Expansive and fixed point free homeomorphisms of the plane.

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Abstract

The aim of this work is to describe the set of fixed point free homeomorphisms of the plane (preserving orientation or not) under certain expansive conditions. We find necessary and sufficient conditions for a fixed point free homeomorphism of the plane to be topologically conjugate to a translation.

1 Introduction.

In [1], [2], necessary and sufficient conditions for a homeomorphism of the plane with one fixed point to be topologically conjugate to a linear hyperbolic automorphism was proved. The discovery of a hypothesis about the behavior of Lyapunov functions at infinity was essential for this purpose. In this work we will describe the set of fixed point free homeomorphisms of the plane which admit a Lyapunov metric function under certain conditions. These homeomorphisms are expansive respect to the Lyapunov metric U, meaning $U : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ continuous and positive (i.e. it is equal to zero only on the diagonal) and $W = \Delta(\Delta U)$ positive with $\Delta U(x, y) = U(f(x), f(y)) - U(x, y)$. As it is well known, expansive homeomorphisms on compact surfaces were classified by Lewowicz in [3] and Hiraide in [8]. As a matter of fact, we began by studying whether some of the results obtained in the previously cited article could be adapted to our new context (i.e. without working in a compact environment but having the local compactness of the plane). These arguments will allow us to construct singular transverse foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. A singular foliation $\mathcal{F}$ on $\mathbb{R}^2$ is a decomposition of $\mathbb{R}^2$ as a disjoint union of leaves. Any point $x \in \mathbb{R}^2$ outside of a discrete set S has a chart $\varphi : U \to \mathbb{R}^2$ carrying the components of $U \cap \mathcal{F}$ to horizontal intervals.
For $x \in S$, x has a chart $\varphi : U \to \mathbb{R}^2$ taking $\mathcal{F} \cap U \to W_k$, where W_k is the standard k-prong singularity or singularity with k separatrices illustrated for $k = 4$ in figure 1.

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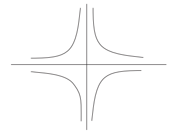

Figure 1:

We call the elements of S by singular points of $\mathcal{F}$.
Two singular foliations are transverse if they have the same singular points and at all other points the leaves are transverse. The main result of this article, theorem 4.0.3, states: *Let f be a homeomorphism of $\mathbb{R}^2$ which is fixed point free and admits a Lyapunov function $U : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that the following properties hold:*

- *U is a metric in $\mathbb{R}^2$ and induces the same topology in the plane as the usual metric,*
- *For each point $x \in \mathbb{R}^2$ and for each $k > 0$ there exist points y and z on the boundary of $B_k(x)$ such that $V(x,y) = U(f(x), f(y)) - U(x,y) > 0$ and $V(x,z) = U(f(x), f(z)) - U(x,z) < 0$, respectively,*
- *f has no singularities ($\mathcal{F}^s$ and $\mathcal{F}^u$ are nonsingular),*

f is topologically conjugate to a translation of the plane if and only if U admits condition **HP** *which states that given any compact set $C \subset \mathbb{R}^2$ the following properties hold:*

- *there exists $k > 0$ such that*
$$|V(x,y) - V(x,z)| \leq k \text{ for all } y, z \text{ in } C \text{ and for all } x \in \mathbb{R}^2,$$
and
- *$W(x,y)$ tends to infinity as $\|x\|$ tends to infinity, uniformly with $y \in C$.*

Condition **HP** arises while searching for sufficient conditions for asserting that every leaf of $\mathcal{F}^s$ intersects every leaf of $\mathcal{F}^u$. This fact will be essential to prove the above result Let us take as an example a translation of the plane and show that admits a Lyapunov metric function with the condition **HP**. For this aim we will adapt some arguments used by White in [7]. Let T be a translation of the plane defined by $T(x,y) = (x+1, y+1)$. Fix $\lambda > 1$ and consider the Riemannian structure defined by

$$<< u, v >>_\sigma = \lambda^{-2y} < u, e^1 >< v, e^1 > + \lambda^{2x} < u, e^2 >< v, e^2 >,$$

where $e^1 = (1,0)$, $e^2 = (0,1)$ and $\sigma = (x,y)$. The metric induced on the plane by this Riemannian metric is defined by

$$d_\sigma(p,q) = \inf_\gamma \int_{[0,1]} \|\gamma'(t)\|_\sigma dt,$$

where γ is an arc joining points p and q. The topology induced by d_σ on the plane is the same that is induced by the usual metric. Let $\gamma = (\gamma_1, \gamma_2) : [0,1] \to I\!R^2$ be a curve joining points p and q of $I\!R^2$. We can consider two pseudo metrics D^s and D^u defined by

$$D^s(p,q) = \inf_\gamma \int_{[0,1]} \lambda^{-\gamma_2(t)} |< \gamma'(t), e^1 >| dt,$$

and

$$D^u(p,q) = \inf_\gamma \int_{[0,1]} \lambda^{\gamma_1(t)} |< \gamma'(t), e^2 >| dt.$$

