GENERIC BI-LYAPUNOV STABLE HOMOCLINIC CLASSES

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Preliminary Draft

Abstract. We study, for C^1 generic diffeomorphisms, homoclinic classes which are Lyapunov stable both for backward and forward iterations. We prove they must admit a dominated splitting and show that under some hypothesis they must be the whole manifold. As a consequence of our results we also prove that in dimension 2 the class must be the whole manifold and in dimension 3, these classes must have nonempty interior.

1. Introduction

The study of the dynamics of chain recurrence classes and their interaction with other chain recurrence classes has become a major problem in C^1-generic dynamics since [BC], where generic dynamics and Conley's theory (see section 10.1 of [R]) were unified. In particular, this has raised interest in the study of homoclinic classes which are generically the chain recurrence classes containing periodic orbits.

So, an interesting problem is to study the possible structures and dynamics a homoclinic class may have, particularly those related to the global dynamics of the diffeomorphism. At the moment, very little is known related to this problem, specially when the class is *wild*, that is, accumulated by infinitely many distinct chain recurrence classes (isolated classes are now quite well understood, see [BDV] Chapter 10). In particular, very natural and simple questions remain wide open such as if one of this homoclinic classes may have nonempty interior and not be the whole manifold, or if one of these homoclinic classes may be Lyapunov stable both for f and for f^{-1} (the progress made so far has to do with [ABCD],[ABD],[PotS]). We shall call *bi-Lyapunov stable* to a compact invariant set which is Lyapunov stable both for f and f^{-1}.

On the other hand, also a lot of work has been done towards the understanding of generic dynamics far from homoclinic tangencies (see for example [PS],[W1], [C3],[Y]), but also very few is known in their presence.

We deal in this paper with this kind of difficulties in a simpler context than just wild homoclinic classes. We study homoclinic classes which are bi-Lyapunov stable (for example, homoclinic classes with non empty interior have this property). We are able, by using new techniques developed in [Gou3], to prove that these classes admit a dominated splitting, a weak form of hyperbolicity which helps to order the dynamics in a class (see [BDV], Appendix B). Also, we handle the case where the class is far from tangencies and we prove that in that case, if a generic diffeomorphism admits a homoclinic class which is bi-Lyapunov stable then the diffeomorphism must be transitive.

Also, our results respond affirmatively to the second part of Problem 5.1 in [ABD] and to the other part also in dimension 2 and 3.

1.1. **Context of the problems.** Let M be a compact connected boundaryless manifold of dimension d and let $\text{Diff}^1(M)$ be the set of diffeomorphisms of M endowed with the C^1 topology.

We shall say that a property is generic if and only if there exists a residual set ($G_\delta-$ dense) $\mathcal{R}$ of $\text{Diff}^1(M)$ for which for every $f \in \mathcal{R}$ satisfies that property. Some well known results in the theory of generic dynamical systems are presented in the appendix of this work and they will be refered to when they are used.

For a hyperbolic periodic point $p \in M$ of some diffeomorphism f we denote its *homoclinic class* by $H(p, f)$. It is defined as the closure of the transversal intersections between the stable and unstable manifolds of the orbit of p.

It is a very important problem to study the structure of homoclinic classes since they are basic pieces of the dynamics (see [BDV], Chapter 10). In particular, a very natural question is if they can admit non-empty interior. The only known examples are robustly transitive diffeomorphisms for which there is (generically) only one homoclinic class which coincides with the whole manifold ([BDV], Chapter 7).

In [ABD] the following conjecture was posed (it also appeared as Problem 1 in [BC])

Conjecture 1 ([ABD])**.** *There exists a residual set $\mathcal{R}$ of $\text{Diff}^1(M)$ of diffeomorphisms such that if $f \in \mathcal{R}$ admits a homoclinic class with nonempty interior, then the diffeomorphism is transitive.*

Some progress has been made towards the proof of this conjecture (see [ABD],[ABCD] and [PotS]), in particular, it has been proved in [ABD] that isolated homoclinic classes as well as homoclinic classes admitting a strong partially hyperbolic splitting verify the conjecture. Also, they proved that a homoclinic class with non empty interior must admit a dominated splitting (see Theorem 8 in [ABD]).

In [ABCD] the conjecture was proved for surface diffeomorphisms, other proof for surfaces can be found in [PotS] where the codimension one case is studied.

Also, from the work of Yang ([Y]) one can deduce the conjecture in the case f is generic and far from homoclinic tangencies (we shall extend this remark in section 5).

When studying some facts about this conjecture, in [ABD] it is proved that if a homoclinic class of a generic diffeomorphism has nonempty interior then this class should be bi-Lyapunov stable. Being bi-Lyapunov stable implies in particular that the class is saturated by stable and unstable sets of its points. In fact, in [ABD] they prove that isolated and strongly partially hyperbolic bi-Lyapunov stable homoclinic classes for generic diffeomorphisms are the whole manifold.

This concept is a priori weaker than having nonempty interior and it is natural to ask the following question.

Question 1 (Problem 1 of [BC])**.** *Is a bi-Lyapunov stable homoclinic class of a generic diffeomorphism necessarily the whole manifold?*

For generic diffeomorphisms, there exists a more general notion of basic pieces of the dynamics, namely, the chain recurrence classes ([1]). It is not difficult no deduce from [BC] that, for generic

[1] The chain recurrent set is the set of points x satisfying that for every $\varepsilon > 0$ there exist an ε-pseudo orbit form x to x, that is, there exist points $x = x_0, x_1, \dots x_k = x$ with $k \geq 1$ such that $d(f(x_i), x_{i+1}) < \varepsilon$. Inside the chain

diffeomorphisms, a chain recurrence class with non empty interior must be a homoclinic class, thus, the answer to Conjecture 1 must be the same for chain recurrence classes and for homoclinic classes.

However, we know that the answer to Question 1 is false if posed for chain recurrence classes. Bonatti and Diaz constructed (see [BD]) open sets of diffeomorphisms in every manifold of dimension ≥ 3 admitting, for generic diffeomorphisms there, uncountably many bi-Lyapunov stable chain recurrence classes which in turn have no periodic points.

Although this may suggest a negative answer for Question 1 we present here some results suggesting an affirmative answer. In particular, we prove that the answer is affirmative for surface diffeomorphisms, and that in three dimensional manifold diffeomorphisms the answer must be the same as for Conjecture 1.

The main reason for which the techniques in [ABCD] or in [PotS] are not able to answer Question 1 for sufraces is because differently from the case of homoclinic classes with interior it is not so easy to prove that bi-Lyapunov stable classes admit a dominated splitting (in fact, the bi-Lyapunov stable chain recurrence classes constructed in [BD] do not admit any). Here we are able, by using new techniques introduced by Gourmelon in [Gou3], to show the existence of a dominated splitting for bi-Lyapunov stable homoclinic classes for generic diffeomorphisms. This will give us the result in dimension 2.

Also, using the same techniques, we are able to extend the results previously obtained in [PotS] to the context of Question 1 which in turn allow us (combined with a new result of Yang, [Y]) also to deduce an affirmative answer to the question in the far from tangencies context.

1.2. **Definitions and statement of results.** Let us first recall the definition of dominated splitting: a compact set H invariant under a diffeomorphism f admits *dominated splitting* if the tangent bundle over H splits into two Df invariant subbundles $T_H M = E \oplus F$ such that there exist $C > 0$ and $0 < \lambda < 1$ such that for all $x \in H$:

$$\|Df^n_{/E(x)}\| \|Df^{-n}_{/F(f^n(x))}\| \leq C\lambda^n.$$

In this case we say that the bundle F dominates E. Let us remark that Gourmelon ([Gou1]) proved that there always exists an adapted metric for which we can take $C = 1$ in the definition. Given a dominated splitting $T_H M = E \oplus F$ we say that $\dim E$ is the *index* of the dominated splitting. Similarly, for a hyperbolic periodic point, its *index* is the dimension of its stable bundle. For a homoclinic class H, we define the *minimal index* of H to the minimum of the indexes of its periodic points.

One can have dominated splittings into more than 2 subbundles (see [BDV], appendix B), in particular, a splitting of the form $T_\Lambda M = \bigoplus_{i=1}^m E^i$ is dominated if for every $j < k$, E^k dominates E^j.

