n}(\Pi^{uu}_{x,z}(h_f^{-1}(\{y\})))$ is sufficiently near $f^{-n}(h_f^{-1}(\{y\}))$. So, we have that

$$\operatorname{diam}(f^{-n}(\Pi^{uu}_{x,z}(h_f^{-1}(\{y\})))) < 2\varepsilon \ll \delta.$$

Assume that $h_f(\Pi^{uu}_{x,z}(h_f^{-1}(\{y\})))$ contains more than one point. These points must differ in the stable coordinate of A, so, after backwards iteration we get that they are at distance bigger than 3δ. Since h_f is ε-C^0-close the identity this represents a contradiction.

$\square$

Remark 1. The second statement of the previous proposition gives that the fibers of h_f are invariant under unstable holonomy.

$\diamondsuit$

3.2.4. The following simple lemma is essential in order to satisfy the properties of Proposition 2.1.

Lemma 3.4. *For every* $f \in \mathcal{U}_1$, *given a disc* D *in* $W^{cs}_{loc}(x)$ *whose image by* h_f *has at least two points, then* $D \cap \mathcal{F}^u(r_f) \neq \emptyset$ *and the intersection is transversal.*

Proof. Given a subset $K \subset \mathcal{F}^{cs}(x)$ we define its *center stable diameter* as the diameter with the metric d_{cs} defined above induced by the metric in the manifold. We shall first prove that there exists n_0 such that $\text{diam}_{cs}(f^{-n_0}(D)) > 100\delta$:

Since D is arc connected so is $h_f(D)$, so, it is enough to suppose that $\text{diam}(D) < \delta$. We shall first prove that $h_f(D)$ is contained in a stable leaf of the stable foliation of A. Otherwise, there would exist points in $h_f(D)$ whose future iterates separate more than 2δ, this contradicts that the center stable plaques are trapped for f (Lemma 3.2).

One now has that, since A is Anosov and that $h_f(D)$ is a connected compact set with more than two points contained in a stable leaf of the stable foliation, there exists $n_0 > 0$ such that $A^{-n_0}(h_f(D))$ has stable diameter bigger than 200δ (recall that $\text{diam}\, \mathbb{T}^3 > 400\delta$). Now, since h_f is close to the identity, one gets the desired property.

We conclude by proving the following:

Claim 1: If there exists n_0 such that $f^{-n_0}(D)$ has diameter larger than 100δ, then D intersects $\mathcal{F}^u(r_f)$.

Proof of the claim. This is proved in detail in section 6.1 of [BV] so we shall only sketch it.

If $f^{-n_0}(D)$ has diameter larger than 100δ, from how we choose δ we have that there is a compact connected subset of $f^{-n_0}(D)$ of diameter larger than 35δ which is outside $B(q, 6\delta)$.

So, $f^{-n_0-1}(D)$ will have diameter larger than 100δ and the same will happen again. This allows to find a point $x \in D$ such that $\forall n > n_0$ we have that $f^{-n}(x) \notin B(q, 6\delta)$.

Now, considering a small disc around x we have that by backward iterates it will contain discs of radius each time bigger and this will continue while the disc does not intersect $B(q, \delta)$. If that happens, since $f^{-n}(x) \notin B(q, 6\delta)$ the disc must have radius at least 3δ.

This proves that there exists m such that $f^{-m}(D)$ contains a center stable disc of radius bigger than 2δ, so, the unstable manifold of r_f intersects it. Since the unstable manifold of r_f is invariant, we deduce that it intersects D and this concludes the proof of the claim.

Transversality of the intersection is immediate from the fact that D is contained in $\mathcal{F}^{cs}$ which is transversal to $\mathcal{F}^u$.

□

3.2.5. We obtain the following corollary which puts us in the hypothesis of Proposition 2.1:

Corollary 3.5. *For every $f \in \mathcal{U}_1$, let $x \in \partial h_f^{-1}(\{y\})$ (relative to the local center stable manifold of $h_f^{-1}(\{y\})$), then, x belongs to the homoclinic class of r_f, and in particular, to Q_f.*

PROOF. Notice first that the stable manifold of r_f coincides with $\mathcal{F}^{cs}(r_f)$ which is dense in $\mathbb{T}^3$. This follows from the fact that when iterating an unstable curve, it will eventually intersect the stable manifold of r_f, since the stable manifold of r_f is invariant, we obtain the density of $\mathcal{F}^{cs}(r_f)$.

