SCALING RATE FOR SEMI-DISPERSING BILLIARDS WITH NON-COMPACT CUSPS

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Abstract. We show that certain billiard tables with non-compact cusps are mixing with respect to the invariant infinite measure, in the sense of Krengel and Sucheston. Moreover, we show that the scaling rate is slower than a certain polynomial rate.

1. Introduction

The *planar billiard* is the dynamical system defined by the free motion of a particle in the interior of a domain $\mathcal{D} \subset \mathbb{R}^2$ (usually called *table*) subjected to elastic collisions to the boundary of $\mathcal{D}$, that is, angle of incidence equals angle of reflection. This motion generates a flow, called the *billiard flow*, defined by the position of the particle on $\mathcal{D}$ and its direction of movement. The *billiard map* is obtained by taking the global cross-section for the flow defined by the bounces on the boundary of $\mathcal{D}$. It is usually defined by the position of the collision at the boundary of $\mathcal{D}$ (seen as an arc-length parametrized curve) and the angle of reflection measured from the velocity vector to the normal vector at the point of collision. It is well known that both the billiard flow and the billiard map have absolutely continuous invariant measures and are dynamical systems with singularities (as consequences of tangent trajectories and the existence of vertices on the boundary).

We are interested in studying mixing properties of the billiard map in tables of the form $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : x \geq 0, 0 \leq y \leq f(x)\}$, where $f : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a three times differentiable bounded convex function, satisfying the hypotheses (H1) to (H5) listed in Section 3. The billiard map for this kind of table has a σ-finite infinite invariant measure.

Billiard tables of this type, that have non-compact cusps, were studied by Lenci in [15]. He proved, as an extension for the infinite measure case of the results of Katok and Strelcyn [13], that the billiard map has an hyperbolic structure. Furthermore, adapting arguments contained in [18], Lenci proved that these billiards maps are ergodic.

Key words and phrases. billiards, scaling rate, infinite measure.

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These billiard tables have dispersing and neutral components of the boundary (given respectively by the curves $y = f(x)$ and $x = 0, \ \ y = 0$). They are the first well studied billiards with infinite measure.

In a seminal work, Sinai [22] proved that the billiard map of a system in a two-dimensional torus with finitely many convex (dispersing) obstacles is a K-automorphism and, thus, mixing with respect to its natural finite measure.

In [2], Bunimovich and Sinai proved a "stretched" exponential decay of correlations for dispersing billiards, that is, billiard tables with finitely many convex pieces, when seen from the interior. Young [25] showed that the decay of correlations is actually exponential. This later result was extended by Chernov [3] for plane billiards with dispersing boundaries and positive-angle corners. In [19], Markarian, based on [26], showed that billiards in the Bunimovich stadium has polynomial decay of correlations. More recently, Chernov and Markarian [6] proved that semi-dispersing billiard tables with compact cusps also have polynomial decay of correlations. Improved estimates for correlations for other types of billiards were also proved by Chernov and Zhang [7].

There is no consensual definition of mixing for systems with infinite measure. Following Krengel and Sucheston [14], we say that a conservative automorphism $\mathcal{F}$ on a σ-finite infinite measure space $(X, \mathcal{B}, \mu)$ is *mixing* if for all measurable set A with $\mu(A) < \infty$,

$$\mu(\mathcal{F}^n A \cap A) \to 0 \quad \text{as} \quad n \to \infty. \tag{1.1}$$

Interesting discussions on different definitions of mixing for systems with infinite measure were recently done in [17] and [10].

Now, let us return to our billiard table $\mathcal{D}$ (see Figure 1). We consider the billiard map $\mathcal{F} : M \to M$ defined by the bounces on the leftmost wall $\mathcal{L}$ (y-axis) and the dispersing part $\mathcal{U}$ of the billiard table. In fact this map represents the whole dynamics on $\mathcal{D}$ (see, for instance Chapter 1 in [5]). For precise definitions of $\mathcal{F}$ and M, see Section 3.

The following are the main results of this paper.

Theorem 1.1. *The billiard map $\mathcal{F}$ is mixing.*

Furthermore we can study the speed of convergence to zero in (1.1). We call it the *scaling rate* (Cf. [11], [12]). The next result shows that the scaling rate is not exponential

Theorem 1.2. *Consider the billiard table $\mathcal{D}$ with a non-compact cusp defined using $f(x) = (x+1)^{-1}$. The billiard map $\mathcal{F}$ is mixing and there exist sets A bounded by singularity lines of the billiard map and a constant $C > 0$, such that for all large enough n, it holds*

$$\mu(\mathcal{F}^n A \cap A) \geq C \frac{1}{n^{7/3}}.$$

This paper is organized as follows. In Section 2 we recall some definitions and results on infinite ergodic theory; in Section 3 we describe the billiard system we are interested in. Section 4 contains the proof of Theorem 1.1. The last Section contains the proof of Theorem 1.2. Subsections 5.1 and 5.2 contain some technical lemmas (proved in the Appendices) that are needed for the proof of Theorem 1.2, which is done in Subsection 5.3.

2. Infinite Ergodic Theory

In this section we review some definitions and basic results on infinite ergodic theory. The main references are [1], [10] and [14].

Let $(X, \mathcal{B}, \mu)$ be a standard measure space equipped with a σ-finite measure. A measurable map $T : X \to X$ is *measure preserving* if $\mu(T^{-1}A) = \mu(A)$ for every $A \in \mathcal{B}$. Measure preserving transformations are called *endomorphisms* for short. An invertible endomorphism T on $(X, \mathcal{B}, \mu)$ is called an *automorphism* if T^{-1} is also an endomorphism on $(X, \mathcal{B}, \mu)$.

The map T is *conservative* if for all sets A of positive measure there exists an integer $n > 0$ such that $\mu(A \cap T^{-n}A) > 0$.

Given a sequence (A_n) of measurable sets, the intersection $\mathcal{R}(A_n)$ of the σ-algebras $\mathcal{B}_k(A_n)$ generated by $A_k, A_{k+1}, \ldots$ is called the *remote σ-algebra* of (A_n). The sequence (A_n) is called *remotely trivial* if $\mathcal{R}(A_n)$ is trivial, i.e., it contains only null sets and their complements. The sequence (A_n) is called *semiremotely trivial* if every subsequence contains a further subsequence which is remotely trivial.

The collection of sets of finite measure is denoted by $\mathfrak{F}$.

Proposition 2.1 ([14] Theorem 1.1)**.** *Assume $\mu(X) = \infty$. The following conditions are equivalent for a sequence of sets A_n such that $\mu(A_n) \le c$ for all n for some constant c:*

(i) *for every $F \in \mathfrak{F}$, $\mu(A_n \cap F) \to 0$ as $n \to \infty$;*
(ii) *for every integer k, $\lim_{n\to\infty} \mu(A_n \cap A_k) = 0$;*
(iii) *the sequence (A_n) is semiremotely trivial.*

Next proposition is a consequence of Proposition 2.1; see [14, Section 2].

Proposition 2.2. *The following properties are equivalent for an endomorphism T:*

(a) *For all $A \in \mathfrak{F}$, the sequence $(T^{-n}A)$ is semiremotely trivial.*
(b) *$f \circ T^n$ tends weakly* to zero in L^2, for every $f \in L^2(X, \mathcal{A}, \mu)$.*

Sucheston [23] proved that a measure-preserving transformation T in a finite measure space is mixing if, and only if, for every measurable set A, the sequence $(T^{-n}A)$ is semiremotely trivial. In view of Proposition 2.1, we present the following definition [14, p. 154]:

Definition 2.3. A non-singular transformation T is called *mixing* if for every $A \in \mathfrak{F}$,

$$\mu(T^{-n}A \cap A) \to 0, \quad \text{as } n \to \infty.$$

Remark 2.4. Note that this definition only makes sense for *conservative* systems. Indeed, even a simple non conservative dynamical system as the translation $T : \mathbb{R} \to \mathbb{R}$, $T(x) = x + 1$, is mixing under this definition (see [14, Theorem 2.1]).

Heuristically, using Proposition 2.2, mixing implies that (after many iterations) the union of the iterated sets of a set of finite measure A becomes a large part of the space. One way to see this is that, since it maintains its measure (by invariance), but starts to occupy less space on subsets of finite measure. This implies that the iterates must intersect "a lot of" subsets of finite measure, thus spreading itself under iterations. Moreover, under hypothesis of σ-finiteness of the measure, if we take a subset with infinite measure, a large subset of it, but with finite measure will start to spread out, implying the same effect on the original subset.

3. The dynamical system

We are interested in tables of the form $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ 0 \leq y \leq f(x)\}$, where $f : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a three times differentiable bounded convex function. Here $\mathbb{R}_0^+ = [0, +\infty)$ and $\mathbb{R}^+$ is its interior.

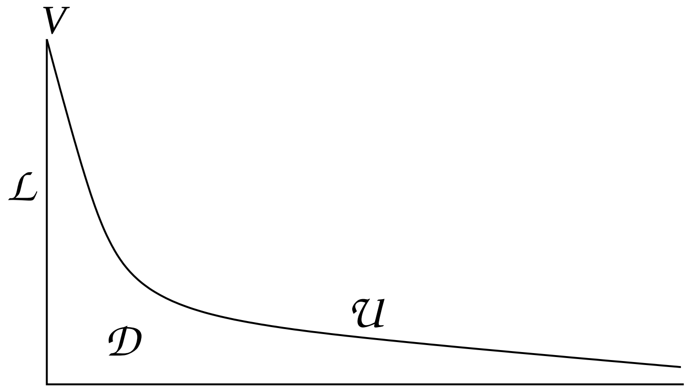

Figure 1: Introducing $\mathcal{D}$, $\mathcal{U}$, $\mathcal{L}$ and the vertex V.

We denote by $\mathcal{U}$ the dispersing part of the table $\mathcal{D}$ and by $\mathcal{L}$ the leftmost vertical wall in $\mathcal{D}$. The angle in the vertex $V = (0, f(0))$ is $\pi/2 + \arctan f'(0^+)$ and it can be zero. So the billiard table might have a compact cusp besides the non-compact one at $x = +\infty$.

Following [15], we introduce two other tables, that will be used in the definitions and many geometrical proofs below:

$$\mathcal{D}_2 = \{(x, y) \in \mathbb{R}_0^+ \times \mathbb{R} : |y| \leq f(x)\},$$

$$\mathcal{D}_4 = \{(x, y) \in I\!R \times I\!R : |y| \le f(|x|)\}.$$

For each x on $\mathcal{D}_2$, define $x_t = x_t(x)$ implicitly by

$$\frac{f(x) + f(x_t)}{x - x_t} = -f'(x_t).$$

One can see in Figure 2 that x_t is the x-coordinate of the tangency point on $\partial\mathcal{D}$ of the straight line passing through $(x, -f(x))$.

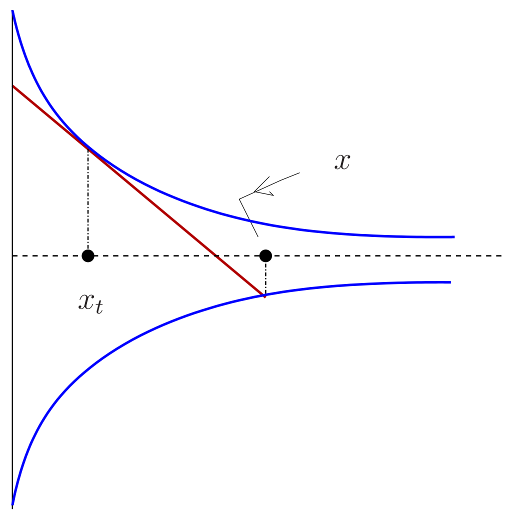

Figure 2: The point x_t in $\mathcal{D}_2$.

