

Translation invariance of weak KAM solutions of the Newtonian n-body problem

Ezequiel Maderna

May 30, 2011

Abstract

We consider the Hamilton-Jacobi equation $H(x, d_x u) = c$, where $c \geq 0$, of the classical N -body problem in some Euclidian space E of dimension at least two. The fixed points of the Lax-Oleinik semigroup are global viscosity solutions for the critical value of the constant ($c = 0$), also called weak KAM solutions. We show that all these solutions are invariant under the action by translations of E in the space of configurations. We also show the existence of non-invariant solutions for the super-critical equations ($c > 0$).

1 Introduction and results

1.1 Preliminaries

Let E be an Euclidean vector space of dimension $k \geq 2$ in which N punctual masses $m_1, \dots, m_N > 0$ are mutually attracted by the Newtonian gravitational forces. At each time, the position vectors of the N bodies determine a configuration of the system $x = (r_1, \dots, r_N) \in E^N$. The Newtonian potential is the function $U : E^N \rightarrow (0, +\infty]$ defined by

$$U(x) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{\|r_i - r_j\|}.$$

It is clear that $U(x) < +\infty$ if and only if x is a configuration *without collisions*, meaning that there are no values of $i < j$ such that $r_i = r_j$. Moreover, if we denote by $\Omega \subset E^N$ the open and dense subset of configurations without collisions, then the Newtonian potential is an analytical function on Ω , and the law of gravitation is equivalent to the second order differential equation $\ddot{x} = \nabla U$ on Ω , where the gradient is taken with respect to the mass inner product.

As well as the equations of motion, the Hamiltonian and the Lagrangian of the system admit a synthetic expression in terms of the mass inner

product. We recall that given two configurations $x = (r_1, \dots, r_N)$ and $y = (s_1, \dots, s_N)$, their mass inner product is given by

$$x \cdot y = \sum_{i=1}^N m_i \langle r_i, s_i \rangle .$$

With the natural identification $TE^N \simeq E^N \times E^N$, the Lagrangian function of the system reads

$$L(x, v) = \frac{1}{2} |v|^2 + U(x)$$

where $|v| = (v \cdot v)^{1/2}$ denotes the induced norm by the mass inner product on E^N . Consequently, if $p = (p_1, \dots, p_N)$ is in the dual space $(E^N)^* \simeq (E^*)^N$, and $|p|$ denotes the operator norm of p for the mass inner product, we have

$$|p|^2 = \sum_{i=1}^N \frac{1}{m_i} \|p_i\|^2$$

where $\|p_i\|$ is the operator norm of $p_i \in E^*$. Since the Hamiltonian H of the system is defined on each fiber of the cotangent bundle T^*E^N as the convex dual of the Lagrangian, we obtain the expression

$$H(x, p) = \frac{1}{2} |p|^2 - U(x).$$

Recall that here the Legendre transform $\mathcal{L}(x, v) = \partial_v L(x, v) \in T^*E^N$ does not depend on the configuration x , hence it can be defined as a map $\mathcal{L} : E^N \rightarrow (E^N)^*$. Thus $p = \mathcal{L}(v)$ will mean that $p(w) = w \cdot v$ for all $w \in E^N$. It is easy to check the Young-Fenchel inequality, which assures that for all $p \in (E^N)^*$ and $v \in E^N$ we have $p(v) \leq H(x, p) + L(x, v)$, and that equality holds if and only if $p = \mathcal{L}(v)$.

Finally, we recall that if $M = m_1 + \dots + m_N$ is the total mass of the system, the center of mass of a configuration is given by the linear map $G : E^N \rightarrow E$

$$G(x) = \frac{1}{M} \sum_{i=1}^N m_i r_i.$$

For any vector $r \in E$ we have $x \cdot (r, \dots, r) = \langle MG(x), r \rangle$, thus it is clear that the space of configurations decomposes as the orthogonal direct sum $E^N = \mathcal{X} \oplus \Delta$, where $\mathcal{X} = \ker G$ is the subspace of all centered configurations, and Δ the diagonal subspace. We also recall that, since the sum of all forces acting on the system is zero, the center of mass of any motion has constant velocity.