By a simple computation we can prove that

$$D^s(T(p), T(q)) = \lambda^{-1} D^s(p,q),$$

and

$$D^u(T(p), T(q)) = \lambda D^u(p,q).$$

Consider the Lyapunov metric D of $I\!R^2$ defined by $D = D^s + D^u$. Lets test some conditions required for D:

(I) **Signs for $\Delta(D)$.**

$$\Delta D(x,y) = D(f(x), f(y)) - D(x,y) =$$
$$D^s(f(x), f(y)) - D^s(x,y) + D^u(f(x), f(y)) - D^u(x,y) =$$
$$(\lambda - 1) D^u(x,y) - (1 - 1/\lambda) D^s(x,y).$$

For every point $x \in I\!R^2$ and for every $k > 0$, there are points y in the boundary of $B_k(x)$ such that $D^u(x,y) = 0$ (this is true because $D^u(x,y) = 0$ if x and y belong to the same horizontal line). Therefore, $\Delta D(x,y) < 0$ as we wanted. A similar argument lets us find points $z \in I\!R^2$ such that $\Delta D(x,z) > 0$.

(II) **Property HP.** Let $V = \Delta D$ and $W = \Delta^2 D$.
Since

$$\Delta^2 D(x,y) = \Delta D(f(x), f(y)) - \Delta D(x,y) =$$
$$(\lambda - 1)^2 D^u(x,y) + (1 - 1/\lambda)^2 D^s(x,y),$$

we can conclude that $\Delta^2 D(x,y)$ tends to infinity when $\|x\|$ tends to infinity. This is true because $D^u(x,y)$ or $D^s(x,y)$ tends to infinity when $\|x\|$ tends to infinity: In fact, if x and y belongs to the same horizontal line then $D^u(x,y) = 0$ and $D^s(x,y) = \lambda^{-k}\|x - y\|$, $k \in I\!R$ tends to infinity as $\|x\|$ tends to infinity through this horizontal line.
Now,

$$|\Delta D(x,y) - \Delta D(x,z)| \leq$$
$$(\lambda - 1)|D^u(x,y) - D^u(x,z)| + (1 - 1/\lambda)|D^s(x,y) - D^s(x,z)| \leq$$
$$(\lambda - 1) D^u(z,y) + (1 - 1/\lambda) D^s(z,y).$$

Therefore $|\Delta D(x,y) - \Delta D(x,z)|$ is uniformly bounded when points y and z lie on a compact set. Hence property **HP** holds.

2 Constructing foliations.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism of the plane that admits a Lyapunov metric function U, meaning $U : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ continuous and positive (i.e. it is equal to zero only on the diagonal) and $W = \Delta(\Delta U)$ positive with $\Delta U(x, y) = U(f(x), f(y)) - U(x, y)$.

Remark 2.0.1 *During this work we require the existence of such a Lyapunov function U, unlike in the compact case where expansiveness is a necessary and sufficient condition for the existence of a Lyapunov function (see [3]). Fathi in [9] proved the existence of a Lyapunov metric function for an expansive homeomorphism on a compact metric space.*

In this section we will resume some results of Lewowicz in [3], and Groisman in [1]. We will work with the topology induced by a Lyapunov function U and define the k-stable set in the following way:

$$S_k(x) = \{y \in \mathbb{R}^2 : \ U(f^n(x), f^n(y)) \leq k, \ n \geq 0\}.$$

Similar definition for the k-unstable set. Let f be a homeomorphism of the plane that admits a Lyapunov function $U : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that the following properties hold:

(1) **U is a metric in $\mathbb{R}^2$ and induces the same topology in the plane as the usual metric.** Observe that given any Lyapunov function it is possible to obtain another Lyapunov function that verifies all the properties of a metric except, perhaps, for the triangular property.

(2) **Existence of both signs for the first difference of U.** For each point $x \in \mathbb{R}^2$ and for each $k > 0$ there exist points y and z on the boundary of $B_k(x)$ such that $V(x, y) = U(f(x), f(y)) - U(x, y) > 0$ and $V(x, z) = U(f(x), f(z)) - U(x, z) < 0$, respectively.