Now, considering $x \in \partial h_f^{-1}(\{y\})$, and $\varepsilon > 0$, we consider a connected component $\tilde{D}$ of $\mathcal{F}^{cs}(r_f) \cap B(x, \varepsilon)$. Clearly, since the fibers are invariant under holonomy and $x \in \partial h_f^{-1}(\{y\})$ we get that $\tilde{D}$ contains a disk D which is sent by h_f to a non trivial connected set. Using the previous lemma we obtain that there is a homoclinic point of r_f inside $B(x, \varepsilon)$ which concludes.

□

3.2.6. The following corollary will allow us to use Theorems 1.1 and 1.2.

Corollary 3.6. *For every $f \in \mathcal{U}_1$ we have that $q_f \in H(r_f)$.*

PROOF. Consider U, a neighborhood of q_f, and D a center stable disc contained in U.

Since the stable manifold of q_f has length bigger than $\delta > \varepsilon$, after backward iteration of D one gets that $f^{-k}(D)$ will eventually have diameter larger than ε, thus $h_f(D)$ will have at least two points, this means that $q_f \in \partial h_f^{-1}(\{h(q_f)\})$. Corollary 3.5 concludes.

□

3.2.7. We finish this section by proving the following theorem which is the topological part of Theorem A.

Theorem 3.7.
(i) *For every $f \in \mathcal{U}_1$ there exists a unique quasi-attractor Q_f which contains the homoclinic class $H(r_g)$ and such that every chain-recurrence class $R \neq Q_f$ is contained in a periodic disc of $\mathcal{F}^{cs}$.*
(ii) *For every $f \in \mathcal{G}_{BC} \cap \mathcal{G}_{BDV} \cap \mathcal{U}_1$ we have that $H(r_f) = Q_f$ and is contained in the closure of the sources of f.*
(iii) *For every $r \geq 2$, there exists a C^1-open dense subset $\mathcal{U}_3$ of $\mathcal{U}_1$ and a residual subset $\mathcal{G}^r \subset \mathcal{U}_3 \cap \mathrm{Diff}^r(\mathbb{T}^3)$ such that for every $f \in \mathcal{G}^r$ the homoclinic class $H(r_f)$ intersects the closure of the sources of f.*
(iv) *For every $f \in \mathcal{U}_1$ there exists a unique Milnor attractor contained in Q_f.*

PROOF. Part (i) follows from Proposition 2.1 since h_f is the desired semiconjugacy: Indeed, Proposition 3.3 and Corollary 3.5 show that the hypothesis of the mentioned proposition are verified (notice that A is clearly expansive).

Part (ii) follows from Theorem 1.1 using Corollary 3.6. Notice that E^{cs} cannot be decomposed in two Df-invariant subbundles since Df has complex eigenvalues in r_f.

Similarly, part (iii) follows from Theorem 1.2 and Corollary 1.2. The need for considering $\mathcal{U}_3$ comes from [BD3] (see Theorem 1.2).

To prove (iv) notice that every point which does not belong to the fiber of a periodic orbit belongs to the basin of Q_f: Since there are only countably many periodic orbits and their fibers are contained in two dimensional discs (which have zero Lebesgue measure) this implies directly that the basin of Q_f has total Lebesgue measure:

Consider a point x whose omega-limit set $\omega(x)$ is contained in a chain recurrence class R different from Q_f. Then, since this chain recurrence class is contained in the fiber $h_f^{-1}(\mathcal{O})$ of a periodic orbit $\mathcal{O}$ of A, which in turn is contained in the local center stable manifold of some point $z \in \mathbb{T}^3$. This implies that some forward iterate of x is contained in $W^{cs}_{loc}(z)$. The fact that the dynamics in W^{cs}_{loc} is trapping (see Lemma 3.2) and the fact that $\partial h_f^{-1}(\mathcal{O}) \subset Q_f$ (see Corollary 3.5) gives that x itself is contained in $h_f^{-1}(\mathcal{O})$ as claimed.

Now, Lemma 1 of [Mi] implies that Q_f contains an attractor in the sense of Milnor.

□

We have just proved parts (a), (b), (c) and (d) of Theorem A hold in $\mathcal{U}_3$. In fact, for the C^1-topology, we have obtain a slightly stronger property than (d) holds in $\mathcal{U}_1$. Also, we have proved that (e) is satisfied.