We shall say that a bundle E is *uniformly contracting (expanding)* if there exists $n_0 > 0$ ($n_0 < 0$) such that $\|Df^{n_0}_{/E(x)}\| < 1/2 \ \forall x \in H$.

recurrence set, the chain recurrence classes are the equivalence classes of the relation given by $x \vdash\!\dashv y$ when for every $\varepsilon > 0$ there exists an $\varepsilon-$pseudo orbit from x to y and one from y to x (see [BC]).

Recall that a homoclinic class H is *Lyapunov stable for f* if for every neighborhood U of H there is V neighborhood of H such that $f^n(\overline{V}) \subset U$ $\forall n \geq 0$; in particular, it is easy to see that Lyapunov stability implies that $W^u(x) \subset H$ for every $x \in H$. We shall say the class is *bi-Lyapunov stable* if it is Lyapunov stable for f and f^{-1}.

Theorem 1.1. *For every f in a residual subset $\mathcal{R}_1$ of $\text{Diff}^1(M)$, if H is a bi-Lyapunov stable homoclinic class for f, then, H admits a dominated splitting. Moreover, it admits at least one dominated splitting with index equal to the index of some periodic point in the class.*

This theorem solves affirmatively the second part of Problem 5.1 in [ABD]. We remark that Theorem 1.1 doesn't imply that the class is not accumulated by sinks or sources. Also, we must remark that the theorem is optimal in the following sense, in [BV] an example is constructed of a robustly transitive diffeomorphism (thus bi-Lyapunov stable) of $\mathbb{T}^4$ admitting only one dominated splitting (into two two-dimensional bundles) and with periodic points of all possible indexes for saddles.

We recall now that a compact invariant set H is *strongly partially hyperbolic* if it admits a three ways dominated splitting $T_H M = E^s \oplus E^c \oplus E^u$, where E^s is non trivial and uniformly contracting and E^u is non trivial and uniformly expanding.

In the context of Question 1 it was shown in [ABD] that generic bi-Lyapunov stable homoclinic classes admiting a strongly partially hyperbolic splitting must be the whole manifold. Thus, it is very important to study whether the extremal bundles of a dominated splitting must be uniform. We are able to prove this in the codimension one case.

Theorem 1.2. *Let f be a diffeomorphism in a residual subset $\mathcal{R}_2$ of $\text{Diff}^1(M)$ with a homoclinic class H which is bi-Lyapunov stable admitting a codimension one dominated splitting $T_H M = E \oplus F$ where $\dim(F) = 1$. Then, the bundle F is uniformly expanding for f.*

This theorem is an extension of a related result from [PotS] (where Conjecture 1 was studied) where the same result was proved in the case the class has nonempty interior.

As a consequence of this theorem we get the following easy corollaries.

Corollary 1.1. *Let H be a bi-Lyapunov stable homoclinic class for a C^1-generic diffeomorphism f such that $T_H M = E^1 \oplus E^2 \oplus E^3$ is a dominated splitting for f and $\dim(E^1) = \dim(E^3) = 1$. Then, H is strongly partially hyperbolic and $H = M$.*

PROOF. The class should be strongly partially hyperbolic because of the previous theorem. Corollary 1 of [ABD] (page 185) implies that $H = M$.

$\square$

Given a hyperbolic periodic point p in a homoclinic class H of a diffeomorphism f we have that its continuation is well defined in a small neighborhood $\mathcal{U}$ of f and we denote p_g to the continuation for $g \in \mathcal{U}$. Thus, we can also define the continuation of the homoclinic class H for every $g \in \mathcal{U}$ as $H_g = H(g, p_g)$ the homoclinic class for g of p_g. It is well known that given a hyperbolic periodic point p of a diffeomorphism f and $\mathcal{U}$ an open set containing f where the continuation of p is well defined, then there exist a residual subset of $\mathcal{U}$ where the map $g \mapsto H(g, p_g)$ is continuous in the Hausdorff topology (see the Appendix).

We say that a periodic point p is *far from tangencies* if there is a neighborhood of f such that there are no homoclinic tangencies associated to the stable and unstable manifolds of the continuation of p. The tangencies are of index i if they are associated to a periodic point of index i, that is, its stable manifold has dimension i. We get the following result following [ABCDW]:

Corollary 1.2. *Let H be a bi-Lyapunov stable homoclinic class for a C^1-generic diffeomorphism f which has a periodic point p of index 1 and a periodic point q of index $d-1$ and such that p and q are far from tangencies . Then, $H = M$.*

Proof. Using Corollary 3 of [ABCDW] we are in the hypothesis of Corollary 1.

□

Using a result of Yang (Theorem 3 of [Y]) and Theorem 1.2 we are able to prove a similar result which is stronger than the previous corollary but which in turn, has hypothesis of a more global nature. We say that a diffeomorphism f is *far from tangencies* if it can not be approximated by diffeomorphisms having homoclinic tangencies for some hyperbolic periodic point.

Proposition 1.1. *There exists a C^1-residual subset of the open set of diffeomorphisms far from tangencies such that if H is a bi-Lyapunov stable homoclinic class for such a diffeomorphism, then, $H = M$.*

With our results we are able to deduce some stronger statements for bi-Lyapunov homoclinic classes for generic diffeomorphisms in manifolds of low dimensions.

In [PotS], our results allowed us to deduce the conjecture in dimension 2 since a homoclinic class with nonempty interior must admit a dominated splitting (this can be deduced from [BDP] in the case the homoclinic class has nonempty interior), so, combining Theorem 1.1 with Theorem 1.2 we get the following Theorem.

Theorem 1.3. *Let f be a C^1-generic surface diffeomorphism having a bi-Lyapunov stable homoclinic class. Then f is conjugated to a linear Anosov diffeomorphism in $\mathbb{T}^2$.*

For the readers convenience, in section 3, we give a proof of Theorem 1.3 which is independent from Theorem 1.1. It contains the ideas of Theorem 1.1 but less technicalities.

Theorem 1.1 together with Theorem 1.2 combine to give the following interesting result which supports the extension of the conjecture to bi-Lyapunov stable classes and solves completely Problem 5.1 in [ABD] for 3 dimensional manifolds.

Proposition 1.2. *Let H be a bi-Lyapunov stable homoclinic class for a C^1-generic diffeomorphism in dimension 3. Then, H has nonempty interior.*

Remark 1. The proof of this Proposition also gives us that if H is a bi-Lyapunov stable homoclinic class for a C^1-generic diffeomorphism in any dimension which admits a codimension one dominated splitting $T_H M = E \oplus F$ with $\dim F = 1$ and a periodic point p of index $d-1$ then the class has nonempty interior.

1.3. **Idea of the proof.** The idea of the proof of Theorem 1.2 is the following.

First we prove that if the homoclinic class is bi-Lyapunov stable, the periodic points in the class (which are all saddles) should have eigenvalues (in the F direction) exponentially (with the period) far from 1. Otherwise we manage to obtain a sink or a source inside the class (by using an improved version of Franks' lemma by Gourmelon, [Gou3], which allows to perturb the derivative controlling the invariant manifolds and also using Lyapunov stability) which is a contradiction. This was done in [PotS] when the class had interior and using the fact that the interior is somehow robust, this is the main difference with [PotS]. After this is done, one can conclude as in [PotS].

To get the dominated splitting in the class (Theorem 1.1) we use a similar idea, if the class doesn't admit a dominated splitting, then, by [BDP] one can create a sink or a source by perturbation, so, using again the Franks' lemma with control of the invariant manifolds, we are able to ensure that the sink or source is contained in the class using also Lyapunov stability and thus reach a contradiction. A similar idea also gives information on the index of the dominated splitting.

To be able to use the improved version of Franks' lemma given by Gourmelon we must first prove a technical result (Theorem 4.1) which may be of interest for other applications as it allows for some applications to use the perturbation results of [BGV] in the context of the main theorem in [Gou3].

1.4. **Organization of the paper.** In section 2 we shall prove Theorem 1.2 and in section 4 we shall prove Theorem 1.1. The main argument used in the paper is presented with full details in the end of the proof of Lemma 2.1, the other places where we use it will refer to it. In section 3 we prove the results in low dimensions, namely Theorem 1.3 (with a proof independent from Theorem 1.1) and Proposition 1.2 using Theorems 1.1 and 1.2. We also present Yang's result (Theorem 3 of [Y]) and its consequences in section 5. We present also a proof of Yang's theorem based in [C3]. In the appendix we recall some generic results which are very well known and shall be used in the course of the paper.

