Preprint PREMAT 2011/131 http://premat.fing.edu.uy/papers/2011/131.pdf
Accepted for publication in the journal "Dynamical Systems"
The final version will appear in http://www.tandfonline.com/toc/cdss20/current#.UikiPX96-ME

On Ilyashenko's Statistical Attractors

Eleonora Catsigeras[a] *

Abstract

We define a minimal α-observability of Ilyashenko's statistical attractors. We prove that the space is always full Lebesgue decomposable into pairwise disjoint sets that are Lebesgue-bounded away from zero and included in the basins of a finite family of minimal α-observable statistical attractors. Among other examples, we analyze the Bowen homeomorphisms with non robust topological heteroclinic cycles. We prove the existence of three types of statistical behaviours for these examples.

Keywords: Statistical attractors; SRB measures; physical measures
MSC 2010: 37A05, 28D05, 37B20, 37B25

1 Introduction

The theory of the Ilyashenko's attractors was principally developed in [1], [2], [7], [9], [10], [11], [12], [19]. These attractors share the advantages of both Milnor's attractors [13], [14] and Pugh-Shub's ergodic attractors [15]. In fact, on the one hand, the sure existence of Milnor's attractors is inherited by Ilyashenko's attractors, since these latter exist for any continuous dynamical system on a compact Riemannian manifold (Theorem 1.10). On the other hand, the fine statistical description of Pugh-Shub's ergodic attractors is also inherited by any (minimal) Ilyashenko's attractor K, since any small neighbourhood of any point of K must be visited in the future with a positive frequency when time goes to infinite. As a counterpart, we notice that in general Pugh-Shub's ergodic attractors do not necessarily exist (Example 5.2, Case C), while most points of Milnor's attractors may be statistically irrelevant; namely, small neighbourhoods of them may receive asymptotically zero-frequent visits in the future (Example 5.1).

*Instituto de Matemática y Estadística "Prof. Rafael Laguardia"(IMERL). Facultad de Ingeniería - Universidad de la República. Address: Av. Herrera y Reissig 565, C.P. 11300, Montevideo, Uruguay. Email: eleonora@fing.edu.uy

Definition 1.1 (Ilyashenko's Statistical Attractor)

Let M be a compact Riemannian manifold. Let $f\colon M \to M$ be a *continuous* map. Let $K \subset M$ be a nonempty compact set. The set K is a *Ilyashenko's statistical attractor* if:

a) the following set $B(K)$, which is called *basin of statistical attraction* of K or in brief *basin* of K, has positive Lebesgue measure:

$$B(K) := \{x \in M : \quad \lim_{n\to+\infty} \frac{1}{n} \#\{0 \leq j \leq n-1 : \quad \text{dist}(f^j(x), K) < \epsilon\} = 1 \ \ \forall \epsilon > 0\}, \tag{1}$$

where $\#A$ denotes the cardinality of the finite set A;

b) K is minimal with respect to the basin $B(K)$, i.e.

$$K' \neq \emptyset \text{ is compact },\ K' \subset K,\ B(K') = B(K) \ \ m-\text{a.e.} \quad \Rightarrow \quad K' = K,$$

where m denotes the Lebesgue measure on M.

From Definition 1.1, the orbit of a point $x \in M$ belongs to the basin of statistical attraction $B(K)$ of K if and only if the asymptotic frequency of its visits in the future to any arbitrarily small neighbourhood of K is 1. The future orbit does not need to remain near K in all the instants of the large future. It is just required that the frequency according to which one may find a future iterate of x far from K, is asymptotically null. So, the attraction to K is not necessarily topological, but statistical.

Also notice that K is a statistical attractor only if its basin $B(K)$ of statistical attraction has positive Lebesgue measure. Thus, a nonempty basin $B(K)$ (as the compact support K of any ergodic measure has) is not enough. It is immediate to check that if there exists an ergodic invariant measure μ that besides is physical, then the compact support K of μ is an statistical attractor. Nevertheless, the converse is false, as Bowen example shows (cf. Example 5.2).

Remark 1.2 Due to the continuity of f and to Condition b) of minimality, it is deduced that any Ilyashenko's statistical attractor K is f-invariant, i.e. $f^{-1}(K) = K$. In fact, take $y \in K$. After the minimality of K with respect to its basin, for any open neighborhood $V \ni y$ there exists a positive-Lebesgue subset $A(y) \subset B(K)$ such that the frequency of visits to V of the orbit of x converges to 1, for all $x \in A(y)$. Conversely, if $y \in M$ is any point of the manifold such that the frequency of visits to V of the orbit of x converges to 1 for all x in certain Lebesgue positive set $A(y) \subset B(K)$, then $y \in K$ (because Equality (1) and because the frequencies of visits to the neighborhoods of two disjoint compact sets, cannot both converge to 1). Since f is continuous, we deduce that the frequency of visits of any orbit to a neighborhood U of a point $z = f(y)$ in the manifold, coincides with the frequency of visits of the same orbit to the neighborhood $V := f^{-1}(U) \ni y$. Thus, we conclude that $y \in K$ if and only if $f(y) \in K$, proving that any atractor K according to Definition 1.1, must be f-invariant. Nevertheless, we are interested to generalize the above definition *for any (non necessarily continuous) Borel measurable map* $f : M \mapsto M$. But the argument above does not work for non continuous f. On the one hand, if we imposed the f-invariance of K in the definition, statistical attractors may not exist for non continuous f (recall for instance the trivial example $f : [0,1] \mapsto [0,1]$ defined by $f(x) = x/2$ if $x > 0$ and $f(0) = 1$). On the other hand, if we did not impose the f-invariance to K, then statistical attractors would still be characterized by means of relevant probability measures that generalize the concept of physical measures (Theorem 1.11), but these measures would not necessarily be f-invariant.

Definition 1.3 (α-Observability and α-obs. Minimality)

For any $0 < \alpha \leq 1$, we say that a nonempty compact set K satisfying condition a) of Definition 1.1 is *α-observable* if $m(B(K)) \geq \alpha$, where m denotes the Lebesgue measure. We abbreviate this property by *α-obs.* We say that K is *minimal α-obs,* if it is α-obs. and no proper compact subset of K is also α-obs for the same value of α.

In Remarks 1.5 and 1.6 we discuss the relation between the α-obs. minimality and the minimality condition (b) of Definition 1.1.

Definition 1.4 (Statistical attractor)

Let M be a compact Riemannian manifold. Let $f\colon M \to M$ be a Borel measurable map. Let $K \subset M$ be a nonempty and compact set. We say that K is a *statistical attractor* if it satisfies condition a) of Definition 1.1 and besides:

b') there exists $\alpha > 0$ such that K is minimal α-obs.

Remark 1.5 Let us prove that Definitions 1.1 and 1.4 are equivalent if f is continuous.

First, any statistical attractor according to Definition 1.4 is a Ilyashenko's statistical attractor according to Definition 1.1. In fact, let us see that Condition b) is satisfied. Take $K' \subset K$ nonempty and compact such that $m(B(K')) = m(B(K))$. The condition $K' \subset K$ immediately implies $B(K') \subset B(K)$. Since K is α-obs. minimal, for some $\alpha > 0$, then $m(B(K)) \geq \alpha$ and $m(B(K'')) < \alpha$ for any compact nonempty set K'' properly contained in K. Since $m(B(K')) = m(B(K)) \geq \alpha$ we conclude that $K' = K$.

Conversely, let us see that any statistical attractor K satisfying Definition 1.1 also satisfies Definition 1.4. In fact, choose $\alpha = m(B(K))$. Assume that K satisfies Definition 1.1 and that $K' \subset K$ is nonempty and compact such that $m(B(K')) \geq \alpha$. Since $K' \subset K$ then $B(K') \subset B(K)$. From $m(B(K')) \geq \alpha = m(B(K))$ we deduce $B(K') = B(K)$ Lebesgue a.e. From Condition b) of Definition 1.1 we deduce that $K' = K$. So, we have shown that K is α-obs. minimal.

Remark 1.6 We notice that we are using the adjective "α-obs. minimal" in the sense of a least set in the chain of inclusion of α-observable nonempty compact sets, for a fixed value of α bounded away from zero.

Let us see that the α-obs. minimality for a *previously specified value of* $\alpha > 0$ is indeed a restriction to the concept of statistical attractor. In other words, let us see that a statistical attractor K satisfying Definition 1.1 (or equivalently Definition 1.4 for some $\alpha' > 0$), is not necessarily $\alpha-$obs. minimal, if one has a previously specified value $0 < \alpha < m(B(K))$. In fact, consider the Bowen example (cf. Example 5.2). It is formed by an eye (homeomorhic to a two-dimensional compact disk) with two saddles p and q in its boundary and such that the interior of the eye is the basin B_1 of statistical attraction of the compact set $K_1 = \{p, q\}$. Now, add other Bowen example, i.e. other eye, whose two saddles are q, r, its interior B_2 is the basin of statistical attraction of $K_2 = \{q, r\}$, and such that the intersection of both eyes is only the saddle q. Assume for instance that $m(B_1) = m(B_2) = 1/2$. Then $K = \{p, q, r\}$ is a Ilyashenko's statistical attractor (satisfying Definition 1.1 and equivalently Definition 1.4 with $\alpha = 1$), whose basin is $B = B_1 \cup B_2$. The statistical attractor K is minimal *with respect to its basin* B and is also 1-obs. minimal. But K is not minimal in an absolute sense. In fact, K_1 and K_2 are also statistical attractors, whose basins are B_1 and B_2 respectively, and do also satisfy Definitions 1.1 and 1.4. They are 1/2-obs. minimal. In this example K_1 and K_2 are the unique $\alpha-$obs. minimal statistical attractors for any $0 < \alpha \leq 1/2$. They are proper compact subsets of K. So, the statistical attractor K is not minimal (in the absolute sense) among all the existing Ilyashenko's attractors.