The following two propositions establish the dynamic consequences of the existence of a Lyapunov function with the above conditions.

Proposition 2.0.1 *A homeomorphism f that admits a Lyapunov function U defined at $\mathbb{R}^2 \times \mathbb{R}^2$ is U-expansive. This means that given two different points of the plane x, y and given any $k > 0$, there exists $n \in \mathbf{Z}$ such that*

$$U(f^n(x), f^n(y)) > k.$$

Proof: Let x and y be two different points of the plane such that $V(x, y) = U(f(x), f(y)) - U(x, y) > 0$. Since $\Delta V > 0$, then $V(f^n(x), f^n(y)) > V(x, y)$ holds for $n > 0$. This means that $U(f^n(x), f^n(y))$ grows to infinity, since

$$U(f^n(x), f^n(y)) = U(x, y) + \sum_{j=0}^{n-1} V(f^j(x), f^j(y)) >$$

$$U(x, y) + nV(x, y).$$

Thus, given $k > 0$ there exists $n \in I\!N$ such that

$$U(f^n(x), f^n(y)) > k.$$

By using similar arguments we can prove the case when $V(x, y) = U(f(x), f(y)) - U(x, y) < 0$. Since $W(x, y) > 0$, if $V(x, y) = 0$, then $V(f(x), f(y)) > 0$ and we fall in the first case considered. $\square$

Definition 2.0.1 *Let $f : I\!R^2 \to I\!R^2$ be a homeomorphism of the plane that admits a Lyapunov metric function U. A point $x \in I\!R^2$ is a stable (unstable) point if given any $k' > 0$ there exists $k > 0$ such that for every $y \in B_k(x)$, it follows that $U(f^n(x), f^n(y)) < k'$ for each $n \geq 0$ ($n \leq 0$).*

Proposition 2.0.2 *Property* (2) *for U implies the non-existence of stable (unstable) points.*

Proof: Given the existence of both signs for $V(x, y) = U(f(x), f(y)) - U(x, y)$ in any neighborhood of x, we can state that for each $k > 0$, there exists a point y in $B_k(x)$ such that $V(x, y) > 0$. Since $\Delta V > 0$, we can state that $V(f^n(x), f^n(y)) > V(x, y)$ for $n > 0$, so $U(f^n(x), f^n(y))$ grows to infinity. Thus, there are no stable points. We can use similar arguments for the unstable case. $\square$

Definition 2.0.2 *Let $C_k(x)$ ($D_k(x)$) be the connected component of the k-stable (k-unstable) set that contains x. We say that $p \in I\!R^2$ has local product structure if a map $h : I\!R^2 \to I\!R^2$ which is a homeomorphism over its image ($p \in Im(h)$) exists and there exists $k > 0$ such that for all $(x, y) \in I\!R^2$ it is verified that $h(\{x\} \times I\!R) = C_k(h(x, y)) \cap Im(h)$ and $h(I\!R \times \{y\}) = D_k(h(x, y)) \cap Im(h)$.*

Proposition 2.0.3 *Except for a discrete set of points, that we shall call singular, every x in $I\!R^2$ has local product structure. The stable (unstable) sets of a singular point y consists of the union of r arcs, with $r \geq 3$ that meet only at y. The stable (unstable) arcs separate unstable (stable) sectors.*

Proof: (See Section 3, [3])

Remark 2.0.2 *The neighborhood's size where there exists a local product structure may become arbitrarily small. However we are able to extend these stable and unstable arcs getting curves that we will denote as $W^s(x)$ and $W^u(x)$, respectively. If two points y and z belong to $W^s(x)$ ($W^u(x)$), then $U(f^n(y), f^n(z) < k$ for some $k > 0$ and for all $n \geq 0$ ($n \leq 0$).*

Finally, we have conditions to state the following theorem:

Theorem 2.0.1 *Let f be a homeomorphism of the plane that admits a Lyapunov function $U : I\!R^2 \times I\!R^2 \to I\!R$ such that the following properties hold:*

- *U is a metric in $I\!R^2$ and induces the same topology in the plane as the usual metric.*
- *For each point $x \in I\!R^2$ and for each $k > 0$ there exist points y and z on the boundary of $B_k(x)$ such that $V(x, y) = U(f(x), f(y)) - U(x, y) > 0$ and $V(x, z) = U(f(x), f(z)) - U(x, z) < 0$, respectively.*

Then, f admits transverse singular foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. Leaves of $\mathcal{F}^s$ ($\mathcal{F}^u$) are the stable (unstable) curves constructed in this section.