Remark 2. The choice of having complex eigenvalues for A was only used to guaranty that E^{cs} admits no Df-invariant subbundles. One could have started with any linear Anosov map A and modify the derivative of a given fixed or periodic point r to have complex eigenvalues and the construction would be the same.

3.3. **Ergodic properties.** In this section we shall work with $f \in \mathcal{U}_2$ so that properties (P1)-(P7) are verified.

3.3.1. We shall briefly explain how it can be deduced from [BV] that there exists a unique SRB measure for every $f \in \mathcal{U}_2$ of class C^2. Let us first consider U a small neighborhood of Q_f and

$$\Lambda_f = \bigcap_{n \geq 0} f^n(\overline{U})$$

which is a (not-necessarily transitive) topological attractor.

We shall show that the hypothesis of Theorem A of [BV] are satisfied for Λ_f (see also Theorem 11.25 in [BDV]), and thus, we get that there are at most finitely many SRB measures such that the union of their (statistical) basins has full Lebesgue measure in the topological basin of Λ_f. Clearly, for Λ_f one has (H1) and (H2). Hypothesis (H3) follows from the following:

Proposition 3.8. *For every $x \in \mathbb{T}^3$ and $D \subset W^{uu}_{loc}(x)$ an unstable arc, we have full measure set of points which have negative Lyapunov exponents in the direction E^{cs}.*

Proof. The proof is exactly the same as the one in Proposition 6.5 of [BV] so we omit it. Notice that conditions (P2), (P4) and (P7) in our construction imply conditions (i) and (ii) in section 6.3 of [BV].

□

3.3.2. The set Λ_f does not verify the hypothesis of Theorem B of [BV] since we do not have minimality of the unstable foliation.

However, the fact that the stable manifold of r_f contains $W^{cs}_{loc}(r_f)$, gives that every unstable manifold intersects $W^s(r_f)$ and so we get that every compact subset of Λ_f saturated by unstable sets must contain $\overline{\mathcal{F}^u(r_f)}$. This implies that for every $x \in \overline{\mathcal{F}^u(r_f)}$ we have that $\overline{\mathcal{F}^u(r_f)} = \overline{\mathcal{F}^u(x)}$ and $\overline{\mathcal{F}^u(r_f)}$ is the only compact set with this property (we say that $\overline{\mathcal{F}^u(r_f)}$ is the unique minimal set of the foliation $\mathcal{F}^u$).

It is not hard to see how the proof of [BV] works in this context[9]. We get thus that f admits an unique SRB measure μ and clearly, the support of this SRB measure is $\overline{\mathcal{F}^u(r_f)}$.

We claim that $\overline{\mathcal{F}^u(r_f)} = H(r_f)$: this follows from the fact that the SRB measure μ is hyperbolic (by Proposition 3.8) and that the partially hyperbolic splitting separates the positive and negative exponents of μ (this is given in Proposition 1.4 of [C1] which states that when one has a hyperbolic measure μ whose supports admits a dominated splitting respecting the exponents of μ then the support is contained in a homoclinic class).

3.3.3. Finally, since the SRB measure has total support and almost every point converges to the whole support, we get that the attractor is in fact a minimal attractor in the sense of Milnor. We have proved:

Proposition 3.9. *If $f \in \mathcal{U}_2$ is of class C^2, then f admits a unique SRB measure whose support coincides with $\overline{\mathcal{F}^u(r_f)} = H(r_f)$. In particular, $\overline{\mathcal{F}^u(r_f)}$ is a minimal attractor in the sense of Milnor for f.*

The importance of considering f of class C^2 comes from the fact that with lower regularity, even if we knew that almost every point in the unstable manifold of r_f has stable manifolds, we cannot assure that these cover a positive measure set due to the lack of absolute continuity in the center stable foliation.

[9]See the first paragraph of section 5 in [BV]. Our Proposition 3.8 implies that (H3) is verified. Moreover, every unstable arc converges after future iteration to the whole $\overline{\mathcal{F}^u(r_g)}$, and since the unstable foliation is minimal in $\overline{\mathcal{F}^u(r_g)}$ we get that there is only one accessibility class there as needed for their Theorem B.

3.3.4. However, the information we gathered for smooth systems in $\mathcal{U}_2$ allows us to extend the result for C^1-generic diffeomorphisms in $\mathcal{U}_2$. Recall that for a C^1-generic diffeomorphisms $f \in \mathcal{U}_2$, the homoclinic class of r_f coincides with Q_f.