The latter example is too simple, because K_1 and K_2 are minimal in the absolute sense, among all the existing Ilyashenko's attractors. But in a general context, there may not exist Ilyshenko's statistical attractors that are minimal among all the existing ones, unless a positive minimum α is previously specified for the Lebesgue measure of the basins of statistical attraction. In fact, consider in a three-dimensional setting the following example. According to a real parameter $\theta \in [0, \pi/2)$ immerse a two-dimensional Bowen example's eye with statistical attractor $K_\theta = \{0, p_\theta\}$, with diameter going to zero when $\theta \to \pi/2$, and such that the two-dimensional basin $B(K_\theta)$ of statistical attraction (the interior of each eye) is contained in a plane forming angle θ with the horizontal plane. In this example, each Ilyashenko's statistical attractor, properly contains infinitely many other attractors, even if each of them is minimal with respect to its basin. In spite of that, for each previously specified value $\alpha > 0$, the space is still Lebesgue-a.e. decomposable into the basins of a *finite number* N_α of α-obs. minimal statistical attractors (in this example, if α goes to zero, then N_α goes to infinite).

For the sake of completeness we include the following definitions:

Definition 1.7 (Milnor's attractor)
Let $f\colon M \to M$ be a Borel measurable map. A compact set $K \subset M$ is a *Milnor's attractor* if the set $A(K) \subset M$ of all the initial states $x \in M$ such that the omega-limit set $\omega(x)$ is contained in K, has positive Lebesgue measure, and if K is the minimal compact set that contains $\omega(x)$ for Lebesgue all the points $x \in A(K)$.

We recall that $\omega(x)$ is the compact nonempty set in M composed by the limits of all the convergent subsequences of the orbit $\{f^n(x)\}_{n\in\mathbf{N}}$. We call $A(K)$ *the basin of topological attraction* of K. We say that a Milnor's attractor is α-observable if $m(A(K)) \geq \alpha$, where m denotes the Lebesgue measure. We say that a Milnor's attractor K is minimal α-obs. if it is α-obs. and no proper subset of K is an α-obs. Milnor's attractor.

Notation: Roughly abusing of the language, we will use the words *statistical attractor* referring to any nonempty compact set K satisfying Condition a) of Definition 1.1, regardless whether K satisfies Condition b) of minimality or not. If besides K is α-obs. minimal for some $\alpha \in (0, 1]$, then K also satisfies Condition b), and conversely (cf. Remark 1.5). The rough use of the language will not produce a conflict with Definitions 1.1 and 1.4 since we will always search for such a compact set K that besides is α-obs. minimal for some α.

Since topological attraction implies statistical attraction, any α-obs. minimal Milnor's attractor is an α-observable (but maybe non minimal) statistical attractor, according to Definitions 1.3 and 1.4. But, as we show in Examples 5.1 and 5.2, not all the minimal α-obs. Milnor attractors are minimal α-obs. statistical attractors. Nevertheless, as a corollary of Theorem 1.12 we prove the following statement: The basin $A(K)$ of topological attraction of any α-obs Milnor's attractor K, is *covered* by the union (up to a zero Lebesgue measure set) of the basins $B(K_i)$ of statistical attraction of a finite family of minimal α_i-obs statistical attractors $K_i \subset K$, for some adequate positive values of α_i. In the above result, the union of all the minimal statistical attractors K_i contained in K, is not necessarily equal to the Milnor's attractor K (see Examples 5.1 and 5.2). Therefore, the statistical attractors are thinner sets than the Milnor's attractors.

To state the following definition, we denote by $\mathcal{M}$ the space of all the Borel probability measures on M endowed with the weak* topology. We denote by $\lim^*$ the limit in $\mathcal{M}$.

Definition 1.8 (EMPIRICAL PROBABILITIES AND THE LIMIT SET IN $\mathcal{M}$)

Let $x \in M$. The *sequence of empirical probabilities* $\{\nu_n(x)\}_{n\geq 1}$ of x is defined by

$$\nu_n(x) := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \in \mathcal{M}, \tag{2}$$

being δ_y the Dirac delta supported on the point $y \in M$. In other words, $\nu_n(x)$ is the probability measure supported, and equally distributed, on the finite piece of the orbit of x between the instant 0 and the instant $n-1$.

The *limit set in the space of probability measures* $\mathcal{M}$ of the orbit of $x \in M$ is:

$$\mathcal{L}^*(x) := \{\mu \in \mathcal{M} : \ \exists \ n_i \to +\infty \text{ such that } \lim^*_{i\to+\infty} \nu_{n_i}(x) = \mu\}. \tag{3}$$

Since $\mathcal{M}$ is compact and sequently compact, then $\mathcal{L}^*(x)$ is nonempty and compact for all $x \in M$. We say that $\mathcal{L}^*(x)$ describes *the asymptotic statistics* of the future orbit of x.

Definition 1.9 (SRB MEASURES AND ERGODIC ATTRACTORS)

A probability Borel measure μ on M is called *SRB or physical* if the set

$$B(\mu) := \{x \in M : \lim^*_{n\to+\infty} \nu_n(x) = \mu\} = \{x \in M : \mathcal{L}^*(x) = \{\mu\}\}$$

has positive Lebesgue measure. We call $B(\mu)$ the *basin of statistical attraction* of μ.

Abusing of the language (regardless whether μ is ergodic), we call *ergodic attractor* to the compact support K of an SRB measure μ (i.e. K is the minimal compact set such that $\mu(K) = 1$).

After Definition 1.9, any SRB measure is f-invariant provided that f is continuous. Nevertheless, SRB measures are not necessarily ergodic (see for instance Bowen's Example 5.2, Case (B) at the end of this paper). We notice that the definition of SRB measure also holds for any Borel-measurable map $f: M \to M$, but in this case μ is not necessarily f-invariant. For instance, $f: [0,1] \to [0,1]$ defined by $f(0) = 1, \ f(x) = (1/2)x$ for all $x \neq 0$, has the SRB measure δ_0 (whose basin is $[0,1]$), which is not f-invariant.

1.1 Statement of the results

The contributions of this paper to the theory of statistical attractors are:

a) Definition 1.3 in which we introduce the concept of minimal α-observability of Ilyashenko's statistical attractors;

b) Theorem 1.10, which slightly strengthen the previously known result of the existence of Ilyashenko's statistical attractors, by proving also their existence under the minimal α-obs. condition for any previously specified value of $\alpha \in (0,1]$.

c) Theorem 1.11, which adds to the previously known results derived from the Krylov-Bogolyubov procedure, the relationship between SRB-like measures, defined in [4], and the statistical attractors of the system;
d) Theorem 1.12, which constructs a natural decomposition of a Lebesgue-full subset of the space into the basins of a finite number of minimal observable statistical attractors;
e) the proof of the existence of three types of statistical behavior in a C^0-version of Bowen's diffeomorphisms (Example 5.2).

Theorem 1.10 (Existence of α-obs. statistical attractors)
Let $f\colon M \to M$ be Borel-measurable. For all $0 < \alpha \leq 1$ there exist minimal α-observable statistical attractors.
Moreover, if $\alpha = 1$, then the minimal α-obs. statistical attractor is unique.

We prove Theorem 1.10 in Section 2, where we also prove Theorems 2.3, 2.4 and 3.5, which are slight generalizations of Theorem 1.10 relative to some previously fixed invariant subsets of the space.

It is straightforward to check that the minimal compact support of an SRB measure, when this measure exists, is an α-obs. minimal statistical attractor for some $\alpha > 0$. The following Theorem 1.11 asserts a kind of converse statement: any α-obs minimal statistical attractor is the minimal compact support of a set of SRB-like measures. The latter measures are obtained after applying the Krylov-Bogolyubov procedure to the empirical probabilities constructed in Equality (2). The method takes any weak* partial limit of the time averages of non necessarily invariant probabilities.

Theorem 1.11 (Characterization of α-obs statistical attractors)
If K is a statistical attractor and if $B(K)$ is its basin, then there exists a unique non empty weak*-compact set $\mathcal{O}_f(K)$ of probability measures (which we call SRB-like measures) *such that:*

(a) *For Lebesgue almost all the initial states $x \in B(K)$, and for all the convergent subsequences of the empirical distributions $\nu_n(x) := (1/n)\sum_{j=0}^{n-1}\delta_{f^j(x)}$, their weak*-limits are probability measures contained in $\mathcal{O}_f(K)$.*

(b) *$\mathcal{O}_f(K)$ is the minimum weak*-compact set of probability measures satisfying* (a).

(c) *$\mu(K) = 1$ for all $\mu \in \mathcal{O}_f(K)$.*

(d) *If besides K is minimal α-obs. for some $0 < \alpha \leq 1$, then K is the minimal compact set in M such that $\mu(K) = 1 \ \forall \ \mu \in \mathcal{O}_f(K)$.*

We prove Theorem 1.11 in Section 3. In the proof we use the SRB-like measures, defined in [4], which we restate in Definition 3.1. In Section 3 we also prove the converse of part (d) of Theorem 1.11. In fact, we show that the minimal compact support of all the measures μ in the weak*-compact set of SRB-like measures, is a minimal α-obs. statistical attractor for some $0 < \alpha \leq 1$ (see Theorem 3.5).

Theorem 1.11 states that for all $0 < \alpha \leq 1$, any α-observable statistical attractor K is provided with a minimal weak*-compact subset $\mathcal{O}_f(K)$ of probability measures with two remarkable properties:

(1) It is a set of f-invariant measures which has, with respect to the attractor K, a "physical" role as SRB measures have, when they exist, with respect to the ergodic attractors. In fact, after the statement (a) of Theorem 1.11, the invariant measures in $\mathcal{O}_f$ completely describe the asymptotic statistics of the time series for Lebesgue-almost all the orbits attracted by K.

