RECURRENCE OF NON-RESONANT HOMEOMORPHISMS ON THE TORUS

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Abstract. We prove that a homeomorphism of the torus homotopic to the identity whose rotation set is reduced to a single totally irrational vector is chain-recurrent. In fact, we show that pseudo-orbits can be chosen with a small number of jumps, in particular, that the nonwandering set is weakly transitive. We give an example showing that the nonwandering set of such a homeomorphism may not be transitive.

1. Introduction

We consider $\text{Homeo}_0(\mathbb{T}^2)$ to be the set of homeomorphisms homotopic to the identity. We shall say that $f \in \text{Homeo}_0(\mathbb{T}^2)$ is *non-resonant* if the rotation set of f is a unique vector (α, β) and the values $1, \alpha, \beta$ are irrationally independent (i.e. α, β and α/β are not rational). This ammounts to say that given any lift F of f to $\mathbb{R}^2$, for every $z \in \mathbb{R}^2$ we have that:

$$\lim_{n\to\infty} \frac{F^n(z) - z}{n} = (\alpha, \beta) (\text{mod}\, \mathbb{Z}^2) \tag{1}$$

In general, one can define the rotation set of a homeomorphism homotopic to the identity (see [MZ]). In fact, although we shall not make it explicit, our constructions work in the same way for homeomorphisms of the torus whose rotation set is contained in a segment of slope (α, β) with α, β and α/β irrational and not containing zero.

Non-resonant torus homeomorphisms[1] have been intensively studied in the last years looking for resemblance between them and homeomorphisms of the circle with irrational rotation number (see [K2], [L], [J1]) and also constructing examples showing some difference between them (see [F], [BCL], [BCJL], [J2]).

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[1] These are called *irrational pseudo-rotations* by several authors, but since some of them use the term exclusively for conservative ones, we adopt the definition used in [K1].

In [K1] the possible topologies of minimal sets these homeomorphisms admit are classified and it is shown that under some conditions, these minimal sets are unique and coincide with the non-wandering set. However, there is one kind of topology of minimal sets where the question of the uniqueness of minimal sets remains unknown. When the topology of a minimal set is of this last kind, [BCJL] constructed an example where the non wandering set does not coincide with the unique minimal set, in fact, they construct a transitive non-resonant torus homeomorphism containing a proper minimal set as a skew product over an irrational rotation.

A natural example of non-resonant torus homeomorphism is the one given by a homeomorphism semiconjugated to an irrational rotation by a continuous map homotopic to the identity. In [J1] it is proved that a non-resonant torus homeomorphism is semiconjugated to an irrational rotation under some quite mild hypothesis.

Under the hypothesis of being semiconjugated by a monotone map[2] which has points whose preimage is a singleton, it is not hard to show the uniqueness of a minimal set (see for example [K1] Lemma 14). However, as shown by Roberts in [R], a continuous monotone map may be very degenerate and thus even if there exist such a semiconjugation, it is not clear whether there should exist a unique minimal set nor the kind of recurrence the homeomorphisms should have. Moreover, for general non-resonant torus homeomorphisms, there does not exist a semiconjugacy to the irrational rotation (even when there is "bounded mean motion", see [J2]).

Here, we give a simple and self-contained proof (based on some ideas of [K1] but not on the classification of the topologies of the minimal sets) of a result which shows that even if there may be more than one minimal set, the dynamics is in some sense irreducible. Clearly, transitivity of f may not hold for a general non-resonant torus homeomorphism (it may even have wandering points, as in the product of two Denjoy counterexamples; some more elaborate examples may be found in [K1]), but we shall show that, in fact, these homeomorphisms are weakly transitive.

Theorem A. *Let* $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ *be a non-resonant torus homeomorphism, then,* $f|_{\Omega(f)}$ *is weakly transitive.*

Recall that for $h : M \to M$ a homeomorphism, and K an $h-$invariant compact set, we say that $h|_K$ is *weakly transitive* if given two open sets U and V of M

[2] A monotone map is a map whose preimages are all compact and connected.

intersecting K, there exists $n > 0$ such that $h^n(U) \cap V \neq \emptyset$ (the difference with being transitive is that for transitivity one requires the open sets to be considered relative to K).