1.2 The Hamilton-Jacobi equation and weak KAM solutions

Since our mechanical system is autonomous his Hamiltonian function H is time independent. Therefore the natural question which arises is the existence and characterization of the stationary solutions of the Hamilton-Jacobi equation $\partial_t S + H(x, \partial_x S) = 0$, that is to say, solutions of the form $S(t, x) = u(x) - ct$. In this paper we will study the global stationary solutions, or in other words, functions $u : E^N \rightarrow \mathbb{R}$ satisfying the Hamilton-Jacobi equation $H(x, d_x u) = c$, more explicitly the equation

$$\frac{1}{2} \sum_{i=1}^N \frac{1}{m_i} \left\| \frac{\partial u}{\partial r_i} \right\|^2 - \sum_{i < j} \frac{m_i m_j}{\|r_i - r_j\|} = c. \quad (1)$$

Considering the mutual distances between the bodies can be arbitrarily large, it is clear that equation (1) only has global solutions for values of the constant $c \geq 0$.

A classical global solution, or global smooth solution, is a continuous function $u : E^N \rightarrow \mathbb{R}$ whose restriction to the open and dense set of configurations without collisions Ω is of class C^1 , and such that (1) is verified at each point of this set. Until now, the only known smooth global solutions are solutions for the case $N = 2$, that is to say, for the Kepler problem in the space E . A direct computation shows that functions $u(r_1, r_2) = \pm \|r_1 - r_2\|^{1/2}$ are global smooth solutions.

Recently, the author has shown in [6] the existence of weak KAM solutions for a family of N -body problems with homogeneous potentials, including the Newtonian case. They are global viscosity solutions of the critical equation $H(x, d_x u) = 0$. The existence was obtained by application of a fixed point theorem to the action of the Lax-Oleinik semigroup on the convex set of viscosity subsolutions.

Moreover, if Γ is a given symmetry group of the system, then the linear space of all invariant functions has nontrivial intersection with the set of subsolutions since clearly constant functions are invariant subsolutions. Considering that the semigroup preserves the set of all invariant subsolutions, which is also convex, the existence of invariant weak KAM solutions is also obtained (theorem 3 in [6]). It is evident that the application of a given isometry of the Euclidian space E to each component of a configuration defines a symmetry of the system. Thus the translation group of E , as well as the orthogonal group $O(E)$, acts naturally on the space of configurations by symmetries. Depending on the values of the masses, other symmetry groups can act on the space of configurations. For example, in the non-generic case in which the masses of some bodies are equal, every permutation of the position vectors of these bodies are symmetries.

We shall remark that for a smooth, convex and superlinear Hamiltonian function $H : T^*M \rightarrow \mathbb{R}$ on a compact manifold M , weak KAM solutions are

always invariant under the action of the identity component of a compact symmetry group (see [5]).

Even if the the orthogonal group is compact, there are always non-invariant solutions for the action of his identity component $SO(E)$. The critical Busemann functions explicitly computed in [6] for the Kepler problem in the plane with equal masses, are examples of such non-invariant weak KAM solutions. Nevertheless, weak KAM solutions are always invariant under translations.

Theorem 1 *Every weak KAM solution of the Newtonian N -body problem in an Euclidean space E of dimension at least two is translation invariant. More precisely, given a weak KAM solution $u : E^N \rightarrow \mathbb{R}$, we have that*

$$u(r_1, \dots, r_N) = u(r_1 + r, \dots, r_N + r)$$

for every configuration $x = (r_1, \dots, r_N) \in E^N$ and every $r \in E$.

The hypothesis $\dim E \geq 2$ will be only used in order to apply Marchal's theorem (whose proof in the case $\dim E > 3$ is due to D. Ferrario and S. Terracini) to the calibrated curves of a given solution. This said, there is no evidence that suggest the theorem is not valid in the one dimensional case.

We will show that the invariance under translations of a weak KAM solution is equivalent to the fact that all his calibrated curves are motions with fixed center of mass (proposition 4). On the other hand, a very strong property of the calibrated curves of a weak KAM solution is the free time minimization property, which we recall below. Therefore, the proof of theorem 1 will follow from lemma 3, where Marchal's theorem is applied. The dynamics when $t \rightarrow +\infty$ of the free time minimizers of the Newtonian N -body problem is described in [1], where is proved that such kind of motions are completely parabolic and asymptotic to some special central configurations.

