Quantitative properties of convex representations

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Abstract

Let Γ be a discrete subgroup of $\mathrm{PGL}(d,\mathbb{R})$ and fix some euclidean norm $\| \ \|$ on $\mathbb{R}^d$. Let $N_\Gamma(t)$ be the number of elements in Γ whose operator norm is $\leq t$. In this article we prove an asymptotic for the growth of $N_\Gamma(t)$ when $t \to \infty$ for a class of Γ's which contains, in particular, Hitchin representations of surface groups and groups dividing a convex set of $\mathbb{P}(\mathbb{R}^d)$. We also prove analogue counting theorems for the growth of the spectral radii.

Contents

1 Introduction

Let $\widetilde{M}$ be a simply connected complete manifold of negative curvature and Γ be a discrete co-compact group of isometries of $\widetilde{M}$. This work consists in studying specific quantitative properties of certain representations $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$.

Recall that the boundary $\partial\Gamma$ of the group Γ is identified with $\widetilde{M}$'s geometric boundary, and that $\partial\Gamma$ has a natural structure of compact metrizable space coming from some Gromov distance (see Ghys-delaHarpe[10]).

Definition 1.1. We say that an irreducible representation $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ is *strictly convex* if there exists a ρ-equivariant Hölder continuous map

$$(\xi,\eta) : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d) \times \mathrm{Gr}_{d-1}(\mathbb{R}^d),$$

where $\mathrm{Gr}_{d-1}(\mathbb{R}^d)$ is the Grassmannian of hyperplanes of $\mathbb{R}^d$, such that $\mathbb{R}^d = \xi(x) \oplus \eta(y)$ whenever $x \neq y$.

We show in lemma 4.1 that strictly convex representations are proximal, that is, every element $\rho(\gamma)$ is a proximal matrix. This implies (cf. corollary 4.2) that for each $x \in \partial\Gamma$ one has $\xi(x) \subset \eta(x)$, and that the equivariant map (ξ, η) is necessarily unique.

Among strictly convex representations we find:

- **Deformations of hyperbolic manifolds in projective structures:** It is consequence of Koszul[13]'s and Benoist[4]'s work that if Γ is the fundamental group of a closed hyperbolic manifold of dimension $d-1$ and $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is a deformation of the embedding $\Gamma \subset \mathrm{PSO}(d-1, 1) \hookrightarrow \mathrm{PGL}(d, \mathbb{R})$ then $\rho(\Gamma)$ leaves invariant an open convex set Ω of $\mathbb{P}(\mathbb{R}^d)$, and the quotient $\rho(\Gamma)\backslash\Omega$ is a compact manifold. This gives an identification $\xi : \partial\Gamma \to \partial\Omega \subset \mathbb{P}(\mathbb{R}^d)$.
  Benoist[3] has shown that Ω is strictly convex and its boundary $\partial\Omega$ is of class $\mathrm{C}^{1+\alpha}$. The identification ξ and the tangent space of $\partial\Omega$ at $\xi(x)$
  $$\eta : \partial\Gamma \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$$
  are thus ρ-equivariant and Hölder. Since $\partial\Omega$ is strictly convex we have $\mathbb{R}^d = \xi(x) \oplus \eta(y)$ if $x \neq y$. These deformations are always irreducible and Zariski dense when the deformation is non trivial. Consequently we have that $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is a strictly convex representation.
- **Groups dividing a convex set of $\mathbb{P}(\mathbb{R}^d)$:** This examples contain the former but we treat them separately because they don't fall exactly in our terminology. Nevertheless the methods of this work apply directly in this setting.
  Consider some open convex set Ω of $\mathbb{P}(\mathbb{R}^d)$ and $\overline{\Omega}$ its closure. Suppose that $\mathbb{P}(V) \cap \overline{\Omega} = \emptyset$ for some hyperplane V of $\mathbb{R}^d$. Assume there is some discrete subgroup Γ of $\mathrm{PGL}(d, \mathbb{R})$ that leaves invariant Ω. Γ's action on Ω is necessarily properly discontinuous, and we assume it is also co-compact. Benoist[3] has shown that if Ω is strictly convex then $\partial\Omega$ is $\mathrm{C}^{1+\alpha}$ and the group Γ is hyperbolic in the sense of Gromov. Following the last example one finds that $\Gamma \subset \mathrm{PGL}(d, \mathbb{R})$ is strictly convex.
- **Hitchin representations of surface groups:**
  Let Σ be a closed orientable hyperbolic surface and let $\pi_1(\Sigma) \subset \mathrm{PSL}(2, \mathbb{R})$ be its fundamental group. Labourie[14] has shown that if $\rho : \pi_1(\Sigma) \to \mathrm{PGL}(d, \mathbb{R})$ is a deformation of the unique irreducible morphism (up to conjugacy) $\mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(d, \mathbb{R})$ then ρ is Zariski dense (when the deformation is non trivial) and there exists a ρ-equivariant Hölder map $\xi : \partial\pi_1(\Sigma) \to \mathscr{F}$ where $\mathscr{F}$ is the space of complete flags of $\mathbb{R}^d$. He shows that this curve is a *Frenet curve*: for $x \in \partial\pi_1(\Sigma)$ set $\xi_i(x)$ to be the i-th space of the flag $\xi(x)$, then if $d = d_1 + \cdots + d_k$ and $x_1, \ldots, x_k$ are pairwise distinct, then
  $$\mathbb{R}^d = \bigoplus_1^k \xi_{d_i}(x_i)$$

and if $n = n_1 + \cdots + n_k \leq d$ then

$$\lim_{(x_i)\to x} \bigoplus_1^k \xi_{n_i}(x_i) = \xi_n(x).$$

The first condition implies that, by considering the first and last coordinate of ξ one obtains a strictly convex representation.

- **Composition:**
  If $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is a Hitchin representation then the composition of ρ with some irreducible representation $\Lambda : \mathrm{PGL}(d, \mathbb{R}) \to \mathrm{PGL}(k, \mathbb{R})$ is strictly convex.

Fix some norm $\|\cdot\|$ in $\mathbb{R}^d$, coming from an inner product. For an element $g \in \mathrm{PGL}(d, \mathbb{R})$ we define its norm $\|g\|$ as the operator norm of some lift $\widetilde{g} \in \mathrm{GL}(d, \mathbb{R})$ such that $\det \widetilde{g} \in \{-1, +1\}$. In the same fashion one can define the spectral radius of g, since these quantities don't depend on the choice of the lift.

The following corollary of theorem A below is the main objective of this work:

Corollary 1. *Let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a strictly convex representation. Then there exist positive real numbers h and c such that*

$$chR^{-h}\#\{\gamma \in \Gamma : \|\rho(\gamma)\| \leq R\} \to 1$$

when R goes to infinity.

The constant h is independent of the norm chosen and is thus invariant under conjugation by elements of $\mathrm{PGL}(d, \mathbb{R})$. This follows from the fact that two euclidean norms in $\mathbb{R}^d$ are equivalent.

We shall now state a stronger result from which this corollary is deduced. The dynamics of each $\gamma \in \Gamma$ on $\partial\Gamma$ is of type north-south, this is, γ has exactly two fixed points, an attractor γ_+ and a repeller γ_-, and the basin of attraction of γ_+ is $\partial\Gamma - \{\gamma_-\}$. Theorem A shows that these fixed points are well distributed on $\partial\Gamma$:

Denote by $C(X)$ the space of continuous real functions over some space X and $C^*(X)$ its dual space.

Theorem A. *Let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a strictly convex representation, then there exist h and c, positive real numbers and two probabilities μ and $\overline{\mu}$ on $\partial\Gamma$ such that*

$$che^{-ht} \sum_{\gamma \in \Gamma : \log \|\rho(\gamma)\| \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \to \overline{\mu} \otimes \mu$$

when $t \to \infty$, in $C^*(\partial\Gamma \times \partial\Gamma)$.

Corollary 1 is deduced directly form theorem A by considering the constant function equal to 1 and the change of parameter $t = \log R$.

For a matrix $g \in \mathrm{PGL}(d, \mathbb{R})$ denote $\lambda_1(g)$ the logarithm of the spectral radius of g. An element g of a given subgroup G is *primitive* if it can't be written as a power of another element of G.

Theorem B. *Let* $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ *be a strictly convex representation. Then there exists* h*, a positive real number, such that*

$$hte^{-ht}\#\{[\gamma] \in [\Gamma] \textit{ primitive} : \lambda_1(\rho(\gamma)) \leq t\} \to 1$$

when $t \to \infty$*, where* $[\Gamma]$ *is the set of conjugacy classes of* Γ*.*

The statement of theorem A is inspired on Roblin[21]'s work, where the distribution of an orbit in a CAT(-1) space is proved, it implies the following corollary which explains how attractive lines of $\rho(\Gamma)$ are distributed in $\mathbb{P}(\mathbb{R}^d)$. Denote g_+ the attractive line of a proximal matrix in $\mathrm{PGL}(d, \mathbb{R})$.

Corollary 2. *Let* $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ *be a strictly convex representation, then there exist* h *and* c*, positive real numbers and a probability* ν *on* $\mathbb{P}(\mathbb{R}^d)$ *such that*

$$che^{-ht} \sum_{\gamma \in \Gamma : \log \|\rho(\gamma)\| \leq t} \delta_{\rho(\gamma)_+} \to \nu$$

when $t \to \infty$*.*

The counting problem in semi-simple Lie groups of higher rank is known for latices and, on the opposite, for Schottky groups. In the case of latices Eskin-McMullen[9] find an asymptotic for the growth of $\#\{\gamma \in \Gamma : |\alpha(\gamma)| \leq t\}$, where $\alpha : \mathrm{PGL}(d, \mathbb{R}) \to \mathfrak{a}$ is the Cartan projection, and Gorodnik-Oh[11] prove a distribution theorem (in the spirit of theorem A) for an orbit on the symmetric space.

For Schottky groups the asymptotic equivalence of $\#\{\gamma \in \Gamma : |\alpha(\gamma)| \leq t\}$ is shown by Quint[20]. For these groups there is also a distribution theorem due to Thirion[24].

In Eskin-McMullen[9]'s work a strong relation between dynamical systems and counting theorems is shown: one should prove a mixing property for an appropriate system. This method is applied by Roblin[21] and by Thirion[24].

In this work we are interested in the growth of α's first coordinate. We still exploit the relation with dynamical systems but in a slightly different manner. We find a symbolic flow and apply counting theorems for periodic orbits due to Parry-Pollicott[18] and the spatial distribution of these due to Bowen[6].

Method and techniques

Recall we have identified the boundary of the group Γ with $\widetilde{M}$'s geometric boundary. Set $B : \partial\Gamma \times \widetilde{M} \times \widetilde{M} \to \mathbb{R}$ to be the Busseman function of $\widetilde{M}$, that is, if $x \in \partial\Gamma$ and $p, q \in \widetilde{M}$ then

$$B_x(p, q) = \lim_{z \to x} d(p, z) - d(q, z).$$

Using the Busseman function one constructs an homeomorphism between $\widetilde{M}$'s unitary tangent bundle $T^1\widetilde{M}$ and $\partial^2\Gamma \times \mathbb{R}$, where

$$\partial^2\Gamma = \{(x, y) \in \partial\Gamma \times \partial\Gamma : x \neq y\}.$$

This is called the *Hopf parametrization* and turns out to be very useful since the geodesic flow on $T^1\widetilde{M}$ is the translation flow on $\partial^2\Gamma \times \mathbb{R}$. This identification shows that invariant measures of the geodesic flow are in correspondence with Γ-invariant measures on $\partial^2\Gamma$.