Acknowledgements: Many of the ideas here came out from conversations with Martin Sambarino, I would like to thank him for that and for reading the first version and making suggestions. I would really like to thank Sylvain Crovisier also for many corrections, both on the writing and on the mathematical aspects, also, he helped improve the results and some proofs, and explained me Yang's results and their relation with [C3] and with Theorem 1.2.

2. Proof of the Theorem 1.2

For $p \in Per(f)$, $\pi(p)$ denotes the period of p. We denote as $Per^{\alpha}(f)$ the set of index α periodic orbits. Let E be a Df invariant subbundle of $T_{\mathcal{O}}M$; $D_{\mathcal{O}}f_{/E}$ denotes the cocycle over the periodic orbit given by its derivative restricted to the invariant subbundle.

We shall make some definitions, let $\mathcal{O}$ be a periodic orbit and A be a linear cocycle over $\mathcal{O}$. We say that A has a *strong stable manifold of dimension* i if the eigenvalues of A, $|\lambda_1| \leq |\lambda_2| \leq \ldots \leq |\lambda_d|$ satisfy that $|\lambda_i| < \min\{1, |\lambda_{i+1}|\}$. If the derivative of $\mathcal{O}$ has strong stable manifold of dimension i then classical results ensure the existence of a local, invariant manifold integrating the eigenvectors of this i eigenvalues and imitating the behaviour of the derivative (see [HPS]).

Let Γ_i be the set of cocycles over $\mathcal{O}$ which have a strong stable manifold of dimension i.

We endow Γ_i with the following distance, $d(\sigma, \tau) = \max\{\|\sigma - \tau\|, \|\sigma^{-1} - \tau^{-1}\|\}$ where the norm is $\|\sigma\| = \sup_{p\in\mathcal{O}}\{\frac{\|\sigma_p(v)\|}{\|v\|}\ ;\ v \in T_pM\backslash\{0\}\}$.

Let g be a perturbation of f such that the cocycles $D_{\mathcal{O}}f$ and $D_{\mathcal{O}}g$ are both in Γ_i, and let U be a neighborhood of $\mathcal{O}$ we shall say that *g preserves locally the $i-$ strong stable manifold of f outside U* if the set of points of the $i-$strong stable manifold of $\mathcal{O}$ outside U whose positive iterates don't leave U once they entered it are the same for f and for g.

We have the following theorem due to Gourmelon which will be the key tool for proving the results here presented.

Theorem 2.1 ([Gou3]). *Let f be a diffeomorphism such that $D_{\mathcal{O}}f \in \Gamma_i$ and let $\gamma : [0,1] \to \Gamma_i$ be a path starting at $D_{\mathcal{O}}f$. Then, given a neighborhood U of $\mathcal{O}$, there is a perturbation g of f such that $D_{\mathcal{O}}g = \gamma(1)$, g coincides with f outside U and preserves locally the $i-$strong stable manifold of f outside U. Moreover, given $\mathcal{U}$ a C^1 neighborhood of f, there exists $\varepsilon > 0$ such that if* $\operatorname{diam}(\gamma) < \varepsilon$ *one can choose $g \in \mathcal{U}$.*

We observe that Franks' lemma (see [F2]) is the previous theorem with Γ_0. Also, we remark that Gourmelon result is more general since it allows to preserve more than one stable and more than one unstable manifolds (of different dimensions, see [Gou3]).

Lemma 2.1. *Let H be a homoclinic class which is Lyapunov stable of a C^1-generic diffeomorphism f and such that the class has only periodic orbits of index smaller or equal to α. So, there exists $K_0 > 0$, $\lambda \in (0,1)$ and $m_0 \in \mathbb{Z}$ such that for every $p \in Per^{\alpha}(f_{/H})$ one has*

$$\prod_{i=0}^{k}\left\|\prod_{j=0}^{m_0-1} Df^{-1}_{/E^u(f^{-im_0-j}(p))}\right\| < K_0\lambda^k \qquad k = \left[\frac{\pi(p)}{m_0}\right].$$

Proof. Let $\mathcal{R}$ be a residual subset of $\operatorname{Diff}^1(M)$ such that if $f \in \mathcal{R}$ and H is a Lyapunov stable homoclinic class of a periodic orbit q, there exists a small neighborhood $\mathcal{U}$ of f where the continuation q_g of q is well defined and such that for every $g \in \mathcal{U} \cap \mathcal{R}$ one has that $H(g, q_q)$ is Lyapunov stable and such that g is a continuity point of the map $g \mapsto H(g, q_g)$ (see the Appendix).

We can also assume that $\mathcal{U}$ and $\mathcal{R}$ were chosen so that for every $g \in \mathcal{U}\cap\mathcal{R}$ every periodic point in $H(g, q_g)$ has index smaller or equal to α. We can also assume that q_g has index α $\forall g \in \mathcal{U}$.

Lemma II.5 of [M] asserts that to prove the thesis is enough to show that there exists $\varepsilon > 0$ such that the set of cocycles $\Theta_\alpha = \{D_{\mathcal{O}}f^{-1}_{/E^u}\ :\ \mathcal{O} = \mathcal{O}(p)\ ,\ p \in Per^{\alpha}(f_{/H})\}$ which all have its eigenvalues of modulus smaller than one, verify that every $\varepsilon-$perturbation of them preserves this property. That is, given $p \in Per^{\alpha}(f_{/H})$ one has that every $\varepsilon-$perturbation $\{A_0, \ldots, A_{\pi(p)-1}\}$ of $D_{\mathcal{O}(p)}f$ verifies that $A_{\pi(p)-1} \ldots A_0$ has all its eigenvalues of modulus bigger or equal to one.

Therefore, assuming by contradiction that the Lemma is false, we get that $\forall\varepsilon > 0$ there exists a periodic point $p \in Per^{\alpha}(f_{/H})$ and a linear cocycle over p, $\{A_0, \ldots, A_{\pi(p)}\}$ satisfying that $\|D_{f^i(p)}f_{/E^u} - A_i\| \le \varepsilon$ and $\|D_{f^i(p)}f^{-1}_{/E^u} - A_i^{-1}\| \le \varepsilon$ and such that $\prod_{i=0}^{\pi(p)-1} A_i$ has some eigenvalue of modulus smaller or equal to 1.

In coordinates $T_{\mathcal{O}(p)}M = E^u \oplus (E^u)^\perp$, since E^u is invariant we have that the form of Df is given by

$$D_{f^i(p)}f = \begin{pmatrix} D_{f^i(p)}f_{/E^u} & K_i^1(f) \\ 0 & K_i^2(f) \end{pmatrix}$$

Since this coordinates vary continuously, there exists a constant $C > 0$ such that for every matrix M which is written in this coordinates as $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we have that $C^{-1}\max\{\|A\|, \|B\|, \|C\|, \|D\|\} < \|M\| < C\max\{\|A\|, \|B\|, \|C\|, \|D\|\}$.

Let $\gamma : [0,1] \to \Sigma_\alpha$ given in coordinates $T_{\mathcal{O}(p)}M = E^u \oplus (E^u)^\perp$ by

$$\gamma_i(t) = \begin{pmatrix} (1-t)D_{f^i(p)}f_{/E^u} + tA_i & K_i^1(f) \\ 0 & K_i^2(f) \end{pmatrix}$$

whose diameter is bounded by $C\varepsilon$.

Now([2]), choose a point x of intersection between $W^s(p,f)$ with $W^u(q,f)$ and choose a neighborhood U of the orbit of p such that:

(i) It doesn't intersect the orbit of q.
(ii) It doesn't intersect the past orbit of x.
(iii) It verifies that once the orbit of x enters U it stays there for all its future iterates by f.

It is very easy to choose U satisfying (i) since both the orbit of p and the one from q are finite. Since the past orbit of x accumulates in q is not difficult to choose U satisfying (ii). To satisfy (iii) one has only to use the fact that x belongs to the stable manifold of p so, after a finite number of iterates, x will stay in the local stable manifold of p, and it is not difficult then to choose a neighborhood U which satisfies (iii) also.

If we now apply Theorem 2.1 we can perturb f to a new diffeomorphism $\hat{g}$ so that the orbit of p has index greater than α and so that it preserves locally its strong stable manifold. This allows to ensure the intersection between $W^u(q_{\hat{g}}, \hat{g})$ and $W^s(p, \hat{g})$.