Theorem 3.10. *There exists a C^1-residual subset $\mathcal{G}_M \subset \mathcal{U}_2$ such that for every $f \in \mathcal{G}_M$ the set $Q_f = H(r_f)$ is a minimal Milnor attractor.*

Proof. Notice that since r_f has a well defined continuation in $\mathcal{U}_2$, it makes sense to consider the map $f \mapsto \overline{\mathcal{F}^u(r_f)}$ which is naturally semicontinuous with respect to the Haussdorff topology. Thus, it is continuous in a residual subset $\mathcal{G}_1$ of $\mathcal{U}_2$. Notice that since the semicontinuity is also valid in the C^2-topology, we have that $\mathcal{G}_1 \cap \text{Diff}^2(\mathbb{T}^3)$ is also residual in $\mathcal{U}_2 \cap \text{Diff}^2(\mathbb{T}^3)$.

It suffices to show that the set of diffeomorphisms in $\mathcal{G}_1$ for which $\overline{\mathcal{F}^u(r_f)}$ is a minimal Milnor attractor is a G_δ set (countable intersection of open sets) since we have already shown that C^2 diffeomorphisms (which are dense in $\mathcal{G}_1$) verify this property.

Given an open set U, we define

$$U^+(f) = \bigcap_{n \leq 0} f^n(\overline{U}).$$

Let us define the set $\mathcal{O}_U(\varepsilon)$ as the set of $f \in \mathcal{G}_1$ such that they satisfy one of the following (disjoint) conditions

- $\overline{\mathcal{F}^u(r_f)}$ is contained in U *or*
- $\overline{\mathcal{F}^u(r_f)} \cap \overline{U}^c \neq \emptyset$ and $Leb(U^+(f)) < \varepsilon$

We must show that this sets are open in $\mathcal{G}_1$ (it is not hard to show that if we consider an countable basis of the topology and $\{U_n\}$ are finite unions of open sets in the basis then $\mathcal{G}_M = \bigcap_{n,m} \mathcal{O}_{U_n}(1/m)$).

To prove that these sets are open, we only have to prove the semicontinuity of the measure of $U^+(f)$ (since the other conditions are clearly open from how we chose $\mathcal{G}_1$).

Let us consider the set $\tilde{K} = \overline{U} \backslash U^+(f)$, so, we can write $\tilde{K}$ as an increasing union $\tilde{K} = \bigcup_{n \geq 1} K_n$ where K_n is the set of points which leave $\overline{U}$ in less than n iterates.

So, if $Leb(U^+(f)) < \varepsilon$, we can choose n_0 such that $Leb(\overline{U} \backslash K_{n_0}) < \varepsilon$, and in fact we can consider K'_{n_0} a compact subset of K_{n_0} such that $Leb(\overline{U} \backslash K'_{n_0}) < \varepsilon$.

In a small neighborhood $\mathcal{N}$ of f, we have that if $f' \in \mathcal{N}$, then $K'_{n_0} \subset \overline{U} \backslash U^+(f')$. This concludes.

□

This completes the proof of part (f) of Theorem A.

3.3.5. To conclude the proof of Theorem A in $\mathcal{U} = \mathcal{U}_2 \cap \mathcal{U}_3$ one only has to check property (g) which follows directly from the main theorem of [BF].

4. Examples in any manifold. Proof of Theorem B

In this section we shall show how to construct an example verifying Theorem B. We shall see that we can construct a quasi-attractor with a partially hyperbolic splitting $E^{cs} \oplus E^u$ such that E^{cs} admits no sub-dominated splitting. In case E^{cs} is volume contracting, it will turn out that this quasi-attractor is in fact a robustly transitive attractor (thus providing examples of robustly transitive attractors with splitting $E^{cs} \oplus E^u$ in every 3−dimensional manifold) and when there is a periodic saddle of stable index 1 and such that the product of any two eigenvalues is greater than one and using Theorems 1.1 and 1.2 we shall obtain that the quasi-attractor will not be isolated for generic diffeomorphisms in a neighborhood.

We shall work only in dimension 3. It will be clear that by multiplying the examples here with a strong contraction, one can obtain examples in any manifold of any dimension.