This allows to re-obtain Corollary E of [J1]:

Corollary 1.1. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be a non-resonant torus homeomorphism such that $\Omega(f) = \mathbb{T}^2$. Then, f is transitive.*

In fact, as a consequence of weak-transitivity, we can obtain also the more well known concept of chain-transitivity for non-resonant torus homeomorphisms.

Corollary 1.2. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be a non-resonant torus homeomorphism, then, f is chain-recurrent.*

Recall that a homeomorphism h of a compact metric space M is *chain-recurrent* if for every pair of points $x, y \in M$ and every $\varepsilon > 0$ there exists an $\varepsilon-$pseudo-orbit $x = z_0, \dots, z_n = y$ with $n \geq 1$ (i.e. $d(z_{i+1}, h(z_i)) < \varepsilon$).

Proof.Consider two points $x, y \in M$ and $\varepsilon > 0$.

We first assume that $x \neq y$ are both nonwandering points which shows the idea in a simpler way. From Theorem A we know that there exists a point z and $n > 0$ such that $d(z, f(x)) < \varepsilon$ and $d(f^n(z), f^{-1}(y)) < \varepsilon$. We can then consider the $\varepsilon-$pseudo-orbit: $\{x, z, \dots, f^n(z), y\}$.

Now, for general $x, y \in \mathbb{T}^2$ we consider $n_0 \geq 1$ such that $d(f^{n_0}(x), \Omega(f)) < \varepsilon/2$ and $d(f^{-n_0}(y), \Omega(f)) < \varepsilon/2$. Now, by Theorem A there exists $z \in \mathbb{T}^2$ and $n > 0$ such that $d(z, f^{n_0}(x)) < \varepsilon$ and $d(f^n(z), f^{-n_0}(y)) < \varepsilon$. Considering the following $\varepsilon-$psudo-orbit $\{x, \dots, f^{n_0-1}(x), z, \dots, f^n(z), f^{-n_0+1}(y), \dots, y\}$ we obtain a pseudo-orbit from x to y and thus proving chain-recurrence.

□

Remark 1. We have proved that in fact, for every $\varepsilon > 0$ the pseudo-orbit can be made with only two "jumps".

As a consequence of our study, we obtain the following result which may be of independent interest:

Proposition B. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be a non-resonant torus homeomorphism and Λ_1 a compact connected set such that $f(\Lambda_1) \subset \Lambda_1$. Then, for every U connected neighborhood of Λ_1, there exists $K > 0$ such that:*

- *If Λ_2 is a compact set which has a connected component in the universal cover of diameter larger than K then*[3]*,*

$U \cap \Lambda_2 \neq \emptyset$.

One could wonder if the stronger property of $\Omega(f)$ being transitive may hold. However, in section 4 we present an example where $\Omega(f)$ is a Cantor set times $\mathbb{S}^1$ where the nonwandering set is not transitive.

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2. Reduction of the proofs of Theorem A and Proposition B

In this section we shall reduce the proofs of Theorem A and Proposition B to Proposition 2.1 and its Addendum 2.2.

We shall use the word *domain* to refer to an open and connected set. We shall say a domain $U \in \mathbb{T}^2$ is *inessential, simply essential or doubly essential* depending on whether the inclusion of $\pi_1(U)$ in $\pi_1(\mathbb{T}^2)$ is isomorphic to $0, \mathbb{Z}$ or $\mathbb{Z}^2$ respectively[4]. If U is simply essential or doubly essential, we shall say it is *essential.*

Remark 2. Notice that if U and V are two doubly essential domains, then it holds that $U \cap V \neq \emptyset$. This is because the intersection number of two closed curves is an homotopy invariant and given two non-homotopic curves in $\mathbb{T}^2$, they have non-zero intersection number, thus, they must intersect. Since clearly, being doubly essential, U and V contain non homotopic curves, we get the desired result.

$\diamondsuit$

So, we get that Theorem 1 can be reduced to the following proposition.

Proposition 2.1. *Given $f \in \text{Homeo}_0(\mathbb{T}^2)$ a non-resonant torus homeomorphism and U an open set such that $f(U) \subset U$ and U intersects $\Omega(f)$, then we have that U has a connected component which is doubly essential.*

Almost the same proof yields also the following statement which will imply Proposition B:

[3]This holds if Λ_2 is a connected set such that $f^i(\Lambda_2) \subset \Lambda_2$ for some $i \in \mathbb{Z}$ for example.