(2) It is the minimal compact set of probability measures that completely describes the asymptotic statistics, as part (b) of Theorem 1.11 states. Therefore, the statistical attractors are the optimal choice, among the compact invariant sets in the ambient manifold M, if one aims to describe the Lebesgue-full asymptotic statistics of the system.

The following Theorem 1.12 states the existence and finitude of a decomposition of the space, up to a zero-Lebesgue subset, into a family of sets, each one contained in the basin of attraction of a statistical attractor satisfying a minimally observable condition.

Theorem 1.12 (Finite decomposition into statistical attractors)
Let $0 < \alpha \leq 1$ be fixed. Let m denote the Lebesgue probability measure.

There exists a finite family $\{K_i\}_{1 \leq i \leq p}$ of α_i-obs statistical attractors K_i with basins $B(K_i)$ such that:

(a) *$\bigcup_{i=1}^{p} B(K_i)$ covers $m-$almost all M.*

(b) *$\alpha_i = \alpha$ for all the values of $i \in \{1, \ldots, p\}$ except at most one, say i_0, for which $0 < \alpha_{i_0} = m(B(K_{i_0})) < \alpha$.* (Therefore $m(B(K_i)) \geq \alpha \;\; \forall\, i \in \{1, \ldots, p\}$ such that $i \neq i_0$.)

(c) *For all $1 \leq i \leq p$ the statistical attractor K_i is α_i-obs. minimal for f restricted to $M \setminus \cup_{j=1}^{i-1} B(K_j)$.* (We denote $\cup_{j=1}^{0} \cdot = \emptyset$.)

We prove Theorem 1.12 in Section 4. The proof is rather natural: roughly speaking, one can take away minimal observable sets (together with what they attract), one by one.

We notice that the statistical attractors K_i of the decomposition in Theorem 1.12 are not necessarily pairwise disjoint. If all the statistical attractors K_i are mutually disjoint, then any pair of them would be at positive distance (since they are compact sets), and so their basins would be also pairwise disjoint. If the additional assumption of mutually disjointness of the attractors holds, Theorem 1.12 asserts that the basins $B(K_i)$ of the finitely many statistical attractors K_i would form a partition of Lebesgue-a.e. the space. Anyway, if the disjointness condition does not hold, the basins of attractions of the finite number of statistical attractors cover Lebesgue-a.e. the space, and are, one by one, Lebesgue-bounded away from zero.

To end this section, we deduce an immediate corollary of Theorem 1.12, which shows that the statistical attractors are thinner than Milnor's attractors: namely, each α-obs. minimal Milnor's attractor contains the union of a finite number of statistical attractors.

Corollary of Theorem 1.12

Let $0 < \alpha \leq 1$, and let K be an α-obs. minimal Milnor's attractor with basin $A(K)$. There exists a finite number of statistical attractors $K_1, \ldots, K_p$ contained in K that satisfy the conditions (a), (b) *and* (c) *of Theorem* 1.12 *for the set $A(K)$ instead of M.*

This corollary is immediate after Theorem 1.12. In fact, along the proof of Theorem 1.12 one does not use the manifold structure of the ambient space M for any purpose except to define its Lebesgue measure m. Therefore, to prove the corollary it is enough to put $f|_{A(K)} \colon A(K) \to A(K)$ in the role of $f \colon M \to M$ and $m|_{A(K)}$ in the role of m, where $m|_{A(K)} := m(B \cap A(K))$ for any Borel set $B \subset M$.

2 Proof of the existence of minimal α-obs. statistical attractors

In this section we prove Theorem 1.10. We also introduce some definitions which impose additional minimal conditions to the statistical attractors (Definitions 2.1 and 2.2). At the end of this section we strengthen Theorem 1.10, proving the existence of statistical attractors satisfying those additional conditions (Theorems 2.3 and 2.4).

Proof of Theorem 1.10

Proof: Let us fix $0 < \alpha \leq 1$. Consider the family $\aleph_\alpha$ of all the α-obs statistical attractors (non necessarily minimal). The family $\aleph_\alpha$ is nonempty since it trivially contains the manifold M.

Define in $\aleph_\alpha$ the partial order $K_1 \leq K_2$ if $K_1 \subset K_2$. Since the attractors are all non empty compact sets, any chain $\{K_a\}_{a \in A} \subset \aleph_\alpha$ (i.e. any totally ordered subset of $\aleph_\alpha$) has a non empty compact intersection: $K := \bigcap_{a \in A} K_a$. Let us prove that $K \in \aleph_\alpha$.

We have to prove that $m(B(K)) \geq \alpha$, where m is the Lebesgue measure and $B(K)$ is the basin of statistical attraction of K, constructed in Definition 1.1.

For any $\epsilon > 0$ and for any nonempty compact set $H \subset M$, define

$$B_\epsilon(H) := \{x \in M : \lim_{n \to +\infty} \omega_{n,H,\epsilon}(x) = 1\}, \quad \text{where} \tag{4}$$

$$\omega_{n,H,\epsilon}(x) := \frac{1}{n}\,\#\{0 \le j \le n-1:\ \mathrm{dist}(f^j(x), H) < \epsilon\} \le 1.$$

It is standard to check that $B_{\epsilon'}(K) \subset B_\epsilon(K)$ if $0 < \epsilon' < \epsilon$. Therefore,

$$B(K) = \bigcap_{\epsilon>0} B_\epsilon(K) = \bigcap_{n\ge 1} B_{1/n}(K),$$

and thus

$$m(B(K)) = \lim_{n\to+\infty} m(B_{1/n}(K)) = \lim_{\epsilon\to 0^+} m(B_\epsilon(K)).$$

So, to deduce that $m(B(K)) \ge \alpha$ it is enough to prove that $m(B_\epsilon(K)) \ge \alpha$ for all $\epsilon > 0$.

Fix $\epsilon > 0$. We assert that there exists $a \in A$ such that $\mathrm{dist}(y, K) < \epsilon$ for all $y \in K_a$. Arguing by contradiction, if the intersection $K_a \bigcap \{y \in M : \mathrm{dist}(y, K) \ge \epsilon\}$ were nonempty for all $a \in A$, since $\{K_a\}_{a\in A}$ is totally ordered, the property of finite intersections of compact sets would imply that $\bigcap_{a\in A} K_a \bigcap \{y \in M : \mathrm{dist}(y, K) \ge \epsilon\} \ne \emptyset$, contradicting the construction of $K = \bigcap_{a\in A} K_a$ and proving the assertion.

Using the triangle property for the value of $a \in A$ satisfying the above assertion, we deduce $\omega_{n,K_a,\epsilon}(x) \le \omega_{n,K,2\epsilon}(x)$ for all $x \in M$ and for all $n \in \mathbb{N}$. Therefore $B_\epsilon(K_a) \subset B_{2\epsilon}(K)$. Since $K_a \in \aleph_\alpha$, we obtain $\alpha \le m(B(K_a)) \le m(B_\epsilon(K_a)) \le m(B_{2\epsilon}(K))$, as wanted. We have proved that $K \in \aleph_\alpha$ and so, any chain in $\aleph_\alpha$ has a minimal element. Applying Zorn Lemma we deduce that there exist minimal elements in $\aleph_\alpha$. This means that there exist α-obs statistical attractors $K \subset M$, that do not contain proper subsets that are also α-obs statistical attractors. So, the existence of minimal α-obs statistical attractors is proved for any previously specified value of $\alpha \in (0, 1]$.

To end the proof of Theorem 1.10 it is left to show that the minimal 1-obs. statistical attractor is unique. In fact, consider K_1 and K_2, minimal 1-obs. statistical attractors, namely their basins $B(K_1)$ and $B(K_2)$ have full Lebesgue measures: $m(B(K_1)) = m(B(K_2)) = 1$. Therefore, $m(B(K_1) \cap B(K_2)) = 1$. Take $x \in B(K_1) \cap B(K_2)$. Thus $1 = \lim_{n\to+\infty} \omega_{n,K_i,\epsilon}(x)$ for all $\epsilon > 0$, for $i = 1, 2$. Denote

$$\omega_{n,K_1,K_2,\epsilon}(x) = \frac{1}{n}\#\{0 \le j \le n-1 : \mathrm{dist}(f^j(x), K_i) < \epsilon \ \text{ for } i = 1, 2\}.$$

We have $\omega_{n,K_1,\epsilon}(x) \le \omega_{n,K_1,K_2,\epsilon}(x) + \big(1 - \omega_{n,K_2,\epsilon}(x)\big)$. Therefore, $\lim_{n\to+\infty} \omega_{n,K_1,K_2,\epsilon}(x) = 1$ for Lebesgue almost all $x \in M$. Besides, if $\mathrm{dist}(y, K_i) < \epsilon$ for $i = 1, 2$, then $\mathrm{dist}\big(y, K_2 \cap \{z \in M : \mathrm{dist}(K_1, z) \le 2\epsilon\}\big) < \epsilon$. Thus,

$$\omega_{n,K_1,K_2,\epsilon}(x) \le \omega_{n,K_2\cap\{z\in M:\ \mathrm{dist}(z,K_1)\le 2\epsilon\},\epsilon}(x) \le \omega_{n,K_2\cap\{z\in M:\ \mathrm{dist}(z,K_1)\le 2\epsilon_0\},\epsilon}(x)$$

for all $0 < \epsilon \leq \epsilon_0$. We deduce that, for each fixed $\epsilon_0 > 0$, the compact set $K_2 \cap \{z \in M : \text{dist}(z, K_1) \leq 2\epsilon_0\}$ is an 1-obs. statistical attractor. Since K_2 is minimal with such a property, we obtain $K_2 \subset \{z \in M : \text{dist}(z, K_1) \leq 2\epsilon_0\}$ for all $\epsilon_0 > 0$. Therefore, $K_2 \subset K_1$. Arguing symmetrically, $K_1 \subset K_2$, and thus $K_1 = K_2$ ending the proof. □

Definition 2.1 Let $0 \leq \alpha \leq 1$ and let $M' \subset M$ be a Borel set such that $M' \subset f^{-1}(M')$ and $m(M') \geq \alpha$. We say that a nonempty, compact and f-invariant set $K \subset M$ is an *α-obs statistical attractor restricted to M'*, or *for $f|_M$*, if its basin $B(K)$, as defined in 1.1, satisfies:

$$m(B(K) \bigcap M') \geq \alpha. \tag{5}$$

We say that an α-obs statistical attractor K is *minimal restricted to M'*, or *for $f|_M$*, if it satisfies the inequality (5) and has not proper, nonempty and compact subsets that satisfy it.