For $f, g : I\!R_0^+ \to I\!R^+$ we use the following notations: $f(x) << g(x)$ indicates that there exists a constant C such that $f(x) \le Cg(x)$, as $x \to \infty$ (analogously for the symbol $>>$).

In [15], Lenci studied tables with $f : I\!R_0^+ \to I\!R^+$ satisfying the following assumptions

(H1) $f''(x) \to 0$ as $x \to +\infty$;
(H2) $|f'(x_t)| << |f'(x)|$;
(H3) $\dfrac{f(x)f''(x)}{(f'(x))^2} >> 1$;
(H4) $\dfrac{|f'''(x)|}{f''(x)} << 1$;
(H5) $|f'(x)| >> (f(x))^\theta$, for some $\theta > 0$.

Observe that $f(x) = \dfrac{1}{x+1}$ satisfies the conditions above.

Following [15] we begin choosing the cross-section defined by the bounces against the dispersing part $\mathcal{U}$ of $\partial\mathcal{D}$. This choice corresponds to an infinite-measure cross-section. More precisely, we consider only unit vectors based in $\mathcal{U}$ and pointing towards the interior of $\mathcal{D}$. We parametrize these line elements as $z = (r, \varphi)$, where $r \in (-\infty, 0]$ is the arc length variable along $\mathcal{U}$ (with $r = 0$ for the vertex V) and $\varphi \in [-\pi/2, \pi/2]$ is the angle between the velocity vector and the normal at the point of collision, as at Figure 3.

The manifold over which Lenci define the dynamical system is $\mathcal{M} = (-\infty, 0) \times (-\pi/2, \pi/2)$. The billiard map defines on $\mathcal{M}$ a Poincaré return map $\mathcal{T}$ which preserves the measure $d\mu = \cos\varphi dr d\varphi$.

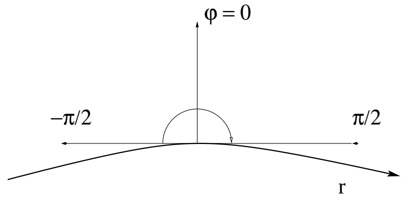

Figure 3: Orientation of r and φ

The map $\mathcal{T}$ is not defined on those points that hit tangentially $\mathcal{U}$ or that would end up in the vertex V. That is, the set $\mathcal{T}^{-1}\partial\mathcal{M}$ is excluded. These points make up the *singularity set* of $\mathcal{T}$, denoted by S: it consists of two curves (see [15, p.138]) $S^+ = S^{1+} \cup S^{2+}$ (as shown in Figure 5). The curve S^{1+} corresponds to tangencies on $\partial\mathcal{D}_4$, recall definition of $\mathcal{D}_4$ above, in the third quadrant (on $\mathcal{D}$, tangencies at $\mathcal{U}$, after a rebound on the vertical side); this curve is as regular as f. As for S^{2+}, its first part corresponds to line elements pointing to V (on $\mathcal{D}$, after a rebound on the horizontal side); as r decreases, these become tangencies at $\partial\mathcal{D}$. the boundary between these two behaviors is the only non-regular point of S^{2+}.

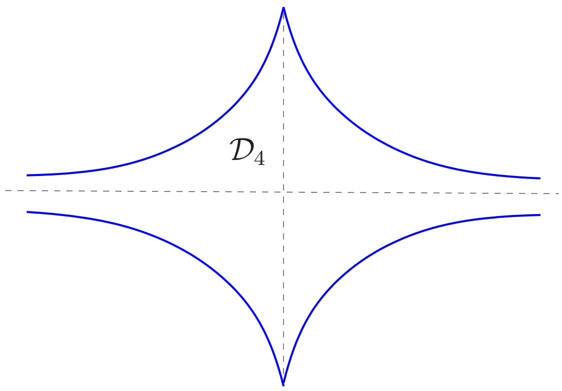

Figure 4: The table $\mathcal{D}_4$.

Analogously define $S^- = S^{1-} \cup S^{2-}$, where $S^{k-}, k = 1, 2$, are the singularity lines of $\mathcal{T}^{-1}$, obtained from S^{i+} using the time-reversal operator $(r, \varphi) \mapsto (r, -\varphi)$.

Denote by $S_n^{\pm} = \bigcup_{i=0}^{n} \mathcal{T}^{\mp i} S^{\pm}$ and $S_\infty^{\pm} = \bigcup_{i=0}^{\infty} \mathcal{T}^{\mp i} S^{\pm}$.

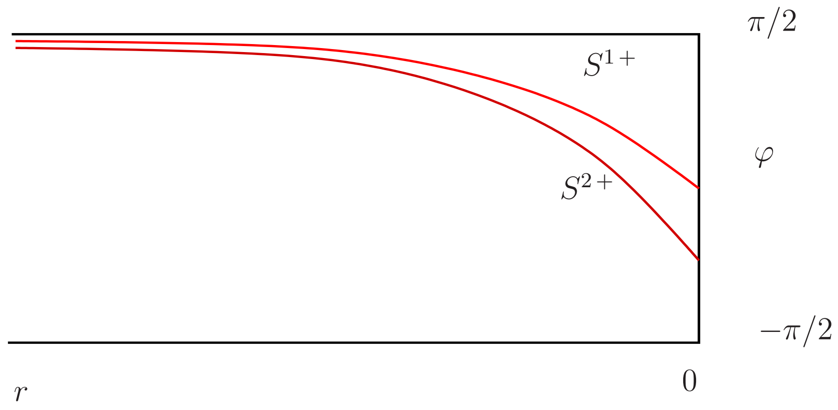

Figure 5: Singularity lines.

Lenci showed that the billiard map $\mathcal{T}$ defined above has a hyperbolic structure, i.e., there exist local stable and unstable manifolds at almost every point and these local foliations are absolutely continuous with respect to the invariant measure [15, Theorem 6.2,Theorem 7.5]. Adapting the formulation of Liverani and Wojtkowski [18] to the infinite measure case, he also proved a local ergodicity property [15, Theorem 8.5] and as a consequence, its global ergodicity [15, Theorem 8.5]. The definition of ergodicity used in these two results is the Boltzmann original formulation: the time average of every integrable function is constant almost everywhere. This is a rather weak notion of ergodicity, as it does not even prevent the existence of two complementary invariant subsets of infinite measure. However, with this definition, he also proved the ergodicity of the (finite-measure) Poincaré map corresponding to the returns onto the vertical side $\mathcal{L}$ of the boundary, which is a much more satisfactory result, and, in a certain way, validates this definition. More precisely, denote by $\mathcal{M}_3$ the region of $\mathcal{M}$ located above S^{2+}. Then $\mu(\mathcal{M}_3) < \infty$; see [15, Remark 3.4]. From the definition of S^{2+}, the line elements of $\mathcal{M}_3$ are precisely the ones that, on $\mathcal{D}_2$, hits the y-axis. Set T_3 for the return map onto $\mathcal{M}_3$. [15, Proposition 8.11] establishes that $(\mathcal{M}_3, T_3, \mu)$ is ergodic. This implies that the billiard map $\mathcal{T}$ is ergodic, in the sense that invariant sets are measurably indecomposable.

We point out that the bouncing at the dispersing part $\mathcal{U}$ contains most of the chaotic behavior of the billiard map $\mathcal{T}$. However, to achieve our results, we consider as well the bouncing at the vertical wall $\mathcal{L}$.

Thus, the manifold M over which we define our dynamical system is the phase space defined by vectors based on $\mathcal{L}$ and $\mathcal{U}$. Let $\mathcal{F} : M \to M$ be the return map to M.

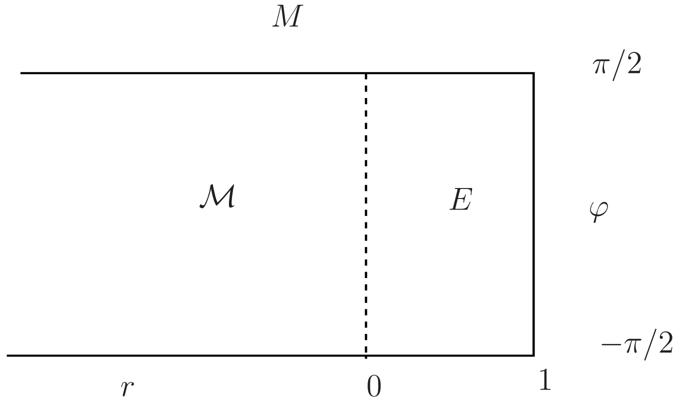

Figure 6: The phase space M. The line $r = 0$, corresponds to the vertex V and $r = 1$ corresponds to the vertex in the point $(0, 0)$ of $\mathcal{D}$.

Now, let $E \subset M$ be the subset of M constituted by vectors based on $\mathcal{L}$ (in our coordinate system, this definition corresponds to elements $z = (r, \varphi)$, with $0 \leq r \leq 1$). Then $\mu(E) < \infty$. Let $\mathcal{F}_E : E \mapsto E$ be the induced map of $\mathcal{F}$ on E. One can see that $(\mathcal{M}_3, T_3, \mu_{\mathcal{M}_3})$ is isomorphic (with respect to μ) to $(E, \mathcal{F}_E, \mu_E)$; [15, Remark 3.4, p. 140]. Hence $\mathcal{F}_E$ is ergodic.

Remark 3.1. Arguing as in [15, Corollary 3.3] we conclude that $\mathcal{F}$ is conservative. Moreover, since the induced map $\mathcal{F}_E : E \to E$ is ergodic and $\cup_{n=1}^{\infty} \mathcal{F}^{-n} E = M$ except for a μ-zero set, we conclude that $\mathcal{F}$ is also ergodic.

The map $\mathcal{F} : M \to M$ defined above describes all the dynamics of the billiard system, and this is the map we shall deal with from now on.

4. Proof of Theorem 1.1

In this section we will present the proof of Theorem 1.1. It is based on the work of Coudene [9] adapted to our definition of mixing. In this section X is a metric space, $\mathcal{A}$ the Borel σ-algebra of X, μ an infinite σ-finite regular measure on X and $T : X \to X$ a μ-measure preserving invertible transformation.

Definition 4.1. ([9, Definition 1]) We define the *stable distribution* of $\mathcal{F}$ of a point $x \in X$ as

$$W^s(x) = \{y \in X : d(T^n(x), T^n(y)) \to 0 \text{ as } n \to \infty\}.$$

A measurable function $f : X \to I\!R$ is called W^s*-invariant* when there exists a set $\Omega \subset X$ with full measure such that for all $x, y \in \Omega$, $y \in W^s(x)$ implies $f(x) = f(y)$. We say that the stable distribution W^s is ergodic if every W^s-invariant function is constant μ-almost everywhere.

Since T is invertible, we define the *unstable distribution* $W^u(x)$ of a point x for T as the stable distribution for T^{-1}. In a similar way , we define a W^u-invariant function and an ergodic W^u distribution.

We remark that, since the measure is regular, Lipschitz functions with compact support are dense in $L^2_\mu(X)$ [21, p.69]. The following propositions are modifications of Theorem 2 and Theorem 3 of [9]. The main tools used in their proofs are Banach-Alaoglu's Theorem [8, p. 130] and Banach-Saks' Theorem [20, p. 80], which are true in infinite measure spaces.