A main difference between this construction and the one done in section 3 is the use of *blenders* instead of the argument à là Bonatti-Viana. Blenders were introduced in [BD1] and constitute a very powerful tool in order to get robust intersections between stable and unstable manifolds of compact sets. We shall only state some of their properties and not enter in their definition or construction for which there are many excellent references (we recommend chapter 6 of [BDV] in particular).

4.1. **Construction of the example.**

4.1.1. Let us consider $P : \mathbb{D}^2 \hookrightarrow \mathbb{D}^2$ the map given by the Plykin attractor in the disk $\mathbb{D}^2$ (see [R]).

We have that $P(\mathbb{D}^2) \subset int(\mathbb{D}^2)$, there exist a hyperbolic attractor $\Upsilon \subset \mathbb{D}^2$ and three fixed sources (we can assume this by considering an iterate).

There is a neighborhood N of Υ which is homeomorphic to the disc with 3 holes removed that satisfies that $P(\overline{N}) \subset N$ and

$$\Upsilon = \bigcap_{n\geq 0} P^n(N).$$

It is well known that given $\varepsilon > 0$, one can choose a finite number of periodic points $s_1, \dots, s_N$ and $L > 0$ such that if $A = \bigcup_{i=1}^N W^u_L(s_i)$, then, for every $x \in \Upsilon \setminus A$ one has that A intersects both connected components of $W^s_\varepsilon(x) \setminus \{x\}$.

We now consider the map $F_0 : \mathbb{D}^2 \times S^1 \hookrightarrow \mathbb{D}^2 \times S^1$ given by $F_0(x,t) = (P(x), t)$ whose chain recurrence classes consist of the set $\Upsilon \times S^1$ which is a (non transitive) partially hyperbolic attractor and three repelling circles.

4.1.2. In [BD1] they make a small C^∞ perturbation F_1 of F_0, for whom the maximal invariant set in $U = N \times S^1$ becomes a C^1-robustly transitive partially hyperbolic attractor Q which remains homeomorphic to $\Upsilon \times S^1$.

This attractor has a partially hyperbolic structure of the type $E^s \oplus E^c \oplus E^u$. One can make this example in order that it fixes the boundary of $\mathbb{D}^2 \times S^1$, this allows to embed this example (and all the modifications we shall make) in any isotopy class of diffeomorphisms of any 3-dimensional manifold (since every diffeomorphism is isotopic to one which fixes a ball, then one can introduce this map by a simple surgery).

4.1.3. We shall now present cs-blenders by its properties: A *cs-blender* K for a diffeomorphism $f : M \to M$ is a compact f-invariant hyperbolic set with splitting $T_K M = E^s \oplus E^u \oplus E^{uu}$ such that the following properties are verified:

- K is the maximal invariant subset in a neighborhood U.
- There exists a cone-field $\mathcal{E}^{uu}$ around E^{uu} defined in all U which is invariant under Df.
- There exists a compact region B with non-empty interior (which is called *activating region*) such that every curve contained in U, tangent to $\mathcal{E}^{uu}$ with length larger than δ and intersecting B verifies that it intersects the stable manifold of a point of K.
- There exists an open neighborhood $\mathcal{U}$ of f such that for every g in $\mathcal{U}$ the properties above are verified for the same cone field, the same set B and for K_g the maximal invariant set of U.

For more properties and construction of cs-blenders, see [BDV] chapters 6 and [BD1]. There one can see a proof of the following:

Proposition 4.1 ([BD1] Lemma 1.9, [BDV] Lemma 6.2)**.** *If the unstable manifold of a periodic point $q \in M$ contains an arc γ tangent to $\mathcal{E}^{uu}$ and intersecting the activating region of a cu-blender K, then, $W^u(p) \subset \overline{W^u(q)}$ for every p periodic point in K.*

4.1.4. In [BD1] the diffeomorphism F_1 constructed verifies the following properties (see [BD1] section 4.a page 391, also one can find the indications in [BDV] section 7.1.3):

(F1) F_1 leaves invariant a C^1-lamination $\mathcal{F}^{cs}$ (see [HPS] chapter 7 for a precise definition) tangent to $E^s \oplus E^c$ whose leaves are homeomorphic to $\mathbb{R} \times \mathbb{S}^1$.