Patterson-Sullivan's measure on $\partial\Gamma$ induces a Γ invariant measure on $\partial^2\Gamma$ whose corresponding measure in $\Gamma\backslash T^1\widetilde{M}$ is the measure of maximal entropy of the geodesic flow. This fact is of particular importance in Roblin[21]'s work where he obtains counting theorems in the negative curvature case.

The main idea of this work is then to construct a flow in a similar fashion of Hopf's parametrization, considering an appropriate cocycle, and give a description of its measure of maximal entropy. We explain now how this flow is built.

Let $\mathbb{R}^{d^*}$ denote the dual space of $\mathbb{R}^d$. The space $\mathbb{P}(\mathbb{R}^{d^*})$ is naturally identified with the space of hyperplanes $\mathrm{Gr}_{d-1}(\mathbb{R}^d)$ of $\mathbb{R}^d$ via $\theta \mapsto \ker\theta$, the action of $\mathrm{PGL}(d,\mathbb{R})$ on $\mathbb{R}^{d^*}$ is then read as $g\cdot\theta \mapsto \theta\circ g^{-1}$.

Given a strictly convex representation $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ with equivariant map (ξ,η) two Hölder cocycles (cf.definition 3.1) $\beta, \overline{\beta} : \Gamma \times \partial\Gamma \to \mathbb{R}$ naturally appear, namely

$$\beta(\gamma,x) = \log\frac{\|\rho(\gamma)v\|}{\|v\|} \text{ and } \overline{\beta}(\gamma,x) = \log\frac{\|\rho(\gamma)\theta\|}{\|\theta\|}$$

where $v \in \xi(x)$ and $\theta \in \mathbb{R}^{d^*}$ verifies $\ker\theta = \eta(x)$.

The periods of a Hölder cocycle are defined as $\ell(\gamma) = \beta(\gamma,\gamma_+)$, where γ_+ is γ's attractive fixed point. The two cocycles defined above are strongly related since the periods of $\overline{\beta}$ are $\overline{\ell}(\gamma) = \ell(\gamma^{-1})$ (this is shown in lemma 4.11).

With the Hölder cocycle β we build an action of the group Γ in the space $\partial^2\Gamma \times \mathbb{R}$ as

$$\gamma(x,y,s) = (\gamma x, \gamma y, s - \beta(\gamma,y)).$$

The translation flow on $\partial^2\Gamma \times \mathbb{R}$ is the flow $\psi_t(x,y,s) = (x,y,s-t)$.

Proposition C1. *Let $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ be a strictly convex representation, then the action of Γ on $\partial^2\Gamma \times \mathbb{R}$ via β is proper and co-compact. Moreover, the translation flow $\psi_t : \Gamma\backslash\partial^2\Gamma \times \mathbb{R} \circlearrowleft$ is conjugated to a Hölder reparametrization of the geodesic flow on $\Gamma\backslash T^1\widetilde{M}$.*

Proposition C1 is cohomology invariant, this is, if one adds to β a cocycle of the form $(\gamma,x) \mapsto U(\gamma x) - U(x)$ for some function $U : \partial\Gamma \to \mathbb{R}$, the statement of the theorem does not change. Namely because with the function U one constructs Γ-equivariant homeomorphisms from one space to the other.

In order to prove theorem A further analysis of the flow ψ_t is needed. Mainly because it is the cocycle β that matters, and not only its cohomology class.

The exponential growth rate of a Hölder cocycle is defined as

$$\limsup_{t\to\infty}\frac{1}{t}\log\#\{[\gamma] : \ell(\gamma) \le t\}.$$

We prove in section §4 that our cocycles have finite exponential growth rate $h > 0$, and following Ledrappier[15] we have two *quasi-invariant* probabilities μ and $\overline{\mu}$ on $\partial\Gamma$, this is μ and $\overline{\mu}$ verify

$$\frac{d\gamma_*\mu}{d\mu}(x) = e^{-h\beta(\gamma,x)} \text{ and } \frac{d\gamma_*\overline{\mu}}{d\overline{\mu}}(x) = e^{-h\overline{\beta}(\gamma,x)}.$$

We define Gromov's product $\mathscr{G} : \mathbb{P}(\mathbb{R}^{d^*}) \times \mathbb{P}(\mathbb{R}^d) - \Delta \to \mathbb{R}$ as

$$\mathscr{G}(\theta, v) = \log \frac{|\theta(v)|}{\|\theta\|\|v\|},$$

where $\Delta = \{(\theta, v) : \theta(v) = 0\}$. Since the representations ρ is strictly convex one has that $\mathbb{R}^d = \xi(x) \oplus \eta(y)$ when $x \neq y$ and thus $\mathscr{G}(\eta(x), \xi(y))$ is a well defined map which we shall call

$$[\cdot, \cdot] : \partial^2\Gamma \to \mathbb{R}.$$

If $d = 2$ the above formula for Gromov's product coincides with the usual Gromov product coming from hyperbolic geometry of surfaces.

We can now describe the measure of maximal entropy of ψ_t.

Proposition C2. *Let ρ be a strictly convex representation, then h is the topological entropy of $\psi_t : \Gamma\backslash\partial^2\Gamma \times \mathbb{R} \circlearrowleft$, the measure $e^{-h[\cdot,\cdot]}\overline{\mu} \otimes \mu \otimes ds$ is Γ-invariant on $\partial^2\Gamma \times \mathbb{R}$ and induces (up to a constant) ψ_t's probability of maximal entropy on the quotient $\Gamma\backslash\partial^2\Gamma \times \mathbb{R}$.*

In section §3 we study Hölder cocycles with finite positive exponential growth rate. We prove there theorem 3.2 which is analogue to propositions C1 and C2. In section §4 we show that for strictly convex representations the cocycle β verifies the hypothesis of theorem 3.2 and Propositions C1 and C2 are then deduced. Section §2 is devoted to the study of reparametrizations of Anosov flows, a characterization of a suspension is there proved and a dynamical property of ψ_t is shown, namely, that it is topologically weakly mixing. This allows us to apply counting theorems for hyperbolic flows in our setting. The last section is devoted to the proofs of theorems A and B.

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2 Cross sections and arithmeticity of periods

The main objectives of this section are lemma 2.4 and corollary 2.10. The former explains how measures of maximal entropy of reparametrizations arise and the latter matters on reparametrizations of geodesic flows on closed negatively curved manifolds.

Let X be a compact metric space and $\phi_t : X \circlearrowleft$ a continuous flow on X without fixed points.

Definition 2.1. We will say that $\phi_t : X \circlearrowleft$ is *topologically weakly mixing* if the only solution to the equation

$$w\phi_t = e^{2\pi iat}w,$$

for $w : X \to S^1$ continuous and $a \in \mathbb{R}$, is $a = 0$ and $w =$constant.

Remark 2.1. Consider some periodic orbit τ of period $p(\tau)$ of the flow $\phi_t : X \circlearrowleft$. If $\phi_t : X \circlearrowleft$ is not weak mixing let $w : X \to S^1$ and $a \in \mathbb{R} - \{0\}$ verify $w\phi_t = e^{2\pi iat}w$. Since $\phi_{p(\tau)}x = x$ for any $x \in \tau$ one then finds $\exp\{2\pi iap(\tau)\} = 1$ which implies that $p(\tau)$ belongs to the discrete group $a^{-1}\mathbb{Z}$. This is, the periods of a non weak mixing flow generate a discrete group of $\mathbb{R}$.

A closed subset K of X is a *cross section* for ϕ_t if the function $T_\phi : K \times \mathbb{R} \to X$ given by $T_\phi(x,t) = \phi_t(x)$ is a surjective local homeomorphism.

Remark 2.2. If $\phi_t : X \circlearrowleft$ admits a cross section then X fibers over the circle and the projection of a periodic orbit (seen as map from $S^1 \to S^1$) has non-zero index.

Remark 2.2 admits a converse due to Schwartzman[23]:

Lemma 2.1 (Schwartzman[23], page 280)**.** *There exists a continuous function $w : X \to S^1$ differentiable in the flow's direction such that its derivative in the flow's direction w' is nowhere zero if and only if the flow admits a cross section.*

We now turn our attention to reparametrizations of flows. Let $F : X \to \mathbb{R}$ be a positive continuous function. Set $\kappa : X \times \mathbb{R} \to \mathbb{R}$ as

$$\kappa(x,t) = \int_0^t F\phi_s(x)ds, \tag{1}$$

if t is positive, and $\kappa(x,t) := -\kappa(\phi_t x, -t)$ for t negative. Thus, κ verifies de cocycle property $\kappa(x, t+s) = \kappa(\phi_t x, s) + \kappa(x,t)$ for every $t, s \in \mathbb{R}$ and $x \in X$.

Since $F > 0$ and X is compact F has a positive minimum and $\kappa(x,\cdot)$ is an increasing homeomorphism of $\mathbb{R}$. We then have an inverse $\alpha : X \times \mathbb{R} \to \mathbb{R}$ that verifies

$$\alpha(x, \kappa(x,t)) = \kappa(x, \alpha(x,t)) = t \tag{2}$$

for every $(x,t) \in X \times \mathbb{R}$.

Definition 2.2. The *reparametrization* of ϕ_t by F is the flow $\psi_t : X \circlearrowleft$ defined as $\psi_t(x) := \phi_{\alpha(x,t)}(x)$. If F is Hölder continuous we shall say that ψ_t is a Hölder reparametrization of ϕ_t.

Remark 2.3. The cocycle property for κ and equation (2) imply that ψ_t is in fact a flow.

The advantage of cross sections is that the definition is invariant via reparametrizations.

Lemma 2.2. *Let ψ_t be a reparametrization of ϕ_t. Then ϕ_t admits a cross section if and only if ψ_t does.*

Proof. Let K be a cross section for ϕ_t, we need to show that the map $T_\psi : X \times \mathbb{R} \to X$ $(x,t) \mapsto \psi_t(x)$ is a surjective local homeomorphism. But this is evident in view of the relation

$$T_\psi = T_\phi \circ \varphi$$

where φ is the homeomorphism $\varphi : K \times \mathbb{R} \circlearrowleft$ $(x,t) \mapsto (x, \alpha(x,t))$. □

One then finds the following corollary.

Corollary 2.3. *A flow $\phi_t : X \circlearrowleft$ does not admit a cross section if and only if every reparametrization of ϕ_t is topologically weakly mixing.*

Proof. Consider some reparametrization ψ_t of ϕ_t and assume ψ_t is not weak mixing, this is, there exists $w : X \to S^1$ such that $w\psi_t(x) = e^{2\pi i a t} w(x)$ for some $a \neq 0$. Such w is differentiable in the flow's direction and

$$\frac{w'(x)}{2\pi i w(x)} = a \neq 0.$$

Applying Schwartzman's lemma 2.1 one obtains a cross section for ψ_t and thus a cross section for ϕ_t.