3 Foliation description.

In this section we describe those foliations introduced in the previous section with one additional condition for the Lyapunov function U:

Property HP. Let $V = \Delta U$ and $W = \Delta^2 U$. Given any compact set $C \subset \mathbb{R}^2$ the following properties hold:

- there exists $k > 0$ such that

  $$|V(x, y) - V(x, z)| \leq k \text{ for all } y, z \text{ in } C \text{ and for all } x \in \mathbb{R}^2,$$

  and

- $W(x, y)$ tends to infinity as $\|x\|$ tends to infinity, uniformly with $y \in C$.

In [2] these foliations were characterized in the case when the homeomorphism f has no singularities and it has a fixed point. Now we will generalize these results for the case when f is fixed point free. The following lemmas refer to these foliations and we will use them in the proof of theorem 3.0.2.

Lemma 3.0.1 *Let f be a homeomorphism of the plane which verifies the conditions of this section. Then stable and unstable curves intersect each other at most once.*

Proof: If they intersect each other more than once, we would contradict U-expansiveness: if two different points x and y belong to the intersection of a stable and an unstable curve, then there exists $k_0 > 0$ such that $U(f^n(x), f^n(y) < k_0$ for all $n \in \mathbf{Z}$. □

Lemma 3.0.2 *Every stable (unstable) curve separates the plane.*

Proof: See section 3 [1]

Theorem 3.0.2 *Let f be a homeomorphism of the plane such that the following conditions hold:*

- *f admits a Lyapunov metric function $U : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. This metric induces in the plane the same topology as the usual distance;*
- *f has no singularities;*
- *for each point $x \in \mathbb{R}^2$ and any $k > 0$, there exist points y and z in the boundary of $B_k(x)$ such that $V(x, y) = U(f(x), f(y)) - U(x, y) > 0$ and $V(x, z) = U(f(x), f(z)) - U(x, z) < 0$, where $B_k(x) = \{y \in \mathbb{R}^2 \; / U(x, y) \leq k\}$.*
- *Property* **HP.**

Then, there exist transverse foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that every leaf of $\mathcal{F}^s$ intersects every leaf of $\mathcal{F}^u$.

Proof: Let $\mathcal{F}^s$ and $\mathcal{F}^u$ be the stable and unstable foliations constructed in the last section. Let x an arbitrarily point of $\mathbb{R}^2$ and denote by $W^s(x)$ and $W^u(x)$ the leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ through x. We will divide this proof in two steps:

- *If $W^u(y) \cap W^s(x) \neq \emptyset$ then $W^s(y) \cap W^u(x) \neq \emptyset$, for every point y in $\mathbb{R}^2$.* If this statement was not true, let z be the first point of the unstable segment of $W^u(y)$ determined by q ($\{q\} = W^u(y) \cap W^s(x)$) and y such that $W^s(z) \cap W^u(x) = \emptyset$ (see fig. 2). This first point exists since f has no singularities, $W^s(q)$ intersects $W^u(x)$ and the fact that $\mathcal{F}^s$ and $\mathcal{F}^u$ are continuous foliations. Let (z_n) be a sequence of

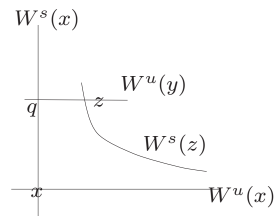

Figure 2:

$W^u(y)$ such that z_n converges to z and $W^s(z_n) \cap W^u(x) \neq \emptyset$, for all n. Let w_n be a sequence defined by $w_n = W^s(z_n) \cap W^u(x)$. As n grows, the behavior of w_n has two possibilities. In the first situation we could find points $w_n = W^s(z_n) \cap W^u(x)$ arbitrarily close to infinity. Since for each n we have that $V(w_n, z_n) < 0$ (because they belong to the same stable leaf) and $V(w_n, x) > 0$ (because they belong to the same unstable leaf), there exists a point q_n, belong to the line segment $z_n x$, such that $V(w_n, q_n) = 0$. Since z_n and q_n belong to a compact set for all n, we can apply