(F2) There are periodic points $p_1, \ldots, p_N$ of stable index 1 such that for every $x \in Q$ one has that the connected component of $\mathcal{F}^{cs}(x) \setminus \overline{(W^u_L(p_1) \cup \ldots \cup W^u_L(p_N))}$ containing x has finite volume for every $x \in Q \setminus \bigcup_{i=1}^N W^u(p_i)_L$. Here $W^u_L(p_i)$ denotes the neighborhood of p_i in its unstable manifold with the metric induced by the ambient.

(F3) There is a periodic point q periodic point of stable index 2 such that its unstable manifold intersects the activation region of a cs-blender K containing the points

p_i. By Proposition 4.1, the unstable manifold of q is dense in the union of the unstable manifolds of p_i.

(F4) The local stable manifold of q intersects every unstable curve of length larger than L.

Before we continue, we shall make some remarks on the properties. The hypothesis (F1) on the differentiability of the lamination $\mathcal{F}^{cs}$ will be used in order to apply the results on normal hyperbolicity of [HPS] (chapter 7, Theorem 7.4). It can be seen in [BD1] that the construction of F_1 is made by changing the dynamics in finitely many periodic circles and this can be done without altering the lamination $\mathcal{F}^{cs}$ which is C^1 before modification. This is in fact not necessary; it is possible to apply the barehanded arguments of the proof of Theorem 3.1 of [BF] in order to obtain that for the modifications we shall make, there will exist a lamination tangent to the bundle E^{cs}.

Hypothesis (F2) is justified by the fact that the Plykin attractor verifies the same property and the construction of F_1 in [BD1] is made by changing the dynamics in the periodic points by Morse-Smale diffeomorphisms which give rise property (F2) (see section 4.a. of [BD1]). Notice that by continuous variation of stable and unstable sets, this condition is C^1-robust.

Property (F3) is the essence in the construction of [BD1], cs-blenders are the main tool for proving the robust transitivity of this examples. As explained in 4.1.3 this is a C^1-open property.

Property (F4) is given by the fact that the local stable manifold of q can be assumed to be $W^s_{loc}(s) \times \mathbb{S}^1$ with a curve removed, where $s \in \Upsilon$ is a periodic point. This is also a C^1-open property.

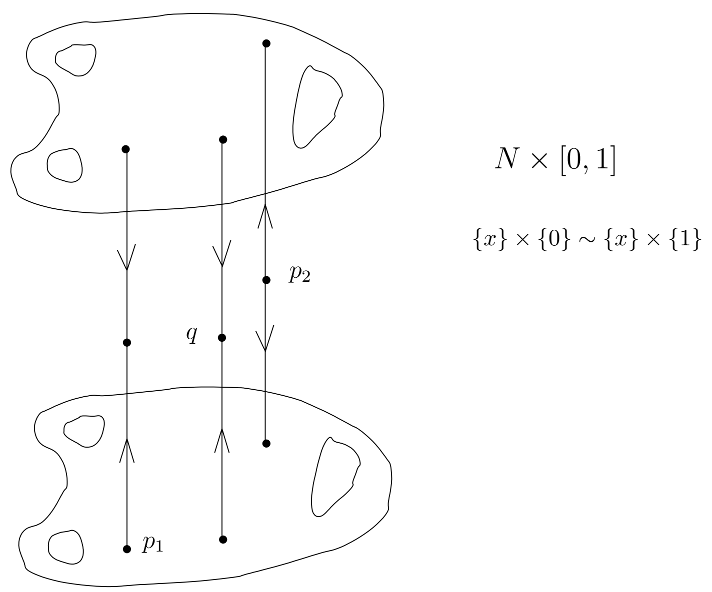

FIGURE 3. How to construct F_1 by small C^∞ perturbations in finitely many circles.

4.1.5. Let us consider a periodic point $r_1 \in Q$ of stable index 1 and another one r_2 of stable index 2. We can assume they are fixed (modulo considering an iterate of F_1). Consider $\delta > 0$ small enough such that $B(r_1, 6\delta) \cup B(r_2, 6\delta)$ is disjoint from:

- the periodic points $p_1, \dots, p_N, q$ defined above,
- the blender K,
- $\overline{(W^u_L(p_1) \cup \dots \cup W^u_L(p_N))}$ and
- from $\mathcal{F}^u_{L'}(q)$ (where L' is chosen such that $\mathcal{F}^u_{L'}(q)$ intersects K).