If ϕ_t admits a cross section one applies Schwartzman's lemma 2.1 and find a continuous function $w : X \to S^1$ whose derivative in the flow's direction is never zero. Set

$$F(x) = \frac{w'(x)}{2\pi i w(x)}$$

and consider ψ_t, the reparametrization of ϕ_t by F. One easily verifies that $w\psi_t = e^{2\pi i t} w$ and thus ψ_t is not topologically weakly mixing. □

If m is a ϕ_t-invariant probability on X then the probability m' defined by $dm'/dm(\cdot) = F(\cdot)/m(F)$ is ψ_t-invariant. In particular, if τ is a periodic orbit of ϕ_t then it is also periodic for ψ_t and the new period is

$$\int_\tau F.$$

This relation between invariant probabilities induces a bijection and Abramov[1] relates the corresponding metric entropies:

$$h(\psi_t, m') = h(\phi_t, m) / \int F dm. \tag{3}$$

Denote $\mathcal{M}^{\phi_t}$ the set of ϕ_t-invariant probabilities. The *pressure* of a continuous function $F : X \to \mathbb{R}$ is defined as

$$P(\phi_t, F) = \sup_{m \in \mathcal{M}^{\phi_t}} h(\phi_t, m) + \int_X F dm.$$

A probability m such that the supremum is attained is called an *equilibrium state* of F.

Lemma 2.4. *Let $\psi_t : X \circlearrowleft$ be the reparametrization of ϕ_t by $F : X \to \mathbb{R}_+^*$. Assume the equation*

$$P(\phi_t, -sF) = 0 \qquad s \in \mathbb{R}$$

has a finite positive solution h, then h is ψ_t's topological entropy h_{top}. Moreover h_{top} is the unique solution to the last equation and the bijection $m \mapsto m'$ induces a bijection between equilibrium states of $-h_{top}F$ and probabilities of maximal entropy for ψ_t.

Proof. Abramov's formula (3) directly implies

$$h(\phi_t, m) - h_{\text{top}} \int F dm = (h(\psi_t, m') - h_{\text{top}}) \int F dm,$$

for any ϕ_t-invariant probability m. This equation, together with the variational principle

$$h_{\text{top}} = \sup_{m' \in \mathcal{M}^{\psi_t}} h(\psi_t, m'),$$

imply

$$P(\phi_t, -h_{\text{top}}F) = \sup_{m \in \mathcal{M}^{\phi_t}} h(\phi_t, m) - h_{\text{top}} \int F dm = 0.$$

If m_F is an equilibrium state of $-h_{\text{top}}F$ then, since $P(\phi_t, -h_{\text{top}}F) = 0$, we must have $h(\psi_t, m'_F) = h_{\text{top}}$. Thus, $m \mapsto m'$ induces a bijection between equilibrium states of $-h_{\text{top}}F$ and probabilities of maximal entropy of ψ_t.

Consider now some $s \in \mathbb{R}$ such that $P(\phi_t, -sF) = 0$. Applying the definition of P and Abramov's formula one deduces

$$0 = \sup_{m \in \mathcal{M}^{\phi_t}} (h(\psi_t, m') - s) \int F dm,$$

and, since F is positive, one has $s = \sup_{m' \in \mathcal{M}^{\psi_t}} h(\psi_t, m')$. Applying again the variational principle one finds $s = h_{\text{top}}$. □

We now restrict our study to hyperbolic flows: Assume from now on that X is a compact manifold and that the flow $\phi_t : X \circlearrowleft$ is C^1. We say that ϕ_t is *Anosov* if the tangent bundle of X splits as a sum of three $d\phi_t$-invariant bundles

$$TX = E^s \oplus E^0 \oplus E^u,$$

and there exist positive constants C and c such that: E^0 is the direction of the flow and for every $t \geq 0$ one has: for every $v \in E^s$

$$\|d\phi_t v\| \leq Ce^{-ct}\|v\|,$$

and for every $v \in E^u$ $\|d\phi_{-t} v\| \leq Ce^{-ct}\|v\|$.

In this setting there is an extra equivalence for the existence of cross sections:

Proposition 2.5. *Let $\phi_t : X \circlearrowleft$ be an Anosov flow. Then ϕ_t admits a cross section if and only if there exists $F : X \to \mathbb{R}_+^*$ Hölder such that the subgroup of $\mathbb{R}$ spanned by*

$$\{\int_\tau F : \tau \text{ periodic}\}$$

is discrete.

Proof. Assume such F exists, and assume (without loss of generality) that $\langle\{\int_\tau F : \tau \text{ periodic}\}\rangle = \mathbb{Z}$. Recall we have defined

$$\kappa(x,t) = \int_0^t F(\phi_s x)ds.$$

The cocycle $\Theta : \mathbb{R} \times X \to S^1$ given by $\Theta(x,t) = e^{2\pi i \kappa(x,t)}$ is, after Livsic[16]'s theorem, cohomologically trivial and thus there exists $w : X \to S^1$ Hölder continuous such that

$$\frac{w\phi_t(x)}{w(x)} = \exp\{2\pi i \int_0^t F(\phi_s x)ds\},$$

one finds a cross section applying Schwartzman's lemma 2.1.

Assume now that ϕ_t admits a cross section. Applying Scwartzman's lemma 2.1 one finds a continuous function $w : X \to S^1$ such that its derivative in the flow's direction is never zero. One can assume that such w is in fact differentiable (by considering another function close to w) and thus the function $F(x) = w'(x)/2\pi i w(x)$ is differentiable with integer periods. □

The following proposition together with lemma 2.4 imply that a Hölder reparametrization of an Anosov flow has a unique probability of maximal entropy.

Proposition 2.6 (Bowen-Ruelle[8])**.** *Let $\phi_t : X \circlearrowleft$ be an Anosov flow. Then given a Hölder potential $G : X \to \mathbb{R}$ there exists a unique equilibrium state for G. Equilibrium states are thus ergodic.*

Corollary 2.7. *Let $\phi_t : X \circlearrowleft$ be an Anosov flow and ψ_t be a Hölder reparametrization of ϕ_t. Then ψ_t has a unique probability of maximal entropy and it's ergodic with respect to this measure.*

We are interested in finding Markov partitions for reparametrizations of Anosov flows.

Definition 2.3. Let $\varphi_t : X \circlearrowleft$ be a flow. We shall say that the triplet (Σ, π, r) is a *Markov coding* for φ_t if Σ is a subshit of finite type, $\pi : \Sigma \to X$ and $r : \Sigma \to \mathbb{R}_+^*$ are Hölder continuous and the function $\pi_r : \Sigma \times \mathbb{R} \to X$ defined as

$$\pi_r(x,t) = \varphi_t \pi(x)$$

verifies the following conditions:

i) π_r is surjective and Hölder,
ii) let $\sigma : \Sigma \circlearrowleft$ be the shift and let $\hat{r} : \Sigma \times \mathbb{R} \circlearrowleft$ be defined as $\hat{r}(x,t) = (\sigma x, t - r(x))$, then π_r is $\hat{r}$-invariant,
iii) $\pi_r : \Sigma \times \mathbb{R}/\hat{r} \to X$ is bounded-to-one and injective on a residual set which is of full measure for every ergodic invariant measure of total support (for σ_t^f),
iv) consider the translation flow $\sigma_t^r : \Sigma \times \mathbb{R}/\hat{r} \circlearrowleft$ then $\pi_r \sigma_t^r = \varphi_t \pi_r$.

Remark 2.4. If a flow $\varphi_t : X \circlearrowleft$ admits a Markov coding then it has a unique probability of maximal entropy and the function $\pi_r : \Sigma \times \mathbb{R}/\hat{r} \to X$ is an isomorphism between the probabilities of maximal entropy of σ_t^r and that of φ_t. In particular the topological entropy of φ_t coincides with that of σ_t^r.

Theorem 2.8 (Bowen[6, 7])**.** *A transitive Anosov flow admits a Markov coding.*

Lemma 2.9. *Let (Σ, π, r) be a Markov coding for a transitive Anosov flow $\phi_t : X \circlearrowleft$. Set $\psi_t : X \circlearrowleft$ to be a Hölder reparametrization of ϕ_t by $F : X \to \mathbb{R}_+^*$ and define $f : \Sigma \to \mathbb{R}_+^*$ as*

$$f(z) = \int_0^{r(z)} F\phi_s(\pi(z))ds.$$

Then (Σ, π, f) is a Markov coding for ψ_t. If moreover ϕ_t does not admit a cross section then the translation flow $\sigma_t^f : \Sigma \times \mathbb{R}/\hat{f} \circlearrowleft$ is topologically weakly mixing.

Proof. We need to check that the function $\pi_f : \Sigma \times \mathbb{R} \to X$ defined as $\pi_f(z,s) := \psi_s(\pi(z))$ is $\hat{f}$ invariant and conjugates the translation flow on $\Sigma \times \mathbb{R}/\hat{f}$ with the flow ψ_t. To prove invariance by $\hat{f}$ we will prove that for every $(z,s) \in \Sigma \times \mathbb{R}$ one has

$$\pi_f(z, s + f(z)) = \pi_f(\sigma z, s).$$

The computation is intricate but direct. Recall that by definition $f(z) = \kappa(\pi(z), r(z))$ (see equation (1)). This immediately implies $\alpha(\pi(z), f(z)) = r(z)$. We then have

$$\pi_f(z, s + f(z)) = \psi_{s+f(z)}(\pi z) = \psi_s \circ \psi_{f(z)}(\pi z) = \psi_s \circ \phi_{\alpha(\pi(z), f(z))}(\pi z)$$

$$= \psi_s \circ \phi_{r(z)}(\pi z) = \psi_s(\pi(\sigma z))$$

since (Σ, π, r) is a Markov coding for ϕ_t. This proves invariance.

The remaining properties of Markov coding then follow.

Suppose now that $\phi_t | X$ does not admit a cross section. We must then show that σ_t^f is weak mixing. Applying proposition 2.5 one has that the periods $\int_\tau F$ generate a dense subgroup of $\mathbb{R}$. Since $\hat{\pi} : \Sigma \times \mathbb{R} \to X$ is surjective, the periods of σ_t^f periodic orbits also generate a dense subgroup of $\mathbb{R}$ and remark 2.1 implies that σ_t^f is weak mixing. □

We find now the following corollary:

Corollary 2.10. *Let Γ be a co-compact group of isometries of a complete simply connected manifold of negative curvature $\widetilde{M}$. Let $\phi_t : \Gamma\backslash T^1\widetilde{M} \circlearrowleft$ be the geodesic flow and $\psi_t : \Gamma\backslash T^1\widetilde{M} \circlearrowleft$ be a Hölder reparametrization of ϕ_t. Consider a Markov coding (Σ, π, f) for ψ_t, then the flow σ_t^f is weak mixing.*

Proof. Since the geodesic flow is a transitive Anosov flow, lemma 2.9 applies. It remains to prove that the geodesic flow on a compact manifold of negative curvature does not admit a cross section. As observed before (remark 2.2) we only need to find a homologically trivial periodic orbit (since such orbit will always have zero index as map $S^1 \to S^1$).