Proposition 4.2. (Based on [9, Theorem 2]) *Let X be a metric space, μ a regular infinite σ-finite measure on X, $T : X \to X$ a μ-measure-preserving transformation and $f \in L^2_\mu(X)$. Then any weak limit of $f \circ T^n$ is W^s-invariant.*

Proof. Let g be a weak limit of $f \circ T^{n_i}$. First assume that f is Lipschitz. Banach-Saks' theorem guarantees that there exist subsequences m_l and n_{i_k} such that

$$\Psi_l(x) = \frac{1}{m_l} \sum_{k=1}^{m_l} f \circ T^{n_{i_k}} \overset{l\to\infty}{\longrightarrow} g \quad \mu - a.e.$$

If $y \in W^s(x)$, then

$$|\Psi_l(x) - \Psi_l(y)| \le \frac{1}{m_l} \sum_{k=1}^{m_l} |f \circ T^{n_{i_k}}(x) - f \circ T^{n_{i_k}}(y)| \overset{l\to\infty}{\to} 0.$$

So g is W^s-invariant.

Let $f \in L^2_\mu$. For all $\varepsilon > 0$, by the previous remark, there exists a Lipschitz function f_0 with compact support such that $\|f - f_0\|_2 < \varepsilon$. Passing to a subsequence, by Banach-Alaoglu's theorem, we can assume that $f_0 \circ T^{n_i}$ converges weakly to a function g_0 which is W^s-invariant. It follows that $(f - f_0) \circ T^{n_i} \longrightarrow g - g_0$ weakly, which implies that

$$\|g - g_0\|_2 \le \liminf \|(f - f_0) \circ T^{n_i}\|_2 \le \|f - f_0\|_2 < \varepsilon.$$

Thus there exists a sequence (g_n) of W^s-invariant functions that converges to g in the L^2_μ-norm and, passing to a subsequence, almost everywhere. Hence, for a set Ω with full measure, if $y, x \in \Omega$, $y \in W^s(x)$, we get that

$$g(y) = \lim g_n(y) = \lim g_n(x) = g(x).$$

This shows that g is W^s-invariant and proves the proposition. □

Using the proposition above, the next one is proved as in [9].

Proposition 4.3. (Based on [9, Theorem 3]) *Let X be a metric space, μ a regular infinite σ-finite measure on X, $T : X \to X$ an invertible μ-measure-preserving transformation and $f \in L^2_\mu(X)$. Then any weak limit of $f \circ T^{-n}$ is W^s-invariant and W^u-invariant.*

Proof of Theorem 1.1. Fix $f \in L^2_\mu(X)$. Let $\psi : X \to I\!R$ be a weak cluster point of $\{f \circ \mathcal{F}^{-n}\}$ in $L^2_\mu(X)$, by the previous proposition, we have that ψ is W^s-invariant and W^u-invariant. By the property of absolute continuity of the local stable and unstable manifolds [15, Theorem 7.5], W^s-invariance and W^u-invariance imply that this function must be constant almost everywhere in the ergodic component of $\mathcal{F}$. However, since $\mathcal{F}$ has only one ergodic component ψ is constant almost everywhere.

Since $\psi \in L^2_\mu(X)$ and the measure μ is infinite, this constant must be zero. Thus $f \circ \mathcal{F}^{-n}$ converges weakly to zero, and this implies that $\mathcal{F}$ is mixing. The proof of Theorem 1.1 is completed. □

The remaining of the paper is dedicated to the proof of Theorem 1.2.

5. Proof of Theorem 1.2

The next two sub-sections contain the auxiliary lemmas used in the proof. Most of the results in these sub-sections are technical and their proofs are done in Appendices A and B.

5.1. **Geometric conditions.** In this subsection and in the next one we analyze the trajectories in a table with a non compact cusp. The main point is to estimate all the geometric behavior of a trajectory that enters into the cusp and comes back. We adapt to our case the analysis in tables with finite cusp developed by Chernov and Markarian [6].

We are in the same setting as in the previous sections and use the geometrical and dynamical results in [15]. We adopt a new system of coordinates from now on. Let $x_n \in [0, \infty)$, $0 \leq n \leq N$, be the x-coordinate associated to the n-th rebound on $\mathcal{U}$, (where $x_0 = 0$, leaving $\mathcal{L}$), and $\gamma_n \in [0, \pi/2]$, $0 \leq n \leq N$, the positive angle between the trajectory and the tangent at the point of collision with coordinate x_n ($\gamma_0 = \pi/2 - |\varphi|$), see Figure 7.

Let us study the behavior of a trajectory that leaving $\mathcal{L}$ (with coordinates (r, φ) in E), enters in the cusp, and comes back after $N > N_0 >> 1$ rebounds. This condition is enough to prove all the estimates in this Section.

Define

$$x_{N_2} := \max\{x_n : n = 1, 2, \ldots, N\},$$

that is, x_{N_2} is the x-coordinate of the *most interior point inside the cusp.*

Let us split the trajectories going through the cusp into three pieces. For this, we choose $\bar{\gamma}$ sufficiently small, the exact value is not important, e.g., take $\bar{\gamma} = 10^{-10}$. This choice allows us to make estimates in three different regions, determined by :

$$\begin{aligned} N_1 &= \max\{n < N_2; \gamma_n \leq \bar{\gamma}\} \\ N_3 &= \min\{n > N_2; \gamma_n \leq \bar{\gamma}\}. \end{aligned}$$

We call the series of rebounds between 1 and N_1 the *entering period*, between N_1 and N_3 the *turning period* and between N_3 and N the *exiting period.* Furthermore,

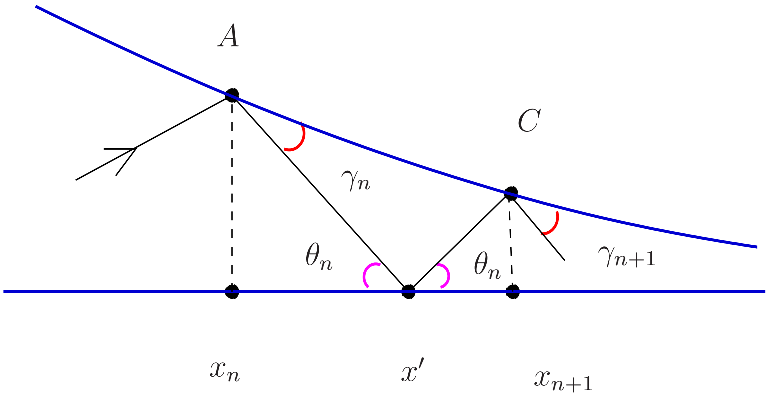

Figure 7: The rebounds on $\mathcal{U}$.

consider x_1 large enough, e.g. $x_1 > 10$. This condition does not change our estimates: it establishes a lower bound for N_0.

If $n \leq N_2 - 1$ then (see Figure 7)

$$\gamma_{n+1} = \gamma_n + \tan^{-1}|f'(x_n)| + \tan^{-1}|f'(x_{n+1})|, \tag{5.1}$$

$$x_{n+1} = x_n + \frac{f(x_n) + f(x_{n+1})}{\tan(\gamma_n + \tan^{-1}|f'(x_n)|)}. \tag{5.2}$$

If $n \geq N_2$ then

$$\gamma_n = \gamma_{n+1} + \tan^{-1}|f'(x_n)| + \tan^{-1}|f'(x_{n+1})|,$$

$$x_n = x_{n+1} + \frac{f(x_n) + f(x_{n+1})}{\tan(\gamma_{n+1} + \tan^{-1}|f'(x_{n+1})|)}.$$

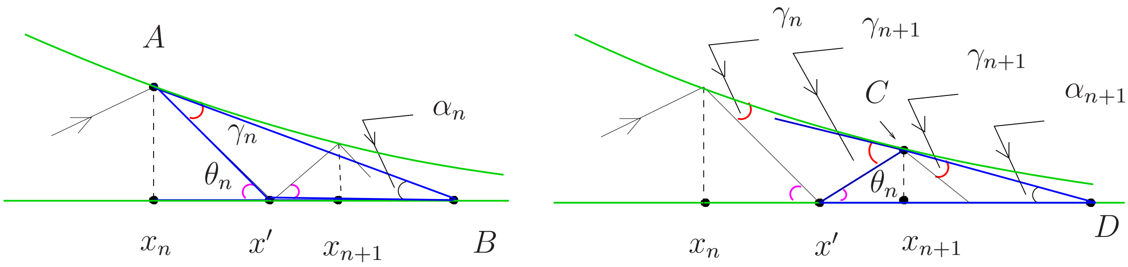

Figure 8: The auxiliary triangles $\widehat{Ax'B}$ and $\widehat{Cx'D}$.

Let us see that relations (5.2) and (5.1) hold.

For this, consider the triangles $\widehat{Ax_nx'}$ and $\widehat{Cx'x_{n+1}}$, as in Figure 7. We obtain $\tan\theta_n = \frac{f(x_n)}{x'-x_n}$ and $\tan\theta_n = \frac{f(x_{n+1})}{x_{n+1}-x'}$. Thus,

$$\tan\theta_n = \frac{f(x_n)+f(x_{n+1})}{x_{n+1}-x_n}. \tag{5.3}$$

Considering the triangles $\widehat{Ax'B}$ and $\widehat{Cx'D}$ as depicted in Figure 8, we obtain that $\theta_n = \gamma_n + \tan^{-1}|f'(x_n)|$, and comparing with (5.3), (5.2) follows.

Now, from the triangles $\widehat{Ax'B}$ and $\widehat{Cx'D}$ depicted in Figure 8, where we denote by $\alpha_i = \tan^{-1}|f'(x_i)|$, $i = n, n+1$, we get $\theta_n = \gamma_n + \alpha_n$ and $\gamma_{n+1} = \theta_n + \alpha_{n+1}$. Thus $\gamma_{n+1} = \gamma_n + \alpha_n + \alpha_{n+1}$, and (5.1) follows.

Convention: We use the following notation: $A \asymp B$ means that $C^{-1} < A/B < C$, for some constant $C = C(\mathcal{D}) > 0$. Also, $A = \mathcal{O}(B)$ means that $|A|/B < C$, for some constant $C = C(\mathcal{D}) > 0$. In both cases the constant C is independent of the values of A and B; it only depends on the billiard table $\mathcal{D}$.

Lemma 5.1. *Using the notation above,* $\left|N_2 - \dfrac{N}{2}\right| = \mathcal{O}(1)$.

Proof. Suppose that, without loss of generality, $x_{N_2+1} \geq x_{N_2-1}$. Then,

$$\gamma_{N_2} = \gamma_{N_2-1} + \tan^{-1}|f'(x_{N_2-1})| + \tan^{-1}|f'(x_{N_2})|.$$

But

$$\gamma_{N_2} = \gamma_{N_2+1} + \tan^{-1}|f'(x_{N_2+1})| + \tan^{-1}|f'(x_{N_2})|.$$

Hence,

$$\gamma_{N_2-1} + \tan^{-1}|f'(x_{N_2-1})| = \gamma_{N_2+1} + \tan^{-1}|f'(x_{N_2+1})| \leq \gamma_{N_2+1} + \tan^{-1}|f'(x_{N_2-1})|,$$

That is, $\gamma_{N_2-1} \leq \gamma_{N_2+1}$.