4.1.6. In the same vein as in section 3.1 we shall first construct a diffeomorphism F_2 modifying F_1 such that:

- F_2 coincides with F_1 outside $B(r_2, \delta)$.
- F_2 preserves the center-stable foliation of F_1.
- DF_2 preserves narrow cones $\mathcal{E}^u$ and $\mathcal{E}^{cs}$ around the unstable direction E^u and the center stable direction $E^s \oplus E^c$ of F_1 respectively. Also, vectors in $\mathcal{E}^u$ are expanded uniformly by DF_2 while every plane contained in $\mathcal{E}^{cs}$ verifies that the volume[10] is contracted by DF_2.
- The point r_2 remains fixed for F_2 but now has complex eigenvalues in r_2.

Before we continue with the construction of the example to prove Theorem B, we shall make a small detour to sketch the following:

Proposition 4.2. *There exists an open C^1-neighborhood $\mathcal{V}$ of F_2 such that for every $f \in \mathcal{V}$ one has that f has a transitive attractor in U.*

Sketch. Notice that one can choose $\mathcal{V}$ such that for every $f \in \mathcal{V}$ one preserves a center-stable foliation close to the original one. Also, one can assume that properties (F2) and (F3) still hold for the continuations $p_i(f)$ and $q(f)$ since F_2 coincides with F_1 outside $B(r_2, \delta)$ and these are C^1-robust properties.

Also, we demand that for every $f \in \mathcal{V}$, the derivative of f preserves the cones $\mathcal{E}^u$ and $\mathcal{E}^{cs}$, contracts volume in $E^{cs} \subset \mathcal{E}^{cs}$ (the plane tangent to the center-stable foliation) and expands vectors in $E^u \subset \mathcal{E}^u$.

Consider now a center stable disk D and an unstable curve γ which intersect the maximal invariant set

$$Q_f = \bigcap_{n>0} f^n(U).$$

Since by future iterations γ will intersect the stable manifold of $q(f)$ (property (F3)) we obtain that by the $\lambda-$lemma it will accumulate the unstable manifold of $q(f)$. Since

[10]This means with respect to the Riemannian metric which allows to define a notion of 2-dimensional volume in each plane.

the unstable manifold of $q(f)$ is dense in the union of the unstable manifolds $W^u(p_1(f)) \cup \ldots \cup W^u(p_N(f))$ we obtain that the union of the future iterates of γ will also be dense there.

Now, iterating backwards the disk D we obtain, using that Df^{-1} expands volume in the center-stable direction that the diameter of the disk grows exponentially with these iterates.

Condition (F2) will now imply that eventually the backward iterates of D will intersect the future iterates of γ. This implies transitivity.

□

4.1.7. Now, we shall modify F_2 inside $B(r_1, \delta)$ in order to obtain an open set to satisfy Theorem B. So we shall obtain F_3 such that:

- F_3 coincides with F_2 outside $B(r_1, \delta)$.
- F_3 preserves the center-stable lamination of F_2.
- DF_2 preserves narrow cones $\mathcal{E}^u$ and $\mathcal{E}^{cs}$ around the unstable direction E^u and the center stable direction E^{cs} of F_2. Also, vectors in $\mathcal{E}^u$ are expanded uniformly by DF_3.
- r_1 is a saddle with stable index 1, the product of any pair of eigenvalues is larger than 1 and the stable manifold of r_1 intersects the complement of $B(r_1, 6\delta)$.

We obtain a C^1 neighborhood $\mathcal{U}_1$ of F_3 where for $f \in \mathcal{U}$, if we denote

$$\Lambda_f = \bigcap_{n \geq 0} f^n(U) :$$

(P1') There exists a continuation of the points $p_1, \ldots, p_N, q, r_1, r_2$ which we shall denote as $p_i(f)$, $q(f)$ and $r_i(f)$. The point $r_1(f)$ is a saddle of stable index 1 and its stable manifold intersects the complement of $B(r_1, 6\delta)$.

(P2') There is a Df-invariant families of cones $\mathcal{E}^u$ in Q_f and for every $v \in \mathcal{E}^u(x)$ we have that

$$\|D_x f v\| \geq \lambda \|v\|.$$

(P3') f preserves a lamination $\mathcal{F}^{cs}$ which is C^0 close to the one preserved by F_3 and which is trapped in the sense that there exists a family $W^{cs}_{loc}(x) \subset \mathcal{F}^{cs}(x)$ such that for every point $x \in Q_f$ the plaque $W^{cs}_{loc}(x)$ is homeomorphic to $(0,1) \times \mathbb{S}^1$ and verifies that

$$f(\overline{W^{cs}_{loc}(x)}) \subset W^{cs}_{loc}.$$

Moreover, the stable manifold of $r_1(f)$ intersects the complement of $W^{cs}_{loc}(r_1(f))$.