In negative curvature we can find two elements in Γ, a and b that don't commute, the closed geodesic associated to the commutator $aba^{-1}b^{-1}$ is then the required periodic orbit. □

3 Cocycles with finite exponential growth rate

Let Γ be a discrete co-compact isometry group of a complete simply connected manifold with negative curvature $\widetilde{M}$. We identify the boundary of the group Γ with the geometric boundary of $\widetilde{M}$.

Definition 3.1. A *Hölder cocycle* is a function $c : \Gamma \times \partial\Gamma \to \mathbb{R}$ such that

$$c(\gamma_0\gamma_1, x) = c(\gamma_0, \gamma_1 x) + c(\gamma_1, x)$$

for any $\gamma_0, \gamma_1 \in \Gamma$ and $x \in \partial\Gamma$, and where $c(\gamma, \cdot)$ is a Hölder map for every $\gamma \in \Gamma$ (the same exponent is assumed for every $\gamma \in \Gamma$).

Given a Hölder cocycle c we define the *periods* of c as the numbers

$$l(\gamma) := c(\gamma, \gamma_+)$$

where γ_+ is the attractive fixed point of γ in $\Gamma - \{e\}$. The cocycle property implies that the period of an element γ only depends on its conjugacy class $[\gamma] \in [\Gamma]$.

Two cocycles c and c' are said to be cohomologous if there exists a Hölder function $U : \partial\Gamma \to \mathbb{R}$ such that for all $\gamma \in \Gamma$ one has

$$c(\gamma, x) - c'(\gamma, x) = U(\gamma x) - U(x).$$

One easily deduces from the definition that the set of periods of a cocycle is a cohomological invariant.

We shall be interested in cocycles whose periods are positive, that is, such that $l(\gamma) > 0$ for every $\gamma \in \Gamma$. The *exponential growth rate* for such cocycle c is defined as:

$$h_c := \limsup_{t\to\infty} \frac{1}{t} \log \#\{[\gamma] : l(\gamma) \leq t\} \in \mathbb{R}_+ \cup \{\infty\}.$$

It is consequence of Ledrappier's work (cf. corollary 3.6) that a Hölder cocycle c with positive periods verifies $h_c > 0$. If moreover c has finite exponential growth rate then, following Patterson's construction, Ledrappier[15] shows the existence of a *quasi-invariant* probability μ over $\partial\Gamma$ of cocycle $h_c c$, that is, μ verifies

$$\frac{d\gamma_*\mu}{d\mu}(x) = e^{-h_c c(\gamma,x)}.$$

Theorem 3.1 (Ledrappier[15] page 102)**.** *Let c be a Hölder cocycle with positive periods. Then c has finite positive exponential growth rate h_c if and only if there exists a quasi-invariant probability of cocycle $h_c c$. If this is the case, the quasi-invariant probability is unique.*

Let $\bar{c}$ be a cocycle such that $\bar{l}(\gamma) = l(\gamma^{-1})$ (this always exists as shown in the next section) and denote by $\overline{\mu}$ the quasi-invariant probability associated to $\bar{c}$. Set $\partial^2\Gamma$ to be the set of pairs $(x, y) \in \partial\Gamma \times \partial\Gamma$ such that $x \neq y$.

The main theorem of this section is the following:

Theorem 3.2. *Let c be a Hölder cocycle with positive periods such that h_c is finite. Then:*

1. *the action of Γ in $\partial^2\Gamma \times \mathbb{R}$*

$$\gamma(x, y, s) = (\gamma x, \gamma y, s - c(\gamma, y))$$

   is proper and co-compact. Moreover, the translation flow $\psi_t : \Gamma\backslash\partial^2\Gamma \times \mathbb{R} \circlearrowleft$

$$\psi_t(x, y, s) = (x, y, s - t)$$

   is conjugated to a Hölder reparametrization of the geodesic flow on $\Gamma\backslash T^1\widetilde{M}$. The conjugating map is also Hölder continuous.

2. *Let μ and $\overline{\mu}$ be the quasi-invariant probabilites of cocycles $h_c c$ and $h_c\bar{c}$ respectively. Then there is a unique ψ_t-invariant probability with the same zero sets as $\overline{\mu} \otimes \mu \otimes ds$, this measure is ψ_t's probability of maximal entropy. The topological entropy of ψ_t is h_c.*

Remark 3.1. The statement of the theorem is cohomology invariant. That is, a change in the choice of the cocycle (in c's cohomology class) doesn't change the statement of theorem 3.2. This is clear for the first item, and for the second item one applies the following result of Ledrappier.

Theorem 3.3 (Ledrappier[15] page 101)**.** *Let c and c' be Hölder cocycles with positive periods and finite exponential growth rate. Let μ and μ' be two quasi-invariant measures of Hölder cocycle $h_c c$ and $h_{c'} c'$ respectively. Then μ and μ' have the same zero sets if and only if $h_c c$ and $h_{c'} c'$ are cohomologous.*

To prove theorem 3.2 we shall find an appropriate cocycle: following Ledrappier[15] we associate to the cocycle c a Γ-invariant Hölder function $F : T^1M \to \mathbb{R}$. The fact that the cocycle is of finite exponential growth rate together with a

Livsic-type theorem due to Lopes-Thieullen[17] will allow us to choose such F to be positive.

One then finishes copying the Hopf parametrization of $T^1\widetilde{M}$. Namely we construct a homeomorphism $T^1\widetilde{M} \to \partial^2\Gamma \times \mathbb{R}$ such that the action of Γ on $T^1\widetilde{M}$ is sent to the action we need (this implies properness of the action) and the action of the geodesic flow will be reparametrized on the right side.

Concerning the proof of the second item: Since the measure of maximal entropy of a reparametrization has the same zero sets as an equilibrium state (lemma 2.4), we will conclude giving a description of the induced measure by this equilibrium state on $\partial^2\Gamma$.

Proof of the first item of theorem 3.2

Identify the unit tangent bundle of $\widetilde{M}$ with $\widetilde{M} \times \partial\Gamma$ and denote $\widetilde{\phi_t}$ the geodesic flow on M. For a given Γ-invariant Hölder function $H : T^1\widetilde{M} \to \mathbb{R}$ Schapira[22] introduced the following geometric cocycle: for $z \in \partial\Gamma$ define $B_z^H : \widetilde{M} \times \widetilde{M} \to \mathbb{R}$ as

$$B_z^H(p,q) = \lim_{s\to\infty} \int_0^{s+B_z(p,q)} H(\phi_t(p,z))dt - \int_0^s H(\phi_t(q,z))dt, \tag{4}$$

where $B_z : \widetilde{M} \times \widetilde{M} \to \mathbb{R}$ is the Busseman function (when $H \equiv 1$ $B_z^1(p,q)$ is exactly $B_z(p,q)$). The expression is convergent since H is Hölder continuous and the geodesic flow is Anosov.

One finds the following properties:

Lemma 3.4. *Let $o, p, q \in \widetilde{M}$ and $z \in \partial\Gamma$, Then*
i) $B_z^H(p,q) = B_{\gamma z}^H(\gamma p, \gamma q)$ *for every* $\gamma \in \Gamma$,
ii) $B_z^H(p,q) = B_z^H(p,o) + B_z^H(o,q)$
iii) if q belongs to the geodesic line from p to z one has

$$B_z^H(p,q) = \int_p^q H$$

where $\int_p^q H$ is the integral of H over the unique oriented geodesic segment that begins in p and finishes in q.

Proof. Property $i)$ follows directly from the Γ-invariance of H and the Busseman function. Property $iii)$ is a direct consequence of the definition. We prove now property $ii)$: by definition

$$B_z^H(p,o) = \lim_{s\to\infty} \int_0^{s+B_z(p,o)} H(\phi_t(p,z))dt - \int_0^s H(\phi_t(o,z))dt.$$

If we consider the change of parameter $s \mapsto s+B_z(o,q)$ the last limit becomes

$$B_z^H(p,o) = \lim_{s\to\infty} \int_0^{s+B_z(p,o)+B_z(o,q)} H(\phi_t(p,z))dt - \int_0^{s+B_z(o,q)} H(\phi_t(o,z))dt$$

and thus, since $B_z(p,o) + B_z(o,q) = B_z(p,q)$ we have

$$B_z^H(p,o) + B_z^H(o,q) = \lim_{s\to\infty} \int_0^{s+B_z(p,q)} H(\phi_t(p,z))dt - \int_0^{s+B_z(o,q)} H(\phi_t(o,z))dt$$

$$+ \lim_{s\to\infty} \int_0^{s+B_z(o,q)} H(\phi_t(o,z))dt - \int_0^{s} H(\phi_t(q,z))dt = B_z^H(p,q).$$

□

Given a Γ-invariant Hölder function $H : T^1\widetilde{M} \to \mathbb{R}$ one can associate to H a Hölder cocycle over the group Γ :

$$c_H(\gamma, z) = B_z^H(\gamma^{-1}o, o), \tag{5}$$

where o is some point on $\widetilde{M}$ fixed from now on.

Two Γ-invariant Hölder functions $H, H' : T^1\widetilde{M} \to \mathbb{R}$ are said to be cohomologous (according Livsic) if there exists a Hölder Γ-invariant function $V : T^1\widetilde{M} \to \mathbb{R}$, differentiable in the direction of the geodesic flow, such that

$$H(p,z) - H'(p,z) = \frac{\partial V \circ \phi_t}{\partial t}(p,z).$$

The conjugacy class $[\gamma]$, of an element $\gamma \in \Gamma$, is naturally identified with the closed geodesic on $\Gamma\backslash T^1\widetilde{M}$ associated to γ. We denote $|\gamma|$ the length of this closed geodesic. The *periods* of the function H are defined to be the numbers

$$\int_{[\gamma]} H.$$

One easily sees that: the periods of H are exactly the periods of c_H; the periods of H are a Livsic-cohomology invariant. We can now state a theorem of Ledrappier.

Theorem 3.5 (Ledrappier[15], page 105)**.** *The map $H \mapsto c_H$ induces a bijection between cohomology classes of Γ-invariant Hölder functions and cohomology classes of Hölder cocycles. The corresponding classes have the same periods.*

Corollary 3.6. *Let c be a Hölder cocycle with positive periods, then the exponential growth rate h_c is positive.*

Proof. Let $F : T^1\widetilde{M} \to \mathbb{R}$ be such that the Hölder cocycles c_F and c are cohomologous. Since c_F has positive periods F must have a positive maximum M and thus $l(\gamma) \le M|\gamma|$, which implies

$$\#\{[\gamma] \in [\Gamma] : l(\gamma) \le t\} \ge \#\{[\gamma] \in [\Gamma] : |\gamma| \le t/M\}.$$

The exponential growth rate of the quantity on the right is known to be strictly positive and the corollary is proved. □

We will need the following lemma.

Lemma 3.7 (Ledrappier[15], page 106)**.** *Let c be a Hölder cocycle with positive periods. Then the exponential growth rate of c is finite if and only if*

$$\inf_{[\gamma]} \frac{l(\gamma)}{|\gamma|} > 0.$$

We shall now state the positive Livsic-type theorem which will allow us to conclude.