Definition 2.2 Let $B \subset M$ be a Borel set such that $B \subset f^{-1}(B)$ and $m(B) \geq \alpha > 0$. We say that a nonempty compact and f-invariant set $K \subset M$ is a *statistical attractor attracting* B if its basin of attraction $B(K)$, as defined in 1.1, satisfies:

$$B(K) \supset B \ m - \text{a.e.} \quad \text{In other words,} \ \ m(B \setminus B(K)) = 0. \tag{6}$$

Since $m(B) \geq \alpha$ any statistical attractor attracting B is $\alpha-$obs.

We say that a statistical attractor is *minimal attracting* B if it satisfies the condition (6) and has not proper, nonempty and compact subsets that satisfy it.

It is standard to check that an α-obs minimal statistical attractor K is also α-obs minimal restricted to its basin, and minimal attracting its basin.

Theorem 2.3 *Let $M' \subset M$ be a Borel set such that $M' \subset f^{-1}(M')$ and $m(M') \geq \alpha > 0$. Then, there exists an α-obs statistical attractor that is minimal restricted to M', according to Definition* 2.1.

Proof: Repeat the proof of Theorem 1.10, using M' in the role of M, $B(K) \cap M'$ in the role of $B(K)$ and $B_\epsilon(H) \cap M'$ in the role of $B_\epsilon(H)$. □

Theorem 2.4 *Let $B \subset M$ be a Borel set such that $B \subset f^{-1}(B)$ and $m(B) > 0$. Then, there exists a statistical attractor that is minimal attracting B, according to Definition* 2.2.

Proof: Repeat the proof of Theorem 1.10, defining the family $\aleph_B$ (instead of $\aleph_\alpha$) of all the statistical attractors $K \subset M$ such that $B(K) \supset B \ \ m-$a.e. □

3 Proof of the probabilistic characterization of statistical attractors

In this section we prove Theorem 1.11. To do that, we first revisit the definition of SRB-like measures, taken from [4]. Let us fix a metric dist^* inducing the weak* topology in the space $\mathcal{M}$ of all the Borel probability measures on M.

Definition 3.1 (SRB-like measures)
Let $B \subset M$ be a forward invariant set (i.e. $B \subset f^{-1}(B)$) that has positive Lebesgue measure. We say that a probability measure μ is *SRB-like or physical-like* for $f|_B$, if for all $\epsilon > 0$ the following set $B_\epsilon(\mu) \subset B$ has positive Lebesgue measure:

$$B_\epsilon(\mu) := \{x \in B : \ \mathrm{dist}^*(\mathcal{L}^*(x), \mu) < \epsilon\},$$

where $\mathcal{L}^*(x)$ is the nonempty weak*-compact set defined in (3).

We call $B_\epsilon(\mu)$ *the basin of ϵ-weak statistical attraction* of the probability μ. We denote by $\mathcal{O}_{f|B}$ the set of all the SRB-like measures μ for $f|_B$. To justify the name "SRB-like measures", compare Definition 3.1 with Definition 1.9.

If f is continuous, then all the measures in $\mathcal{O}_{f|B}$ are f-invariant. In other words $\mathcal{O}_{f|B} \subset \mathcal{M}_f$. In fact, $\mathcal{L}^*(x) \subset \mathcal{M}_f$ for all $x \in M$, so any $\mu \in \mathcal{O}_{f|B}$ belongs to the weak*-closure of $\mathcal{M}_f$. But if f is continuous, then $\mathcal{M}_f$ is weak*-closed. Thus, $\mu \in \mathcal{M}_f$ as wanted.

The lemmas 3.2 and 3.3 below, are reformulations of results communicated in [4].

Lemma 3.2 *$\mathcal{O}_{f|_B}$ is weak*-compact and nonempty.*

Proof: It is immediate that $\mathcal{O}_{f|B}$ is weak*-compact, because it is closed in the space $\mathcal{M}$, which is a compact metric space for any metric inducing its weak* topology. Let us prove that it is nonempty. Assume by contradiction that no measure in $\mathcal{M}$ is SRB-like. Then for all $\mu \in \mathcal{M}$ there exists $\epsilon > 0$ such that $m(B_\epsilon(\mu)) = 0$, where m denotes the Lebesgue measure on M. Since $\mathcal{M}$ is compact, there exists a finite covering of $\mathcal{M}$ with balls $\{\mathcal{B}_i\}_{i=1,\ldots,s}$ of radii ϵ_i, centred at μ_i and such that $m(B_{\epsilon_i}(\mu_i)) = 0$ for all $i = 1, \ldots, s$. Since $m(\bigcup_{i=1}^s B_{\epsilon_i}(\mu_i)) = 0$ and $\bigcup_{i=1}^s B_{\epsilon_i}(\mu_i) \supset \{x \in B(K) : \mathcal{L}^*(x) \bigcap \mathcal{M} \neq \emptyset\}$, we conclude that for Lebesgue almost all $x \in B(K)$ the limit set $\mathcal{L}^*(x)$ (which by definition is always contained in the space $\mathcal{M}$), is empty. This is a contradiction since the space $\mathcal{M}$ is sequentially compact when endowed with the weak* topology, and thus, $\mathcal{L}^*(x) \neq \emptyset$ for all $x \in B(K)$. $\square$

Lemma 3.3 *The set $\mathcal{O}_{f|B}$ is the minimum weak* compact set in the space $\mathcal{M}$ of Borel probability measures such that $\mathcal{L}^*(x) \subset \mathcal{O}_{f|B}$ for Lebesgue almost all $x \in B$.*

Proof: Let us first prove that for m-almost all $x \in B$ the limit set $\mathcal{L}^*(x)$ is contained in $\mathcal{O}_{f|B}$. Assume by contradiction that the set of such points x has m-measure smaller than $m(B)$. Then $\lim_{\epsilon \to 0} m(A_\epsilon) < m(B)$, where

$$A_\epsilon := \{x \in B : \quad \max\{\text{dist}^*(\nu, \mu) : \ \nu \in \mathcal{L}^*(x), \, \mu \in \mathcal{O}_{f|B}\} \ < \ \epsilon\}.$$

Then, for some $\epsilon_0 > 0$ small enough $m(B \setminus A_{\epsilon_0}) > 0$. In other words, for a Lebesgue positive set of points $x \in B$, the limit set $\mathcal{L}^*(x)$ intersects the weak*-compact set $\mathcal{K} := \{\mu \in \mathcal{M} : \ \text{dist}^*(\mu, \mathcal{O}_{f|B}) \geq \epsilon_0\}$. Therefore, at least one of the measures $\mu \in \mathcal{K}$ satisfies $m(B_\epsilon(\mu)) > 0$ for all $0 < \epsilon \leq \epsilon_0$, where

$$B_\epsilon(\mu) := \{x \in B : \quad \text{dist}^*(\mathcal{L}^*(x), \mu) < \epsilon\}.$$

In fact, if the latter assertion were not true, we would cover $\mathcal{K}$ with a finite number of balls $\{\mathcal{B}_i\}_{i=1,\dots,s}$ such that for Lebesgue almost all point $x \in B$, $\mathcal{L}^*(x) \bigcap \mathcal{B}_i = \emptyset$ for all $i = 1, \dots, s$. Thus $\mathcal{L}^*(x) \bigcap \mathcal{K} = \emptyset$ for Lebesgue almost all $x \in B$, contradicting the construction of the set $\mathcal{K}$.

Thus, there exists $\mu \in \mathcal{K}$ such that $m(B_\epsilon(\mu)) > 0$ for all $0 < \epsilon \leq \epsilon_0$ Then, after Definition 3.1 the probability measure μ is SRB-like for $f|B$. Therefore $\mathcal{K} \bigcap \mathcal{O}_{f|B} \neq \emptyset$, contradicting the construction of the compact set $\mathcal{K}$. This ends the proof of the first assertion: for m-almost all $x \in B$, $\mathcal{L}^*(x) \subset \mathcal{O}_{f|B}$.