This intersection is transversal so it persist by small perturbations, the same occurs with the index of p so we can assume that $\hat{g}$ is in $\mathcal{R} \cap \mathcal{U}$ and using Lyapunov stability of $H(\hat{g}, q_{\hat{g}})$ we obtain that $p \in H(\hat{g}, q_{\hat{g}})$. This is because a Lyapunov stable homoclinic class is saturated by unstable sets, so, since $q_{\hat{g}} \subset H(\hat{g}, q_{\hat{g}})$ we have that $\overline{W^u(\hat{g}, q_{\hat{g}})} \subset H(\hat{g}, q_{\hat{g}})$ and since $W^s(p, \hat{g}) \cap W^u(\hat{g}, q_{\hat{g}}) \neq \emptyset$, we get that $p \in \overline{W^u(\hat{g}, q_{\hat{g}})}$ and we get what we claimed.

This contradicts the choice of $\mathcal{U}$ since we find a diffeomorphism in $\mathcal{U}$ with a periodic point with index bigger than α in the continuation of H, and so the lemma is proved.

□

Remark 2. One can recover Lemma 2 in [PotS] in this context. In fact, if there is a codimension one dominated splitting of the form $T_H M = E \oplus F$ with $\dim F = 1$ then (using the addapted

[2] The following argument will be used repeatedly along the paper.

metric given by [Gou1]) for a periodic point of maximal index one has $\|Df^{-1}_{/F(p)}\| \leq \|Df^{-1}_{/E^u(p)}\|$ so,

$$\prod_{i=0}^{k} \left\| \prod_{j=0}^{m_0-1} Df^{-1}_{/F(f^{-im_0-j}(p))} \right\| < K_0 \lambda^k \qquad k = \left[\frac{\pi(p)}{m_0} \right]$$

And since F is one dimensional one has $\prod_i \|A_i\| = \| \prod_i A_i\|$ so $\|Df^{-\pi(p)}_{/F(p)}\| \leq \tilde{K}_0 \tilde{\lambda}^{\pi(p)}$.

In fact, there is $\gamma \in (0,1)$ such that for every periodic point of maximal index and big enough period one has $\|Df^{-\pi(p)}_{/F(p)}\| \leq \gamma^{\pi(p)}$

Also, it is not hard to see that if the class admits a dominated splitting of index bigger or equal than the index of all the periodic points in the class then periodic points should be hyperbolic in the period along F (for a precise definition and discussion on this topics one can read [BGY], [W2]).

$\diamondsuit$

Remark 3. In the proof of the lemma we in fact proved that we can perturb in an invariant subspace of a cocycle in order to change the eigenvalues without altering the eigenvalues in the rest of the subspaces and with a perturbation of similar size to the size of the perturbation in the cocycle. See Lemma 4.1 of [BDP]. Notice also that we could have perturbed the cocycle $\{K_i^2(f)\}_i$ without altering the eigenvalues in of the cocycle $D_{\mathcal{O}(p)}f_{/E^u}$.

$\diamondsuit$

One can now conclude the proof of Theorem 1.2 with the same techniques as in the proof of the main Theorem of [PotS].

We have that $T_H M = E \oplus F$ with $\dim F = 1$. We first prove that the center unstable curves integrating F should be unstable and with uniform size (this is Lemma 3 of [PotS]). To do this, we first use Lemma 2.1 to get this property in the periodic points and then uses the results from [PS2] to extend the property to the rest of the points.

Assuming the bundle F is not uniformly expanded, one has two cases, one can apply Liao's selecting lemma or not (see [W2]).

In the first case one gets periodic points inside the class which contradict the thesis of Lemma 2.1. The second case is similar, if Liao's selecting lemma doesn't apply, one gets a minimal set inside H where E is uniformly contracting and then classical arguments give a periodic point close to this minimal set. Since the stable manifold of this periodic point will be uniform, it will intersect the unstable manifold of a point in H, and then Lyapunov stability implies the point is inside the class and again contradicts Lemma 2.1.

For more details see [PotS].

3. Low dimensional consequences

We shall first prove Theorem 1.3. We remark that it is an easy corollary of Theorems 1.1 and 1.2 since toghether they give hyperbolicity of the homoclinic class. However, this proof doesn't use Theorem 1.1 and we believe it may illuminate the idea of the more general proof.

Proposition 3.1. *Let H be a bi-Lyapunov stable homoclinic class of a C^1-generic surface diffeomorphism f. Then, H admits a dominated splitting and thus $H = \mathbb{T}^2$ and f is an Anosov diffeomorphism.*

Proof . Consider a periodic point $q \in H$ fixed such that for a neighborhood $\mathcal{U}$ of f the class $H(q_g, g)$ is bi-Lyapunov stable for every g in a residual subset of $\mathcal{U}$.

If the class doesn't admit a dominated splitting, arguments from [M] allow us to deduce that after a small perturbation, for any $\delta > 0$ we have a periodic point p such that the angle between the stable and unstable bundle is smaller than δ. Let us assume for example that the determinant of $Df_p^{\pi(p)}$ is smaller than one.

By classical arguments (see for example Lemma 7.7 of [BDV]) one knows that composing $D_{\mathcal{O}(p)}f$ with a rotation of angle smaller than δ one has that the resulting cocycle, A, has complex eigenvalues of modulus smaller than 1 (since the rotation has determinant equal to 1). Also, the same argument of the determinant implies that we can join $D_{\mathcal{O}(p)}f$ with A by a curve with small diameter and such that every cocycle in the curve has one eigenvalue with modulus smaller than one.

This implies that we can apply Theorem 2.1 to perturb the periodic point p preserving its strong stable manifold locally.

Arguing as in Lemma 2.1 we can do the perturbation so that $W^u(q_g, g)$ intersects $W^s(p, g)$ and thus, using Lyapunov stability we obtain that $p \in H(q_g)$ which is absurd since p is a sink.

The rest follows by applying Theorem 1.2 which implies that H is hyperbolic and thus using local product structure we get that $H = M$. So, f is an Anosov diffeomorphism, and by Franks' theorem ([F1]), it must be conjugated to a linear Anosov diffeomorphism of $\mathbb{T}^2$.

□

Now, we shall deduce with the results of this paper Proposition 1.2. This gives that Question 1 has the same answer as Conjecture 1 for diffeomorphisms in 3 dimensional manifolds. The results from [PotS] also imply that for studying such Conjecture one has to look only at homoclinic classes with interior admitting a dominated splitting of the form $E \oplus E^u$ where E^u is uniformly expanded of dimension 1 and there are periodic points of index 1 and of index 2 (some of them with complex eigenvalues) inside the homoclinic class. Also, one can assume volume hyperbolicity in the period for the bundle E.

Proof of Proposition 1.2. Applying Theorem 1.1 one can assume that the class H admits a dominated splitting of the form $E \oplus F$, and without loss of generality one can assume that $\dim F = 1$.

Theorem 1.2 thus implies that F is uniformly expanded so the splitting is $T_H M = E \oplus E^u$.

Assume first that there exist a periodic point p in H of index 2. Thus, this periodic point has a local stable manifold of dimension 2 which is homeomorphic to a 2 dimensional disc.

Since the class is Lyapunov stable for f^{-1} the stable manifold of the periodic point is completely contained in the class.

Now, using Lyapunov stability for f and the foliation by strong unstable manifolds given by [HPS] one gets (saturating by unstable sets the local stable manifold of p) that the homoclinic class contains an open set. This implies the thesis under this assumption.

So, we must show that if all the periodic points in the class have index 1 then the class is the whole manifold. As we have been doing, using the genericity of f we can assume that there is a residual subset $\mathcal{R}$ of $\text{Diff}^1(M)$ and an open set $\mathcal{U}$ of f such that for every $g \in \mathcal{U} \cap \mathcal{R}$ all the periodic points in the class have index 1.

We have 2 situations, on the one hand, we consider the case where E admits two invariant subbundles $E = E^1 \oplus E^2$ with a dominated splitting and thus we get that E^1 should be uniformly contracting (using Theorem 1.2) and thus proving that the homoclinic class is the whole manifold (Corollary 1.1).

If E admits no invariant subbundles then, using Theorem 4.1 bellow we get that we can perturb the derivative in a periodic point in the class in such a way that the cocycle over the periodic point restricted to E has all its eigenvalues contracting thus we can construct a periodic point of index 2 inside the class.

□

Remark 4. It is very easy to adapt the proof of this proposition to get that if a bi-Lyapunov stable homoclinic class of a generic diffeomorphism admits a codimension one dominated splitting $T_H M = E \oplus F$ with $\dim F = 1$ and has a periodic point of index $d-1$ then the class has nonempty interior.