Next we show that $x_{N_2-i} \leq x_{N_2+i}$ and $\gamma_{N_2-i} \leq \gamma_{N_2+i}$, for all $i = 1, 2, \ldots$ while the collisions remain inside the cusp. We do it by induction. Suppose that the assertion is true for i and let us prove that is so for $i+1$. We have

$$x_{N_2-i} = x_{N_2-(i+1)} + \frac{f(x_{N_2-i}) + f(x_{N_2-(i+1)})}{\tan(\gamma_{N_2-(i+1)} + \tan^{-1}|f'(x_{N_2-(i+1)})|)},$$

$$x_{N_2+i} = x_{N_2+(i+1)} + \frac{f(x_{N_2+i}) + f(x_{N_2+(i+1)})}{\tan(\gamma_{N_2+(i+1)} + \tan^{-1}|f'(x_{N_2+(i+1)})|)},$$

$$\gamma_{N_2-i} = \gamma_{N_2-(i+1)} + \tan^{-1}|f'(x_{N_2-(i+1)})| + \tan^{-1}|f'(x_{N_2-i})|,$$

$$\gamma_{N_2+i} = \gamma_{N_2+(i+1)} + \tan^{-1}|f'(x_{N_2+(i+1)})| + \tan^{-1}|f'(x_{N_2+i})|.$$

By the induction hypothesis, the following holds

$$\begin{aligned} & \gamma_{N_2-(i+1)} + \tan^{-1}|f'(x_{N_2-(i+1)})| + \tan^{-1}|f'(x_{N_2-i})| \\ \leq\ & \gamma_{N_2+(i+1)} + \tan^{-1}|f'(x_{N_2+(i+1)})| + \tan^{-1}|f'(x_{N_2+i})| \\ \leq\ & \gamma_{N_2+(i+1)} + \tan^{-1}|f'(x_{N_2+(i+1)})| + \tan^{-1}|f'(x_{N_2-i})|. \end{aligned}$$

Thus,

$$\gamma_{N_2-(i+1)} + \tan^{-1}|f'(x_{N_2-(i+1)})| \le \gamma_{N_2+(i+1)} + \tan^{-1}|f'(x_{N_2+(i+1)})|. \tag{5.4}$$

Using the hypothesis of induction and (5.4) we get

$$\begin{aligned} & x_{N_2-(i+1)} + \frac{f(x_{N_2-i}) + f(x_{N_2-(i+1)})}{\tan(\gamma_{N_2-(i+1)} + \tan^{-1}|f'(x_{N_2-(i+1)})|)} \\ \le \;& x_{N_2+(i+1)} + \frac{f(x_{N_2+i}) + f(x_{N_2+(i+1)})}{\tan(\gamma_{N_2-(i+1)} + \tan^{-1}|f'(x_{N_2-(i+1)})|)}. \end{aligned}$$

Hence, $x_{N_2-(i+1)} + f(x_{N_2-(i+1)}) \le x_{N_2+(i+1)} + f(x_{N_2+(i+1)})$. Since $x_i \ge 1$, for all $1 \le i \le N$, we get $x_{N_2-(i+1)} \le x_{N_2+(i+1)}$ and $\gamma_{N_2-(i+1)} \le \gamma_{N_2+(i+1)}$, as we wish to demonstrate.

These estimates show that the number of collisions into the cusp before N_2 and after N_2 differ by no more than one, that is, $|N_2 - N/2| \le 2$. Thus $|N_2 - N/2| = \mathcal{O}(1)$, finishing the proof of Lemma 5.1. □

From now on, we use the table $\mathcal{D}$ defined by $f(x) = (x+1)^{-1}$. Until the end of this section, we use the following change of variables:

$$t_n = x_n + 1, \ \ \forall\, 1 \le n \le N.$$

For each $n = 1, 2, \dots, N_1$, let $\omega_n = \dfrac{\gamma_n}{|f'(t_n)|} = \gamma_n t_n^2$ and $u_n = \dfrac{t_n}{t_{n+1}}$.

The proof of the next two lemmas are detailed in the Appendix A.

Lemma 5.2. *For $n < N_2$, the following estimates hold:*

$$\omega_n = 6n + \mathcal{O}(\ln n), \quad \omega_{n+1} = 6 + \omega_n + \mathcal{O}\left(\frac{1}{n} + \gamma_n^2 + t_n^{-4}\right),$$

$$u_n = 1 - \frac{1}{3n} + \mathcal{O}\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right), \quad \frac{1}{u_n} = 1 + \frac{1}{3n} + \mathcal{O}\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right).$$

Lemma 5.3. *We have*

$$N_1 \asymp N_2 - N_1 \asymp N_3 - N_2 \asymp N - N_3 \asymp N,$$

so each period (entering, turning and exiting) has size of order N. Moreover,

$$x_1 \asymp N^{\frac{1}{6}}; \quad x_{N_2} \asymp N^{\frac{1}{2}} \text{ and } x_n \asymp n^{\frac{1}{3}} N^{\frac{1}{6}}; \quad \forall n = 2, \dots, N_1. \tag{5.5}$$

Also

$$\gamma_1 = \mathcal{O}(N^{-1/3}); \quad \gamma_2 \asymp N^{-1/3} \text{ and } \gamma_n \asymp n^{\frac{1}{3}} N^{-\frac{1}{3}}; \quad \forall n = 2, \dots, N_1. \tag{5.6}$$

For $1 \le n \le N_2$, let τ_n be the time between the n-th and the $n+1$-st collision in the billiard table:

$$\tau_n = \frac{f(t_n) + f(t_{n+1})}{\sin(\gamma_n + \tan^{-1}(|f'(t_n)|))}.$$

Lemma 5.4. *For all* $1 \le n < N_2$, $\tau_n \asymp n^{-2/3}N^{1/6}$.

Proof. Using the estimates at Lemma 5.2 we obtain, for $1 \le n < N_2$,

$$
\begin{aligned}
\tau_n &= \frac{f(t_n)+f(t_{n+1})}{\sin(\gamma_n + \tan^{-1}(|f'(t_n)|))} \\
&= \frac{\dfrac{1}{t_n}+\dfrac{1}{t_{n+1}}}{\gamma_n + \tan^{-1}(|f'(t_n)|) + \mathcal{O}\left((\gamma_n + \tan^{-1}(|f'(t_n)|))^3\right)} \\
&= \frac{\dfrac{1}{t_n}+\dfrac{1}{t_{n+1}}}{\left(\gamma_n + \dfrac{1}{t_n^2}\right) + \mathcal{O}\left(\left(\gamma_n + \dfrac{1}{t_n^2}\right)^3\right)} = \frac{t_n + \dfrac{t_n^2}{t_{n+1}}}{\omega_n + 1 + \mathcal{O}(t_n^2\gamma_n^3)} \\
&= \frac{t_n(1+u_n)}{\omega_n(1+\omega_n^{-1}+\mathcal{O}(\gamma_n^2))} = \frac{t_n}{\omega_n}\frac{2+\mathcal{O}(n^{-1})}{1+\mathcal{O}(n^{-1})+\mathcal{O}(\gamma_n^2)} \\
&= = \frac{2t_n}{\omega_n}(1+\mathcal{O}(n^{-1})+\mathcal{O}(\gamma_n^2)) \asymp n^{-2/3}N^{1/6},
\end{aligned}
$$

completing the proof of Lemma 5.4. □

Now, let K_n be the curvature of the dispersing part of the table at the point of collision (r_n, φ_n). Then

$$
K_n = \frac{f''(t_n)}{(1+(f'(t_n))^2)^{3/2}} = \frac{\dfrac{2}{t_n^3}}{\left(1+\left(\dfrac{1}{t_n^2}\right)^2\right)^{3/2}} = \frac{\dfrac{2}{t_n^3}}{\left(\dfrac{t_n^4+1}{t_n^4}\right)^{3/2}} = \frac{2}{t_n^3}.
$$

The proof of the next result can be found in Appendix A.

Lemma 5.5. *For every* $1 \le n \le N_2$ *the following statements hold:*

(a) $\dfrac{\tau_n K_n}{\sin(\gamma_n)} = \dfrac{1}{9n^2} + \mathcal{O}\left(\dfrac{\log n}{n^3} + \dfrac{\gamma_n^2}{n^2}\right)$,

(b) $\dfrac{\tau_{n+1}}{\tau_n} = 1 - \dfrac{2}{3n} + \mathcal{O}\left(\dfrac{\log n}{n^2} + \dfrac{\gamma_n^2}{n} + \dfrac{t^{-4}}{n}\right)$.

Remark 5.6. Due to the reversibility property of the billiard map, all the formulas obtained above hold for the exiting period as well. In particular,

$$
x_N \asymp N^{1/6} \quad \text{and} \quad \gamma_N = \mathcal{O}(N^{-1/3}). \tag{5.7}
$$

During the exiting period we can use the countdown index $m = N+1-n$ obtaining asymptotic rates for $m = 1, 2 \ldots, N - N_3$, as for example, $x_m \asymp m^{1/3}N^{1/6}$, $\tau_m \asymp m^{-2/3}N^{1/6}$, etc.

5.2. **Hyperbolicity.** Let $\mathcal{F} : M \to M$ be the map described in Section 3, that gives the dynamics of the billiard system.

If $(r_1, \varphi_1) = z_1 = \mathcal{F}(z) = \mathcal{F}(r, \varphi)$, the derivative $D\mathcal{F}$ at a point z is given by ([5, Ch. 2])

$$D_z\mathcal{F} = \pm\frac{1}{\cos\varphi_1}\begin{bmatrix} \tau K + \cos\varphi & \tau \\ \tau K K_1 + K\cos\varphi_1 + K_1\cos\varphi & \tau K_1 + \cos\varphi_1 \end{bmatrix}, \tag{5.8}$$

where $\tau = \tau(z)$ is the traveling time (equivalently, the distance in $\mathcal{D}_2$) between the two collision points, K (resp. K_1) is the curvature of $\partial\mathcal{D}$ at z (resp. z_1). We adopt the convention that in our billiard the curvature is non-negative. The sign in front of the above matrix is plus if the the two consecutive collisions occur at $\mathcal{U}$; and minus if one of the collisions occurs at $\mathcal{U}$ and the other at $\mathcal{L}$.

On TM we define the *unstable* and *stable* cones by

$$\mathcal{C}^u(z) = \{(dr, d\varphi) \in T_zM : dr d\varphi \geq 0\}, \quad \mathcal{C}^s(z) = \{(dr, d\varphi) \in T_zM : dr d\varphi \leq 0\}.$$

Using (5.8), we obtain that the unstable cones $\mathcal{C}^u$ are strictly invariant under the action of $D\mathcal{F}$ [24, 18, 15].

From now on we use the *p-norm*, defined by

$$\|dz\|_p = \cos\varphi |dr|,$$

for vectors $dz \in T_zM$ at a point $z = (r, \varphi)$. The advantage of using it is that for billiard maps $\mathcal{F}$, the expansion rate of unstable vectors (i.e., in an unstable cone) in the p-norm is given by

$$\frac{\|D_z\mathcal{F}^{n+1}(dz)\|_p}{\|dz\|_p} = \prod_{i=0}^{n} |1 + \tau_i B_i^+|,$$

where B_i^+ denotes the curvature of a small arc transverse to the wave front. For further details we suggest Chernov and Markarian's book [4, Chapter IV] and also [5, Chapter 3 and 4].

The values B_i^+ can be calculated inductively as

$$B_{n+1}^+ = \frac{2K_{n+1}}{\sin\gamma_{n+1}} + \frac{B_n^+}{1 + \tau_n B_n^+}. \tag{5.9}$$

For any $z \in E$ we call

$$R(z) = \min\{n \geq 1 \mid \mathcal{F}^n(z) \in E\}$$

the *return function.*

Let $z \in E$ be a point whose trajectory $\{\mathcal{F}^i(z)\}_{i=1}^N$ is going down the cusp and comes back after N reflections. In that case, the return function assume the value $R(z) = N + 1$. We define the set E_N by

$$E_N = \{z \in E \mid R(z) = N + 1\}.$$

In another words, E_N is precisely the set of points z that enters and leaves the cusp after N reflections. This set is bounded by the curves S^*, S^*_{N-1}, S^*_N and $r = 1$, as depicted in Figure 9.