Second, let us prove that $\mathcal{O}_{f|B}$ is minimal among all the weak* compact sets containing $\mathcal{L}^*(x)$ for Lebesgue almost all $x \in B$. In fact, if $\emptyset \neq \mathcal{K} \subset \mathcal{O}_{f|B}$, and $\mathcal{K}$ is compact, any measure $\mu \in \mathcal{O}_{f|B} \setminus \mathcal{K}$ is at positive distance, say $\epsilon > 0$ (depending on μ), from $\mathcal{K}$. After Definition 3.1 there exists a m-positive set of points $x \in B$ such that $\text{dist}^*(\mathcal{L}^*(x), \mu) < \epsilon$. Therefore $\mathcal{L}^*(x) \not\subset \mathcal{K}$ for those points x. We conclude that $\mathcal{O}_{f|B}$ has not a proper, nonempty and compact subset $\mathcal{K}$ containing $\mathcal{L}^*(x)$ for Lebesgue almost all $x \in B$. This ends the proof that $\mathcal{O}_{f|B}$ is minimal with such a property. $\square$

Lemma 3.4 *If $K \subset M$ is a compact set such that $\mu(K) = 1$ for all $\mu \in \mathcal{O}_{f|B}$, then K is a statistical attractor whose basin $B(K)$ contains B Lebesgue a.e.*

Proof: Fix $\epsilon > 0$ and choose any continuous function $\psi \in C^0(M, [0, 1])$ such that $\psi|K = 1$ and $\psi(y) = 0$ for all $y \in M$ such that $\text{dist}(y, K) \geq \epsilon$. Choose and fix $x \in B$, and a sequence of natural numbers $n_i \to +\infty$ such that the following limits exist:

$$L = \lim_{i \to +\infty} \frac{1}{n_i} \#\{0 \leq j \leq n_i - 1 : \ \text{dist}(f^j(x), K) < \epsilon\}.$$

$$\mu = \lim_{i \to +\infty}{}^{*} (1/n_i) \sum_{j=0}^{n_i - 1} \delta_{f^j(x)} \in \mathcal{L}^*(x).$$

On the one hand, $\mathcal{L}^*(x) \subset \mathcal{O}_{f|B}$ for m-a.e. $x \in B$, due to Lemma 3.3. Besides, by hypothesis, $\mu(K) = 1$ for all $\mu \in \mathcal{O}_{f|B}$. Therefore, the expected value of ψ respect to the probability μ is equal to 1. In fact: $1 \geq \int \psi \, d\mu \geq \int_K \psi \, d\mu = \mu(K) = 1$. On the other hand, since ψ is continuous, the weak*-limit in the space of probability measures implies:

$$1 = \int \psi \, d\mu = \lim_{i \to +\infty} \int \psi \, d\left(\frac{1}{n_i} \sum_{j=0}^{n_i - 1} \delta_{f^j(x)} \right) = \lim_{i \to +\infty} \frac{1}{n_i} \sum_{j=0}^{n_i - 1} \psi(f^j(x)),$$

and, by construction of ψ:

$$\frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x)) \leq (1/n) \# \{0 \leq j \leq n-1 : \ \mathrm{dist}(f^j(x), K) < \epsilon\} \leq 1.$$

Then, $1 = \lim_{n \to +\infty} \frac{1}{n} \# \{0 \leq j \leq n-1 : \ \mathrm{dist}(f^j(x), K) < \epsilon\}$ for m-a.e. $x \in B$. We deduce that $x \in B(K)$ for Lebesgue almost all $x \in B$, and so K is a statistical attractor whose basin contains Lebesgue a.e. B. □

End of the proof of Theorem 1.11.

Proof: Consider the basin of attraction $B(K)$ of a given minimal α-obs. statistical attractor K. By hypothesis $m(B(K)) \geq \alpha > 0$. It is straightforward to check that if $x \in B(K)$ then $f(x) \in B(K)$ (even if f is only a measurable map that is not continuous). Then, we can apply Definition 3.1, and consider the set $\mathcal{O}_{f|B(K)}$ of all the SRB-like measures for $f|_{B(K)}$. After Lemmas 3.2 and 3.3 (denoting $\mathcal{O}_f(K)$ to $\mathcal{O}_{f|B(K)}$), Assertions (a) and (b) of Theorem 1.11 are proved.

Now, let us prove Assertion (c). We shall prove that $\mu(K) = 1$ for all $\mu \in \mathcal{O}_f(K)$. Fix $\mu \in \mathcal{O}_f(K)$, choose an arbitrarily small $\epsilon > 0$ and denote

$$V_\epsilon = \{x \in M : \ \mathrm{dist}(x, K) < \epsilon\}.$$

Construct a continuous real function $\psi \in C^0(M, [0,1])$ such that $\psi|_K = 1$ and $\psi(x) = 0$ if $x \notin V_\epsilon$. After the continuity of ψ there exists $0 < \epsilon' < \epsilon$ such that $\psi(x) > 1 - \epsilon \ \ \forall \ x \in V_{\epsilon'}(K)$. Let us compute the expected value of ψ respect to the probability μ:

$$\int \psi \, d\mu = \int_{V_\epsilon} \psi \, d\mu \leq \mu(V_\epsilon). \tag{7}$$

Recall Equality (3) which defines $\mathcal{L}^*(x)$ for all $x \in M$ and Definition 1.1 of the basin $B(K)$ of the statistical attractor K. Taking into account Equality (4) which defines the set $B_\epsilon(K) \subset M$ for all $\epsilon > 0$, we deduce that $B(K) = \bigcap_{\epsilon > 0} B_\epsilon(K)$. From the statements (a) and (b) of Theorem 1.11 and Definition 3.1, we deduce that there exists $x \in B(K) \subset B_{\epsilon'}(K))$ and $\tilde{\mu} \in \mathcal{L}^*(x)$ such that $|\int \psi\, d\mu - \int \psi\, d\tilde{\mu}| < \epsilon$. Therefore, there exists a subsequence $\{\nu_{n_i}(x)\}_{i\geq 1}$ convergent to $\tilde{\mu}$ in the weak* topology of $\mathcal{M}$, where $\nu_n(x) := (1/n)\sum_{j=0}^{n-1} \delta_{f^j(x)}$. Thus:

$$\int \psi\, d\tilde{\mu} = \lim_{i\to+\infty} \int \psi\, d\left(\frac{1}{n_i}\sum_{j=0}^{n_i-1} \delta_{f^j(x)}\right) = \lim_{i\to+\infty} \frac{1}{n_i}\sum_{j=0}^{n_i-1} \psi(f^j(x)) \geq$$

$$(1-\epsilon)\lim_{i\to+\infty} \frac{1}{n_j} \#\{0 \leq j \leq n_j - 1 : f^j(x) \in \mathcal{V}_{\epsilon'}\}.$$

Since $x \in B_{\epsilon'}(K)$, the limit of the right term in the above inequality, is equal to 1 (recall Equality (4)). Therefore, $\int \psi\, d\tilde{\mu} \geq 1-\epsilon$, and thus $\int \psi\, d\mu \geq 1 - 2\epsilon$. Joining this latter result with Inequality (7), we deduce that $\mu(V_\epsilon) \geq 1 - 2\epsilon\ \forall\ \epsilon > 0$. Taking $\epsilon \to 0^+$ and taking into account that the compact set K is the decreasing intersection of the open sets V_ϵ, we obtain:

$$1 \geq \mu(K) = \lim_{\epsilon\to 0^+} \mu(V_\epsilon) = 1.$$

We have proved that $\mu(K) = 1$ for all $\mu \in \mathcal{O}_f(K)$, as wanted.

Finally, it is left to prove that if K is minimal α-obs with basin $B(K)$, then K is the minimum compact set such that $\mu(K) = 1$ for all $\mu \in \mathcal{O}_f(K)$. Take a nonempty and compact set $K' \subset K$ such that $K \setminus K' \neq \emptyset$. We shall prove that $\mu(K') < 1$ for some $\mu \in \mathcal{O}_f(K)$. The minimality hypothesis on K implies that the set $B(K')$ (according to Definition 1.1), excludes a Lebesgue-positive set of points of $B(K)$. In other words, $m(C) > 0$, where $C := B(K) \setminus B(K') = \bigcup_{\epsilon>0} B(K) \setminus B_\epsilon(K') \subset B(K)$, with $B_\epsilon(K')$ satisfying Equality (4). Fix a point $x \in C$ and fix $\epsilon > 0$ such that $x \notin B_\epsilon(K')$. Choose a continuous real function $\psi \in C^0(M, [0,1])$ such that $\psi|_{K'} = 1$ and $\psi(y) = 0$ for all y such that $\text{dist}(y, K') \geq \epsilon$. After Equality (4), we obtain $\liminf_{N\to+\infty} \omega_{\epsilon,N}(x, K', \epsilon) < 1$ for all $x \in C$. In other words, there exists a sequence $n_i \to +\infty$ such that

$$\lim_{i\to+\infty} \frac{1}{n_i} \#\{0 \leq j \leq n_i - 1 : \quad \text{dist}(f^j(x), K') < \epsilon\} < 1\ .$$

Therefore,

$$\limsup_{i\to+\infty} \int \psi\, d\nu_{n_i}(x) := \limsup_{i\to+\infty} \int \psi\, d\left(\frac{1}{n_i}\sum_{j=1}^{n_i} \delta_{f^j(x)}\right) = \limsup_{i\to+\infty} \frac{1}{n_i}\sum_{j=0}^{n_i-1} \psi(f^j(x))$$

$$\leq \limsup_{i\to+\infty} \frac{1}{n_i} \#\{0 \leq j \leq n_i - 1 : \quad \text{dist}(f^j(x), K') < \epsilon\} \leq 1 - \epsilon.$$

Taking if necessary a subsequence of $\{n_i\}_{i\geq 0}$ (which we still denote $\{n_i\}_{i\geq 0}$) such that $\{\nu_{n_i}(x)\}_{i\geq 0}$ is convergent in the weak* topology to a probability measure $\mu \in \mathcal{L}^*(x)$, we obtain: $\int \psi \, d\mu = \lim_{i\to+\infty} \int \psi \, d\nu_{n_i}(x) < 1$. But, on the other hand, $\int \psi \, d\mu \geq \int_{K'} \psi \, d\mu = \mu(K')$. So $\mu(K') < 1$.