♢

4. Existence of a dominated splitting

We prove here Theorem 1.1 which states that a bi-Lyapunov stable homoclinic class of a generic diffeomorphism admits a dominated splitting. The idea is the following: in case H doesn't admit any dominated splitting we shall find a sink or a source in the class. For that we shall use Theorem 2.1 and the techniques from [BDP] and [BGV] to ensure that the stable or unstable manifold of a periodic point in the class intersects the unstable or stable set of the source or the sink respectively and reach a contradiction.

In fact, this generalizes to the case where the class doesn't admit a dominated splitting with index in between the indexes of the periodic points in the class since by using these kind of results we may construct either a periodic point of smaller index than those on the class or one of bigger index and manage to relate it to the points in the class using Lyapunov stability.

This section is divided in two, in the first part we prove a technical result which allows us to use the results of [BGV] together with the result of Gourmelon (Theorem 2.1) which needs that the perturbations over the derivative are made over a specific kind of paths. This will be done by using the ideas in [BGV], luckily we are able to use also their result so that we do not need to rewrite all the proof to adapt it to our context. This result may have independent interest for other applications. We remark that in [Gou3], Section 6, it is claimed that the perturbations made in [BGV] can be done along small paths.

In the second part, we prove Theorem 1.1.

4.1. **Perturbations of cocycles over small paths.** We shall first give some definitions taken from [BGV].

Let $\mathcal{A} = (\Sigma, f, E, A)$ be a *large period linear cocycle* of dimension d bounded by K over an infinite set Σ, that is

- $f : \Sigma \to \Sigma$ is a bijection such that all points in Σ are periodic and such that given $n > 0$ there are only finitely many with period less than n.
- E is a fiber bundle over Σ, that is, there is $p : E \to \Sigma$ such that $E_x = p^{-1}(x)$ is a vector space of dimension d endowed with an euclidean metric $\langle , \rangle_x$.
- $A : x \in \Sigma \mapsto GL(E_x, E_{f(x)})$ is such that $\|A_x\| \leq K$ and $\|A_x^{-1}\| \leq K$.

For every $x \in \Sigma$ we denote by $\pi(x)$ its period and $M_x^A = A_{f^{\pi(x)-1}} \circ \ldots \circ A_x$.

We shall say a subbundle $F \subset E$ is invariant if $\forall x \in \Sigma$ we have $A_x(F_x) = F_{f(x)}$.

We shall say that $\mathcal{A}$ admits *dominated splitting* if one can decompose $E = F \oplus G$ in two non trivial invariant subbundles such that there exists $\ell > 0$ verifying that for every $x \in \Sigma$, $v \in F_x$, $w \in G_x$ one has

$$\frac{\|A_x^\ell(v)\|}{\|v\|} \leq \frac{1}{2} \frac{\|A_x^\ell(w)\|}{\|w\|} \ .$$

We say that $\mathcal{A}$ is *strictly without domination* if for every $\Sigma' \subset \Sigma$ which is a $f-$invariant subset admiting a dominated splitting we have that Σ' is finite.

Let Γ_x be the set of cocycles over the orbit of x with the topology given in section 2.

Given $\{A_x, A_{f(x)}, \ldots, A_{f^{\pi(p)-1}(x)}\} \in \Gamma_x$, we say that $\{B_x, B_{f(x)}, \ldots, B_{f^{\pi(p)-1}(x)}\} \in \Gamma_x$ is a *perturbation of the cocycle over the orbit of x along a path of diameter $\leq \varepsilon$* if there is a continuous path $\gamma : [0,1] \to \Gamma_x$ of diameter $\leq \varepsilon$ such that $\gamma(0) = \{A_x, A_{f(x)}, \ldots, A_{f^{\pi(p)-1}(x)}\}$ and $\gamma(1) = \{B_x, B_{f(x)}, \ldots, B_{f^{\pi(p)-1}(x)}\}$. For simplicity, we shall abuse notation and write $\gamma(0) = A$, $\gamma(1) = B$.

We shall prove in this subsection the following Theorem which allows us to use the results from [BGV] together with the improved version of Franks' Lemma given by Gourmelon.

Theorem 4.1. *Let $\mathcal{A} = (\Sigma, f, E, A)$ be a large period linear cocycle of dimension d bounded by K strictly without domination. Assume also that all the matrices M_x^A have stable index (associated to the eigenvalues of modulus < 1) bigger or equal to i and such that $|det(M_x^A)| < 1$ for every x. Then, given $\varepsilon > 0$ there exists a point $x \in \Sigma$ and a path γ_x of diameter smaller than ε such that $\gamma_x(0) = A$, $M_x^{\gamma_x(1)}$ has all its eigenvalues of the same modulus, smaller than one, and the stable index of $M_x^{\gamma_x(t)}$ is bigger or equal to i for every $t \in [0,1]$.*

Following [BGV] we say that $\mathcal{B}$ is a perturbation of $\mathcal{A}$ ($\mathcal{B} \sim \mathcal{A}$) if for every $\varepsilon > 0$ the set of points $x \in \Sigma$ such that $\{B_x, \ldots, B_{f^{\pi(x)-1}(x)}\}$ is not $\varepsilon-$close to the cocycle $\{A_x, \ldots, A_{f^{\pi(x)-1}(x)}\}$ os finite.

Similarly, we say that $\mathcal{B}$ is a *path perturbation* of $\mathcal{A}$ if for every $\varepsilon > 0$ one has that the set of points $x \in \Sigma$ such that $\{B_x, B_{f(x)}, \dots, B_{f^{\pi(p)-1}(x)}\}$ is not a perturbation of the cocycle over the orbit of x along a path of diameter $\leq \varepsilon$ is finite. We shall denote this relation by $\mathcal{B} \sim^* \mathcal{A}$.

We have that $\sim$ and $\sim^*$ are equivalence relations and clearly $\sim^*$ is contained in $\sim$.

Remark 2.13 of [BGV] applies in the context of path perturbation, we get then that if $\mathcal{B} \sim^* \mathcal{A}$ then, there exists $\mathcal{C} \sim^* \mathcal{A}$ such that for every $x \in \Sigma$ we have $det(M_x^C) = det(M_x^A)$ and a real valued function $\alpha : \Sigma \to \mathbb{R}$ such that $C_x = \alpha(x)B_x$.

Corollary 2.15 of [BGV] asserts that if $\mathcal{A}$ is strictly without domination then the same happens to every $\mathcal{B} \sim \mathcal{A}$. Since $\sim^*$ is contained in $\sim$ we get the same statement for path perturbations of $\mathcal{A}$.

We shall first give a sketch of the proof of Theorem 2.1 of [BGV] to show how its proof easily adapts to our context and gives the following:

Lemma 4.1. *For every $\mathcal{A}$ cocycle, there exists a path perturbation $\mathcal{B}$ such that for every $x \in \Sigma$ the eigenvalues of M_x^B have all different modulus and their modulus is arbitrarily near the original one in M_x^A. Also, we can choose the perturbation such that given $\varepsilon > 0$, the set of points for which the perturbation is bigger than ε is finite.*

SKETCH We proceed by induction. In dimension 2 the result is the same as in Proposition 3.7 of [BGV] (the only perturbations done there can be made along paths without any difficulty).

If we assume the result holds in dimension $< d$, one can prove it also holds in dimension d by noticing that there always exists an invariant subspace of dimension 2, so you can make independent perturbations and change the eigenvalues as required using the induction hipothesis (see Remark 3), for this, one should perturb in the invariant subspace and in the quotient (see the proof of Theorem 2.1 in [BGV]).

□

We shall prove Theorem 4.1 also by induction. We first prove it in dimension 2.

Lemma 4.2. *Let $\mathcal{A}$ be a cocycle of dimension 2 strictly non dominated such that $|det(M_x^A)| < 1$ for every $x \in \Sigma$. Then, there exists $\mathcal{B} \sim^* \mathcal{A}$ such that the path that joins $\mathcal{A}$ with $\mathcal{B}$ mantains or increases the stable index of every $x \in \Sigma$ and such that $\mathcal{B}$ has the following properties:*

1. *The set of points with both eigenvalues with the same modulus is infinite.*
2. *$|det(M_x^A)| = |det(M_x^B)|$ for every $x \in \Sigma$.*

PROOF. This is very standard (see [M] or section 7.2.1 of [BDV]). With a small perturbation which can be done along arbitrarily small paths one can make the angle between the stable and unstable spaces arbitrarily small. After other perturbation one creates complex eigenvalues. It is easy to see that the perturbation can be made in order to not change the index along the path, since the fact that the determinant is smaller than one guaranties that there is always one eigenvalue of modulus smaller than one.