Theorem 3.8 (Lopes-Thieullen[17])**.** *Let N be a compact manifold with a C^1 Anosov flow φ_t, and let $f : N \to \mathbb{R}$ be a Hölder continuous function. Then there exist $U, V : N \to \mathbb{R}$ Hölder continuous with $U \geq 0$ and V of class C^1 in the direction of the flow such that for every point p in N one has*

$$f(p) = m(f) + \frac{\partial V \circ \varphi_t}{\partial t}(p) + U(p),$$

where

$$m(f) = \inf\{\frac{1}{p(\tau)} \int_\tau f : \tau \ \varphi_t\text{-periodic}\}.$$

Proof of first item of theorem 3.2. We begin with a Hölder cocycle c with positive periods and finite exponential growth rate. After Ledrappier's theorem 3.5 we find a Γ-invariant Hölder function $H : T^1\widetilde{M} \to \mathbb{R}$ whose periods coincide with those of c.

Ledrappier's lemma 3.7 then implies that

$$\inf_{[\gamma]} \frac{\int_{[\gamma]} H}{|\gamma|} > 0.$$

Applying Lopes-Thieullen's theorem we find that $F := U + m(H)$ is positive and its cocycle c_F (defined by the formula (5)) is cohomologous to c.

We shall prove the statement for the cocycle c_F. The idea is to construct a parametrization of $T^1\widetilde{M}$ using F's geometric cocycle B^F_z (equation (4)) as following:

Fix some point $o \in \widetilde{M}$ and for a geodesic through (p, v) denote $v_{-\infty}$ and v_∞ its origin and end points in $\partial\Gamma$, then define

$$E : (p, v) \mapsto (v_{-\infty}, v_\infty, B^F_{v_\infty}(p, o)).$$

Consider some geodesic $a(t)$ in $T^1\widetilde{M}$ with endpoints $a(-\infty) = v_{-\infty}$ and $a(\infty) = v_\infty$. Applying lemma 3.4 we have, for every $t \in \mathbb{R}$, that

$$E(a(t)) = (v_{-\infty}, v_\infty, B^F_{v_\infty}(a(0), o) - \int_0^t F(a(s))ds).$$

Since $F > 0$ we deduce that E is inyective when restricted to the geodesic $\{a(t) : t \in \mathbb{R}\}$, and since F has a positive minimum it is surjective over the set $\{(v_{-\infty}, v_\infty)\} \times \mathbb{R}$. This implies that E is an homeomorphism from $T^1\widetilde{M}$ to $\partial^2\Gamma \times \mathbb{R}$.

E is Γ-equivariant: Write $E(p,v) = (x, y, B_y^F(p,o))$ and consider some $\gamma \in \Gamma$, then by definition

$$E(\gamma(p,v)) = (\gamma x, \gamma y, B^F_{\gamma y}(\gamma p, o)).$$

Applying lemma 3.4 one has

$$B^F_{\gamma y}(\gamma p, o) = B^F_{\gamma y}(\gamma p, \gamma o) + B^F_{\gamma y}(\gamma o, o) = B^F_y(p,o) - c_F(\gamma, y).$$

One concludes that E is a Γ-equivariant homeomorphism between $\Gamma \curvearrowright T^1\widetilde{M}$ and the action $\Gamma \curvearrowright \partial^2\Gamma \times \mathbb{R}$ via c_F. Since $\Gamma \curvearrowright T^1\widetilde{M}$ is proper (and co-compact), so is the action on $\partial^2\Gamma \times \mathbb{R}$ via c_F.

The geodesic flow is reparametrized: If $(p,v) \mapsto (v_{-\infty}, v_\infty, B^F_{v_\infty}(p,o))$ and $q \in \widetilde{M}$ is the base point of $\phi_t(p,v)$ then by definition

$$E(\phi_t(p,v)) = (v_{-\infty}, v_\infty, B^F_{v_\infty}(q,o)),$$

applying again lemma 3.4

$$B^F_{v_\infty}(q,o) = B^F_{v_\infty}(p,o) - \int_0^t F\phi_t(p,v)dt.$$

This means exactly,

$$E(\phi_t(p,v)) = \psi_{\int_0^t F\phi_s(p,v)ds} E(p,v),$$

in other words, the flow $E^{-1}\psi_t E$ is the reparametrization of the geodesic flow by F (see definition 2.2). □

Proof of the second item of theorem 3.2

In the last subsection we showed that the flow $\psi_t : \Gamma\backslash\partial^2\Gamma \times \mathbb{R} \circlearrowleft$ is Hölder conjugated to a Hölder reparametrization of the geodesic flow. We also recall that the statement of theorem 3.2 is cohomology invariant.

We can thus place ourselves in the following situation: $F : T^1\widetilde{M} \to \mathbb{R}^*_+$ is a Γ-invariant positive Hölder function and $\psi_t : T^1\widetilde{M} \circlearrowleft$ is the reparametrization of the geodesic flow by F. We fix from now on the cocycle c_F associated to F.

Lemma 3.9 (Ledrappier[15] page 106)**.** *If there exists h such that $P(-hF) = 0$ then h is c_F's exponential growth rate. Conversely, if the exponential growth rate h of c_F is finite and positive then $P(-hF) = 0$.*

Corollary 3.10. *The topological entropy of the flow ψ_t is the exponential growth rate of the cocycle c_F.*

Proof. Let h be c_F's exponential growth rate. Then Ledrappier's lemma 3.9 implies $P(-hF) = 0$. Lemma 2.4 states that this condition determines ψ_t's topological entropy h_{top}, and thus $h = h_{\text{top}}$. $\square$

We now give a precise description of the measure induced on $\partial^2\Gamma$ by the equilibrium state of $-hF$. Denote $a : T^1M \to T^1M$ the antipodal map. The periods of the function $\overline{F} : (p, w) \mapsto F(a(p, w))$ are the numbers $l(\gamma^{-1})$. We shall then call $\overline{c}_F$ the cocyle associated to $\overline{F}$.

Lemma 3.11. *The cocycles c_F and $\overline{c}_F$ have the same exponential growth rate.*

Proof. The function $\gamma \mapsto \gamma^{-1}$ induces a bijection between the sets $\{\gamma \in \Gamma : l(\gamma) \leq t\}$ and $\{\gamma \in \Gamma : l(\gamma^{-1}) \leq t\}$. $\square$

Denote μ_F and $\overline{\mu}_F$ as the quasi-invariant measures whose cocycles are c_F and $\overline{c}_F$ respectively. The second item of the theorem is then deduced from the following proposition of Schapira[22] (proposition 2.4).

Proposition 3.12. *Identify $T^1\widetilde{M}$ with $\partial^2\widetilde{M} \times \mathbb{R}$ via the Hopf parametrization. Let*

$$B^F, B^{\overline{F}} : \partial\widetilde{M} \times \widetilde{M} \times \widetilde{M} \to \mathbb{R}$$

be the geometric cocycles defined by (4) for the functions F and $\overline{F}$ respectively. Then the measure

$$m_F := e^{-h(B_w^{\overline{F}}(o,u)+B_z^F(o,u))} d\overline{\mu}_F(w) d\mu_F(z) ds$$

(where u is any point on the geodesic that joins z and w) induces in the quotient $\Gamma\backslash T^1\widetilde{M}$ the Gibbs state of $-hF$.

In order to finish the proof of the second item of theorem 3.2 we remark that, as observed in section §2 (proposition 2.7), ψ_t has a unique probability of maximal entropy ν. After lemma 2.4 ν has the same zero sets as the equilibrium state of $-hF$, and thus, after the last proposition the lift of ν to $\partial^2\Gamma \times \mathbb{R}$ has the same zero sets as $\overline{\mu} \otimes \mu \otimes ds$.

Conversely, a ψ_t-invariant measure with the same zero sets as ν is, since ν is ergodic, a multiple of ν. This finishes the proof of the theorem.

4 The proof of propositions C1 and C2

Recall we have a ρ-equivariant Hölder map $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and we have defined the cocycle

$$\beta(\gamma, x) = \log \frac{\|\rho(\gamma)v\|}{\|v\|}$$

for $v \in \xi(x) - \{0\}$ and a fixed euclidean norm on $\mathbb{R}^d$.

In order to apply theorem 3.2 we need to prove that the our cocycle β is of positive periods and of finite exponential growth rate. We shall first show that the period $\beta(\gamma, \gamma_+)$ is exactly $\lambda_1(\rho(\gamma))$, the log of the spectral radius of some lift of $\rho(\gamma)$ with determinant $\in \{-1, 1\}$.

We say that $g \in \mathrm{PGL}(d, \mathbb{R})$ is *proximal* if it has a unique complex eigenvalue of maximal modulus, and its generalized eigenspace is one dimensional. This eigenvalue is necessarily real and its modulus is equal to $\exp \lambda_1(g)$. We will denote g_+ the g-fixed line of $\mathbb{R}^d$ consisting of eigenvectors of this eigenvalue and denote g_- the g-invariant complement of g_+ (this is $\mathbb{R}^d = g_+ \oplus g_-$). g_+ is an attractor on $\mathbb{P}(\mathbb{R}^d)$ for the action of g and g_- is a repelling hyperplane.

Lemma 4.1. *Let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a strictly convex representation. Then for every $\gamma \in \Gamma$ $\rho(\gamma)$ is proximal and $\xi(\gamma_+)$ is its attractive fixed line.*

Proof. Consider $\gamma_0 \in \Gamma$ and write, to simplify the notation, $a = \exp \lambda_1(\rho(\gamma_0))$. We consider a lift of $\rho(\gamma_0)$ to $\mathrm{SL}(d, \mathbb{R})_\pm$ which we still call $\rho(\gamma_0)$.

Let V_0 be the sum of all generalized $\rho(\gamma_0)$-eigenspaces of eigenvalues with modulus equal to a. We will show that $V_0 = \xi(\gamma_{0+})$. Set $V = V_0 \cap \eta(\gamma_{0-})$, and recall that $\mathbb{R}^d = \xi(\gamma_{0+}) \oplus \eta(\gamma_{0-})$.

Since V is a sum of generalized eigenspaces in $\eta(\gamma_{0-})$, it has a $\rho(\gamma_0)$-invariant complement $W \subset \eta(\gamma_{0-})$ and thus $\mathbb{R}^d = \xi(\gamma_{0+}) \oplus W \oplus V$.