We have proved that for all $x \in C$ there exists a measure $\mu = \mu_x \in \mathcal{L}^*(x)$ such that $\mu_x(K') < 1$. Recall that $C \subset B(K)$ and $m(C) > 0$. After the statement (a) of Theorem 1.11 (which we have already proved), $\mathcal{L}^*(x) \subset \mathcal{O}_f(K)$ for m-a.e. $x \in B(K)$. So, in particular, the above inclusion holds for m-a.e. $x \in C$. We conclude that $\mu(K') < 1$ for some $\mu \in \mathcal{O}_f(K)$ as wanted. □

Theorem 2.4 states that, for any given forward invariant set B with positive Lebesgue measure, there exists a statistical attractor that is minimal attracting B. We will show how this attractor can be constructed:

Theorem 3.5 *Let $B \subset M$ be a nonempty and forward invariant set* (i.e. $B \subset f^{-1}(B)$) *such that $m(B) > 0$. Construct the set $\mathcal{O}_{f|B}$ of all the SRB-like measures of $f|_B$. Then, the minimal compact set $K \subset M$ such that $\mu(K) = 1$ for all $\mu \in \mathcal{O}|_{f|B}$ is a statistical attractor, its basin of attraction contains B Lebesgue a.e., and K is minimal attracting B.*

Proof: After Theorem 2.4 there exists a statistical attractor K' that is minimal attracting B, i.e. $B(K') \supset B$ m-a.e. It is enough to prove that $K' = K$.

Applying Lemma 3.4 we have that $B \subset B(K)$ Lebesgue a.e. Since K' is minimal attracting B (see Definition 2.2), we deduce that $K' \subset K$.

Now, let us prove that $K \subset K'$. Notice that the set $\mathcal{O}_f(K')$ of probability measures satisfying Assertions (a) and (b) of Theorem 1.11, coincides with the set $\mathcal{O}|_{f|B(K')}$ of Lemma 3.3. After Assertion (c) of Theorem 1.11 $\mu(K') = 1$ for all $\mu \in \mathcal{O}_{f|B(K')}$. Since $B \subset B(K')$ then $\mathcal{O}_{f|B} \subset \mathcal{O}|_{f|B(K')}$. Therefore $\mu(K') = 1$ for all $\mu \in \mathcal{O}_{f|B}$. By hypothesis K is the minimal compact subset of the space such that $\mu(K) = 1$ for all $\mu \in \mathcal{O}_{f|B}$. We conclude that $K \subset K'$ as wanted. □

4 Proof of the Lebesgue-decomposition of the space

In this section we prove Theorem 1.12.

Proof:

After Theorem 1.10, there exists an α-minimal observable statistical attractor K_1. Then $m(B(K_1)) \geq \alpha$ Denote $\alpha_1 = \alpha$.

1st. Step.

Denote $r_1 = m(B(K_1)) \geq \alpha$. Either $r_1 = 1$ or $1 - \alpha < r_1 < 1$ or $\alpha \leq r_1 \leq 1 - \alpha$.

(1) In the first case, Theorem 1.12 becomes trivially proved with $p = 1$.

(2) In the second case denote $\alpha_2 = 1 - m(B(K_1))$. Then $0 < \alpha_2 < \alpha$. Consider the set $B := M \setminus B(K_1)$. After Definition 1.1 it is standard to check that

$f^{-1}(B(K_1)) = B(K_1)$. Therefore $f^{-1}(B) = B$. After Theorem 2.4 there exists a statistical attractor K_2 which is minimal attracting B. In other words, K_2 is minimal such that $B \subset B(K_2)$ m-a.e. As $\alpha_2 = m(B)$, the attractor K_2 is α'_2-obs minimal for $f|B$. Therefore, in this second case Theorem 1.12 is proved with $p = 2$.

(3) In the third case, the set $B := M \setminus B(K_1)$ has Lebesgue measure $m(B) = \alpha_2 \geq \alpha$. After Theorem 2.3 there exists a statistical attractor K_2 that is α-obs. minimal for $f|B$. Now we go to the second step by discussing again into three sub-cases:

2nd. Step.

Denote $r_2 := m(B(K_1) \cup B(K_2)) = m(B(K_1)) + m(B(K_2) \setminus B(K_1)) \geq 2\alpha$.

$$\text{Either } r_2 = 1, \text{ or } 1 - \alpha < r_2 < 1 \text{ or } 2\alpha \leq r_2 \leq 1 - \alpha.$$

(1') In the first case, Theorem 1.12 becomes trivially proved with $p = 2$.

(2') In the second case, Theorem 1.12 becomes proved with $p = 3$, after the construction of a statistical attractor K_3 following the same arguments that were used in (2) to construct K_2.

(3') In the third case, we can construct a minimal α-obs statistical attractor K_3 for $f|_{M \setminus (B(K_1) \cup B(K_2))}$, by applying the same arguments that we used above in (3) to construct K_2. Now, we go to the following step, by discussing about the value of $r_3 := m(B(K_1) \cup B(K_2) \cup B(K_3))$.

Last Step.

After $p \geq 1$ steps as above, define the number

$$r_p = m(\bigcup_{i=1}^{p} B(K_i)) = \sum_{i=1}^{p} m(B(K_i) \setminus \cup_{j=1}^{i-1} B(K_j)) \geq p\,\alpha.$$

Since $r_p \leq 1$, the last step p satisfies $p \leq 1/\alpha$ and $1 - \alpha < r_p$. So, $p =$ Integer part$(1/\alpha)$. Therefore, in the last step p we always eventually drop in the cases (1) or (2). We conclude that there exists a finite number (p or $p + 1$) of statistical attractors satisfying the statements (a), (b) and (c) of Theorem 1.12. □

5 Examples

Example 5.1 (Hu-Young Diffeo)

Consider the topologically transitive C^2 diffeomorphism f studied in [8]: it acts in the 2-torus $\mathbf{T}^2$, and is obtained by an isotopy from a linear Anosov in such a way that the eigenvalues of df at a fixed point x_0 are modified. Along the contracting subspace the eigenvalue is still smaller than 1, while in the eigendirection tangent to a topologically expansive (topologically unstable) C^1 submanifold, the eigenvalue is weakened to become equal to 1. In [8] it is proved, for such an f, that the sequence in Equality (2) of empirical probabilities converges to δ_{x_0} in the space $\mathcal{M}$ of all the

Borel probability measures (endowed with the weak*-topology), for Lebesgue a.e. $x \in \mathbf{T}^2$. In other words, δ_{x_0} is a physical measure, the ergodic attractor is $K = \{x_0\}$ and its basin of attraction covers $\mathbf{T}^2$ Lebesgue-a.e. Therefore, the frequency of visits to any arbitrarily small neighbourhood of the fixed point is asymptotically equal to 1, for Lebesgue almost all the initial states. The asymptotic frequency of visits to all the rest of the space is zero. In other words, $\{x_0\}$ is the unique $\alpha-$ observable minimal statistical attactor, for all $0 < \alpha \leq 1$.

Nevertheless, since f is transitive, the unique (and thus minimal) Milnor's attractor is the whole torus.

Example 5.2 (Bowen homeomorphism)

This example is attributed to Bowen in [18] and [6], and was also posed in [17]. Consider in a two dimensional manifold a non singular homeomorphism f (namely $m(f^{-1}(B)) = 0$ if and only if $m(B) = 0$, where m is the Lebesgue measure). Construct such an f so that:

(i) f has three fixed points x_1, x_2 and x_3.

(ii) When restricted to the union of three small compact and pairwise disjoint neighbourhoods U_1, U_2 and U_3 of x_1, x_2 and x_3 respectively, f is a diffeo onto $f(U_1 \cup U_2 \cup U_3)$, and the fixed points x_1 and x_2 are hyperbolic saddles, while x_3 is a hyperbolic source.

(iii) $W_1^s \setminus \{x_1\} = W_2^u \setminus \{x_2\}$, $W_1^u \setminus \{x_1\} = W_2^s \setminus \{x_2\}$. We denote $W_{1,2}^{s,u}$ to half-branches of the global one-dimensional stable and unstable manifolds of $x_{1,2}$ respectively. They are embedded topological arcs of C^1 type in a neighbourhood of the saddles $x_{1,2}$. So $W_1^s \cup W_2^s$ is a compact, simple and closed arc which is the boundary of an open set V homeomorphic to a 2-ball.

(iv) The hyperbolic source x_3 is in V and the orbits in $V \setminus \{x_3\}$ include x_1 and x_2 in their ω-limit sets and have $\{x_3\}$ as α-limit set.

Note that such a C^0 map f can be constructed for any previously specified values of the eigenvalues of df at the two saddles x_1 and x_2, and after an adequate choice of the values $f(x)$ for $x \in V \setminus (U_1 \cup U_2 \cup U_3)$.