[4]In [K1] these concepts are called *trivial, essential* and *doubly-essential.*

Addendum 2.2. *For f as in Proposition 2.1, if Λ is a compact connected set such that $f(\Lambda) \subset \Lambda$, then, for every U connected neighborhood of Λ, we have that U is doubly-essential.*

Notice that the fact that $f(\Lambda) \subset \Lambda$ for Λ compact implies that $\Lambda \cap \Omega(f) \neq \emptyset$.

Proof of Theorem A and Proposition B. Let us consider two open sets U_1 and V_1 intersecting $\Omega(f)$, and we consider the sets $U = \bigcup_{n>0} f^n(U_1)$ and $V = \bigcup_{n<0} f^n(V_1)$. These sets verify that $f(U) \subset U$ and $f^{-1}(V) \subset V$ and both intersect the nonwandering set.

Proposition 2.1 (applied to f and f^{-1}) implies that both U and V are doubly essential, so, they must intersect. This implies that for some $n > 0$ and $m < 0$ we have that $f^n(U_1) \cap f^m(V_1) \neq \emptyset$, so, we have that $f^{n-m}(U_1) \cap V_1 \neq \emptyset$ and thus $\Omega(f)$ is weakly transitive.

Proposition B follows directly from Addendum 2.2 since given a doubly-essential domain U in $\mathbb{T}^2$, there exists $K > 0$ such that its lift $p^{-1}(U)$ intersects every connected set of diameter larger than K.

□

Remark 3. Notice that in higher dimensions, Remark 2 does not hold, in fact, it is easy to construct two open connected sets containing closed curves in every homotopy class which do not intersect. So, even if we could show a result similar to Proposition 2.1, it would not imply the same result.

♢

3. Proof of Proposition 2.1

Consider $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ a non-resonant torus homeomorphism, and let us assume that U is an open set which verifies that $f(U) \subset U$ and such that $U \cap \Omega(f) \neq \emptyset$.

Since $U \cap \Omega(f) \neq \emptyset$, for some $N > 0$ we have that there is a connected component of U which is f^N-invariant. We may thus assume from the start that U is a domain such that $f(U) \subset U$ and $U \cap \Omega(f) \neq \emptyset$.

Let $p : \mathbb{R}^2 \to \mathbb{T}^2$ be the canonical projection. Consider $U_0 \subset p^{-1}(U)$ a connected component. We can choose F a lift of f such that $F(U_0) \subset U_0$.

We shall denote $T_{p,q}$ to the translation by vector (p, q), that is, the map from the plane such that $T_{p,q}(x) = x + (p, q)$ for every $x \in \mathbb{R}^2$.

Lemma 3.1. *The domain U is essential.*

Proof.Consider $x \in U_0$ such that $p(x) \in \Omega(f)$. And consider a neighborhood $V \subset U_0$ of x. Assume that there exists $n_0 > 0$ and $(p,q) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $F^{n_0}(V) \cap (V + (p,q)) \neq \emptyset$. Since U_0 is F-invariant, we obtain two points in U_0 which differ by an integer translation, and since U_0 is connected, this implies that U contains a non-trivial curve in $\pi_1(\mathbb{T}^2)$ and thus, it is essential.

To see that there exists such n_0 and (p,q), notice that otherwise, since x is not periodic (because f is a non-resonant torus homeomorphism) we could consider a basis V_n of neighborhoods of $p(x)$ such that $f^k(V_n) \cap V_n = \emptyset$ for every $0 < k \leq n$. Since x is non-wandering, there exists some $k_n > n$ such that $f^{k_n}(V_n) \cap V_n \neq \emptyset$, but since we have that $F^{k_n}(V_n) \cap (V_n + (p,q)) = \emptyset$ for every $(p,q) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, we should have that $F^{k_n}(V_n) \cap V_n \neq \emptyset$ for every n. Since $k_n \to \infty$, we get that f has zero as rotation vector, a contradiction.