□

We shall now give a definition motivated by [BGV]. Let $\sigma^+(x,\mathcal{A})$ (resp. $\sigma^-(x,\mathcal{A})$) the largest Lyapunov exponent (resp. smallest) of M_x^A, that is $\frac{log|\lambda^+|}{\pi(x)}$ (resp. $\frac{log|\lambda^-|}{\pi(x)}$) where λ^+ and λ^- are the eigenvalues of largest and smallest modulus of M_x^A. The *Lyapunov diameter* of the cocycle $\mathcal{A}$ is defined as $\delta(\mathcal{A}) = \underline{\lim}_{\pi(x)\to\infty}(\sigma^+(x,\mathcal{A}) - \sigma^-(x,\mathcal{A}))$.

We define $\delta_{min}(\mathcal{A}) = \inf_{\mathcal{B}\sim\mathcal{A}}\{\delta(\mathcal{B})\}$ and $\delta^*_{min}(\mathcal{A})$ defined as the infimum of $\delta(\mathcal{B})$ among path perturbations of $\mathcal{A}$ which mantain or increase the stable index of periodic points.

Lemma 4.3 of [BGV] proves that the infimum is in fact a minimum. The proof applies verbatim to path perturbations, that is, $\forall\mathcal{A}$ we have that there exists $\mathcal{B}$ a path perturbation of $\mathcal{A}$ along a path which maintains or increases the index and such that $\delta(\mathcal{B}) = \delta^*_{min}(\mathcal{A})$.

Theorem 4.1 of [BGV] implies that a cocycle is strictly without dominated splitting if and only if $\delta_{min}(\mathcal{A}) = 0$.

To finish our Theorem it is enough to prove that if $\mathcal{A}$ is strictly without domination, then $\delta^*_{min}(\mathcal{A}) = 0$. We have proved that in dimension 2 this is the case (Lemma 4.2), we must prove now that if the assertion holds in dimension smaller than d then it also holds in dimension d.

Let $\mathcal{B}$ be a path perturbation of $\mathcal{A}$ maintaining or increasing the index such that $\delta(\mathcal{B}) = \delta^*_{min}(\mathcal{A})$. We shall prove that if $\delta(B) > 0$ we can decrease it with a small perturbation. This will give a contradiction so we can conclude that $\delta(\mathcal{B}) = 0$ as wanted.

First, we use Lemma 4.1 so we can assume that for every $x \in \Sigma$ the eigenvalues of M_x^B are all real and of different modulus without changing $\delta(\mathcal{B})$ (if there is some eigenvalue equal to one we perturb first with a small contracting homothety to allow the perturbation to maintain or increase the index).

So, let $E_1 \oplus \ldots \oplus E_d$ be the decomposition of E in the invariant one dimensional eigenspaces associated to the eigenvalues in increasing order of modulus. Since $\mathcal{A}$ is strictly without domination, Theorem 4.1 of [BGV] implies that there is no subdominated splitting between these subspaces since that would imply that $\delta_{min}(\mathcal{A}) > 0$.

Now, the induction hypothesis allows us to ensure that the eigenvalues associated to $E_1, \ldots E_{d-1}$ are all very near (notice that the determinant there must be smaller than 1). Otherwise, there would be a perturbation which decreases the distance between eigenvalues there and not altering the other eigenvalue (see Remark 3).

Now, since $\delta(\mathcal{B}) > 0$ we know that the eigenvalue in E_d is strictly bigger than the rest. So if we consider the invariant subspace given by $E_{d-1} \oplus E_d$ we have that it cannot admit a dominated splitting. So, we can perturb along a path in order to decrease $\delta(\mathcal{B})$ without changing the index of the points in Σ.

Notice that if the determinant in $E_{d-1} \oplus E_d$ is smaller than one we can use the induction hypothesis and if it is bigger than one, we can apply the technique in Lemma 4.2 and stop before we have an eigenvalue of modulus one, since we can preserve the determinant, necessarily the eigenvalue associated to E_d must have decreased and we are done.

□

4.2. **Proof of Theorem 1.1.** Let H be a generic bi-Lyapunov stable homoclinic class. Let us assume that H contains periodic points of index α and we consider $\Delta_\alpha \subset \{p \in Per^\alpha(f_{/H})\}$ the set of index α periodic points in H with determinant smaller than one (if there are finitely many of these, we choose the ones with determinant bigger than one and make symmetric arguments).

If H doesn't admit a dominated splitting, Theorem 4.1 implies that there is a periodic point $p \in \Delta_\alpha$ which can be turned into a sink C^1 small perturbation done along a path which maintains or increases the index.

Now we are able to use Theorem 2.1 and reach a contradiction. Consider a periodic point $q \in H$ fixed such that for a neighborhood $\mathcal{U}$ of f the class $H(q_g, g)$ is bi-Lyapunov stable for every $g \in \mathcal{U} \cap \mathcal{R}$ (recall the first paragraph of the proof of Lemma 2.1).

Suppose the class doesn't admit any dominated splitting, so, we have a periodic point $p \in \Delta_\alpha$ such that f can be perturbed in an arbitrarily small neighborhood of p to a sink (or a source) for a diffeomorphism $g \in \mathcal{U}$ and preserving locally the strong stable manifold of p (resp. the strong unstable manifold of p). So, we choose a neighborhood of p such that it doesn't meet the orbit of q nor the past orbit of some intersection of its unstable manifold with the local stable manifold of p with the same argument as in Lemma 2.1.

Thus, we get that $W^u(q_g, g) \cap W^s(p, g) \neq \emptyset$ and using Lyapunov stability we reach a contradiction since it implies that $p \in H(q_g)$ which is absurd since p is a sink.

The same argument would work if Δ_α consisted of points with determinant bigger than one, but we should have used Lyapunov stability for f^{-1} in this case.

Now we consider the finest dominated splitting of the class (that is, such that the subbundles admit no sub-dominated splitting, see [BDV] Appendix B) which we denote as $T_H M = E_1 \oplus \ldots \oplus E_k$.

Assume that all the indexes of the homoclinic class belong to the segment $[\alpha, \beta]$ and that there exists l such that $\sum_{j=1}^{l-1} \dim E_j < \alpha$ and that $\sum_{j=1}^{l} \dim E_j > \beta$.

Then, Theorem 4.1 allows to conclude that there exists a periodic point of index α such that an arbitrarily small perturbation of the cocycle $Df_{/E_l}$ has all its eigenvalues of the same modulus. Assume that for example the modulus are smaller than one, so the previous argument gives also a contradiction by finding a periodic point in a perturbation of the class of index bigger or equal to $\sum_{j=1}^{l} \dim E_j$.

□

Remark 5.
- Also the same ideas give that periodic points in the class must be volume hyperbolic in the period (not necessarily uniformly, see [BGY] for a discussion on the difference between hyperbolicity in the period and uniform hyperbolicity) for the extremal subbundles of the finest dominated splitting (see [BDV] Appendix B for definitions).
- The same proof allows to prove the following sharper result which may be useful in other contexts. If a homoclinic class H of a generic diffeomorphism f is Lyapunov stable and the class has one periodic point p such that $|det(Df_p^{\pi(p)})| \leq 1$ then the class admits a dominated splitting. Moreover, this dominated splitting can be considered having index bigger than the minimal index of the the class. One should only notice that using transitions (see

[BDP]) one can prove that periodic points with such a property are dense in the class and carry on with the same arguments.

♢

5. Bi-Lyapunov stable homoclinic classes far from tangencies

We prove here that if f is a generic diffeomorphism far from homoclinic tangencies and admits a homoclinic class which is bi-Lyapunov stable, then f must be transitive.

For this we shall use Theorem 1.2 and a recent result of [Y]. Similar results can be found in [C3]. In fact, we present here a proof of this result using the techniques in [C3].