(P4') Properties (F2),(F3) and (F4) are satisfied for f and every curve γ tangent to $\mathcal{E}^u$ of length larger than L intersects the stable manifold of $q(f)$.

Notice that (P4') implies that there exists a unique quasi-attractor Q_f in U for every $f \in \mathcal{U}$ which contains the homoclinic class $H(q(f))$ of $q(f)$ (the proof is the same as Lemma 3.1).

4.2. **The example verifies the mechanism of Proposition 2.1.** We shall show that every $f \in \mathcal{U}$ is in the hypothesis of Proposition 2.1 which will conclude the proof of Theorem B as in Theorem 3.7. We shall only sketch the proof since it has the same ingredients as the proof of Theorem A, the main difference is that instead of having an a priori semiconjugacy we must construct one.

To construct the semiconjugacy, one uses property (P3'), specifically the fact that $f(\overline{W^{cs}_{loc}(x)}) \subset W^{cs}_{loc}(x)$ (compare with Lemma 3.2) to consider for each point $x \in \Lambda_f$ the set:

$$A_x = \bigcap_{n \geq 0} f^n(\overline{W^{cs}_{loc}(f^{-n}(x))})$$

(compare with Proposition 3.3 (1)). One easily checks that the sets A_x constitute a partition of Λ_f into compact connected sets contained in local center stable manifolds and that the partition is upper-semicontinuous. It is not hard to prove that if $h_f : \Lambda_f \to \Lambda_f/_\sim$ is the quotient map, then, the map $g : \Lambda_f/_\sim \to \Lambda_f/_\sim$ defined such that

$$h_f \circ f = g \circ h_f$$

is expansive (in fact, $\Lambda_f/_\sim$ can be seen to be homeomorphic to Υ and g conjugated to P). See [D] for more details on this kind of decompositions and quotients.

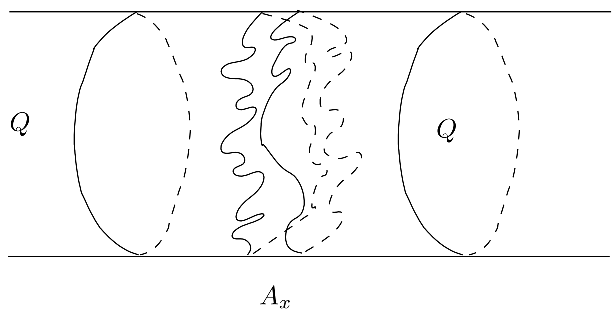

FIGURE 4. The set A_x is surrounded by points in $\overline{W^u(q)} \subset Q$

Since fibers are contained in center stable sets, we get that h_f is injective on unstable manifolds and one can check that the fibers are invariant under unstable holonomy (see the proof of Proposition 3.3 (2)). Stable sets of g are dense in $\Lambda_f/_\sim$.

The point $r_1(f)$ will be in the boundary of $h_f^{-1}(\{h_f(r(f))\})$ since its stable manifold is not contained in $W^{cs}_{loc}(r_1(f))$.

4.2.1. We claim that the boundary of the fibers restricted to center-stable manifolds is contained in the unique quasi-attractor Q_f. This is proven as follows:

Assume that $x \in \partial h_f^{-1}(\{h_f(x)\})$ and consider a small neighborhood V of x. Consider a disk D in $W^{cs}_{loc}(x)$, since x is a boundary point, we get that $h_f(D)$ is a compact connected set containing at least two points in the stable set of $h_f(x)$ for g, so by iterating backwards, and using (F2) (guaranteed for f by (P4')) we get that there is a backward iterate of D which intersects $\mathcal{F}^u(q) \subset Q_f$ which concludes.

Now, Theorem B follows with the same argument as for Theorem 3.7, using Proposition 2.1 and the fact that $r_1(f)$ is contained in Q_f.

□

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CMAT, Facultad de Ciencias, Universidad de la República, Uruguay

LAGA; Institute Galilee, Universite Paris 13, Villetaneuse, France

E-mail address: `rpotrie@cmat.edu.uy`