The following corollary will be easily deduced from the theorem. It shows as a counterpart, that the super-critical equations (1), that is for the values of $c > 0$, can be non-invariant under translations.

Corollary 2 *If $u_0 : E^N \rightarrow \mathbb{R}$ is a weak KAM solution of the N -body problem, and $r \in E$ is an arbitrary vector, then the function*

$$u(x) = u_0(x) + \langle G(x), r \rangle$$

is a global viscosity solution of the Hamilton-Jacobi equation

$$H(x, d_x u) = \frac{1}{2} \|r\|^2.$$

The following section is devoted to prove the above results. It is interesting to remark that all the statements and proofs remains valid if we replace the Newtonian potential by homogeneous potential of degree $\alpha \in (-2, 0)$.

2 Proofs

We start recalling some basic notions of the calculus of variations that will be used. The Lagrangian action of an absolutely continuous curve $\gamma : [a, b] \rightarrow E^N$

$$A(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt$$

is a well defined function of γ taking values in $(0, +\infty]$. For two given configurations $x, y \in E^N$, the set of all absolutely continuous curves starting at x and arriving to y in time $T > 0$ will be denoted $\mathcal{C}(x, y, T)$. It is well known that the Lagrangian action is lower semicontinuous, which implies that the infimum of its restriction to any set $\mathcal{C}(x, y, T)$ is always reached. In that follows $\phi(x, y, T)$ will denote the minimum of the Lagrangian action restricted to $\mathcal{C}(x, y, T)$.

Let us recall Marchal's theorem on the fixed endpoint problem (cf. [2],[4]). It says that minimizers of the Lagrangian action in a set $\mathcal{C}(x, y, T)$ not suffers collisions in any interior time. In other words, if an absolutely continuous curve $\gamma : [a, b] \rightarrow E^N$ is such that $A(\gamma) \leq A(\sigma)$ for every $\sigma \in \mathcal{C}(\gamma(a), \gamma(b), b - a)$, then the curve γ avoid collisions except perhaps in the prescribed endpoints, that is to say, $\gamma(t) \in \Omega$ for all $t \in (a, b)$. In particular, the restriction of a such curve to the open interval (a, b) satisfies the equation of motion, hence it must be of class C^∞ . Furthermore, if one of its endpoints, say $\gamma(a)$, is a configuration without collisions, then the solution can be extended to an interval of the form $(a - \epsilon, b)$ for some $\epsilon > 0$, which implies that $\dot{\gamma}(a)$ exists and coincide with $\lim_{t \rightarrow a^+} \dot{\gamma}(t)$.

2.1 Action potential, free time minimizers and calibrated curves

We will recall some results from [6] as well as some well known features of weak KAM theory.

2.1.1 The action potential

The set of all absolutely continuous curves starting at x and ending at y without restriction of time will be denoted $\mathcal{C}(x, y)$. The action potential, also called the Mañé critical potential in Aubry-Mather theory, is the function $\phi : E^N \times E^N \rightarrow \mathbb{R}^+$

$$\phi(x, y) = \inf \{ A(\gamma) \mid \gamma \in \mathcal{C}(x, y) \} = \inf_{T>0} \phi(x, y, T).$$

For convenience, we will use the maximum norm in E^N ,

$$\| (r_1, \dots, r_N) \| = \max_{i=1, \dots, N} \| r_i \|_E.$$

The action potential is a distance function in the space of configurations. Moreover, it is Hölder continuous in each variable with respect to any norm in E^N . In fact, there is a constant $\eta > 0$ such that

$$\phi(x, y) \leq \eta \|x - y\|^{1/2}$$

for every pair of configurations. The constant η only depends on the number of bodies N and the total mass M . Since several of the here considered notions are easily characterized in terms of the action potential, it will play a central role.

Another important property of the action potential that we will also use, is that it is locally Lipschitz in $\Omega \times \Omega$. More precisely, if $C \subset \Omega$ is a compact set, then there is a constant $K > 0$ such that $\phi(x, y) \leq K \|x - y\|$ whenever $(x, y) \in C \times C$. Of course the Lipschitz constant grows as well as the compact set C approaches some given collision configuration.