The curve S^* is a singularity line for $\mathcal{F}$. It is constituted of points $z \in E$ whose trajectory hit the dispersing part $\mathcal{U}$ tangentially at the first rebound. As for S^*_N, this is a singularity line for $\mathcal{F}^N$ and made up of points $z \in E$ whose trajectory is going down the cusp and returns after N rebounds, but the N-th rebound is grazing. Since all these curves are singularity lines, they are smooth curves whose slope in the $r\varphi$ coordinates is negative and bounded away from infinity [5, Chapter 4], i.e.,

$$-\infty < C_1 \leq \frac{d\varphi}{dr} \leq 0,$$

for some constant $C_1 < 0$. One can easily check (see Figure 10(a)) that the slope of the curve S^* at $z = (1, 0)$ is $\frac{d\varphi}{dr} = 0$.

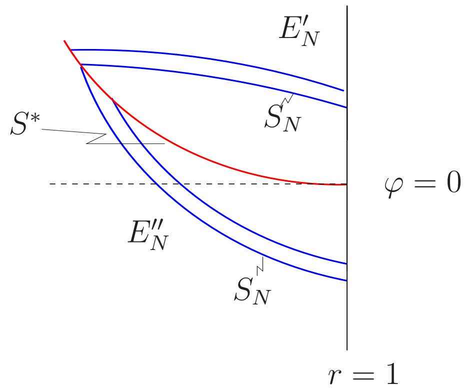

Figure 9: The sets E'_N , E''_N and E_N.

The images $F_N = \mathcal{F}^{N+1}(E_N)$ are sets bounded by singularity lines for $\mathcal{F}^{-i}$, $i = N-1, N$. Due to the time-reversibility of the billiard map, $(r, \varphi) \in E_N$ if, and only if, $(r, -\varphi) \in F_N$. So, F_N is obtained reflecting E_N along the line $\varphi = 0$.

Each domain E_N is split in two strips: E'_N and E''_N. The inferior strip E''_N consists of points $z \in E$ whose first collision in $\mathcal{D}$ occurs at the dispersing part $\mathcal{U}$. The superior strip E'_N consists of points whose first collision in D occurs at the horizontal part of the table. (Recall the orientation of the table, depicted in Figure 3.) The sets E_N, $N > N_0$, make a nested structure that shrink to (1,0) as N goes to infinity.

From the equations (5.6) and (5.7), we know that, for trajectories in E_N, $\gamma_1 = \mathcal{O}(N^{-1/3})$ and $\gamma_N = \mathcal{O}(N^{-1/3})$. Hence they can be arbitrarily close to zero, which implies that the expansion rate would be extremely high. However B^+_{n+1} is an increasing function of B_n^+ and $\dfrac{1}{\sin \gamma_{n+1}}$. So, if γ_n increases, B_n^+ decreases. In this way, we can obtain a lower bound for the expansion rate taking upper bounds for γ_1 and γ_N. Thus we study the expansion on points of E_N satisfying

$$\gamma_1 \asymp N^{-1/3} \quad \text{and} \quad \gamma_N \asymp N^{-1/3}. \tag{5.10}$$

The main goal of this section is to prove the following result:

Theorem 5.7. *For all $z \in E_N$, satisfying $\gamma_1 \asymp N^{-1/3}$ and $\gamma_N \asymp N^{-1/3}$,*

$$\frac{\|D_z \mathcal{F}^{N+1}(dz)\|_p}{\|dz\|_p} \asymp N.$$

Let us denote $\tau_i B_i^+$ by λ_i. For $n \geq 1$,

$$\lambda_{n+1} = \frac{2\tau_{n+1}K_{n+1}}{\sin \gamma_{n+1}} + \frac{\tau_{n+1}}{\tau_n} \cdot \frac{\lambda_n}{1+\lambda_n}. \tag{5.11}$$

The proof of Theorem 5.7 uses sharp estimates on λ_n and τ_n described in the next three technical lemmas, whose proofs are in Appendix B.

Lemma 5.8. *For all $z \in E_N$ satisfying (5.10) we have*

$$\lambda_n \asymp \frac{1}{n} \quad \textit{for} \quad 1 \leq n \leq N_1,$$
$$\lambda_n \asymp \frac{1}{n} \asymp \frac{1}{N} \quad \textit{for} \quad N_1 \leq n \leq N_3,$$
$$\lambda_n \asymp \frac{1}{(N-n)} \quad \textit{for} \quad N_3 \leq n < N.$$

Lemma 5.8 implies that $\displaystyle\sum_{n=1}^{N-1} \lambda_n = \mathcal{O}(1)$. Therefore, for $1 \leq N' < N'' \leq N$,

$$\prod_{n=N'}^{N''-1}(1+\lambda_n) = \exp\left(\sum_{n=N'}^{N''-1} \ln(1+\lambda_n)\right) \asymp \exp\left(\sum_{n=N'}^{N''-1} \lambda_n\right). \tag{5.12}$$

In the turning period, we have that $\sum_{n=N_1+1}^{N_3-1} \lambda_n \asymp 1$, showing that the expansion during this period is negligible.

Lemma 5.9. *For all $z \in E_N$ satisfying (5.10), $\prod_{n=1}^{N_1}(1+\lambda_n) \asymp N^{2/3}$.*

Lemma 5.10. *For all $z \in E_N$ satisfying (5.10), $\prod_{n=N_3}^{N-1}(1+\lambda_n) \asymp N^{1/3}$.*

Proof of Theorem 5.7. Let dz be an unstable vector. From the preceding lemmas the proof is almost complete, we just need to estimate the expansion rate at the beginning of the trajectory and at its end ($n = 0, n = N$)

At the exiting period, $\lambda_m \asymp 1/m$ and $\tau_m \asymp m^{-2/3}N^{1/6}$, for $m = 2, \ldots, N - N_3$. Therefore $B_m^+ = \frac{\lambda_m}{\tau_m} \asymp m^{-1/3}N^{-1/6}$, $m = 2, \ldots, N - N_3$. Lemma 5.4 implies that for $m = 1$, $\tau_N \asymp N^{1/6}$. For points $z \in E_N$ satisfying (5.10) we can still use (5.9), that give us

$$B_N^+ \asymp B_{N-1}^+ \asymp N^{-1/6}.$$

So unstable vectors are additionally expanded by $1 + \tau_N B_N^+ \asymp 1$.

For $n = 0$, $\tau_0 \asymp x_1 \asymp N^{1/6}$ and B_0^+ is the last B^+ of a trajectory that is coming from the cusp. From Figure 12 and (5.8) we can deduce that this trajectory has also done $\asymp N$ bounces in the cusp. So $B_0^+ \asymp N^{-1/6}$, and $1 + \tau_0 B_0^+ \asymp 1$.

Hence, from Lemma 5.9 and Lemma 5.10, we get

$$\frac{\|D_z \mathcal{F}^{N+1}(dz)\|_p}{\|dz\|_p} \asymp 1 \times N^{2/3} \times N^{1/3} \times 1 \asymp N,$$

for all $z \in E_N$ satisfying $\gamma_1 \asymp N^{-1/3}$ and $\gamma_N \asymp N^{-1/3}$, finishing the proof. □

5.3. **Singularity lines and scaling rate: proof of Theorem 1.2.** In this section we estimate the scaling rate for the billiard map $\mathcal{F}$. To do so, we analyze the set $E_N = \{z \in E \mid R(z) = N + 1\}$ defined above.

Let us estimate the "size" of each E_N. The intersection of E_N with S^* occurs at a distance $\asymp N^{-1/6}$, since $x_1 \asymp N^{1/6}$. See Figure 10(a). The distance of E_N'' to S^* at $r = 1$ is $\asymp N^{-1/3}$. See Figure 10(b). Since the lines S^*, S_{N-1}^* and S_N^* are decreasing and S^* has a horizontal tangency at $z = (1, 0)$, the "length" of each strip E_N is $\asymp N^{-1/6}$.

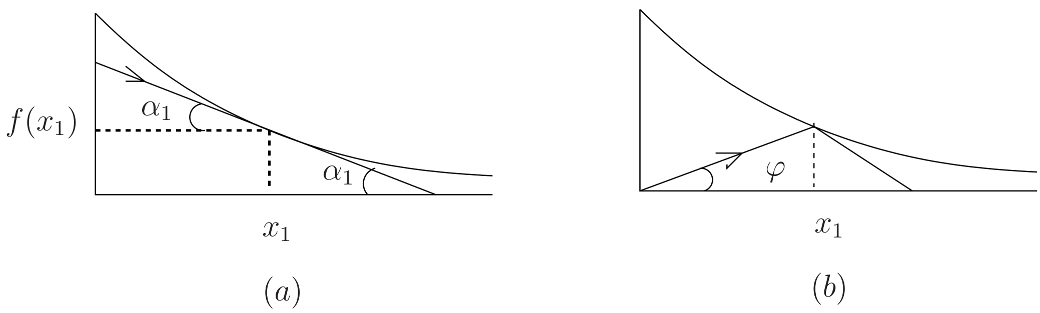

Figure 10: In part (a), since $x_1 \asymp N^{1/6}$, $f(x_1) \asymp N^{-1/6}$ and $\alpha_1 = \tan^{-1}|f'(x_1)|$. So, the distance between the starting point of the grazing trajectory and the right-angled vertex is $\asymp N^{-1/6}$. In part (b), we obtain that $\varphi = \tan^{-1}\left(\frac{f(x_1)}{x_1}\right) \asymp N^{-1/3}$.

Due to the symmetry, $F_N = \mathcal{F}^{N+1}(E_N)$ consists of two strips F_N' and F_N'' of length $\asymp N^{-1/6}$, see Figure 11. If $W \subset E_N'$ is an unstable curve stretching across E_N' (from S_N to S_{N-1}), then its image $W' = \mathcal{F}^{N+1}(W)$ is made up of curves stretching "from top to bottom" of each strip of F_N, so its "length" is $\asymp N^{-1/6}$. Using the fact that the derivative of $\mathcal{F}^{N+1}$ has an expansion rate of $\asymp N$ for

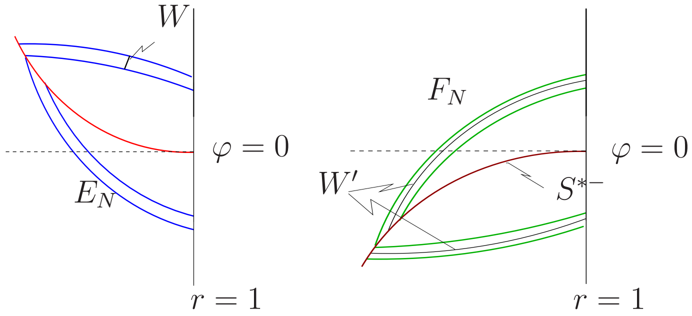

Figure 11: The unstable curve W and its image $W' = \mathcal{F}^{N+1}(W)$ and the time-reversed of S^*, S^{*-}.

unstable vectors, given by Theorem 5.7, we get that $|W| \asymp N^{-1/6}/N = N^{-7/6}$. This is the "width" of each of the strips of E_N.