We claim that $\xi(\gamma_{0+}) \oplus W$ contains a $\rho(\Gamma)$-invariant subspace. Since ρ is irreducible we obtain $V = \{0\}$ and $V_0 = \xi(\gamma_{0+})$, which implies the lemma. For this we will show that

$$\xi(\partial\Gamma) \subset \mathbb{P}(\xi(\gamma_{0+}) \oplus W).$$

Let $x \in \partial\Gamma - \{\gamma_{0-}\}$. Since $\gamma_0^n x \to \gamma_{0+}$ the same occurs via ξ, this is

$$\rho(\gamma_0{}^n)\xi(x) \to \xi(\gamma_{0+}) \tag{6}$$

in $\mathbb{P}(\mathbb{R}^d)$. Take some u_x in the line $\xi(x)$ and write, following the decomposition $\mathbb{R}^d = \xi(\gamma_{0+}) \oplus V \oplus W$,

$$u_x = u_+ + v + w$$

for some $u_+ \in \xi(\gamma_{0+})$, $v \in V$ and $w \in W$. We consider now the sequence

$$\frac{\rho(\gamma_0^n)u_x}{a^n} = \frac{\rho(\gamma_0^n)(u_+ + v + w)}{a^n}.$$

Since the spectral radius of $\rho(\gamma_0)|W$ is strictly smaller than a (by definition of V) we have

$$\rho(\gamma_0^n)w/a^n \to 0,$$

also, since u_+ is an eigenvector of $\rho(\gamma_0)$ we must have either $\rho(\gamma_0)u_+/a = \pm u_+$ or $\rho(\gamma_0^n)u_+/a^n \to 0$.

On the other hand, since $\rho(\gamma_0)|V$ consists of Jordan blocks of eigenvalue of modulus a we have $a^n \leq c\|\rho(\gamma_0^n)v\|$ for some $c > 0$ and all n sufficiently large. This implies that the sequence

$$\frac{\rho(\gamma_0^n)v}{a^n}$$

is far from zero (when $v \neq 0$).

Consequently: if $\rho(\gamma_0^n)u_+/a^n \to 0$ the limit line of $\rho(\gamma_0^n)\xi(x)$ is contained in $\mathbb{P}(V)$, this contradicts equation (6) and convexity of ρ. We then have that $\rho(\gamma_0)u_+/a = \pm u_+$ and, since $\rho(\gamma_0^n)v/a^n$ is far from zero, in order that (6) holds we must have $v = 0$. Thus $\xi(\partial\Gamma) \subset \mathbb{P}(\xi(\gamma_{0+}) \oplus W)$ which implies $V = 0$. This finishes the proof. □

We find then the following corollaries.

Corollary 4.2. *Let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a strictly convex representation, then the equivariant maps $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\eta : \partial\Gamma \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ are unique and for every $x \in \partial\Gamma$ one has $\xi(x) \subset \eta(x)$.*

Proof. The fact that $\xi(\gamma_+)$ is $\rho(\gamma)$'s attractive line and the fact that attractors $\{\gamma_+ : \gamma \in \Gamma\}$ form a dense subset of $\partial\Gamma$ prove uniqueness of ξ, and by analogue reasoning, uniqueness of η.

Since $\eta(\gamma_-)$ is the repeller hyperplane of $\rho(\gamma)$ and $\xi(\gamma_-)$ is $\rho(\gamma^{-1})$'s attractive line we must have $\xi(\gamma_-) \subset \eta(\gamma_-)$. Again, density of repellers implies that $\xi(x) \subset \eta(x)$ for every $x \in \partial\Gamma$. □

Corollary 4.3. *Let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a strictly convex representation, then the period $\beta(\gamma, \gamma_+)$ is $\lambda_1(\rho(\gamma))$. Moreover $\lambda_1(\rho(\gamma)) > 0$ for every $\gamma \in \Gamma$.*

Proof. After lemma 4.1 we have $\xi(\gamma_+)$ is the fixed attractive line of $\rho(\gamma)$. We then have

$$\beta(\gamma, \gamma_+) = \log \frac{\|\rho(\gamma)u_+\|}{\|u_+\|} = \lambda_1(\rho(\gamma))$$

were $u_+ \in \xi(\gamma_+)$.

The fact the the periods are positive is also consequence of the fact that $\rho(\gamma)$ is proximal. If $\lambda_1(\rho(\gamma)) = 0$ then considering some lift of $\rho(\gamma)$ with determinant in $\{-1, 1\}$ one sees that every eigenvalue of this lift would be of modulus 1 and thus $\rho(\gamma)$ would not be proximal. □

Since β is a cocycle with positive periods, corollary 3.6 implies that the exponential growth rate of β is positive. We must then prove the following lemma:

Lemma 4.4. *Let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a strictly convex representation. Then*

$$\limsup_{t\to\infty} \frac{1}{t} \log \#\{[\gamma] \in [\Gamma] : \lambda_1(\rho(\gamma)) \leq t\}$$

is finite.

The objective now is the proof of lemma 4.4. The following lemma is a general property of hyperbolic groups for which we refer the reader to Tukia[25]. For the second assertion of the lemma one can apply explicitly lemma 1.6 of Bowditch[5].

Lemma 4.5. *Let Γ be a hyperbolic group and let $\{\gamma_n\}$ be a sequence in Γ going to infinity, then there exists a subsequence $\{\gamma_{n_k}\}$ and two points $x_0, y_0 \in \partial\Gamma$ (not necessarily distinct) such that $\gamma_{n_k} x \to x_0$ uniformly on compact sets of $\partial\Gamma - \{y_0\}$. Moreover one can assume that $\gamma_{n_k+} \to x_0$ and $\gamma_{n_k-} \to y_0$.*

In order to prove lemma 4.4 we need some quantified version of proximality. Recall that Gromov's product $\mathscr{G} : \mathbb{P}(\mathbb{R}^{d^*}) \times \mathbb{P}(\mathbb{R}^d) - \Delta \to \mathbb{R}$ is defined as

$$\mathscr{G}(\theta, v) = \log \frac{|\theta(v)|}{\|\theta\|\|v\|},$$

where $\Delta = \{(\theta, v) : \theta(v) = 0\}$, and $[x, y] = \mathscr{G}(\eta(x), \xi(y))$ for $x, y \in \partial\Gamma$ distinct.

We say that a linear transformation g is (r, ε)-proximal for some $r \in \mathbb{R}_+$ and $\varepsilon > 0$ if it is proximal,

$$\exp \mathscr{G}(g_-, g_+) > r,$$

and the complement of an ε-neighborhood of g_- is sent by g to an ε-neighborhood of g_+. The following lemmas (4.6 and 4.7) will also be used in the proof of theorem A.

Lemma 4.6 (Benoist[2])**.** *Let r and δ be positive numbers. Then there exists ε such that for every (r, ε)-proximal transformation g one has*

$$|\log \|g\| - \lambda_1(g) + \mathscr{G}(g_-, g_+)| < \delta.$$

Proof. Consider the compact sets

$$P_{r,\varepsilon} = \{(r, \varepsilon)\text{-proximal linear transformations with norm } 1\}.$$

For a fixed r consider $P_r = \bigcap_\varepsilon P_{r,\varepsilon}$. An element $T \in P_r$ is a rank one operator with the constraint $\|T\| = 1$ and such that $\operatorname{im} T \cap \ker T = \{0\}$. One explicitly writes

$$Tw = \frac{\theta(w)}{\|\theta\|\|v\|} v$$

where $v \in \mathbb{R}^d$ and $\theta \in \mathbb{R}^{d^*}$ are such that $\theta(v) \neq 0$. It is easy to verify that the above formula for T gives a rank one operator with norm equal to 1.

One finishes with the remark that the function $g \mapsto \lambda_1(g)$ is continuous and $\lambda_1(T) = \mathscr{G}(\theta, v)$ for $T \in P_r$. □

Lemma 4.7. *Let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a strictly convex representation. Fix $r \in \mathbb{R}_+$ and $\varepsilon > 0$. Then the set*

$$\{\gamma \in \Gamma : \exp([\gamma_-, \gamma_+]) > r \textit{ and } \rho(\gamma) \textit{ is not } (r, \varepsilon)\textit{-proximal}\}$$

is finite.

Proof. Let $\gamma_n \to \infty$ be a sequence in Γ such that $\exp[\gamma_{n-}, \gamma_{n+}] > r$. Since ξ and η are uniformly continuous we have that $d_o(\gamma_{n-}, \gamma_{n+}) > \kappa$ for some $\kappa > 0$ and some Gromov distance d_o in $\partial\Gamma$. By applying lemma 4.5 we find a subsequence (still called γ_n) and two points x_0, y_0 such that γ_{n-} and γ_{n+} converge to y_0 and x_0 respectively, and such that $\gamma_n x \to x_0$ for every $x \neq y_0$.

We have $x_0 \neq y_0$ since $d_o(\gamma_{n-}, \gamma_{n+}) > \kappa$.

By considering again a subsequence we assume that

$$\frac{\rho(\gamma_n)}{\|\rho(\gamma_n)\|} \to T$$

for some linear transformation T of $\mathbb{R}^d$. We will prove that T is a proximal rank one operator, which implies that for sufficiently large n $\rho(\gamma_n)$ is (r, ε)-proximal.

Since $\xi(\gamma_{n+})$ is $\rho(\gamma_n)$-invariant for all n we have that $\xi(x_0)$ is T-invariant and by analogue reasoning we also have that $\eta(y_0)$ is T-invariant (recall we also have $\mathbb{R}^d = \xi(x_0) \oplus \eta(y_0)$ since ρ is strictly convex and $x_0 \neq y_0$).

Consider now a point $x \in \partial\Gamma - \{y_0\}$ and u_x a vector in the line $\xi(x)$. Write $u_x = u + v$ for some $u \in \xi(x_0)$ and $v \in \eta(y_0)$. Since $\rho(\gamma_n)\xi(x) \to \xi(x_0)$ we must have $Tu_x \in \xi(x_0)$ and thus $Tv = 0$ (since $\eta(y_0)$ is T-invariant).

Consequently $\xi(\partial\Gamma) \subset \mathbb{P}(\xi(x_0) + \ker T)$. Irreducibility of ρ implies $\mathbb{R}^d = \xi(x_0) + \ker T$. In order to finish we remark that since $\|T\| = 1$ we must have $T|\xi(x_0) \neq 0$ and thus $T \neq 0$. We have then a rank one operator whose image is not contained in its kernel. □

The following lemma states that strictly convex representations are discrete and, using the fact that the fundamental group of a negatively curved manifold is torsion free, they are also inyective.

Lemma 4.8. *Let Γ be a non elementary hyperbolic group and $\rho_0 : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a non trivial representation such that there exists a ρ_0-equivariant continuous map $\xi_0 : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$. Then ρ_0 is discrete with finite kernel. In particular strictly convex representations are discrete and inyective.*

Proof. Assume there exists a divergent sequence $\gamma_n \to \infty$ in Γ such that $\rho_0(\gamma_n)$ converges to $g \in \mathrm{PGL}(d, \mathbb{R})$. Consider a subsequence (which we still call γ_n) and the points $x_0, y_0 \in \partial\Gamma$ given by lemma 4.5. We then have that for any $x \in \partial\Gamma$ different from y_0 one has $\gamma_n x \to x_0$.

Since ξ_0 is ρ_0-equivariant we have that $\rho_0(\gamma_n)\xi_0(x) \to \xi_0(x_0)$ and thus

$$g\xi_0(x) = \xi_0(x_0)$$

for every $x \neq y_0$. Since ρ_0 is non trivial ξ_0 is non constant and thus g is not injective. This is a contradiction.

We proved that ρ_0 is proper and thus has finite kernel. This finishes the proof of the lemma. □

We can now prove that the exponential growth rate of the cocycle β is finite.

