ON THE FREE TIME MINIMIZERS OF THE NEWTONIAN N-BODY PROBLEM

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Abstract. In this paper we study the existence and the dynamics of a very special class of motions, which satisfy a strong global minimization property. More precisely, we call a free time minimizer a curve which satisfies the least action principle between any pair of its points without the constraint of time for the variations. An example of a free time minimizer defined on an unbounded interval is a parabolic homothetic motion by a minimal central configuration. The existence of a large amount of free time minimizers can be deduced from the weak KAM theorem. In particular, for any choice of x_0, there should be at least one free time minimizer $x(t)$ defined for all $t \geq 0$ and satisfying $x(0) = x_0$. We prove that such motions are completely parabolic. Using Marchal's theorem we deduce as a corollary that there are no entire free time minimizers, i.e. defined on $\mathbb{R}$. This means that the Mañé set of the Newtonian N-body problem is empty.

1. Introduction and results

Let E be a finite dimensional Euclidean space, and let $m_1, \ldots, m_N > 0$ be the masses of N punctual bodies in E. The Newtonian N-body problem consists in the study of the dynamics of these bodies when the law governing the motion is given by the Newtonian potential $U : E^N \to (0, +\infty]$

$$U(x) = \sum_{1 \leq i < j \leq N} m_i \, m_j \; \| \, r_{ij} \, \|^{-1}$$

where $x = (r_1, \ldots, r_N) \in E^N$ is a configuration and $r_{ij} = r_i - r_j$. This means that a curve $x : (a, b) \to E^N$, $x(t) = (r_1(t), \ldots, r_N(t))$, such that $r_{ij}(t) \neq 0$ whenever $i \neq j$ is the position vector of a true motion of the bodies (in a fixed inertial frame) if an only if their components satisfy the Newton's equations of motion

$$\ddot{r}_i = \sum_{j \neq i} m_j \; \| \, r_{ij} \, \|^{-3} \; r_{ij}.$$

Newton's equations of motion can be easily derived from the Hamilton's principle of stationary action, which states that the dynamics is determined by a variational property of the trajectories. More precisely, according to Hamilton's principle, the trajectories must be extremal curves of the Lagrangian action, thus they must satisfy the corresponding Euler-Lagrange equation. But in fact, as it is well known, every extremal curve of the Lagrangian action is locally minimizing, in the sense that it must solve the least action principle. This viewpoint, in the study of the dynamics of a given mechanical system, is doubtlessly deep and fruitful. Nevertheless, during all the last century, a major problem in the case of Newtonian gravitational model, prevented the use of the direct method of the calculus of variations to prove the existence of particular solutions. Namely, the problem is that the Newtonian potential allows the existence of curves with singularities (collisions) and finite Lagrangian action. A big breakthrough in this problem was done by the

Date: January 28, 2013.

discovery essentially due to C. Marchal, of the fact that minimizing orbits always avoid collisions (assuming the obviously necessary hypothesis $\dim E > 1$).

Until now the mathematicians agree upon the fact that we only dispose of a little information about the dynamics of an arbitrary trajectory of the Newtonian N-body problem, except in the case $N \leq 3$. After the pioneer works of J. Chazy and K. Sundman at the beginning of the last century, C. Marchal, H. Pollard and D. Saari (see for instance [15], [17] and [18]) were among the first in continuing the systematic study of the general case, that is to say, without no restriction on the number of bodies nor on the values of the masses. A common factor in these works is the *a priori* assumption that the motion is well defined for all the future. In other words, no singularity is encountered in any future time. Now, Marchal's theorem enable us to apply all this general theory to minimizing solutions on unbounded intervals which is the subject of this paper.

Recently, important advances were obtained in the study of these trajectories in a more general context. More precisely, the work of Barutello, Terracini and Verzini ([2], [3]) on parabolic trajectories, extends the analysis to a big class of homogeneous potentials.