□

Remark 4. Clearly, this result holds equally for any domain containing a compact connected forward invariant set in its interior since we used only the fact that U had points which where non-wandering and thus this Lemma works also in the hypothesis of Addendum 2.2

♢

We conclude the proof of by showing the following Lemma which has some resemblance with Lemma 11 in [K1].

Lemma 3.2. *The domain U is doubly-essential.*

Proof.Assume by contradiction that U is simply-essential.

Since the inclusion of $\pi_1(U)$ in $\pi_1(\mathbb{T}^2)$ is non-trivial by the previous lemma, there exists a closed curve η in U such that when lifted to $\mathbb{R}^2$ joins a point $x \in U_0$ with $x + (p,q)$ (which will also belong to U_0 because η is contained in U and U_0 is a connected component of $p^{-1}(U)$).

We claim that in fact, we can assume that η is a simple closed curve and such that $g.c.d(p,q) = 1$ (the greatest common divisor). In fact, since U is open, we can assume that the curve we first considered is in general position, and by considering a subcurve, we get a simple one (maybe the point x and the vector (p,q) changed, but we shall consider the curve η is the simple and closed curve from the start). Since it is simple, the fact that $g.c.d(p,q) = 1$ is trivial.

If η_0 is the lift of η which joins $x \in U_0$ with $x + (p,q)$, we have that it is compact, so, we get that

$$\tilde{\eta} = \bigcup_{n\in\mathbb{Z}} T_{np,nq}\eta_0$$

is a proper embedding of $\mathbb{R}$ in $\mathbb{R}^2$. Notice that $\tilde{\eta} \subset U_0$.

By extending to the one point compactification of $\mathbb{R}^2$ we get by using Jordan's Theorem (see [M] chapter 4) that $\tilde{\eta}$ separates $\mathbb{R}^2$ in two disjoint unbounded connected components which we shall call L and R and such that their closures $L \cup \tilde{\eta}$ and $R \cup \tilde{\eta}$ are topologically a half plane (this holds by Shönflies Theorem, see [M] chapter 9).

Consider any pair a, b such that[5] $\frac{a}{b} \neq \frac{p}{q}$, we claim that $T_{a,b}(\tilde{\eta}) \cap U_0 = \emptyset$. Otherwise, the union $T_{a,b}(\tilde{\eta}) \cup U_0$ would be a connected set contained in $p^{-1}(U)$ thus in U_0 and we could find a curve in U_0 joining x to $x + (a, b)$ proving that U is doubly essential (notice that the hypothesis on (a, b) implies that (a, b) and (p, q) generate a subgroup isomorphic to $\mathbb{Z}^2$), a contradiction.

Translations are order preserving, this means that $T_{a,b}(R) \cap R$ and $T_{a,b}(L) \cap L$ are both non-empty and either $T_{a,b}(R) \subset R$ or $T_{a,b}(L) \subset L$ (both can only hold in the case $\frac{a}{b} = \frac{p}{q}$). Also, one can easily see that $T_{a,b}(R) \subset R$ implies that $T_{-a,-b}(L) \subset L$.

Now, we choose (a, b) such that there exists a curve γ from x to $x + (a, b)$ satisfying:

- $T_{a,b}(\tilde{\eta}) \subset L$.
- γ is disjoint from $T_{p,q}(\gamma)$.
- γ is disjoint from $T_{a,b}(\tilde{\eta})$ and $\tilde{\eta}$ except at its boundary points.

We consider $\tilde{\eta}_1 = T_{a,b}(\tilde{\eta})$ and $\tilde{\eta}_2 = T_{-a,-b}(\tilde{\eta})$. Also, we shall denote $\tilde{\gamma} = \gamma \cup T_{-a,-b}(\gamma)$ which joins $x - (a, b)$ with $x + (a, b)$.

We obtain that U_0 is contained in $\Gamma = T_{a,b}(R) \cap T_{-a,-b}(L)$ a band whose boundary is $\tilde{\eta}_1 \cup \tilde{\eta}_2$.

Since U_0 is contained in Γ and is F-invariant, for every point $x \in U_0$ we have that $F^n(x)$ is a sequence in Γ, and since f is a non-resonant torus homeomorphism, we have that $\lim \frac{F^n(x)}{n} = \lim \frac{F^n(x)-x}{n} = (\alpha, \beta)$ is totally irrational.