Since the sets E_N are away from $\varphi = \pm\pi/2$, the measure μ is equivalent to the Lebesgue measure on $\mathbb{R}^2$. Thus,

$$\mu(E_N) \asymp N^{-1/6} \times N^{-7/6} = N^{-4/3} \quad \Longrightarrow \quad \mu\left(\bigcup_{n=N}^{\infty} E_n\right) \asymp N^{-1/3}.$$

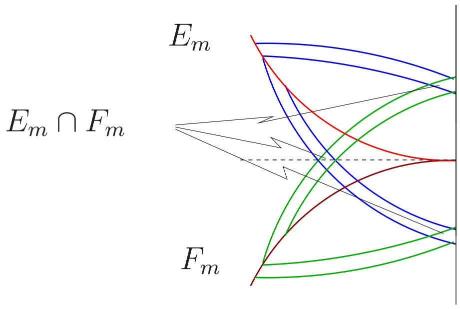

Figure 12: The sets E_m and $F_m = \mathcal{F}^{m+1}(E_m)$, and their intersections.

The measure of the intersection $E_m \cap \mathcal{F}^{m+1} E_m$, for $m > N$, can be estimated using the symmetry of the sets E_m and F_m:

$$\mu\left(E_m \cap \mathcal{F}^{m+1} E_m\right) \geq const \cdot m^{-7/6} \times m^{-7/6} = const \cdot m^{-7/3}.$$

Thus,

$$\begin{aligned} \mu\left(A \cap \mathcal{F}^{m+1} A\right) &= \mu\left(\bigcup_{n=N}^{\infty} E_n \cap \mathcal{F}^{m+1}\left(\bigcup_{n=N}^{\infty} E_n\right)\right) \\ &\geq \mu\left(E_m \cap \mathcal{F}^{m+1} E_m\right) \geq const \cdot m^{-7/3}, \end{aligned}$$

for all $m > N$, showing that the scaling rate is at most polynomial. This concludes the proof of Theorem 1.2.

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Appendix A. Proofs of Lemmas 5.2, 5.3 and 5.5

Proof of Lemma 5.2

Recall that $\omega_n = \dfrac{\gamma_n}{|f'(t_n)|} = \gamma_n t_n^2$ and $u_n = \dfrac{t_n}{t_{n+1}}$. Multiplying (5.1) by t_{n+1}^2 and expanding $\tan^{-1}$ in its Taylor's series we get

$$\omega_{n+1} = \frac{\omega_n + 1}{u_n^2} + 1 + \mathcal{O}(t_n^{-4}). \tag{A.1}$$

From (5.1), we get

$$\gamma_1 + \frac{1}{t_1^2} + \frac{2}{t_2^2} + \ldots + \frac{2}{t_{n-1}^2} + \frac{1}{t_n^2} + \mathcal{O}\left(\sum_{i=1}^{n} t_i^{-6}\right) = \gamma_n \leq \frac{\pi}{2}. \tag{A.2}$$

Thus,

$$\sum_{i=1}^{n} t_i^{-2} = \mathcal{O}(1). \tag{A.3}$$

From equation (A.1), we obtain

$$\omega_n > 2n - 2. \tag{A.4}$$

From (5.2) and using the fact that $\tan x > x$, we have

$$\frac{1}{u_n} < 1 + \frac{2}{\omega_n + 1}(1 + \mathcal{O}(t_n^{-6})). \tag{A.5}$$

Replacing (A.5) in (A.1):

$$\begin{aligned}\omega_{n+1} &< 1+(\omega_n+1)\left(1+\frac{2}{\omega_n+1}(1+\mathcal{O}(t_n^{-6}))\right)^2+\mathcal{O}(t_n^{-4})\\ &= 6+\omega_n+\frac{4}{\omega_n+1}+\mathcal{O}(t_n^{-4})<6+\omega_n+\frac{4}{2n-1}+\mathcal{O}(t_n^{-4}).\end{aligned}$$

In particular,

$$\omega_n < 6n + 2\ln n + \mathcal{O}(1). \tag{A.6}$$

Next we proceed as follows. For $n = 1, 2, \ldots, N_1$, we have

$$t_{n+1} = t_n + \frac{\frac{1}{t_n}+\frac{1}{t_{n+1}}}{\gamma_n+\frac{1}{t_n^2}+\mathcal{O}\left(\left(\gamma_n+\frac{1}{t_n^2}\right)^3\right)} = t_n + \frac{t_n+\frac{t_n^2}{t_{n+1}}}{\omega_n+1+t_n^2\mathcal{O}\left(\left(\gamma_n+\frac{1}{t_n^2}\right)^3\right)}.$$

Dividing by t_n we obtain

$$\frac{1}{u_n} = 1+\frac{1+u_n}{w_n+1}\left(\frac{1}{1+\frac{t_n^2}{\gamma_n t_n^2+1}\mathcal{O}\left(\left(\gamma_n+\frac{1}{t_n^2}\right)^3\right)}\right).$$

Since the choices of $\gamma_n < 10^{-10}$ and $x_1 > 10^6$ imply that $\mathcal{O}\left(\left(\gamma_n+\frac{1}{t_n^2}\right)^3\right) = \mathcal{O}(\gamma_n^3)$ and because $\frac{t_n^2}{\gamma_n t_n^2+1}\mathcal{O}\left(\gamma_n^3\right) = \mathcal{O}\left(\gamma_n^2\right)$, we obtain

$$\frac{1}{u_n} = 1+\frac{1+u_n}{w_n+1}\left(\frac{1}{1+\mathcal{O}(\gamma_n^2)}\right) = 1+\frac{1+u_n}{w_n+1}(1+\mathcal{O}(\gamma_n^2)),$$

since $\mathcal{O}(\gamma_n^2)$ is sufficiently small. From (A.5) we get

$$\begin{aligned}\frac{1}{u_n} &> 1+\left(\frac{1}{\omega_n+1}+\frac{1}{\omega_n+3+O(t_n^{-6})}\right)(1+\mathcal{O}(\gamma_n^2))\\ &> 1+\frac{2}{6n+2\ln n+\mathcal{O}(1)}+\mathcal{O}\left(\frac{\gamma_n^2}{n}\right)\\ &> \exp\left(\frac{2}{6n+2\ln n+\mathcal{O}(1)}-\frac{4}{(6n+2\ln n+\mathcal{O}(1))^2}+\mathcal{O}\left(\frac{\gamma_n^2}{n}\right)\right). \qquad \text{(A.7)}\end{aligned}$$

In the last inequality above we use the fact that $1 + x > \exp(x - x^2)$ for small x.

However

$$\begin{aligned}\omega_{n+1} &= 1+\frac{\omega_n+1}{u_n^2}+\mathcal{O}(t_n^{-4})\\ &> 1+(\omega_n+1)\left(1+\frac{2}{\omega_n+3}+\mathcal{O}\left(\frac{\gamma_n^2}{n}\right)\right)^2+\mathcal{O}(t_n^{-4})\\ &> \omega_n+6-\frac{4}{2n+1}-\frac{4}{(2n+1)^2}+\mathcal{O}(\gamma_n^2)+\mathcal{O}(t_n^{-4}).\end{aligned}$$

This implies that

$$\omega_n > 6n-2\ln n+\mathcal{O}(1). \tag{A.8}$$

Thus

$$\begin{aligned}\frac{1}{u_n^2} &= \frac{\omega_{n+1}-1+\mathcal{O}(t_n^{-4})}{\omega_n+1}<\frac{\omega_n+1+4+\frac{4}{\omega_n+1}+\mathcal{O}(t_n^{-4})}{\omega_n+1}\\ &< 1+\frac{4}{6n-2\ln n+\mathcal{O}(1)}+\frac{4}{(6n-2\ln n+\mathcal{O}(1))^2}+\mathcal{O}\left(\frac{t_n^{-4}}{n}\right).\end{aligned}\tag{A.9}$$

From (A.6), (A.8), (A.7) and (A.9), we get the desired results.

Proof of Lemma 5.3

From (A.4) and (A.6) we conclude that $\omega_n=\gamma_n t_n^2 \asymp n$. Since $\gamma_{N_2}\approx \pi/2$, it follows that $x_{N_2}^2 \asymp N_2 \asymp N$, by Lemma 5.1.

Multiplying (A.7), from $i=1$ to $n-1$, we get

$$\begin{aligned}\prod_{i=1}^{n-1} u_i^{-1} &> \exp\left(\sum\frac{2}{6i+2\ln i+\mathcal{O}(1)}-\sum\frac{4}{(6i+2\ln i+\mathcal{O}(1))^2}+\mathcal{O}\left(\sum\frac{\gamma_i^2}{i}\right)\right)\\ &> \exp(\ln n^{1/3}-C)=C'n^{1/3},\end{aligned}$$

because

$$\sum_{i=1}^{n-1}\frac{\gamma_i^2}{i}\le 12\sum_{i=1}^{n-1}\frac{\gamma_i^2}{\omega_i}=12\sum_{i=1}^{n-1}\frac{\gamma_i}{x_i^2}=\mathcal{O}(1),$$

since $\gamma_i<\pi/2$ and $\sum_{i=1}^{n} t_i^{-2}=\mathcal{O}(1)$, as obtained in (A.3). We obtain

$$\frac{t_n}{t_1}>C'n^{1/3}. \tag{A.10}$$

Multiplying (A.9), from $i=1$ to $n-1$,

$$\begin{aligned}\prod_{i=1}^{n-1} u_i^{-2} &< \exp\left(\sum\frac{4}{6i-2\ln i+\mathcal{O}(1)}+\sum\frac{4}{(6i-2\ln i+\mathcal{O}(1))^2}+\mathcal{O}\left(\sum\frac{x_i^{-4}}{i}\right)\right)\\ &< \exp(\ln n^{2/3}+C)=C'n^{2/3}.\end{aligned}$$

And we obtain

$$\left(\frac{t_n}{t_1}\right)^2 < C' n^{2/3}. \tag{A.11}$$

From (A.10) and (A.11) we get $\frac{t_n}{t_1} \asymp n^{1/3}$ and so $n \asymp \gamma_n t_n^2 \asymp \gamma_n n^{2/3} t_1^2$. However, $\gamma_{N_1} \approx \bar{\gamma} = \text{const}$, and hence

$$N_1 \asymp \gamma_{N_1} N_1^{2/3} x_1^2 \Rightarrow x_1 \asymp N_1^{1/6}.$$

Thus

$$t_n \asymp n^{1/3} N_1^{1/6} \quad \text{and} \quad \gamma_n \asymp n^{1/3} N_1^{-1/3}, \quad \text{for all} \quad n = 2, \dots, N_1.$$

To show that $N_1 \asymp N$, just notice that at the turning period, i.e., $N_1 \leq n \leq N_2$, the angle γ_n increases from $\bar{\gamma}$ to approximately $\pi/2$ and

$$\frac{1}{t_n^2} = \frac{\gamma_n}{\omega_n} > \frac{\bar{\gamma}}{6n + 2\ln n + C}.$$

It follows from (A.2) and (A.3) that

$$\sum_{n=N_1}^{N_2} (\gamma_n - \gamma_{n-1}) \geq \sum_{n=N_1}^{N_2} \frac{C'}{6n + 2\ln n + C} \geq C'' \ln \frac{N_2}{N_1},$$

for some constants $C', C'' > 0$. This implies that $N_1 < N_2 < C''' N_1$, for some $C''' > 0$, completing the proof.