2.1.2 Free time minimizers

We call a free time minimizer, or semistatic curve, an absolutely continuous curve such that the action of each compact segment equals the action potential between the endpoints of the segment. In other words, a curve $\gamma : I \rightarrow E^N$ is a free time minimizer if we have $A(\gamma|_{[a,b]}) = \phi(\gamma(a), \gamma(b))$ whenever $[a, b] \subset I$ is a compact subinterval. When $\dim E \geq 2$, Marchal's theorem implies that a free time minimizer defined on an interval $[a, b]$ avoid collisions for $t \in (a, b)$.

2.1.3 Viscosity subsolutions as dominated functions

On the other hand we will be concerned with the notion of weak solution of (1) introduced by M. Crandall and P.-L. Lions in [3], namely the notion of viscosity solution. The classical way to define them is in terms of test functions, as functions which are at the same time viscosity subsolutions and viscosity supersolutions. It is well known that the set of viscosity subsolutions of the critical Hamilton-Jacobi equation $H(x, d_x u) = 0$, which we call the set of dominated functions, is exactly the convex set

$$\mathcal{H} = \{ u : E^N \rightarrow \mathbb{R} \mid u(y) - u(x) \leq \phi(x, y) \text{ for all } x, y \in E^N \}.$$

By the previous considerations, we know in particular that functions in \mathcal{H} are Hölder continuous and locally Lipschitz in Ω . Since Ω is a set of total measure, by Rademacher's theorem we also know that they are differentiable almost everywhere.

2.1.4 Calibrated curves

For a given dominated function $u \in \mathcal{H}$, we say that an absolutely continuous curve $\gamma : I \rightarrow E^N$ is calibrated for u , if for each compact subinterval $[a, b] \subset I$

we have that $u(\gamma(b)) - u(\gamma(a)) = A(\gamma |_{[a,b]})$. Note that a calibrated curve is always a free time minimizer. On the other hand, if $u \in \mathcal{H}$ and γ is a calibrated curve for u , defined on $I = (a, b)$ and of class C^1 , then u is differentiable at $\gamma(t)$ for all $t \in (a, b)$, and $d_{\gamma(t)}u = \mathcal{L}(\dot{\gamma}(t))$. It follows, using the Young-Fenchel inequality, that $H(\gamma(t), d_{\gamma(t)}u) = 0$ for all $t \in (a, b)$.

2.1.5 Characterization of weak KAM solutions

The action of the Lax-Oleinik semigroup $(T_t)_{t \geq 0}$ is well defined in $[0, +\infty) \times \mathcal{H}$ by

$$T_t u(x) = \inf \{ u(y) + \phi(y, x, t) \mid y \in E^N \}$$

for $t > 0$ and $T_0 = id_{\mathcal{H}}$. A weak KAM solution is a fixed point of the Lax-Oleinik semigroup, that is to say, a function $u \in \mathcal{H}$ such that $T_t u = u$ for all $t \geq 0$. It is proved in [6] that such fixed points exist and that in fact, they are global viscosity solutions of the critical Hamilton-Jacobi equation. Moreover they are characterized between dominated functions as follows: $u \in \mathcal{H}$ is a weak KAM solution if and only if for each configuration $x \in E^N$ there is a calibrated curve $\gamma_x : [0, +\infty) \rightarrow E^N$ such that $\gamma_x(0) = x$. By all the above considerations, we deduce that if u is a weak KAM solution differentiable at a configuration $x \in \Omega$, then there is only one calibrated curve, since it must coincide with the motion generated by the initial conditions $\gamma(0) = x$ and $\dot{\gamma}(0) = \mathcal{L}^{-1}(d_x u)$.

2.2 Proof of theorem 1

As we have said, the proof is a direct consequence of the following lemma and proposition.

Lemma 3 *If $\dim E \geq 2$ and $x : [0, +\infty) \rightarrow E^N$ is a free time minimizer then the center of mass $G(x(t))$ is constant.*

Proof. First, we must observe that Marchal's theorem implies $x(t) \in \Omega$ for all $t > 0$. Therefore x satisfies the equation of motion on $(0, +\infty)$, from which we deduce that $G(x(t))$ has constant velocity, even at $t = 0$. With the notation $x_0 = x(0)$ and $G_0 = G(x_0)$, we have that $G(x(t))$ is of the form $G_0 + tv$ for some vector $v \in E$. We will prove that $v = 0$.