However, we notice that Γ can be written as:

$$\Gamma = \bigcup_{n\in\mathbb{Z}} T_{np,nq}(\Gamma_0)$$

[5] We accept division by 0 as being infinity.

where Γ_0 is a compact set in $\mathbb{R}^2$. Indeed, if we consider the curve $\tilde{\gamma} \cup T_{a,b}(\eta_0) \cup T_{p,q}(\tilde{\gamma}) \cup T_{-a,-b}(\eta_0)$ we have a Jordan curve. Considering Γ_0 as the closure of the bounded component we have the desired fundamental domain.

So, if we consider a sequence of points $x_n \in \Gamma$ such that $\lim \frac{x_n}{n}$ exists and is equal to v it will verify that the coordinates of v have the same proportion as p/q, thus cannot be totally irrational. This is a contradiction and concludes the proof of the Lemma.

□

Remark 5. Considering Λ as in Addendum 2.2 we see that this proof works equally well since we only used that U_0 contained points which remained there to create the non wanted rotation vector and not that the whole U_0 was invariant.

♢

4. An example where $f|_{\Omega(f)}$ is not transitive

The example is similar to the one in section 2 of [J2], however, we do not know a priori if these specific examples admit or not a semiconjugacy.

Consider $g_1 : \mathbb{S}^1 \to \mathbb{S}^1$ and $g_2 : \mathbb{S}^1 \to \mathbb{S}^1$ Denjoy counterexamples with rotation numbers ρ_1 and ρ_2 which are irrationally independent and have minimal invariant sets M_1 and M_2 properly contained in S^1. We shall consider the following skew-product map $f_\beta : \mathbb{T}^2 \to \mathbb{T}^2$ given by:

$$f_\beta(x, y) = (g_1(x), \beta(x)(y))$$

where $\beta : \mathbb{S}^1 \to \mathrm{Homeo}_+(\mathbb{S}^1)$ is continuous and such that $\beta(x)(y) = g_2(y)$ for every $(x, y) \in M_1 \times \mathbb{S}^1$.

The same proof as in Lemma 2.1 of [J2] yields:

Lemma 4.1. *The map f_β is a non-resonant torus homeomorphism and $M_1 \times M_2$ is the unique minimal set.*

Proof.The proof is the same as the one in Lemma 2.1 of [J2]. Indeed any invariant measure for f must be supported in $M_1 \times M_2$ and the dynamics there is the product of two Denjoy counterexamples and thus uniquely-ergodic. Since rotation vectors can be computed with ergodic measures, we also get that f_β has a unique rotation vector (ρ_1, ρ_2) which is totally irrational by hypothesis.

□

Clearly, if we restrict the dynamics of f_β to $M_1 \times \mathbb{S}^1$ it is not hard to see that the nonwandering set will be $M_1 \times M_2$ (it is a product system there). So, we shall prove that if β is properly chosen, we get that $\Omega(f_\beta) = M_1 \times \mathbb{S}^1$. In fact, instead of constructing a specific example, we shall show that for "generic" β in certain space, this is satisfied, this will give the existence of such a β.

First, we define $\mathcal{B}$ to be the set of continuous maps $\beta : \mathbb{S}^1 \to \text{Homeo}_+(\mathbb{S}^1)$ such that $\beta(x) = g_2$ for every $x \in M_1$. We endow $\mathcal{B}$ with the topology given by restriction from the set of every continuous map from $\mathbb{S}^1$ to $\text{Homeo}_+(\mathbb{S}^1)$. With this topology, $\mathcal{B}$ is a closed subset of the set of continuous maps from $\mathbb{S}^1 \to \text{Homeo}_+(\mathbb{S}^1)$ which is a Baire space, thus, $\mathcal{B}$ is a Baire space.