Proof of Lemma 5.5

To prove (a), using the similar calculations as in Lemma 5.4, we obtain

$$\begin{aligned} \frac{\tau_n K_n}{\sin \gamma_n} &= \frac{2 t_n \omega_n^{-1}(1 + O(n^{-1}) + \mathcal{O}(\gamma_n^2))}{\gamma_n + \mathcal{O}(\gamma_n^3)} \frac{2}{t_n^3} = \frac{4}{t_n^2} \frac{(1 + \mathcal{O}(1/n) + \mathcal{O}(\gamma_n^2))}{\omega_n \gamma_n (1 + \mathcal{O}(\gamma_n^2))} \\ &= \frac{1}{9n^2} + \mathcal{O}\left(\frac{\ln n}{n^3} + + \frac{\gamma_n^2}{n^2}\right). \end{aligned}$$

Moreover,

$$\frac{\tau_{n+1}}{\tau_n} = \frac{f(t_{n+1}) + f(t_{n+2})}{f(t_n) + f(t_{n+1})} \frac{\sin(\gamma_n + \tan^{-1}(|f'(t_n)|))}{\sin(\gamma_{n+1} + \tan^{-1}(|f'(t_{n+1})|))} = F_1 F_2. \tag{A.12}$$

To obtain F_1, we first notice that

$$F_1 = \frac{\frac{1}{t_{n+1}} + \frac{1}{t_{n+2}}}{\frac{1}{t_{n+1}} + \frac{1}{t_n}} = \frac{\frac{1}{t_{n+1}}\left(1 + \frac{t_{n+1}}{x_{n+2}}\right)}{\frac{1}{x_n}\left(1 + \frac{t_n}{t_{n+1}}\right)} = u_n \frac{(1 + u_{n+1})}{(1 + u_n)}.$$

A long and straightforward calculation gives

$$\frac{1 + u_{n+1}}{1 + u_n} = 1 + \frac{5}{36n^2} + \mathcal{O}\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right).$$

And to obtain F_2,

$$\begin{aligned} F_2 &= \frac{\gamma_n + \frac{1}{t_n^2} + \mathcal{O}(\gamma_n^3)}{\gamma_{n+1} + \frac{1}{t_{n+1}^2} + \mathcal{O}(\gamma_{n+1}^3)} = \frac{t_n^2}{t_n^2}\frac{t_{n+1}^2}{t_{n+1}^2}\frac{\gamma_n + \frac{1}{t_n^2} + \mathcal{O}(\gamma_n^3)}{\gamma_{n+1} + \frac{1}{t_{n+1}^2} + \mathcal{O}(\gamma_{n+1}^3)} \\ &= \frac{1}{u_n^2}\left(1 - \frac{1}{n} + \frac{7}{6n^2} + \mathcal{O}\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right)\right). \end{aligned}$$

Replacing the values of F_1 and F_2 found above in (A.12) we obtain

$$\begin{aligned} \frac{\tau_{n+1}}{\tau_n} &= \left(1 + \frac{1}{3n}\right)\left(1 + \frac{5}{36n^2}\right)\left(1 - \frac{1}{n} + \frac{7}{6n^2}\right) + \mathcal{O}\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right) \\ &= 1 - \frac{2}{3n} + \mathcal{O}\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right), \end{aligned}$$

completing the proof of (b).

Appendix B. Proof of Lemmas 5.8, 5.9 and 5.10

Proof of Lemma 5.8.

For $1 \le n \le N_1$, $\lambda_{n+1} > \dfrac{a}{n^2} + \left(1 - \frac{b}{n}\right)\dfrac{\lambda_n}{1+\lambda_n}$, for some $a, b > 0$. Suppose that $\lambda_n > c/n$. Then

$$\lambda_{n+1} > \frac{a}{n^2} + \left(1 - \frac{b}{n}\right)\frac{c/n}{1 + c/n} = \frac{c + (a - bc + ac/n)/n}{n + c}.$$

If $c > 0$ is small enough, the expression in parenthesis is positive and so $\lambda_{n+1} > \dfrac{c}{n+c} > \dfrac{c}{n+1}$. Similarly, $\lambda_{n+1} < \dfrac{A}{n^2} + \left(1 - \dfrac{B}{n}\right)\dfrac{\lambda_n}{1+\lambda_n}$, with $A > 0, B > 0$. Supposing $\lambda_n < C/n$, we get $\lambda_{n+1} < \dfrac{C + (A - BC + AC/n)/n}{n + C}$. If $C > 0$ is large enough, the expression in parenthesis is negative for N large and thus $\lambda_{n+1} < \dfrac{C}{n+C} < \dfrac{C}{n+1}$, completing the induction.

For $N_1 \le n \le N_3$, $\lambda_{N_1} \asymp \dfrac{1}{N}$ e $\tau_n \asymp n^{-2/3}N^{1/6} \asymp N^{-1/2}$. So, $B^+_{N_1} = \dfrac{\lambda_{N_1}}{\tau_{N_1}} \asymp N^{-1/2}$. Thus,

$$\begin{aligned} K_{n+1} \asymp N^{-3/2} &\Rightarrow \exists\, a, A > 0 \text{ such that } aN^{-3/2} \le K_{n+1} \le AN^{-3/2}, \\ \tau_n \asymp N^{-1/2} &\Rightarrow \exists\, b, B > 0 \text{ such that } bN^{-1/2} \le \tau_n \le BN^{-1/2}, \\ B^+_{N_1} \asymp N^{-1/2} &\Rightarrow \exists\, c, C > 0 \text{ such that } cN^{-1/2} \le B^+_{N_1} \le CN^{-1/2}. \end{aligned}$$

Moreover

$$2 \le \frac{2}{\sin\gamma_{n+1}} \le \frac{2}{\sin\bar{\gamma}} =: G, \quad \forall N_1 \le n \le N_3.$$

Hence,

$$B^+_{N_1+1} = \frac{2K_{N_1+1}}{\sin\gamma_{N_1+1}} + \frac{B^+_{N_1}}{1+\tau_{N_1}B^+_{N_1}} \le GAN^{-3/2} + CN^{-1/2};$$

$$B^+_{N_1+2} = \frac{2K_{N_1+2}}{\sin\gamma_{N_1+2}} + \frac{B^+_{N_1+1}}{1+\tau_{N_1+1}B^+_{N_1+1}} \le GAN^{-3/2} + B^+_{N_1+1} \le 2GAN^{-3/2} + CN^{-1/2}.$$

Thus,

$$\begin{aligned} B^+_n &= \frac{2K_n}{\sin\gamma_n} + \frac{B^+_{n-1}}{1+\tau_{n-1}B^+_{n-1}} \le (n-N_1)GAN^{-3/2} + CN^{-1/2} \\ &\le DNGAN^{-3/2} + CN^{-1/2} = (DGA+C)N^{-1/2} = EN^{-1/2}. \end{aligned}$$

But

$$B^+_{N_1+1} = \frac{2K_{N_1+1}}{\sin\gamma_{N_1+1}} + \frac{B^+_{N_1}}{1+\tau_{N_1}B^+_{N_1}} \ge 2aN^{-3/2} + \frac{cN^{-1/2}}{1+BEN^{-1}}.$$

$$\begin{aligned} B^+_{N_1+2} &= \frac{2K_{N_1+2}}{\sin\gamma_{N_1+2}} + \frac{B^+_{N_1+1}}{1+\tau_{N_1+1}B^+_{N_1+1}} \\ &\ge 2aN^{-3/2}\left(1 + \frac{1}{(1+BEN^{-1})}\right) + \frac{cN^{-1/2}}{(1+BEN^{-1})^2}. \end{aligned}$$

$$\begin{aligned} B^+_n &= \frac{2K_n}{\sin\gamma_n} + \frac{B^+_{n-1}}{1+\tau_{n-1}B^+_{n-1}} \\ &\ge 2aN^{-3/2}\left(\sum_{i=0}^{n-N_1-1} \frac{1}{(1+BEN^{-1})^i}\right) + \frac{cN^{-1/2}}{(1+BEN^{-1})^{n-N_1}}. \end{aligned}$$

There are $\ell > 0$, $h > 0$ such that $\sum_{i=0}^{n-N_1-1} \frac{1}{(1+BEN^{-1})^i} \ge \ell N$ and $\frac{1}{(1+BEN^{-1})^{n-N_1}} \ge h$, because $\sum_{i=0}^{n-N_1-1} \frac{1}{(1+BEN^{-1})^i} \ge \sum_{i=0}^{n-N_1-1} \frac{1}{(1+BEN^{-1})^{N_3-N_1}} \asymp N$, and $\frac{1}{(1+BEN^{-1})^{n-N_1}}$ is bounded. So,

$$B^+_n \ge 2a\ell N^{-1/2} + chN^{-1/2} = (2a\ell + ch)N^{-1/2} = eN^{-1/2}.$$

Thus $B^+_n \asymp N^{-1/2}$, and therefore $\lambda_n = B^+_n\tau_n \asymp N^{-1/2}N^{-1/2} = N^{-1}$.

For $N_3 \le n < N$, using the reversibility property of the billiard map (see Remark 5.6),

$$\lambda_{m-1} = \frac{2\tau_{m-1}K_{m-1}}{\sin\gamma_{m-1}} + \frac{\tau_{m-1}}{\tau_m}\cdot\frac{\lambda_m}{1+\lambda_m},$$

for $m = N+1-n$. In particular, there are $0 < a < A < \infty$ and $0 < b < B < \infty$, such that

$$\frac{a}{m^2} < \frac{2\tau_{m-1}K_{m-1}}{\sin\gamma_{m-1}} < \frac{A}{M^2} \quad \text{and} \quad 1 + \frac{b}{m} < \frac{\tau_{m-1}}{\tau_m} < 1 + \frac{B}{m}.$$

Supposing $\lambda_m > c/m$,

$$\begin{aligned}\lambda_{m-1} &> \frac{a}{m^2} + \left(1 + \frac{b}{m}\right)\frac{c/m}{1+c/m} \\ &= \frac{c + [a + bc - c - c^2 + (ac - a - bc - ac/m)/m]/(m+c)}{m-1}.\end{aligned}$$

If $c > 0$ is small enough, the expression between brackets is positive for m large, and we obtain that $\lambda_{m-1} > c/(m-1)$.

Supposing that $\lambda_m < C/m$,

$$\begin{aligned}\lambda_{m-1} &< \frac{A}{m^2} + \left(1 + \frac{B}{m}\right)\frac{C/m}{1+C/m} \\ &= \frac{C + [A + BC - C - C^2 + (AC - A - BC - AC/m)/m]/(m+C)}{m-1}.\end{aligned}$$

If $C > 0$ is large enough, the expression between brackets is negative for m large, and we obtain $\lambda_{m-1} < C/(m-1)$, completing the proof.