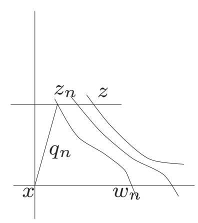

Figure 3:

condition **HP** and find $n \in \mathbb{N}$ such that

$$\frac{|V(w_n, z_n) - V(w_n, q_n)|}{W(w_n, z_n)} < 1,$$

which implies that

$$W(w_n, z_n) + V(w_n, z_n) > 0,$$

and then

$$V(f(w_n), f(z_n)) > 0.$$

This yields a contradiction since points $f(w_n), f(z_n)$ are in the same stable leaf. In the second situation the set $\{w_n : n \in \mathbb{N}\}$ is bounded. Let us consider, as figure 4 shows, the line segment $w_n z_n$ and the compact arc b_n of $W^s(z_n)$ determined by the points w_n and z_n. Let h_n be a point of the arc b_n such that $\|h_n\| \to \infty$ when

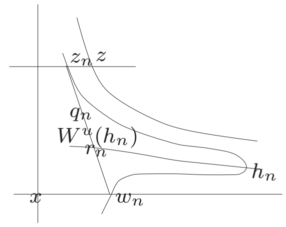

Figure 4:

$n \to \infty$ (such sequence (h_n) exists since $W^s(z)$ separates the plane and $\mathcal{F}^s$ is a continuous foliation). $W^u(h_n)$ must intersect the line segment $w_n z_n$. Otherwise, it would cut $W^s(z_n)$ more than once. Let r_n be that intersection point. We want to apply our condition **HP**. Observe that points r_n, z_n would be in a compact set for all $n \in \mathbb{N}$ and h_n tends to infinity when z_n tends to z. $V(h_n, r_n) > 0$ because they belong to the same unstable leaf, and $V(h_n, z_n) < 0$ because they belong to the same stable leaf. So, there exists a point q_n that belongs to line segment $z_n r_n$ such that $V(h_n, q_n) = 0$. Then

$$\lim_{n\to\infty} \frac{|V(h_n, z_n) - V(h_n, q_n)|}{W(h_n, z_n)} = 0.$$

Therefore, we can choose h_n such that

$$W(h_n, z_n) + V(h_n, z_n) > 0,$$

which implies that

$$V(f(h_n), f(z_n)) > 0.$$

This contradicts the fact that points $f(h_n), f(z_n)$ are in the same stable leaf.

- *$W^u(y) \cap W^s(x) \neq \emptyset$ and $W^s(y) \cap W^u(x) \neq \emptyset$, for every point y in $\mathbb{R}^2$.* Let us consider the set A consisting of the points whose stable (unstable) leaf intersects the unstable (stable) leaf of point x. It is clear that A is open. Let us prove that it is also closed. Let (q_n) be a sequence of A, convergent to some point q (see figure 5). Let $V(q)$ be a neighborhood of q with local product structure. Let us consider

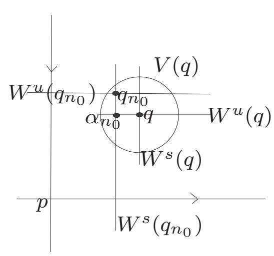

Figure 5: Coordinates

$q_{n_0} \in V(q)$. So, we have that $W^s(q_{n_0}) \cap W^u(q) = \alpha_{n_0}$ as a consequence of the local product structure and $W^s(q_{n_0}) \cap W^u(p) \neq \emptyset$ since $q_{n_0} \in A$. But then α_{n_0} is a point in $W^s(q_{n_0})$ that cuts the unstable leaf of point x, and then, applying the previous step we have that $W^u(q) = W^u(\alpha_{n_0})$ must cut the stable leaf of point x. A similar argument lets us prove that the stable leaf of q must cut the unstable leaf of point x. Therefore q belongs to the set A and consequently A is closed. Then A is the whole plane.

Since x is an arbitrary point, this proof is finished. $\square$

4 Main section.

Let f be a fixed point free homeomorphism of $\mathbb{R}^2$. Brouwer's translation theorem (see [4], [5]) asserts that if f preserves orientation, then every $x_0 \in \mathbb{R}^2$ is contained in a domain of translation for f, i.e. an open connected subset of $\mathbb{R}^2$ whose boundary is $L \cup f(L)$ where L is the image of a proper embedding of $\mathbb{R}$ in $\mathbb{R}^2$, such that L separates $f(L)$ and $f^{-1}(L)$. The purpose of this section is to find some domain of translation of a fixed point free homeomorphism of $\mathbb{R}^2$ which admits a Lyapunov metric function U such that properties presented in the last section hold. This situation will allow us to prove that the homeomorphisms discussed in theorem 3.0.2 are topologically conjugate to a translation of the plane.