Let us consider the restricted dynamical system $f|_{\overline{V}}$. On the one hand and from the topological viewpoint, all the orbits of $V \setminus \{x_3\}$ are attracted to (i.e. have ω-limit set contained in) the boundary ∂V. This closed arc is the unique 1-obs. minimal Milnor's attractor of $f|_V$. On the other hand, from the statistical viewpoint the behavior of the system is much more delicate (i.e. when looking the asymptotic behavior of the sequence of empirical probability measures defined in Equality (2)). In fact, necessarily one and only one of the following properties (A), (B) or (C) holds, and any of the three is realizable if the eigenvalues of x_1 and x_2 are adequately chosen and the C^0 map $f|_{V \setminus (U_1 \cup U_2)}$ is well constructed:

(A) There exists a unique SRB measure attracting $V \setminus \{x_3\}$ which is δ_{x_1} or δ_{x_2}. In this case either $\{x_1\}$ or $\{x_2\}$ is an ergodic attractor, it is the unique statistical attractor

and the physical measure δ_{x_i} is ergodic. We prove that this case is nonempty (see the argument following the end of Example 5.3 in this section).
(B) There exists a unique SRB measure μ attracting $V \setminus \{x_3\}$, which is $\mu = t\delta_{x_1} + (1-t)\delta_{x_2}$ for some constant $0 < t < 1$. In this case (B), the set $\{x_1, x_2\}$ is an ergodic attractor for $f|_V$, it is the unique statistical attractor, and the physical measure μ is non ergodic. Moreover, for an adequate choice of the eigenvalues of x_1 and x_2 one can obtain this property for any previously specified value of $t \in (0, 1)$. The existence of examples in this case (B) is stated for instance in Lemma Part (i) of page 457 in [17]. For the detailed construction of an example in this case, consider $\lambda = 1/\sigma$ in the Equalities of Theorem 1 of [18], and construct f such that it preserves area in both the disjoint compact neighbourhoods U_1 and U_2 of the saddles, and is adequately C^0-chosen outside $U_1 \cup U_2$ to have the two saddles in the omega-limit of all the orbits of $V \setminus \{x_3\}$. We note that, after a standard computation that we sketch in the proof at the end of this section, one should construct f contracting outside $U_1 \cup U_2$, so the sequence (2) is convergent according to formulae of Theorem 1 of [18] (with the parameters $\lambda = 1/\sigma$ in those formulae).
(C) There does not exist any physical measure, since for Lebesgue almost all the points $x \in V$, the limit set $\mathcal{L}^*(x)$ of the empirical distributions of Equality (3) is a segment in the space $\mathcal{M}$ of probabilities. In other words, the sequence (2) of empirical probabilities for $f|_{\overline{V}}$ does not converge for Lebesgue a.e. initial state. Thus, there does not exist any ergodic attractor. The existence of C^2 examples in this case (C) is proved in [18] and [6] for which the set $\mathcal{O}_{f|\overline{V}}$ of SRB-like measures is a segment which is always properly contained in $[\delta_{x_1}, \delta_{x_2}] \subset \mathcal{M}$. Nevertheless, one can construct f of C^0 class in $\overline{V}$ such that the set of SRB-like measures for $f|_{\overline{V}}$ is exactly the segment $[\delta_{x_1}, \delta_{x_2}]$ (see the remark at the end of this section).

In this case (C), there exist uncountably many SRB-like measures for $f|_{\overline{V}}$ (after Theorem 1.7 of [4]). All of them are supported on $\{x_1, x_2\}$, due to the Poincaré Recurrence Theorem. After Theorem 3.5 the set $\{x_1, x_2\}$ is a statistical attractor. Besides, since the common minimal compact support of all the measures in $\mathcal{L}^*(x)$ is $\{x_1, x_2\}$ for Lebesgue a.e. $x \in V$, this statistical attractor is the unique α-obs minimal one, for all $0 < \alpha \leq 1$. In other words, in this case (C) of example 5.2, the unique α-obs. minimal Milnor's attractor ∂V, and the unique α-obs. minimal statistical attractor, are different, while Pugh-Shub's ergodic attractors do not exist.

Let us exhibit now an example that shows that if the purpose is to find the (always existing) statistical attractors of a C^1 map, even under the strong hypothesis of uniform hyperbolicity, then the classic approach of searching for the invariant probability measures that are absolutely continuous with respect to Lebesgue may become noneffective. Nevertheless, as a consequence of Theorem 1.11, there exists an optimal nonempty subset of probability measures that describe the statistics of Lebesgue almost all the orbits (see Definition 3.1). In other words, for C^1 mappings that are not C^1 plus Hölder, the optimal probability measures are not necessarily absolutely continuous with respect to Lebesgue.

Example 5.3 (Campbell and Quas expanding maps)
Let us consider a one-dimensional C^1 map $f: S^1 \mapsto S^1$ on the circle S^1, which is expanding, i.e. $|f'(x)| > 1$ for all $x \in S^1$. In Theorem 1 of [3], Campbell and Quas proved that C^1-generically there exists a unique physical measure μ, that this measure μ is mutually singular with respect to Lebesgue, and that its basin of attraction covers Lebesgue almost all the points. This measure μ is supported on a compact subset $K \subset S^1$ (non necessarily properly contained in S^1). So, this compact support K is by definition an ergodic attractor. It is the unique statistical attractor and it is α-obs minimal for all $0 < \alpha \leq 1$, since the basin of statistical attraction of μ covers Lebesgue almost all the space. It is described by a single SRB-like measure which, in this case, is SRB.

Example 5.4 (Quas expanding maps)
In [16] Quas gave a C^1-non generic example, of an expanding map f on the circle S^1 (which is C^1 but non C^1-plus-Hölder), exhibiting a statistical behavior that is rather opposite to that of the generic case of Campbell and Quas in Example 5.3. He constructed such an f preserving the Lebesgue measure m, but for which m is non ergodic. So, after Birkhoff Theorem and after the Ergodic Decomposition Theorem, for m-almost every point $x \in S^1$ the set $\mathcal{L}^*(x)$ (defined in Equality (3)) consists of one ergodic component of m. In this example, as in the general case, there exists a unique 1-obs. minimal statistical attractor (Theorem 1.10). But in this example, the set of all the SRB-like measures that describe completely the statistical behavior has more than one probability. On purpose, all the SRB-like measures describing the statistics of the Ilyashenko's attractors of C^1 expanding maps on the circle, have also other good ergodic properties, from the viewpoint of the thermodynamic formalism. In fact, in [5] it is proved that they all satisfy the Pesin's Entropy Formula.

Proof of the existence of Case (A) in Example 5.2
There exists an homeomorphism f as in Example 5.2 such that

$$\mathcal{O}|_{f|\overline{V}} = \{\delta_{x_2}\}.$$

Proof: Choose f and the eigenvalues of x_1 and x_2 so that $f|_{U_1 \cup U_2}$ is C^1 and area conservative. Construct first an area preserving map in a small neighbourhood of ∂V. Then perturb f near ∂V, in the C^0 topology, without changing $f|_{\partial V \cup U_1 \cup U_2}$, to become hyper dissipative in a small neighbourhood of a fundamental domain of $W^s_{x_2} \setminus (U_1 \cup U_2)$, and not too much dissipative in a small neighbourhood of a fundamental domain of $W^s_{x_1} \setminus (U_1 \cup U_2)$. Precisely, construct this perturbation f such that it satisfies the following property:

At each return time $n_i(x)$ to U_2 (of any orbit with initial state $x \in V \setminus \{x_3\}$), and at each return time $n'_i(x)$ to U_1 such that $n_i(x) < n'_i(x) < n_{i+1}(x)$, consider the distances

$$d_i(x) := \mathrm{dist}(f^{n_i}(x), W^s_{x_2}), \quad d'_i(x) := \mathrm{dist}(f^{n'_i}(x), W^s_{x_1}) \tag{8}$$

Make f to be C^0 in $V \setminus (U_1 \cup U_2)$ near ∂V, so the above distances satisfy the following inequalities:

$$0 < d_{i+1}(x) < d'_i(x)^{-\log d'_i(x)}, \qquad \frac{d'_i(x)}{3} \leq d'_{i+1}(x) \leq \frac{d'_i(x)}{2}.$$

At each visit i to the set U_2, denote $N_i(2)$ (depending on x) to the time length that the orbit of x spends inside U_2, and denote $N_i(1)$ to the time length that it spends inside U_1 after its i-th. visit to U_1. Up to a constant $k > 0$, the number of iterates between the $i-$th. and the $(i+1)-$th. visit to U_2 is $N_i(2) + N_i(1)$. Besides, after a standard computation, we obtain

$$N_i(2) \geq -c_2 \cdot \log d_i > c_2(-\log d'_i)^2, \quad N_i(1) \leq -c_1 \cdot \log d'_i,$$

for some positive constants c_1 and c_2. So, there exists $c > 0$ such that

$$N_i(2) \geq c\,(N_i(1))^2 \ \ \forall \ \ i \geq 1.$$

Consider the accumulated time average $\omega_n(U_1)$ inside U_1 of the finite piece of orbit from instant 0 up to instant $n \geq 1$ (namely, the relative frequency of staying in U_1).

First, if n is exactly the end instant of the staying time inside U_1 at the m-th. visit to U_1, then $\omega_n(U_1)$ is computed as follows:

$$\omega_n(U_1) = \frac{\sum_{i=1}^m N_i(1)}{km + \sum_{i=1}^m N_i(2) + \sum_{i=1}^m N_i(1)}$$

$$\frac{1}{\omega_n(U_1)} = \frac{km + \sum_{i=1}^m N_i(2)}{\sum_{i=1}^m N_i(1)} + 1 \geq \frac{\sum_{i=1}^m [N_i(1)]^2}{\sum_{i=1}^m N_i(1)}.$$

Since $N_i(1) \to +\infty$ when $i \to +\infty$, then $1/\omega_n(U_1) \to +\infty$ when $m \to +\infty$ and so $\omega_n(U_1) \to 0$.

Second, if n is larger than the end instant n' of the staying time inside U_1 at the m-th visit, but smaller than the next return time to U_1, then $\omega_n(U_1) = (n'/n)\,\omega_{n'}(U_1) \leq \omega_{n'}(U_1) \to 0$ when $m \to +\infty$.

Third and finally, if n is a stoping time such that $f^n(x) \in U_1$ during the m-th. visit of the orbit to N_1, but n is smaller than the end instant n' of the staying time N_m inside U_1, then $0 < n' - n < N_m \leq c_1(-\log d'_m)$. Since $d'_{i+1} \geq d'_i/3$ for all $i \geq 1$, we have $d'_m \geq (1/3^m)\,d'_1(x)$ for all $m \geq 1$. Thus, there exists a constant $K(x) > 0$ such that $-\log d'_m \le K(x) \cdot m$ for all $m \ge 1$. This implies that $0 < n' - n < N_m \le c'(x) \cdot m$ where $c'(x) = c_1 \cdot K(x)$. On the other hand $n \ge m$. Therefore

$$\omega_n(U_1) = \frac{n'}{n}\, \omega_{n'}(U_1) = \omega_{n'}(U_1) \left(1 + \frac{n' - n}{n}\right) \le \omega_{n'}(U_1)\,(1 + c'(x)) \to 0$$

when $m \to +\infty$.