Proof of Lemma 5.9

According to the equation (5.12), it is sufficient to show that

$$\lambda_n = \frac{2}{3n} + \chi_n; \quad \text{where} \quad \sum_{n=1}^{N_1} \chi_n = \mathcal{O}(1).$$

We have, by (5.11), that

$$\lambda_{n+1} = \frac{2}{9n^2} + a_n + \left(1 - \frac{2}{3n} + b_n\right)\frac{\lambda_n}{1+\lambda_n},$$

where

$$a_n = \mathcal{O}\left(\frac{\ln n}{n^3} + \frac{\gamma_n^2}{n^2}\right) \quad \text{and} \quad b_n = \mathcal{O}\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{x_n^{-4}}{n}\right),$$

are relative to the equations (A.12) and (A.13). Note that $|a_n| \leq c/n^2$ and $|b_n| \leq c/n$, for some $c > 0$ small enough.Writing $\lambda_n = 2\frac{1+Z_n}{3n}$. we get

$$\begin{aligned}2\frac{1+Z_{n+1}}{3(n+1)} &= \frac{2}{9n^2} + a_n + \left(1 - \frac{2}{3n} + b_n\right) \times \\ &\quad \times \left(\frac{2}{3n} + \frac{2Z_n}{3n}\right)\left(1 - \frac{2}{3n} - \frac{2Z_n}{3n} + \mathcal{O}\left(\frac{1}{n^2} + \frac{Z_n^2}{n^2}\right)\right) \\ &= \frac{2}{9n^2} + a_n + X_1 \cdot X_2 \cdot X_3,\end{aligned}$$

where

$$X_2 \cdot X_3 = \frac{2}{3n} + \frac{2Z_n}{3n} - \frac{4}{9n^2} - \frac{8Z_n}{9n^2} - \frac{4Z_n^2}{9n^2} + \mathcal{O}\left(\frac{1}{n^3} + \frac{Z_n}{n^3} + \frac{Z_n^2}{n^3} + \frac{Z_n^3}{n^3}\right)$$

and

$$\begin{aligned} X_1 \cdot X_2 \cdot X_3 &= \frac{2}{3n} - \frac{8}{9n^2} + \frac{2b_n}{3n} + \frac{2Z_n}{3n} - \frac{12Z_n}{9n^2} + \\ &+ \frac{2b_nZ_n}{3n} - \frac{4Z_n^2}{9n^2} + \mathcal{O}\left(\frac{1}{n^3} + \frac{Z_n}{n^3} + \frac{Z_n^2}{n^3} + \frac{Z_n^3}{n^3}\right), \end{aligned}$$

Therefore,

$$Z_{n+1} = R_n + Z_n \times \left(1 - \frac{1}{n} + b_n + \mathcal{O}\left(\frac{1}{n^2}\right) - Z_n\left(\frac{2}{3n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) + \mathcal{O}\left(\frac{Z_n^2}{n^2}\right)\right),$$

where

$$R_n = \frac{3}{2}na_n + b_n + \mathcal{O}\left(\frac{1}{n^2}\right).$$

If we fix a small $\delta > 0$, then for n large enough

$$|Z_{n+1}| \le |R_n| + |Z_n|\left(1 - \frac{\delta}{n}\right).$$

Without affecting the asymptotic behavior of Z_n, we can assume that the upper bound holds for all n. Using it recurrently we get that

$$|Z_n| \le |R_n| + \sum_{k=1}^{n-1} |R_k| \prod_{i=k}^{n-1}\left(1 - \frac{\delta}{i+1}\right) \le \text{const} \sum_{k=1}^{n} |R_k|(k/n)^\delta.$$

Then

$$\begin{aligned} \sum_{n=1}^{N_1} |\chi_n| &\le \sum_{n=1}^{N_1} |Z_n|/n \le \text{const} \sum_{n=1}^{N_1}\sum_{k=1}^{n} |R_k| k^\delta / n^{\delta+1} \\ &\le \text{const} \sum_{k=1}^{N_1} |R_k| \sum_{n=k}^{N_1} k^\delta/n^{\delta+1} \le \text{const} \sum_{k=1}^{N_1} |R_k|. \end{aligned}$$

The last sum is uniformly bounded on N, which completes the proof of Lemma 5.9.

Proof of Lemma 5.10.

It is sufficient to show that, for $m = N - n + 1$, it holds

$$\lambda_m = \frac{1}{3m} + \chi_m; \qquad \text{where} \qquad \sum_{m=2}^{N-N_3} \chi_m = \mathcal{O}(1).$$

We have that

$$\lambda_{m-1} = \frac{2}{9m^2} + a_m + \left(1 + \frac{2}{3m} + b_m\right)\frac{\lambda_m}{1+\lambda_m},$$

where

$$a_m = \mathcal{O}\left(\frac{\ln m}{m^3} + \frac{\gamma_m^2}{m^2}\right) \quad \text{and} \quad b_m = \mathcal{O}\left(\frac{\ln m}{m^2} + \frac{\gamma_m^2}{m} + \frac{x_m^{-4}}{m}\right).$$

Note that $|a_m| \le c/m^2$ and $|b_m| \le c/m$, for some $c > 0$ small enough. Writing $\lambda_m = \frac{1+Z_m}{3m}$. we get

$$\begin{aligned}
\frac{1+Z_{m-1}}{3(m-1)} &= \frac{2}{9m^2} + a_m + \left(1 + \frac{2}{3m} + b_m\right) \times \\
&\quad \times \left(\frac{1}{3m} + \frac{Z_m}{3m}\right)\left(1 - \frac{1}{3m} - \frac{Z_m}{3m} + \mathcal{O}\left(\frac{1}{m^2}\right) + \mathcal{O}\left(\frac{Z_m^2}{m^2}\right)\right) \\
&= \frac{2}{9m^2} + a_m + X_1 \cdot X_2 \cdot X_3.
\end{aligned}$$

with

$$\begin{aligned}
X_2 \cdot X_3 &= \frac{1}{3m} + \frac{Z_m}{3m} - \frac{1}{9m^2} - \frac{2Z_m}{9m^2} - \\
&\quad - \frac{Z_m^2}{9m^2} + \mathcal{O}\left(\frac{1}{m^3} + \frac{Z_m}{m^3} + \frac{Z_m^2}{m^3} + \frac{Z_m^3}{m^3}\right),
\end{aligned}$$

and

$$\begin{aligned}
X_1 \cdot X_2 \cdot X_3 &= \frac{1}{3m} - \frac{1}{9m^2} + \frac{b_m}{3m} + \frac{Z_m}{3m} + \frac{b_m Z_m}{3m} - \\
&\quad - \frac{Z_m^2}{9m^2} + \mathcal{O}\left(\frac{1}{m^3} + \frac{Z_m}{m^3} + \frac{Z_m^2}{m^3} + \frac{Z_m^3}{m^3}\right),
\end{aligned}$$

Therefore,

$$Z_{m-1} = R_m + Z_m \times \left(1 - \frac{1}{m} + b_m + \mathcal{O}\left(\frac{1}{m^2}\right) - Z_m\left(\frac{1}{3m} + \mathcal{O}\left(\frac{1}{m^2}\right)\right) + \mathcal{O}\left(\frac{Z_m^2}{m^2}\right)\right),$$

where

$$R_m = 3ma_m + b_m + \mathcal{O}\left(\frac{1}{m^2}\right).$$

If we fix a small $\delta > 0$, then for n large enough

$$|Z_{m-1}| \le |R_m| + |Z_m|\left(1 - \frac{\delta}{m}\right).$$

Without affecting the asymptotic behavior of Z_m, we can assume that the bound above holds for all $m \ge 3$. Using it recurrently we get

$$|Z_m| \le \sum_{k=m}^{N-N_3} |R_k| \prod_{i=m}^{k}\left(1 - \frac{\delta}{i}\right) \le \text{const} \sum_{k=m}^{N-N_3} |R_k| (m/k)^{\delta}.$$

Then

$$\sum_{m=2}^{N-N_3} |\chi_m| \le \sum_{m=2}^{N-N_3} |Z_m|/m \le \text{const} \sum_{m=2}^{N-N_3} \sum_{k=m}^{N-N_3} |R_k| m^{\delta-1}/k^{\delta} \le$$

$$\text{const} \sum_{k=2}^{N-N_3} |R_k| \sum_{m=2}^{k} m^{\delta-1}/k^{\delta} \le \text{const} \sum_{k=2}^{N} |R_k|.$$

The last sum is uniformly bounded on N, which completes the proof of Lemma 5.10.

References

[1] J. Aaronson, "An introduction to infinite ergodic theory", Mathematical Surveys and Monographs, **50**, American Mathematical Society, Providence, RI, 1997.

[2] L. A. Bunimovich and Ya. G. Sinai, *Statistical properties of Lorentz gas with periodic configuration of scatterers*, Commun. Math. Phys. **78** (1981), 479–497.

[3] N. I. Chernov, *Decay of correlations and dispersing billiards*, J. Statist. Phys. **94**(3-4) (1999), 513–556.

[4] N. Chernov and R. Markarian, "Introduction to the Ergodic Theory of Chaotic Billiards", 24^o Colóquio Brasileiro de Matemática, Publicações Matemáticas, IMPA, 2003.

[5] N. Chernov and R. Markarian, "Chaotic billiards", Mathematical Surveys and Monographs, **127**, American Mathematical Society, Providence, RI, 2006.

[6] N. Chernov and R. Markarian, *Dispersing billiards with cusps: slow decay of correlations*, Commun. Math. Phys. **270** no 3 (2007), 727–758.
2000, 89-120.

[7] N. Chernov and H.-K. Zhang, *Improved estimates for correlations in billiards*, Commun. Math. Phys. **277** no 2 (2008), 305–321.

[8] J. B. Conway, "A course in functional analysis", Graduate Texts in Mathematics, **96**, 2nd edition, Springer-Verlag, New York, 1990.

[9] Y. Coudene, *On invariant distributions and mixing*, Ergod. Th. and Dynam. Sys. **27** (2007), 109–112.

[10] A. I. Danilenko and C. E. Silva, *Ergodic theory: nonsingular transformations*, Encyclopedia of complexity and systems science, Springer, 2009, 3055–3083.

[11] S. Isola, *Renewal sequences and intermittency*, J. Statist. Phys. **97**(1-2) (1999), 263–280.

[12] S. Isola, *On systems with finite ergodic theory*, Far East J. Dyn. Syst. **5**(1) (2003), 1–62.

[13] A. Katok, J.-M. Strelcyn, (in collaboration with F. Ledrappier and F.Przytycki), "Invariant manifolds, entropy and billiards; smooth maps with singularities", Lect. Notes in Math., 1222, Spinger-Verlag, 1986.

[14] U. Krengel and L. Sucheston, *On mixing in infinite measure spaces*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **13** (1969), 150–164.

[15] M. Lenci, *Semi-dispersing billiards with an infinite cusp I*, Commun. Math. Phys. **230** 1 (2002), 133–180.

[16] M. Lenci, *Semidispersing billiards with an infinite cusp II*, Chaos **13** 1 (2003), 105–111.

[17] M. Lenci, *On infinite-volume mixing*, Commun. Math. Phys. **298**(2) (2010), 485–514.

[18] C. Liverani and M. Wojtkowski, *Ergodicity in Hamiltonian systems*, Dynamics reported, 130–202, Dynam. Report. Expositions Dynam. Systems (N.S.), 4, Springer, Berlin, 1995.

[19] R. Markarian, *Billiards with polynomial decay of correlations*, Ergod. Th. Dynam. Sys. **24** (2004), 177–197.

[20] F. Riesz and B. Sz. Nagy, "Functional Analysis", Translated from the 2^{nd} french edition by Leo F. Boron, Dover Publications, Inc., New York, 1990.
[21] W. Rudin, "Real and Complex Analysis", WCB/McGraw-Hill, 3^{rd} edition, 1987.
[22] Ya. G. Sinai, *Dynamical systems with elastic reflections*, Russ. Math. Surveys **25** (1970), 137–189.
[23] L. Sucheston, *On mixing and the zero-one law*, J. Math. Analysis Appl. **6** (1963), 447–456.
[24] M. Wojtkowski, *Principles for the design of billiards with nonvanishing Lyapunov exponents*, Commun. Math. Phys. **105** (1986), 391–414.
[25] L. S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Ann. of Math.(2) **147** no. 3 (1998), 585–650.
[26] L. S. Young, *Recurrence times and rates of mixing*, Israel J. Math. **110** (1999), 153–188.

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