So, the existence of the desired β is a consequence of:

Lemma 4.2. *There exists a dense G_δ (residual) subset of $\mathcal{B}$ of maps such that the induced map f_β verifies that $\Omega(f_\beta) = M_1 \times \mathbb{S}^1$.*

Proof.First, we will prove the Lemma assuming the following claim:

Claim. *Given $\beta \in \mathcal{B}$, $x \in M_1 \times \mathbb{S}^1$, $\varepsilon > 0$ and $\delta > 0$ there exists $\beta' \in \mathcal{B}$ which is $\delta-$close to β such that there exists $k > 0$ with $f^k_{\beta'}(B(x,\varepsilon)) \cap B(x,\varepsilon) \neq \emptyset$.*

Assuming this claim, the proof of the Lemma is a standard Baire argument: Consider $\{x_n\} \subset M_1 \times S^1$ a countable dense set. Using the claim, we get that the sets $\mathcal{B}_{n,N}$ consisting of the functions $\beta \in \mathcal{B}$ such that there exists a point y and a value $k > 0$ such that y and $f^k_\beta(y)$ belong to $B(x_n, 1/N)$ is a dense set. Also, the set $\mathcal{B}_{n,N}$ is open, since the property is clearly robust for C^0 perturbations of f_β. This implies that the set $\mathcal{R} = \bigcap_{n,N} \mathcal{B}_{n,N}$ is a residual set, which implies, by Baire's theorem that it is in fact dense.

For $\beta \in \mathcal{R}$ we get that given a point $x \in M_1 \times S^1$ and $\varepsilon > 0$, we can choose $x_n \in B(x, \varepsilon/2)$ and N such that $1/N < \varepsilon/2$, so, since $\beta \in \mathcal{B}_{n,N}$ we have that there exists $k > 0$ such that $f^k_\beta(B(x,\varepsilon)) \cap B(x,\varepsilon) \neq \emptyset$ proving that $M_1 \times \mathbb{S}^1$ is nonwandering for f_β as desired.

Proof of the Claim. The point $x \in M_1 \times \mathbb{S}^1$ can be written as (s,t) in the canonical coordinates.

Consider the curve $\gamma = (s-\varepsilon, s+\varepsilon) \times \{t\} \subset B(x,\varepsilon)$. It is easy to see[6], using the properties of g_1 that there exists $(a,b) \subset (s-\varepsilon, s+\varepsilon)$ such that

[6]Consider for example, (a,b) contained in a wandering interval contained in $(s-\varepsilon, s+\varepsilon)$.

$g^{k_n}(a,b) \subset (s-\varepsilon, s+\varepsilon)$ for $k_n \to \infty$. We can also assume that both a and b have disjoint orbits under g_1 and do not belong to M_1. We shall call $\gamma' \subset \gamma$ to the curve $\gamma' = (a,b) \times \{t\}$.

We can assume that $f_\beta^{k_n}(\gamma') \cap B(x,\varepsilon) = \emptyset$ for every $n > 0$, otherwise, there is nothing to prove.

We shall thus consider a $\delta-$perturbation of β such that it does not modify the orbit of (a,t) but moves the orbit of (b,t) in one direction making it give a complete turn around $\mathbb{S}^1$ and thus an iterate of γ' will intersect $B(x,\varepsilon)$.

Notice that considering $x_n = g_1^n(b)$ we have that $\beta^n = \beta(x_{n-1}) \circ \dots \beta(x_0)$ has rotation number ρ_2 as the distance between β^n and g_2^n grows sublinearly (c.f. Lemma 4.1). On the other hand, if we consider R_θ the rotation of angle $\theta \in (0,\delta)$, the map

$$\beta_\theta^n = R_\theta \circ \beta(x_{n-1}) \circ R_\theta \circ \beta_{x_{n-2}} \circ \dots \circ R_\theta \circ \beta(x_0)$$

approaches sublinearly with n going to infinity to $(R_\theta \circ g_2)^n$ which has rotation number larger than ρ_2. This implies that there exists n_0 such that for $n > n_0$ we have that if $\tilde{\beta}_\theta$ and $\tilde{\beta}$ denote the lifts of β_θ and β to $\mathbb{R}$ one has

$$|\tilde{\beta}_\theta^n(t) - \tilde{\beta}^n(t)| > 1$$

So, if we consider $k_n > n_0$ and we choose β' such that:

- It coincides with β in the g_1-orbit of a.
- It coincides with $R_\theta \circ \beta$ in the points $\{b, g_1(b), \dots, g_1^{k_n}(b)\}$.
- Is at distance smaller than δ from β.

We have that $f_{\beta'}^{k_n}(\gamma') \cap B(x,\varepsilon) \neq \emptyset$ as desired.

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□

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