Lemma 4.0.3 *Let f be a homeomorphism of $\mathbb{R}^2$ which admits a Lyapunov function $U : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that the following properties hold:*

- *U is a metric in $\mathbb{R}^2$ and induces the same topology in the plane as the usual metric;*
- *for each point $x \in \mathbb{R}^2$ and for each $k > 0$ there exist points y and z on the boundary of $B_k(x)$ such that $V(x,y) = U(f(x), f(y)) - U(x,y) > 0$ and $V(x,z) = U(f(x), f(z)) - U(x,z) < 0$, respectively.*

Let W^s be an arbitrary non-invariant leaf of $\mathcal{F}^s$ such that W^s separates $f(W^s)$ and $f^{-1}(W^s)$. Define D as the open connected subset of $\mathbb{R}^2$ whose boundary is $W^s \cup f(W^s)$ and the open set $U = \bigcup f^k(\overline{D})$ with $k \in \mathbf{Z}$. If $U \neq \mathbb{R}^2$ then the boundary of U consists of the union of leaves of $\mathcal{F}^s$. Similarly for the unstable case.

Proof: If $U \neq \mathbb{R}^2$ then there exists a point p and a sequence (x_n) such that $x_n \in f^n(W^s)$ $n > 0 (n < 0)$ and $\lim x_n = p$. Let $W^s(p)$ be the leaf of $\mathcal{F}^s$ through p. Using continuity of $\mathcal{F}^s$ respect to the initial point we could state that every point of $W^s(p)$ is the limit of a sequence (y_n) such that $y_n \in f^n(W^s)$ $n > 0 (n < 0)$. Thus, $W^s(p)$ is included in the boundary of U. If $W^s(p)$ is non invariant then all the iterates of $W^s(p)$ are also included in the boundary of U. $\square$

Lemma 4.0.4 *Let f be a homeomorphism of $\mathbb{R}^2$ which is fixed point free and admits a Lyapunov function $U : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that the following properties hold:*

- *U is a metric in $\mathbb{R}^2$ and induces the same topology in the plane as the usual metric,*

- *for each point $x \in \mathbb{R}^2$ and for each $k > 0$ there exist points y and z on the boundary of $B_k(x)$ such that $V(x,y) = U(f(x), f(y)) - U(x,y) > 0$ and $V(x,z) = U(f(x), f(z)) - U(x,z) < 0$, respectively,*

- *U admits condition* **HP**,

- *f has no singularities.*

If W^s (W^u) is an arbitrary non invariant leaf of $\mathcal{F}^s$ ($\mathcal{F}^u$), then W^s (W^u) separates $f^2(W^s)$ ($f^2(W^u)$) and $f^{-2}(W^s)$ ($f^{-2}(W^u)$).

Proof: Let W^s be an arbitrary non invariant leaf of $\mathcal{F}^s$ and consider the leaves $f(W^s)$ and $f^{-1}(W^s)$. Let us suppose that none of these three leaves separates the others (see fig. 6). Take any point x in W^s. Applying theorem 3.0.2 we have that $W^u(x)$ must

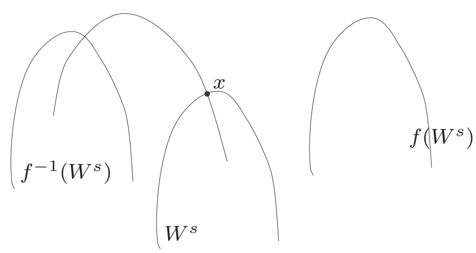

Figure 6:

intersect transversally $f(W^s)$ and $f^{-1}(W^s)$. Also, recall that in lemma 3.0.1 we proved that stable and unstable leaves intersect each other at most once. Then $W^u(x)$ goes from one component to the other determined by W^s only once (remember that W^s separates the plane). Since we are assuming that W^s does not separate $f(W^s)$ and $f^{-1}(W^s)$, then these two leaves are in the same component determined by W^s. So, if $W^u(x)$ intersects $f(W^s)$ then it can not intersect $f^{-1}(W^s)$ (because if it does, we would have either more than one intersection between $W^u(x)$ and $f(W^s)$ or an auto intersection of $W^u(x)$). That yields a contradiction. Then, one of the three stable leaves separates the other two. Then W^s must separate $f^2(W^s)$ and $f^{-2}(W^s)$. $\square$

Lemma 4.0.5 *Let f be a homeomorphism of $\mathbb{R}^2$ which is fixed point free and admits a Lyapunov function $U : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that the following properties hold:*

- *U is a metric in $\mathbb{R}^2$ and induces the same topology in the plane as the usual metric,*