Let $y : [0, +\infty) \rightarrow E^N$ be the internal motion associated with x , that is to say, the curve defined by $x(t) = y(t) + \delta(G_0 + tv)$, where $\delta : E \rightarrow E^N$ is the map $\delta(r) = (r, \dots, r)$. It is clear that for all $t \geq 0$ we have that $U(x(t)) = U(y(t))$ and $G(y(t)) = 0$. On the other hand, a computation shows that $\dot{x} \cdot \dot{x} = \dot{y} \cdot \dot{y} + M \|v\|^2$ for all $t > 0$. Hence we deduce that for all $T > 0$ we have

$$A(x |_{[0,T]}) = A(y |_{[0,T]}) + \frac{1}{2} T M \|v\|^2.$$

Since x is a free time minimizer, and ϕ verify the triangle inequality, we also obtain

$$A(x|_{[0,T]}) = \phi(x_0, x(T)) \leq \phi(x_0, x_0 + T\delta(v)) + \phi(x_0 + T\delta(v), x(T)).$$

Let y_T be the curve obtained translating the restriction of y to the interval $[0, T]$ in such a way that the center of mass of y_T becomes fixed at $G(x(T))$. Thus, y_T is the curve defined for $t \in [0, T]$ by $y_T(t) = y(t) + \delta(G_0 + Tv)$. Both curves have the same action, but now we have $\gamma_T \in \mathcal{C}(x_0 + T\delta(v), x(T))$. Therefore,

$$\phi(x(0) + T\delta(v), x(T)) \leq A(y|_{[0,T]}).$$

We use now the Hölder estimation for the action potential, and we get the bound

$$\phi(x(0), x(0) + T\delta(v)) \leq \eta \|T\delta(v)\|^{1/2} = T^{1/2} \eta \|v\|^{1/2}.$$

We can conclude that the inequality

$$\frac{1}{2} T M \|v\|^2 \leq T^{1/2} \eta \|v\|^{1/2}$$

holds for all $T > 0$, but this is possible only in case $v = 0$. \square

Proposition 4 *A weak KAM solution $u : E^N \rightarrow \mathbb{R}$ is invariant under translations if and only if all calibrated curves have constant center of mass.*

Proof. We start showing the necessity of the condition. Let u be a weak KAM solution which is invariant under translations. At each point of differentiability $x \in E^N$ we must have $\Delta \subset \ker d_x u$. Let $x \in E^N$ be any configuration, and let $\gamma_x : [0, +\infty)$ be a calibrated curve for u starting at x . We know that u is differentiable at $\gamma_x(t)$ for all $t > 0$ and that $d_{\gamma_x(t)} = \mathcal{L}(\dot{\gamma}_x(t))$, that is to say, $d_x u(w) = w \cdot \dot{\gamma}_x(t)$ for all $w \in E^N$. It follows that $\dot{\gamma}_x(t) \in \Delta^\perp = \mathcal{X}$ for $t > 0$, hence that $G(\gamma_x(t))$ is constant.

Let now u be a weak KAM solution such that all calibrated curves have constant center of mass. Let $x \in E^N$ be some given configuration, and choose a non zero vector $r \in E$. We will prove that $u(x) = u(x + \delta(r))$, where $\delta(r) = (r, \dots, r) \in \Delta$.

Since u is continuous and the set Ω is dense and translation invariant, it suffices to consider configurations $x \in \Omega$. Let

$$B(\epsilon, r) = \{ z \in E^N \mid z \cdot \delta(r) = 0, \|z\| \leq \epsilon \}$$

be the closed ball of radius $\epsilon > 0$ in the orthogonal complement of $\delta(r) \in \Delta$. Since $x \in \Omega$, we can assume that ϵ is so small that the set

$$B = \{ x + z \mid z \in B(\epsilon, r) \}$$

is contained in Ω , which implies that also the compact cylinder

$$C = \{ y + t\delta(r) \mid y \in B, t \in [0, 1] \}$$

is contained in Ω . Let us call $C' \subset C$ the set of points where u is differentiable. Recall that u is a dominated function, hence it is a Lipschitz function on C , and C' has total measure in C . For $q \in C'$, let $\gamma_q : [0, +\infty) \rightarrow E^N$ be the only calibrated curve for u with $\gamma_q(0) = q$. We know that $d_q u(w) = w \cdot \dot{\gamma}_q(0)$ for all $w \in E^N$. By using the hypothesis we deduce that $\dot{\gamma}_q(0) \in \mathcal{X} = \ker G = \Delta^\perp$, and therefore that $\Delta \subset \ker d_q u$ for all $q \in C'$.