We have proved that $\lim_{n \to +\infty} \omega_n(U_1) = 0$ for all $x \in V \setminus \{x_3\}$. Besides, $\lim_n \omega_n(U_2) + \omega_n(U_1) = 1$. We deduce that $\lim_n \omega_n(U_2) = 1$. Since the argument above also holds (for the same f) for any arbitrary neighbourhood U'_2 of the saddle x_2, we obtain that the sequence (2) converges to δ_2, as wanted. □

Remark about Case (C) of Example 5.2

There exists an homeomorphism f as in Example 5.2, *for which*

$$\mathcal{O}_{f|\overline{V}} = [\delta_{x_1}, \delta_{x_2}].$$

Sketch of the proof. Let us apply similar arguments to those of the proof of case (A), making f hyper dissipative near $W^s(x_2) \setminus (N_1 \cup N_2)$ but also hyper dissipative near $W^s(x_1) \setminus (N_1 \cup N_2)$. We deduce, adapting the computations in the proof of case (A), that the empirical sequence (2) will have at least two convergent subsequences, one converging to δ_{x_1} and the other to δ_{x_2}. Fix a metric dist* in the space $\mathcal{M}$ inducing the weak* topology. After the convex-like property stated and proved in Theorem 2.1 of [4], for all $t \in [0, 1]$ the limit set $\mathcal{L}^*(x)$ contains an invariant measure $\mu_t(x)$ such that

$$\text{dist}^*(\mu_t(x), \delta_{x_1}) = t\text{dist}^*(\delta_{x_2}, \delta_{x_1}). \tag{9}$$

From Poincaré Recurrence Theorem μ_t is supported on $\{x_1, x_2\}$, so it is a convex combination of δ_{x_1} and δ_{x_2}. But the unique such a convex combination satisfying Equality (9), is $\mu_t = t\delta_{x_1} + (1 + t)\delta_{x_2}$, if the metric dist* is chosen to depend linearly on t for the measures in the segment $[\delta_{x_1}, \delta_{x_2}]$. So $\mathcal{O}_{f|\overline{V}} = [\delta_{x_1}, \delta_{x_2}]$, as wanted. □

Existence of Case (B) of Example 5.2

For all $0 < t < 1$ *there exists an homeomorphism f as in Example* 5.2, *for which*

$$\mathcal{O}_{f|\overline{V}} = \{t\delta_{x_1} + (1 - t)\delta_{x_2}\}.$$

Proof:

Applying similar arguments to those of the proof of case (A), let us construct a weakly dissipative map f near $W^s(x_2) \setminus (N_1 \cup N_2)$, such that it is also weakly dissipative near $W^s(x_1) \setminus (N_1 \cup N_2)$. Precisely, let us denote $d_i(x)$ and $d'_i(x)$ the distances defined in Equalities (8) in the proof of case (A). We can perturb a map f in the C^0 topology, in $V \setminus (U_1 \cap U_2)$, so that

$$\frac{d'_i}{3} \le d_{i+1} \le \frac{d'_i}{2}, \qquad \frac{d_i}{3} \le d'_i \le \frac{d_i}{2} \quad \forall\ i \ge 1.$$

Recall that $f|_{U_1 \cup U_2}$ is area preserving. Adapting standard computations obtained by applying Hartman-Grossman Theorem inside the neighbourhoods U_1 and U_2 of the two saddles, we deduce that the staying times $N_i(1)$ and $N_i(2)$ (during the $i-th$ visit to U_1 and U_2 respectively) satisfy the following inequalities, for some positive constants c and $k'(x)$:

$$N_i(1) \le c\frac{\log d'_i}{\log \sigma_1} \le k'(x)\frac{i}{\log \sigma_1} \le N_i(1) + 1 \ \ \forall\ i \ge 1,$$

$$N_i(2) \le c\frac{\log d_i}{\log \sigma_1} \le k'(x)\frac{i}{\log \sigma_2} \le N_i(2) + 1 \ \ \forall\ i \ge 1,$$

where $\sigma_{1,2} > 1$ are the expanding eigenvalues of the saddles $x_{1,2}$ respectively.

After similar computations to those in the proof of case (A), we deduce that the frequencies $\omega_n(U_1)$ and $\omega_n(U_2)$ of visits of the finite piece of orbit up to any stoping time $n \ge 1$, to the neighbourhoods U_1 and U_2 respectively, can be computed as follows:

$$\omega_n(U_{1,2}) \sim \frac{\sum_{i=1}^m N_i(1,2)}{km + \sum_{i=1}^m N_i(2) + \sum_{i=1}^m N_i(1)}$$

where k is a constant and m is the number of visits to U_2 up to time n. Thus,

$$\frac{1}{\omega_n(U_1)} \sim 1 + \frac{km \log \sigma_1}{k'(x) \sum_{i=1}^m i} + \frac{\log \sigma_1}{\log \sigma_2} \to 1 + \frac{\log \sigma_1}{\log \sigma_2}$$

and analogously $\dfrac{1}{\omega_n(U_2)} \to 1 + \dfrac{\log \sigma_2}{\log \sigma_1}$.

After checking that $1 = (1 + \log \sigma_1 / \log \sigma_2)^{-1} + (1 + \log \sigma_2 / \log \sigma_1)^{-1}$ we deduce that the empirical sequence (2) will be convergent to

$$t\delta_1 + (1-t)\delta_2, \ \text{where } t = \frac{1 + \log \sigma_2 / \log \sigma_1}{2 + \log \sigma_2 / \log \sigma_1 + \log \sigma_1 / \log \sigma_2}.$$

Since the eigenvalues $\sigma_{1,2} > 1$ can be arbitrarily chosen, the parameter t can be equalled to any previously specified value in the open interval $(0, 1)$. □

Acknowledgements: We thank A. Gorodetsky and Y. Zhao for their valuable comments to an early version of this paper. We thank ANII and CSIC of the Universidad de la República, Uruguay, for their partial financial support.

References

[1] V. I. Arnold, V. S. Afrajmovich, Yu. Ilyashenko and L. P. Shilnikov: *Bifurcation Theory and Catastrophe Theory,* Springer-Verlag, Berlin, Heidelberg, 1999

[2] P. S. Bachurin: *The connection between time averages and minimal attractors* Russ. Math. Surv. **54** (1999) 1233–1235.

[3] J. Campbell, A. Quas: *A Generic C^1 Expanding Map has a Singular S-R-B Measure* Commun. Math. Phys **221** (2001) 335–349

[4] E. Catsigeras, H. Enrich: *SBR-like measures for C^0 dynamics.* Bull. Polish Acad. of Sci. Math. **59** (2011) 151–164

[5] E. Catsigeras, H. Enrich: *Equilibrium States and SRB-like measures of C^1 Expanding Maps of the Circle.* Preprint arXiv:1202.6584v1 [math.DS] (2012) To appear in Portugaliae Math.

[6] T. Golenishcheva-Kutuzova, V. Kleptsyn: *Convergence of the Krylov-Bogolyubov procedure in Bowan's example.* (Russian) Mat. Zametki **82** (2007) 678–689; Translation in *Math. Notes* **82** (2007) 608–618

[7] A. Gorodetski, Yu. S. Ilyashenko: *Minimal and strange attractors* Int. Journ. of Bif. and Chaos **6** (1996) 1177–1183.

[8] H. Hu, L. S. Young: *Nonexistence of SRB measures for some diffeomorphisms that are almost Anosov.* Ergod. Th. and Dyn. Sys. **15** (1995) 67–76.

[9] Yu. S. Ilyashenko: *The concept of minimal attractors and maximal attractors of partial differential equations of the Kuramoto-Sivashinski type.* Chaos **1** (1991) 168-173.

[10] Yu. S. Ilyashenko: *Minimal attractors.* In Proceedings of EQUADIFF 2003, 421–428. World Scientific Publishing, Singapore, 2005

[11] O. Karabacak, P. Ashwin: *On statistical attractors and the convergence of time averages* Math. Proc. Camb. Phil. Soc. **150** (2011) 353–365.

[12] V. Kleptsyn: *An example of non-coincidence of minimal and statistical attractors* Erg. Theor. & Dyn. Sys. **26** (2006) 759–768.

[13] J. Milnor: *On the concept of attractor* Commun. in Math. Phys. **99** (1985), 177–195

[14] J. Milnor: *On the concept of attractor: Correction and remarks* Comm. Math. Phys. **102** (1985), 517–519

[15] C. Pugh, M. Shub: *Ergodic Attractors.* Trans. Amer. Math. Soc. **312** (1989) 1–54.

[16] A. Quas: *Non-ergodicity for C^1 expanding maps and g-measures.* Ergod. Th. & Dyn. Sys. **16** (1996) 531–543.

[17] Y. Takahashi: *Entropy Functional (free energy) for Dynamical Systems and their Random Perturbations.* In K. Itô (Editor) Stochastic Analysis, North-Holland Math. Library **32** (1984) 437–467

[18] F. Takens: *Heteroclinic attractors: time averages and moduli of topological conjugacy.* Bol. Soc. Brasil. Mat. **25** (1994) 107–120.

[19] D. Volk, Yu. S. Ilyashenko: *Cascades of e-invisibility.* Journ. of Fixed Point Theor. and Appl. **7** (2010) 161–188.