Proof of lemma 4.4. Since the action of Γ on $T^1\widetilde{M}$ is co-compact, one has a compact fundamental domain D. Since a conjugacy class $[\gamma] \in [\Gamma]$ is identified with a closed geodesic, one can always find a representative $\gamma_0 \in [\gamma]$ such that the γ_0-invariant geodesic on $T^1\widetilde{M}$ intersects D. The fact that the fundamental domain is compact implies that γ_0's fixed points on $\partial\Gamma$ are necessarily far away by some constant independent of $[\gamma]$.

In other words, there exists some constant $k > 0$ such that every conjugacy class of $[\gamma]$ has a representative γ_0 with $d_o(\gamma_{0-}, \gamma_{0+}) > k$ for some Gromov distance d_o on $\partial\Gamma$.

Since the equivariant maps ξ and η are uniformly continuous one has that every conjugacy class $[\gamma]$ has a representative γ_0 such that $\exp[\gamma_{0-}, \gamma_{0+}] > r$ for some r independent of $[\gamma]$.

We shall fix some number $\delta > 0$ from now on and consider $\varepsilon > 0$ given by lemma 4.6. Thus, applying lemma 4.7 all γ's with $\exp[\gamma_-, \gamma_+] > r$ (but a finite number depending only on r and ε) are (r, ε)-proximal and thus verify, after Benoist's lemma 4.6,

$$\log \|\rho(\gamma)\| + \log r - \delta \leq \lambda_1(\rho(\gamma)).$$

One concludes, by choosing for each conjugacy class $[\gamma]$ a representative γ_0 with $\exp[\gamma_{0-}, \gamma_{0+}] > r$, that $\#\{[\gamma] \in [\Gamma] : \lambda_1(\rho(\gamma)) \leq t\} \leq$

$$\begin{aligned} &\#\{\gamma : [\gamma_-, \gamma_+] > \log r \text{ and } \log \|\rho(\gamma)\| \leq t + \delta - \log r\} \\ &\qquad + \#\{\text{finite set independent of } t\} \\ &\leq \#\{\gamma \in \Gamma : \log \|\rho(\gamma)\| \leq t + \delta - \log r\} + \#\{\text{finite set independent of } t\}. \end{aligned}$$

Since the cardinal of the finite set is neglectable when computing the exponential growth rate, one is led to study the exponential growth rate of the quantity $\#\{\gamma \in \Gamma : \log \|\rho(\gamma)\| \leq t\}$ when $t \to \infty$.

Lemma 4.8 states that $\rho(\Gamma)$ is discrete and injective and thus the fact that the exponential growth rate of $\#\{\gamma \in \Gamma : \log \|\rho(\gamma)\| \leq t\}$, is finite when $t \to \infty$, is implied by the following general fact. □

Lemma 4.9. *Let G be a discrete subgroup of* $\mathrm{PGL}(d, \mathbb{R})$*, then*

$$\limsup_{t \to \infty} \frac{\log \#\{g \in G : \log \|g\| \leq t\}}{t} < \infty.$$

Proof. This is consequence of the following estimation of the Haar measure of $\mathrm{PGL}(d, \mathbb{R})$ which can be found in Helgason[12]:

$$\limsup_{R \to \infty} \frac{\log \mathrm{Haar}\{g \in \mathrm{PGL}(d, \mathbb{R}) : \|g\| \leq R\}}{\log R} < \infty.$$

□

Proof of propositions C1 and C2

Recall we have two ρ-equivariant Hölder maps $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\eta : \partial\Gamma \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ such that $\xi(x) \notin \eta(y)$ if $x \neq y$. We have defined the cocycles

$$\beta(\gamma, x) = \log \frac{\|\rho(\gamma)v\|}{\|v\|}$$

for any $v \in \xi(x) - \{0\}$ and

$$\overline{\beta}(\gamma, x) = \log \frac{\|\rho(\gamma)\theta\|}{\|\theta\|}$$

for any $\theta \in \mathbb{R}^{d^*}$ such that $\ker\theta = \eta(x)$.

Lemma 4.4 then implies that the cocycle β is of finite exponential growth rate and proposition C1 is immediatly deduced from theorem 3.2. That is the translation flow $\psi_t : \Gamma\backslash\partial^2\Gamma \times \mathbb{R} \circlearrowleft$ is Hölder conjugated to a Hölder reparametrization of the geodesic flow on $\Gamma\backslash T^1\widetilde{M}$.

In order to prove proposition C2 we need to verify certain things.

Lemma 4.10. *Let $g \in \mathrm{GL}(d, \mathbb{R})$ be proximal with maximal eigenvalue a, and let $\theta \in \mathbb{R}^{d^*}$ such that $\ker\theta = g_-$, then $g\theta = a^{-1}\theta$.*

Proof. Since $\ker\theta = g_-$ one has $g\theta = b\theta$ for some real b. Consider now some $u_+ \in g_+$. One has

$$b\theta(u_+) = g\theta(u_+) = \frac{1}{a}\theta(u_+)$$

and, since $\theta(u_+) \neq 0$, we have $b = a^{-1}$ □

One trivially deduces the following lemma.

Lemma 4.11. *The cocycles β and $\overline{\beta}$ verify $\ell(\gamma) = \overline{\ell}(\gamma^{-1})$ for all $\gamma \in \Gamma$.*

Lemma 4.12. *The measure $m_\rho := e^{-h[\cdot,\cdot]}\overline{\mu} \otimes \mu \otimes ds$ is Γ-invariant and ψ_t-invariant on $\partial^2\Gamma \times \mathbb{R}$, where $\overline{\mu}$ is the quasi-invariant probability of cocycle $h\overline{\beta}$ and μ is that of $h\beta$.*

Proof. One easily verifies that for every $g \in \mathrm{PGL}(d, \mathbb{R})$ one has

$$\mathscr{G}(g\theta, gv) - \mathscr{G}(\theta, v) = -(\log \frac{\|g\theta\|}{\|\theta\|} + \log \frac{\|gv\|}{\|v\|}),$$

(recall that the action of $\mathrm{PGL}(d, \mathbb{R})$ on $\mathbb{P}(\mathbb{R}^{d^*})$ coherent with the identification $\mathbb{P}(\mathbb{R}^{d^*}) \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ is $\theta \mapsto \theta \circ g^{-1}$) this means exactly that for every $\gamma \in \Gamma$ one has

$$[\gamma x, \gamma y] - [x, y] = -(\overline{\beta}(\gamma, x) + \beta(\gamma, y)).$$

The quasi-invariance of the measures $\overline{\mu}$ and μ, and the fact that the exponential growth rate is h for both β and $\overline{\beta}$ (lemma 3.11), imply that the measure $e^{-h[\cdot,\cdot]}\overline{\mu} \otimes \mu$ on $\partial^2\Gamma$ is Γ-invariant and thus m_ρ is Γ-invariant.

Since the flow ψ_t acts by translations we have that m_ρ is also ψ_t-invariant and the lemma is proved. □

After theorem 3.2 the measure of maximal entropy of ψ_t is the unique ψ_t-invariant probability with the same zero sets as $\overline{\mu}\otimes\mu\otimes ds$. The measure stated in proposition C2 verifies this condition and the proposition is proved.

5 The distribution of fixed points: theorems A and B

Counting the growth of the spectral radii

We prove now theorem B.

Recall that $\ell(\gamma):=\beta(\gamma,\gamma_+)=\lambda_1(\rho(\gamma))$. After proposition C1 the translation flow $\psi_t:\Gamma\backslash\partial^2\Gamma\times\mathbb{R}\circlearrowleft$ is well defined and is a reparametrization of the geodesic flow. If τ is a periodic orbit of ψ_t, then any lift to $\partial^2\Gamma\times\mathbb{R}$ is of the form (γ_-,γ_+,s) for some primitive $\gamma\in\Gamma$ and $s\in\mathbb{R}$. One checks that

$$\gamma(\gamma_-,\gamma_+,s)=(\gamma_-,\gamma_+,s-\ell(\gamma))$$

which implies that the period $p(\tau)$ of τ is $\ell(\gamma)$. One then has

$$\#\{\gamma\in[\Gamma]\text{ primitive}:\ell(\gamma)\leq t\}=\#\{\tau\text{ periodic}:p(\tau)\leq t\}.$$

We are led to count the number of periodic orbits of period $\leq t$.

Applying corollary 2.10 we have a weak mixing Markov coding (Σ,π,f) associated to ψ_t. Recall that ψ_t's topological entropy coincides with the topological entropy of σ_t^f. One finishes applying the following theorem of Parry-Pollicott[18] (see also [19]). This completes the proof of theorem B.

Theorem 5.1 (Prime Orbit Theorem[18]). *Let Σ be a sub-shift of finite type and let $f:\Sigma\to\mathbb{R}_+^*$ be Hölder continuous. Suppose that the suspension flow σ_t^f is weak mixing, and set $p(\tau)$ the period of a σ_t^f periodic orbit, then*

$$hte^{-ht}\#\{\tau\textit{ periodic}:p(\tau)\leq t\}\to 1$$

when $t\to\infty$, where h is the topological entropy of the suspension flow σ_t^f.

The proof of theorem A

In order to prove the asymptotic growth of the norm we use the following distribution theorem of Bowen:

Theorem 5.2 (Bowen[6, 7]). *Let Σ be a sub-shift of finite type and $f:\Sigma\to\mathbb{R}_+^*$ be Hölder continuous. Then*

$$\#\{\tau\ \sigma_t^f\textit{-periodic}:p(\tau)\leq t\}^{-1}\sum_{\tau:p(\tau)\leq t}\frac{1}{p(\tau)}\operatorname{Leb}_\tau$$

converges to the probability of maximal entropy of σ_t^f, where Leb_τ is the Lebesgue measure on τ of length $p(\tau)$.

We shall write $\|m_\rho\|$ for the total mass of the measure $m_\rho = e^{-h[\cdot,\cdot]}\overline{\mu}\otimes\mu\otimes ds$ on the compact quotient $\Gamma\backslash\partial^2\Gamma\times\mathbb{R}$.

Proposition 5.3. *Let*

$$\nu_t = \|m_\rho\| h e^{-ht} \sum_{\gamma\in\Gamma:\ell(\gamma)\leq t} \delta_{\gamma_-}\otimes\delta_{\gamma_+}$$

then $\nu_t \to e^{-h[\cdot,\cdot]}\overline{\mu}\otimes\mu$ *in* $C_c^*(\partial^2\Gamma)$ *when* $t\to\infty$*, where* $C_c(\partial^2\Gamma)$ *is the space of real continuous functions with compact support.*

Proof. As observed in section §2 we have a weak mixing Markov coding for the flow $\psi_t : \Gamma\backslash\partial^2\Gamma\times\mathbb{R} \circlearrowleft$ (corollary 2.10). Applying Parry-Pollicott's Prime Orbit theorem 5.1 together with Bowen's result we find the convergence of

$$hte^{-ht}\sum_{\tau:p(\tau)\leq t}\frac{1}{p(\tau)}\,\mathrm{Leb}_\tau$$

to the probability of maximal entropy of ψ_t on $\Gamma\backslash\partial^2\Gamma\times\mathbb{R}$, when $t\to\infty$. After proposition C2 this measure is lifted to $\partial^2\Gamma\times\mathbb{R}$ as

$$\frac{e^{-h[\cdot,\cdot]}\overline{\mu}\otimes\mu\otimes ds}{\|m_\rho\|}.$$

Since periodic orbits of ψ_t are of the form (γ_-,γ_+,s) for some $\gamma\in\Gamma$ primitive, and the period of such orbit is $\ell(\gamma)$ we have the convergence

$$hte^{-ht}\sum_{\gamma\text{ primitive: }\ell(\gamma)\leq t}\frac{1}{\ell(\gamma)}\delta_{\gamma_-}\otimes\delta_{\gamma_+}\otimes ds \to \frac{e^{-h[\cdot,\cdot]}\overline{\mu}\otimes\mu\otimes ds}{\|m_\rho\|}$$

in $\partial^2\Gamma\times\mathbb{R}$.