We consider now the Lipschitz continuous function $f : B \times [0, 1] \rightarrow \mathbb{R}$ defined by $f(y, t) = u(y + t\delta(r)) - u(y)$. It is clear that $\partial_t f$ is zero almost everywhere. Therefore, applying Fubini's theorem $\partial_t f$ on a cylinder of the form $A \times [0, 1]$, where $A \subset B$ is an open set, we obtain

$$\int_A f(y, 1) dy = 0.$$

Since the open subset $A \subset B$ is arbitrary, we must have $f(y, 1) = 0$ for all $y \in B$. Hence, we have $f(x, 1) = 0$, that is to say, $u(x + \delta(r)) = u(x)$. \square

2.3 Viscosity solutions of the supercritical equations

Proof of corollary 2 Let u_0 be a weak KAM solution and $r \in E$. We will use the fact that u_0 is a global solution of $H(x, d_x u) = 0$ in the viscosity sense. As before, we call $\delta : E \rightarrow \Delta$ the trivial isomorphism. Thus function u_r can be written as $u_r = u_0 + a_r$, where a_r is the linear function $a_r(x) = x \cdot \delta(r)$.

Let x_0 be a given configuration, and suppose that $\varphi \in C^1(E^N)$ is such that $\varphi - u_r$ has a local minimum at x_0 . Since u_0 is a viscosity solution of $H(x, d_x u) = 0$ and we have that $(\varphi - a_r) - u_0$ has a local minimum at x_0 , we have $H(x_0, d_{x_0}(\varphi - a_r)) \leq 0$.

On the other hand, by theorem 1 we know that u_0 is translation invariant, hence $\Delta \subset \ker d_{x_0}(\varphi - a_r)$. Since $d_{x_0} a_r = a_r$, it is clear that $\mathcal{X} \subset \ker d_{x_0} a_r$ and that $|d_{x_0} a_r|^2 = M^* \|r\|^2$. Therefore we deduce that

$$|d_{x_0} \varphi|^2 = |d_{x_0}(\varphi - a_r)|^2 + M^* \|r\|^2$$

and we conclude that

$$H(x_0, d_{x_0} \varphi) = H(x_0, d_{x_0}(\varphi - a_r)) + \frac{1}{2} M^* \|r\|^2 \leq \frac{1}{2} M^* \|r\|^2,$$

proving that u_r is a viscosity subsolution of the supercritical equation at x_0 . With a symmetric argument it can be proved that u_r is a viscosity supersolution of the same equation. \square

References

- [1] A. Da Luz and E. Maderna, *On the free time minimizers of the Newtonian N -body problem*, preprint, <http://premat.fing.edu.uy/2009.htm>
- [2] A. Chenciner, *Action minimizing solutions of the Newtonian n -body problem: from homology to symmetry*, Proceedings of the ICM, Vol. III (Beijing, 2002), 279–294, Higher Ed. Press, Beijing, 2002.
- [3] M.G. Crandall and P.-L. Lions, *Viscosity Solutions of Hamilton-Jacobi Equations*, Trans. Amer. Math. Soc. **277** (1983), 1–42.
- [4] D. Ferrario and S. Terracini, *On the existence of collisionless equivariant minimizers for the classical n -body problem*, Invent. Math. **155** (2004), no. 2, 305–362.
- [5] E. Maderna, *Invariance of global solutions of the Hamilton-Jacobi equation*, Bull. Soc. Math. France **130** (2002) no.4, 493–506.
- [6] E. Maderna, *On weak KAM theory for N -body problems*, Ergodic Theory Dynam. Systems. Available on CJO 2011 doi:10.1017/S0143385711000046