- *for each point $x \in \mathbb{R}^2$ and for each $k > 0$ there exist points y and z on the boundary of $B_k(x)$ such that $V(x,y) = U(f(x), f(y)) - U(x,y) > 0$ and $V(x,z) = U(f(x), f(z)) - U(x,z) < 0$, respectively,*

- *U admits condition* **HP**,

- *f has no singularities.*

If W^s (W^u) is an arbitrary non invariant leaf of $\mathcal{F}^s$ ($\mathcal{F}^u$) such that W^s (W^u) does not separate $f(W^s)$ ($f(W^u)$) and $f^{-1}(W^s)$ ($f^{-1}(W^u)$), then $f^{2n}(W^s)$ or $f^{-2n}(W^s)$, $n > 0$, is a sequence of stable leaves converging to a f^2-invariant stable leaf.

Proof: Let us suppose that $f(W^s)$ separates W^s and $f^{-1}(W^s)$. As fig. 7 shows let us consider an unstable arc a joining a point x of $f(W^s)$ with a point y of W^s. Since

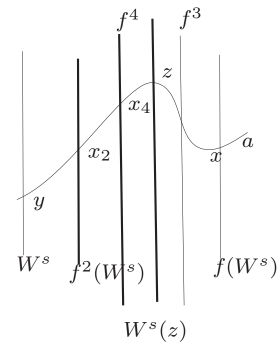

Figure 7:

$f^n(W^s)$ must separate $f^{n-1}(W^s)$ and $f^{n-2}(W^s)$ we conclude that $f^n(W^s)$ must intersect the compact arc a in a point x_n. Using lemma 4.0.4, we can state that sequence x_{2n} is monotone and bounded. Let us denote by z its limit. Since $W^s(z)$ separates $f^{-2}(W^s)$ and $f^2(W^s)$ we conclude that $W^s(z)$ is f^2-invariant. The other case is analogous.

Lemma 4.0.6 *Let f be a homeomorphism of $\mathbb{R}^2$ which is fixed point free and admits a Lyapunov function $U : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that the following properties hold:*

- *U is a metric in $\mathbb{R}^2$ and induces the same topology in the plane as the usual metric,*
- *for each point $x \in \mathbb{R}^2$ and for each $k > 0$ there exist points y and z on the boundary of $B_k(x)$ such that $V(x,y) = U(f(x), f(y)) - U(x,y) > 0$ and $V(x,z) = U(f(x), f(z)) - U(x,z) < 0$, respectively,*
- *U admits condition* **HP**.

Then $U^* = U$ is a Lyapunov function for f^2 which verifies the conditions established for U.

Proof: By a simple computation, we have that:

$$V^*(x,y) = U^*(f^2(x), f^2(y)) - U^*(x,y) = V(f(x), f(y)) + V(x,y),$$

$$\begin{aligned} W^*(x,y) &= V^*(f^2(x), f^2(y)) - V^*(x,y) \\ &= W(f^2(x), f^2(y)) + 2W(f(x), f(y)) + W(x,y). \end{aligned}$$

For each point $x \in \mathbb{R}^2$ and for each $k > 0$, there exists $y \in W^s_f(x) \cap \partial B_k(x)$ and $z \in W^u_f(x) \cap \partial B_k(x)$ (this is true because W^s and W^u separate the plane). Then

$$V^*(x,y) = V(f(x), f(y)) + V(x,y) < 0$$

and

$$V^*(x,z) = V(f(x), f(z)) + V(x,z) > 0.$$

$$|V^*(x,y) - V^*(x,z)|$$
$$\leq |V(f(x), f(y)) - V(f(x), f(z))| + |V(x,y) - V(x,z)|.$$

Therefore $|V^*(x,y) - V^*(x,z)|$ is uniformly bounded when points y and z lie on a compact set.
Since

$$W^*(x,y) = W(f^2(x), f^2(y)) + 2W(f(x), f(y)) + W(x,y),$$

then $W^*(x,y)$ tends to infinity as $\|x\|$ tends to infinity and y lies in a compact set. Hence property **HP** holds for U^*. □

Lemma 4.0.7 *Let f be a homeomorphism of $\mathbb{R}^2$ which is fixed point free and admits a Lyapunov function $U : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ which verifies the conditions established in lemma 4.0.6. Then, f does not have any invariant stable or unstable leaf.*