First we state Yang's result for generic Lyapunov stable homoclinic classes far from tangencies (we shall denote $\overline{\text{Tang}}$ as the set of diffeomorphisms which can be C^1 perturbed in order to get a homoclinic tangency between the stable and unstable manifold of a hyperbolic periodic point):

Theorem 5.1 ([Y] Theorem 3)**.** *Let $f \in \mathcal{R}$ a residual subset of $\text{Diff}^1(M)\setminus\overline{\text{Tang}}$ and let H be a Lyapunov stable homoclinic class for f of minimal index α. Let $T_HM = E \oplus F$ be a dominated splitting for H with $\dim E = \alpha$, so one has the following two options:*

1. *E is uniformly contracting.*
2. *E decomposes as $E^s \oplus E^c$ where E^s is uniformly contracting and E^c is one dimensional and H is the Haussdorff limit of periodic orbits of index $\alpha - 1$.*

With this theorem and Theorem 1.2, Proposition 1.1 follows easily. We must remark that this theorem alone is enough to prove the same result for homoclinic classes with interior since they are not compatible with being accumulated in the Hausdorff topology with periodic points of index smaller than the class, but Theorem 1.2 is strictly necessary in the bi-Lyapunov stable case as will be seen clearly in the proof.

Proof of Proposition 1.1. First of all, if the class has all its periodic points with index between α and β we know that it admits a 3 ways dominated splitting of the form $T_HM = E \oplus G \oplus F$ where $\dim E = \alpha$ and $\dim F = d - \beta$. This is because we can apply the result of [W1] which says that far from homoclinic tangencies there is an index i dominated splitting over the closure of the index i periodic points together with the fact that index α and β periodic points should be dense in the class since the diffeomorphism is generic (see [BC]).

Now, we will show that H admits a strongly partially hyperbolic splitting. If E or is one dimensional, then it must be uniform because of Theorem 1.2. If not, suppose $\dim E > 1$ then, if it is not uniform, Theorem 5.1 implies that it can be decomposed as a uniform bundle together with a one dimensional central bundle, since $\dim E > 1$ we get a uniform bundle of positive dimension. The same argument applies for F so we get a strong partially hyperbolic splitting.

Corollary 1 of [ABD] finishes the proof.

□

We now introduce a proof of Yang's result based on [C3]. We shall assume certain familiarity of the reader with the notation and definitions in [C3], in particular, the definitions presented in Section 2 of [C3].

Proof of Theorem 5.1. Let $\mathcal{R} \subset \text{Diff}^1(M)\backslash\overline{\text{Tang}}$ a residual subset such that for every $f \in \mathcal{R}$ and every periodic point p of f, there exists a neighborhood $\mathcal{U}$ of f, where the continuation p_g of p is well defined, such that f is a continuity point of the map $g \mapsto H(g, p_g)$ and such that if $H(p, f)$ is a Lyapunov stable homoclinic class for f, then $H_g = H(p_g, g)$ is also Lyapunov stable for every $g \in \mathcal{U} \cap \mathcal{R}$. Also, we can assume that for every $g \in \mathcal{U} \cap \mathcal{R}$, the minimal index of H_g is α (see the Appendix).

Assume the class admits a dominated splitting of the form $T_H M = E \oplus F$ for which we assume that the subbundle E is not uniformly contracted and of $\dim E = \alpha$.

We shall use first this result from [C3]

Theorem 5.2 (Theorem 1 of [C3]). *Let f be a diffeomorphism in* $\text{Diff}^1(M)\backslash\overline{\text{Tang}}$ *and K_0 an invariant compact set with dominated splitting $T_{K_0}M = E \oplus F$. If E is not uniformly contracted, then, one of the following cases occurs.*

(1) *K_0 intersects a homoclinic class whose minimal index is strictly less than* $\dim E$.
(2) *K_0 intersects a homoclinic class whose minimal index is* $\dim E$ *and containing weak periodic orbits (for every δ there is a sequence of hyperbolic periodic orbits homoclinically related which converge Hausdorff to a set $K \subset K_0$ whose index is* $\dim E$ *but its maximal exponent in E is in $(-\delta, 0)$). Also, this implies that every homoclinic class H intersecting K_0 verifies that it admits a dominated splitting of the form $T_H M = E' \oplus E^c \oplus F$ with* $\dim E^c = 1$.
(3) *There exists a compact invariant set $K \subset K_0$ with minimal dynamics and which has a partially hyperbolic structure of the form $T_K M = E^s \oplus E^c \oplus E^u$ where* $\dim E^c = 1$ *and* $\dim E^s < \dim E$. *Also, any measure supported on K has zero lyapunov exponent along E^c.*

We shall apply this Theorem to H, and since the minimal index of H is α, option 1) of the Theorem cannot occur.

We shall prove that if option 3) in the Theorem occurs, then option 2) also happens, so, option 2) occurs always. This is enough to prove Theorem 5.1 since if we apply Theorem 5.2 to E' given by option 2) we get that since $\dim E' = \alpha - 1$ option 1) and 2) cannot happen, and since option 3) implies option 2) we get that E' must be uniformly contracted thus proving Theorem 5.1 (observe that the statement on the Haussdorff convergence of periodic orbits to the class can be deduced from option 2) also).

Since we are far from tangencies, to get option 2) in Theorem 5.2 is enough to find one periodic orbit of index α in H which is weak (that is, it has one Lyapunov exponent in $(-\delta, 0)$).

This follows using the fact that being far from tangencies there is a dominated splitting in the orbit given by [W1] (see Corollary 1.1 of [C3]) with a one dimensional central bundle associated with the weak eigenvalue. Let $\mathcal{O}$ be the weak periodic orbit, so we have a dominated splitting of the form $T_{\mathcal{O}}M = E^s \oplus E^c \oplus E^u$.

Using transitions (see [BDP]), we can find a dense subset in the class of periodic orbits that spend most of the time near the orbit we found, say, for a small neighborhood U of $\mathcal{O}$, we find a dense subset of periodic points p_n such that the cardinal of the set $\{i \in \mathbb{Z} \cap [0, \pi(p_n) - 1] \ : \ f^i(p_n) \in U\}$ is bigger than $(1 - \varepsilon)\pi(p_n)$.

Since we can choose U to be arbitrarily small, we can choose ε so that the orbits of all p_n admit the same dominated splitting (this can be done using cones for example) and maybe by taking ε smaller to show that p_n are also weak periodic orbits.

So, it rests to prove that option 3) implies that option 2) occurs also. To do this, we shall discuss depending on the structure of the partially hyperbolic splitting using the results and classification given in [C3]. There are 2 different cases, we shall name them as in [C3].

5.0.1. *Case A: There exist a chain recurrent central segment.* Assume that the set $K \subset H$ admits a chain recurrent central segment, that is, curve γ integrating E^c contained in H and such that γ is contained in chain transitive set contained in a small neighborhood of K. In this case, the results of [C2] (see also Proposition 4.2 of [C3]) imply that there is a periodic orbit which is in the same chain recurrence class as K (i.e. H) with index $\dim E^s \leq \alpha - 1$, a contradiction.

This is because the partially hyperbolic splitting can be passed to γ and then one can approximate K' (the chain transitive set containing γ and contained in a small neighborhood of K) by periodic orbits (see [C1]) and since the class is saturated by unstable sets, one can show that the stable set of the periodic orbits intersects H and thus, the periodic orbits are in the class (for more details see Proposition 4.2 of [C3]).

5.0.2. *Case B: Cases (N), (H), and (P).* For cases (H) and non twisted (P) one can apply Proposition 4.4 of [C3] which implies that there is a weak periodic orbit in H giving 2) of Theorem 5.2.

Cases (N) and (P_{SN}) give a family of central curves γ_x $\forall x \in K$ (integrating E^c, see [C3]) which satisfy that $f(\gamma_x) \subset \gamma_{f(x)}$. It is not difficult to see that there is a neighborhood U of K such that for every invariant set in U the same property will be satisfied (see remark 2.3 of [C2]).

Consider a set $\hat{K} = K \cup \bigcup_n \mathcal{O}_n$ where $\mathcal{O}_n$ are close enough periodic orbits converging in Hausdorff topology to K (these are given, for example, by [C1]) which we can suppose are contained in U.

So, since for some $x \in K$, we have that $W^{uu}_{loc}(x)$ will intersect $W^{cs}_{loc}(p_n)$ in a point z (for a point x, the local center stable set, $W^{cs}_{loc}(x)$ is the union of the local strong stable leaves of the points in γ_x).

Since the ω-limit set of z must be a periodic point (see Lemma 3.13 of [C2]) and since H is Lyapunov stable we get that there is a periodic point of index α which is weak, or a periodic point of smaller index in H which implies we are in option 2) of Theorem 5.2 or gives a contradiction.