We can delete de $\mathbb{R}$-component by comparing the measures of sets of the form $A\times B\times I$ for some interval I, and we find:

$$\sigma_t := \|m_\rho\| hte^{-ht}\sum_{\gamma\text{ primitive: }\ell(\gamma)\leq t}\frac{1}{\ell(\gamma)}\delta_{\gamma_-}\otimes\delta_{\gamma_+}\to e^{-h[\cdot,\cdot]}\overline{\mu}\otimes\mu$$

when $t\to\infty$.

In order to finish the proof of the proposition we shall delete the terms $t/\ell(\gamma)$ with the restriction "γ primitive". We will follow a method of Roblin([21], page 71)

The integer part of $t\ell(\gamma)^{-1}$ is the number of powers of γ such that $\ell(\gamma^n)\leq t$, this is

$$\left[\frac{t}{\ell(\gamma)}\right] = \#\{n\in\mathbb{N} : \ell(\gamma^n) = n\ell(\gamma)\leq t\}.$$

We then have

$$\nu_t = \|m_\rho\| h e^{-ht} \sum_{\gamma \text{ primitive: } \ell(\gamma)\le t} \left[\frac{t}{\ell(\gamma)}\right] \delta_{\gamma_-} \otimes \delta_{\gamma_+}$$

and we find $\nu_t \le \sigma_t$.

For a complementary inequality. Fix some $\kappa > 0$. Now, if $e^{-\kappa} t < \ell(\gamma) \le t$ we have $[t/\ell(\gamma)] \ge e^{-\kappa} t/\ell(\gamma)$ and

$$\nu_t \ge \|m_\rho\| e^{-\kappa} h t e^{-ht} \sum_{\substack{\gamma \text{ primitive} \\ e^{-\kappa} t < \ell(\gamma) \le t}} \frac{1}{\ell(\gamma)} \delta_{\gamma_-} \otimes \delta_{\gamma_+}$$

$$= e^{-\kappa} \sigma_t - e^{-\kappa} h t e^{-ht} \sum_{\substack{\gamma \text{ primitive} \\ \ell(\gamma) \le e^{-\kappa} t}} \frac{1}{\ell(\gamma)} \delta_{\gamma_-} \otimes \delta_{\gamma_+}.$$

Since the second term goes to zero when $t \to \infty$ we find

$$\limsup \nu_t \ge e^{-k} \limsup \sigma_t.$$

Since κ is arbitrary, these two inequalities show the proposition. □

We can now prove theorem A.

Proof of theorem A. Choose some positive δ and let $A, B \subset \partial\Gamma$ be two disjoint open subsets small enough such that $[\cdot,\cdot] : A \times B \to \mathbb{R}$ is constant r modulo δ, this is $|[x,y] - r| \le \delta$ for every $(x,y) \in A \times B$.

Lemma 4.7 allows us to assume (excluding a finite set of Γ, that depends on r and δ) that if $\gamma_- \in A$ and $\gamma_+ \in B$ then $\rho(\gamma)$ is $(\exp r, \varepsilon)$-proximal, where ε comes from lemma 4.6 for $\exp r$ and δ.

We have then, after lemma 4.6, that $|\log\|\rho(\gamma)\| - \ell(\gamma) + r| \le 2\delta$. This is

$$\ell(\gamma) - r - 2\delta \le \log\|\rho(\gamma)\| \le \ell(\gamma) - r + 2\delta.$$

Set

$$\theta_t := \|m_\rho\| h e^{-ht} \sum_{\log\|\rho(\gamma)\| \le t} \delta_{\gamma_-} \otimes \delta_{\gamma_+}$$

The last inequalities imply that for all $t > 0$:

$$e^{-2\delta} e^{hr} \|m_\rho\| h e^{-h(t+r-2\delta)} \sum_{\ell(\gamma) \le t+r-2\delta} \delta_{\gamma_-}(A) \delta_{\gamma_+}(B) \le \theta_t(A \times B)$$

$$\le e^{2\delta} e^{hr} \|m_\rho\| h e^{-h(t+r+2\delta)} \sum_{\ell(\gamma) \le t+r+2\delta} \delta_{\gamma_-}(A) \delta_{\gamma_+}(B)$$

Applying proposition 5.3 we find when $t \to \infty$ that,

$$e^{-2\delta}e^{h(r-[\cdot,\cdot])}\overline{\mu}\otimes\mu(A\times B)\leq\liminf_{t\to\infty}\theta_t(A\times B)$$
$$\leq\limsup_{t\to\infty}\theta_t(A\times B)\leq e^{2\delta}e^{h(r-[\cdot,\cdot])}\overline{\mu}\otimes\mu(A\times B),$$

one has, since $|r-[x,y]|\leq\delta$ for every $(x,y)\in A\times B$, that

$$e^{-3\delta}\overline{\mu}(A)\mu(B)\leq\liminf_{t\to\infty}\theta_t(A\times B)\leq\limsup_{t\to\infty}\theta_t(A\times B)\leq e^{3\delta}\overline{\mu}(A)\mu(B).$$

Since δ is arbitrary this argument proves the convergence of $\theta_t\to\overline{\mu}\otimes\mu$ outside the diagonal, this is, subsets of $\partial\Gamma\times\partial\Gamma-\{(x,x):x\in\partial\Gamma\}$. In order to finish we will prove the following: Given ε_0 there exists an open covering $\mathcal{U}$ of $\partial\Gamma$ such that $\sum_{U\in\mathcal{U}}\theta_t(U\times U)\leq\varepsilon_0$ for all t large enough. The following argument was personally communicated by Thomas Roblin.

Since $\overline{\mu}$ and μ have no atoms and $\gamma_*\mu\ll\mu$ for every $\gamma\in\Gamma$, one has that the diagonal has measure zero for $\overline{\mu}\otimes\gamma_*\mu$ (for every $\gamma\in\Gamma$).

Fix two elements γ_0 and γ_1 in Γ and fix some $\varepsilon_0>0$. We can assume that γ_0 and γ_1 have no common fixed point in $\partial\Gamma$. Choose an open covering $\mathcal{U}$ of $\partial\Gamma$ such that for every $i=0,1$ one has

$$\sum_{U\in\mathcal{U}}\overline{\mu}(U)\times\mu(\gamma_i(U))<\varepsilon_0.$$

By refining $\mathcal{U}$ we can assume that for every $U\in\mathcal{U}$ there exists $i\in\{0,1\}$ such that $\gamma_i\overline{U}\cap\overline{U}=\emptyset$ where $\overline{U}$ is U's closure.

Since $\partial\Gamma$ is compact we may assume that the covering $\mathcal{U}$ is finite and thus, by enlarging the U's, we can consider a new covering $\mathcal{V}$ verifiyng the following:

1. for each $U\in\mathcal{U}$ there exists $V\in\mathcal{V}$ such that $\overline{U}\subset V$, and for each $V\in\mathcal{V}$ there exists a unique U verifying this condition.
2. If $\gamma_i\overline{U}\cap\overline{U}=\emptyset$ for some $i\in\{0,1\}$ then $\gamma_i\overline{V}\cap\overline{V}=\emptyset$ for the unique V such that $\overline{U}\subset V$,
3. $\sum_{V\in\mathcal{V}}\mu\otimes\gamma_{i*}\mu(V\times V)<\varepsilon_0$ for every $i\in\{0,1\}$.

Consider some $U\in\mathcal{U}$ and suppose that $\gamma_0\overline{U}\cap\overline{U}=\emptyset$. We study the set $\Gamma_U:=\{\gamma\in\Gamma:(\gamma_-,\gamma_+)\in U\times U\}$.

Lemma 5.4. *Consider $V\in\mathcal{V}$ such that $\overline{U}\subset V$. Except for a finite number of $\gamma\in\Gamma_U$, the repeller $(\gamma_0\gamma)_-$ of $\gamma_0\gamma$ belongs to V and the attractor $(\gamma_0\gamma)_+\in\gamma_0V$.*

Proof. Consider a sequence $\gamma_n\in\Gamma_U$ and the point x_0,y_0 given by lemma 4.5. Since $\gamma_{n-}\to y_0$ and $\gamma_{n+}\to x_0$ we have that x_0 and y_0 belong to $\overline{U}\subset V$, for a unique $V\in\mathcal{V}$. Thus, since $y_0\notin\gamma_0\overline{V}$, one has $\gamma_n(\gamma_0V)\to x_0$ uniformly.

This implies that the set

$$F_U=\{\gamma\in\Gamma_U:\gamma(\gamma_0V)\subsetneq V\}$$

is finite.

Consider now some $\gamma \in \Gamma_U - F_U$. The sequence $(\gamma_0\gamma)^n\gamma_+$ is contained in $\gamma_0 V$ and thus (since γ_+ is not the repeller of $\gamma_0\gamma$) the attractor of $\gamma_0\gamma$ also belongs to $\gamma_0 V$.

Analogue reasoning gives the remaining statement of the lemma. $\square$

After the lemma one has that $\theta_t(U \times U) \leq$

$$\|m_\rho\| h e^{-ht} \sum_{\gamma : \log\|\rho(\gamma)\| \leq \log\|\rho(\gamma_0)\| + t} \delta_{\gamma_-}(V) \otimes \delta_{\gamma_+}(\gamma_0 V)$$

$$+\|m_\rho\| h e^{-ht} \#\{\text{finite set independent of } t\},$$

where $V \in \mathcal{V}$ is such that $\overline{U} \subset V$. Since $V \times \gamma_0 V$ is far from the diagonal and the cardinal of the finite set does not depend on t, the right side of the formula converges to $\|\gamma_0\|\overline{\mu}(V) \times \mu(\gamma_0 V)$ when $t \to \infty$.

One then has, since V is unique for each given $U \in \mathcal{U}$, that $\sum_{U \in \mathcal{U}} \theta_t(U \times U) \leq \sum_{i \in \{0,1\}} \sum_{V \in \mathcal{V}} \|\gamma_i\| \overline{\mu}(V) \mu(\gamma_i V) \leq 2\varepsilon_0 \max\{\|\gamma_0\|, \|\gamma_1\|\}$. Since γ_0 and γ_1 are fixed and ε_0 is arbitrary small the theorem is proved. $\square$

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