Proof: Let us reason by contradiction and suppose that there exists an invariant stable leaf W^s. Let x be a point of W^s. Since f is a fixed point free homeomorphism, W^s separates the plane and is invariant under f, we can conclude that $\|f^n(x)\|$ tend to infinity as n grows and that there exists $k > 0$ such that $U(f^n(x), f^{n+1}(x)) \leq k$, for all $n \geq 0$. We claim that $W(x, f^n(x))$ is uniformly bounded for all $n \geq 0$. By definition we have that

$$W(x, f^n(x)) \leq |V(f(x), f^{n+1}(x))| + |V(x, f^n(x))|.$$
$$|V(x, f^n(x))| = U(x, f^n(x)) - U(f(x), f^{n+1}(x))$$
$$\leq U(x, f(x)) + U(f(x), f^{n+1}(x)) + U(f^{n+1}(x), f^n(x)) - U(f(x), f^{n+1}(x))$$
$$= U(x, f(x)) + U(f^{n+1}(x), f^n(x)) \leq 2k.$$

Therefore, $W(x, f^n(x))$ is uniformly bounded by $4k$, and this contradicts condition **HP**.
□

Theorem 4.0.3 *Let f be a homeomorphism of $\mathbb{R}^2$ which is fixed point free and admits a Lyapunov function $U : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that the following properties hold:*

- *U is a metric in $\mathbb{R}^2$ and induces the same topology in the plane as the usual metric,*
- *for each point $x \in \mathbb{R}^2$ and for each $k > 0$ there exist points y and z on the boundary of $B_k(x)$ such that $V(x,y) = U(f(x), f(y)) - U(x,y) > 0$ and $V(x,z) = U(f(x), f(z)) - U(x,z) < 0$, respectively,*
- *f has no singularities.*

Then, f is topologically conjugate to a translation of the plane if and only if U admits condition **HP**.

Proof: If f admits a Lyapunov function which verifies the conditions established in the statement then f must preserve orientation. This is true because if f does not preserve orientation there must exist a non invariant stable (or unstable) leaf W^s such that W^s does not separate $f^{-1}(W^s)$ and $f(W^s)$. Applying lemma 4.0.5 the existence of a f^2-invariant leaf S of $\mathcal{F}^s$ is guaranteed and applying lemmas 4.0.6 and 4.0.7 we arrive to a contradiction. Therefore we have that f preserves orientation and every stable (or unstable) leaf W^s must separate $f^{-1}(W^s)$ and $f(W^s)$. Let W^s be an arbitrary non invariant leaf of $\mathcal{F}^s$. Define D as the open connected subset of $\mathbb{R}^2$ whose boundary is $W^s \cup f(W^s)$ and the open set $U = \bigcup f^k(\overline{D})$ with $k \in \mathbf{Z}$. U is an open set invariant under f such that the restriction of f to U is topologically conjugate to a translation of $\mathbb{R}^2$. If $U = \mathbb{R}^2$, then the theorem is proved. If not, applying lemmas 4.0.3 and 4.0.4 there exists a f^2-invariant stable leaf S in the boundary of U (since S separates $f^2(S)$ and $f^{-2}(S)$, it is easy to prove that S is f^2-invariant). Then applying lemmas 4.0.6 and 4.0.7 we arrive once more to a contradiction.

Reciprocally, lets consider the translation of the plane T defined by $T(x, y) = (x + 1, y + 1)$. It was proved at the introduction that T admits a Lyapunov metric function D with the conditions required in this theorem. Now, lets see the case when f is conjugated to a translation T. Let us define a Lyapunov function for f such as

$$L(p_1, p_2) = D(H(p_1), H(p_2)),$$

where D is the previous defined Lyapunov metric function for T and H is a homeomorphism from $\mathbb{R}^2$ over $\mathbb{R}^2$ such that $H \circ f = T \circ H$. It follows easily that L is a Lyapunov metric function for f such that property **HP** holds. This conclude the proof. $\square$

Using the results of [1], [2] and the last theorem we can state the following general characterization theorem:

Theorem 4.0.4 *Let f be a homeomorphism of $\mathbb{R}^2$ which admits a Lyapunov function $U : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that the following properties hold:*

- *U is a metric in $\mathbb{R}^2$ and induces the same topology in the plane as the usual metric,*
- *For each point $x \in \mathbb{R}^2$ and for each $k > 0$ there exist points y and z on the boundary of $B_k(x)$ such that $V(x, y) = U(f(x), f(y)) - U(x, y) > 0$ and $V(x, z) = U(f(x), f(z)) - U(x, z) < 0$, respectively,*
- *f has no singularities.*

Then,

- *If f admits a fixed point, then f is conjugated to a linear hyperbolic automorphism if and only if U admits condition* **HP***;*
- *If f is fixed point free then f is topologically conjugate to a translation of the plane if and only if U admits condition* **HP**.

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