5.0.3. *Case C: Twisted (P_{UN}) or (P_{SU}).* One has a minimal set K which is contained in a Lyapunov stable homoclinic class and it admits a partially hyperbolic splitting with one dimensional center with zero exponents and twisted returns (this is defined in [C3], we shall not use it).

This gives that given a compact neighborhood U of K, there exists a family of curves $\gamma_x : [0,1] \to U$ tangent to the central bundle such that $f^{-1}(\gamma_x([0,1])) \subset \gamma_{f^{-1}(x)}([0,1))$. They also verify that their chain unstable set of K restricted to U (that is the set of points that can be reached from K by arbitrarily small pseudo orbits contained in U) which we denote as $pW^u(K,U)$, contains these curves.

Also, we know that for every ε there exists a continuous function $h : K \to [0, \varepsilon]$ such that $f^{-1}(\gamma_x([0, h(x)]) \subset \gamma_{f^{-1}(x)}([0, h(x))).$ We shall call this property the *trapping property.*

Assume we could extend the partially hyperbolic splitting from K to a dominated splitting $T_{K'}M = E_1 \oplus E^c \oplus E_3$ in a chain transitive set $K' \subset H$ containing $\gamma_x([0, t))$ for some $x \in K$ and for some $t \in (0, 1)$.

Since the orbit of $\gamma_x([0, 1])$ remains near K for past iterates, we can assume (by choosing U sufficiently small) that the bundle E_1 is uniformly contracted there. A similar argument gives that the central bundle cannot be contracted, so, we get that E_3 is uniformly expanded. This would allow us to carry on with the same arguments as in case (R) to conclude.

We shall argue as in section 1.4 of [C3] to extend the dominated splitting.

Consider a point $y = \gamma_z(t)$ with $z \in K$ and $t > 0$. We first argue as in Lemma 1.11 of [C3].

For every $\varepsilon > 0$ we consider a pseudo orbit $X_\varepsilon = z_0 \in K, z_1, \ldots, z_n = y$ from K to y contained in U. using the trapping property we get that for ε small enough we have that $B_\nu(y) \cap X_\varepsilon = \{y\}$. So if we consider a Hausdorff limit of the sequence $X_{1/n}$ we get a set Z^- for which y is isolated and such that it is contained in the chain unstable set of K restricted to U.

If we now consider the pair (Δ^-, y) where $\Delta^- = Z^- \backslash \{f^{-n}(y)\}_{n \geq 0}$ we get a pair as the one obtained in Lemma 1.11 of [C3] where y plays the role of x^-.

Now we use the techniques of the proof of Proposition 1.10 in [C3] to extend the splitting to y. We shall only sketch the proof since is almost the same as the proof given in [C3].

First we notice that if property (I) of [C3] is verified (that is, there is an open neighborhood V such that every chain transitive set $K \subset K' \subset H \cap V$ satisfies that $K' = K$) then the proof is exactly the same as in Proposition 1.10.

If this property is not satisfied, then we can assume that that there is a chain transitive set $K' \subset U$ strictly containing K and we can also assume that it doesn't intersect y since in that case we could finish the proof.

Consider a point x^+ in $K' \backslash K$. We get that the future orbit of x^+ doesn't intersect the orbit of y.

So, we can consider neighborhoods $W^+ \subset \hat{W}^+$ of x^+ and $W^- \subset \hat{W}^-$ of y such that

- All the iterates $f^i(\hat{W}^+)$ and $f^{-j}(\hat{W}^-)$ for $0 \leq i, j \leq N$ are pairwise disjoint and their closures are disjoint from K (N given by Hayashi's connecting lemma, see Theorem 5 of [C1], this is recalled in the appendix of this paper).
- The iterates $f^i(\hat{W}^+)$ for $0 \leq i \leq N$ are disjoint from the past orbit of y.

Since there is arbitrarily small pseudo orbits going from y to x^+ contained in H and f is generic, Theorem 7 of [C1] (see the appendix) gives us an orbit $\{x_0, \ldots, f^l(x_0)\}$ in a small neighborhood of H and such that $x_0 \in W^-$ and $f^l(x_0) \in W^+$.

The same argument gives us an orbit $\{x_1, \ldots, f^L(x_1)\}$ contained in U such that $x_1 \in f^N(W^+)$ and $f^L(x_1) \in B_\nu(y)$ for any ν. We can suppose that L is much bigger than l.

Using Hayashi's connecting lemma (Theorem 5 of [C1]) we can then create a periodic orbit $\mathcal{O}$ for a diffeomorphism g close to f which passes through $\hat{W}^-$ (in the proof of Proposition 1.10 is explained how one can compose two perturbations in order to close the orbit).

One should also take care that after closing the orbit, this passes more time inside U than outside. This is not obvious a priori since Hayashi's connecting lemma may erase some of the iterates. This can be done since any orbit going from $f^N(\hat{W}^+)$ to $B_\nu(y)$ has at least k iterates where $k \to +\infty$ as $\nu \to 0$.

Since the orbit spends most of the time inside U, and all the exponents of E^c in K are zero, we can assume it has one small exponent along E^c and since we are far from tangencies, Wen's theorem (see Corollary 1.1 of [C3]) gives us a uniform dominated splitting extending the partially hyperbolic splitting.

Since we can consider $\hat{W}^-$ arbitrarily small we finally get the dominated splitting on y as we wanted.

□

Appendix. Some generic properties

The following theorem gives some well known generic properties we shall be using in the proof of our results. The main reference is [BDV] (and the references therein) and mainly [CMP], [C2] and [BC].

Theorem 5.3. *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that if $f \in \mathcal{R}$*

- a1) *f is Kupka Smale (that is, all its periodic points are hyperbolic and their invariant manifolds intersect transversally).*
- a2) *The periodic points of f are dense in the chain recurrent set of f. Moreover, if a chain recurrence class C contains a periodic point p then $C = H(p)$.*
- a3) *Given $p \in Per(f)$ there exists $\mathcal{U}_1$ a neighborhood of f such that for every $g \in \mathcal{U}_1 \cap \mathcal{R}$, g is a continuity point for the map $g \mapsto H(g, p_g) = H_g$ where p_g is the continuation of p for g. The continuity is with the Hausdorff distance between compact subsets of M.*
- a4) *If a homoclinic class H is Lyapunov stable for $f \in \mathcal{R}$ then there exists $\mathcal{U}_2 \subset \mathcal{U}_1$ a neighborhood of f such that H_g is Lyapunov stable for every $g \in \mathcal{U}_2 \cap \mathcal{R}$.*
- a5) *If a homoclinic class H for $f \in \mathcal{R}$ has all its periodic points of index bigger or equal to α, then there exists $\mathcal{U}_3 \subset \mathcal{U}_1$ such that for every $g \in \mathcal{U}_3$ all the periodic points of g in H_g have index bigger or equal to α.*
- a6) *Let $X \subset M$, and $x, y \in X$ such that for every $\varepsilon > 0$ there exists an $\varepsilon-$pseudo orbit inside X from x to y. Then, for every $\delta > 0$ there exists a segment of orbit $z, \dots f^n(z)$ such that is contained in an δ neighborhood of X and such that $d(x, z) < \delta$ and $d(y, f^n(z)) < \delta$.*

We also recall the following perturbation result which has been used in the paper, it is called the Hayashi's connecting lemma, the statement we give is Theorem 5 of [C2]:

Theorem 5.4. *Let $f \in \mathrm{Diff}^1(M)$ and $\mathcal{U}$ a neighborhood of f. Then, there exists $N > 0$ such that every non periodic point $x \in M$ admits two neighborhoods $W \subset \hat{W}$ with the following property: for every $p, q \in M \backslash (f(\hat{W}) \cup \dots \cup f^{N-1}(\hat{W}))$ such that p has a forward iterate $f^{n_p} \in W$ and q has a backward iterate $f^{-n_q}(q) \in W$, we have that there exists $g \in \mathcal{U}$ which coincides with f in $M \backslash (f(\hat{W}) \cup \dots \cup f^{N-1}(\hat{W}))$ and such that for some $m > 0$ we have $g^m(p) = q$.*

Moreover, $\{p, g(p), \dots, g^m(p)\}$ *is contained in the union of the orbits* $\{p, \dots, f^{n_p}(p)\}$, $\{f^{-n_q}(q), \dots, q\}$ *and the neighborhoods* $\hat{W}, \dots, f^N(\hat{W})$. *Also, the neighborhoods* $\hat{W}, W$ *can be chosen arbitrarily small.*

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