1.1. **The variational setting of the N-body problem.** In order to explain our main results, let us recall before some usual notations. The Lagrangian is the function $L : TE^N \to (0, +\infty]$

$$L(x, v) = T(v) + U(x) = \frac{1}{2} \sum_{i=1}^{N} m_i \parallel v_i \parallel^2 \; + \; U(x),$$

thus the Lagrangian action of an absolutely continuous curve $\gamma : [a, b] \to E^N$ is

$$A(\gamma) = \int_a^b L(\gamma(t), \dot\gamma(t))\, dt$$

and takes values in $(0, +\infty]$. We will denote by $\mathcal{C}(x, y, \tau)$ the set of curves binding two given configurations $x, y \in E^N$ in time $\tau > 0$, that is to say,

$$\mathcal{C}(x, y, \tau) = \left\{ \gamma : [a, b] \to E^N \text{ absolutely continuous } \mid b - a = \tau,\, \gamma(a) = x, \gamma(b) = y \right\},$$

and $\mathcal{C}(x, y)$ will denote the set of curves binding two configurations $x, y \in E^N$ without any restriction on time,

$$\mathcal{C}(x, y) = \bigcup_{\tau > 0} \mathcal{C}(x, y, \tau) \; .$$

In all that follows we will consider curves which minimize the action on these sets, so we need to define the function $\phi : E^N \times E^N \times (0, +\infty) \to \mathbb{R}$,

$$\phi(x, y, \tau) = \inf \{ A(\gamma) \mid \gamma \in \mathcal{C}(x, y, \tau) \} ,$$

and the *critical action potential*, or the *Mañé critical potential*

$$\phi(x, y) = \inf \{ A(\gamma) \mid \gamma \in \mathcal{C}(x, y) \} = \inf \{ \phi(x, y, \tau) \mid \tau > 0 \} .$$

defined on $E^N \times E^N$. It is important to say that in the first definition, the infimum is reached for every pair of configurations $x, y \in E^N$. In the second one the infimum is reached if and only if $x \neq y$. As we will see, these facts are essentially due to the lower semicontinuity of the Lagrangian action.

We can now introduce the object of study of this work.

Definition 1. *A free time minimizer defined on an interval $J \subset \mathbb{R}$ is an absolutely continuous curve $\gamma : J \to E^N$ which satisfies $A(\gamma \mid_{[a,b]}) = \phi(\gamma(a), \gamma(b))$ for all compact subinterval $[a, b] \subset J$.*

There is a more or less evident way to give an example of a free time minimizer defined on an unbounded interval. we need before to define the minimal configurations of the problem. Recall that the moment of inertia (about the origin) of a given configuration $x \in E^N$ is

$$I(x) = \sum_{i=1}^{N} m_i \parallel r_i \parallel^2 .$$

We say that $a \in E^N$ is a (normal) minimal configuration of the problem when $I(a) = 1$ and $U(a) = \min \left\{ U(x) \mid x \in E^N,\ I(x) = 1 \right\}$. Also recall that a central configuration is a configuration $a \in E^N$ which admits homothetic motions i.e. of the form $x(t) = \lambda(t)\, a$. This happens if and only if a is a critical point of $\tilde{U} = I^{1/2} U$ and λ satisfies the Kepler equation $\ddot{\lambda}\, \lambda^2 = -U(a)\, I(a)^{-1}$. Thus minimal configurations are in particular central configurations. For a given central configuration a, we can choose a constant $\mu > 0$ such that $x(t) = \mu\, t^{2/3}\, a$ is an homothetic motion. We will see that such motions are free time minimizers when the configuration a is minimal.

A less trivial way to show the existence of free time minimizers can be obtained using the weak KAM theory. It was proved by the second author (see [10]) that the critical action potential is a Hölder continuous distance function on E^N. From this it is shown that the Hamilton-Jacobi equation of the Newtonian N-body problem has global critical solutions in a weak sense. These solutions are viscosity solutions, and to each one it can be associated a lamination of the space of configurations by free time minimizers. More precisely, if the Hamiltonian of the system is $H : T^*M \to [-\infty, +\infty)$, then given a weak solution $u : E^N \to \mathbb{R}$ of the critical Hamilton-Jacobi equation $H(x, d_x u) = 0$, and any configuration $x_0 \in E^N$, there is a curve $x : [0, +\infty) \to E^N$ which calibrates u and such that $x(0) = x_0$. The fact that the curve is calibrating for the weak KAM solution means that $A(x\mid_{[0,t]}) = u(x_0) - u(x(t))$ for all $t > 0$. Therefore the curve must be a free time minimizer, since u is a weak subsolution of the critical Hamilton-Jacobi equation, which can be expressed in terms of the action potential saying that $u(x) - u(y) \leq \phi(x, y)$ for any pair of configurations $x, y \in E^N$.

1.2. **Main results.** In this paper we study the asymptotic behavior of a free time minimizer, so we will assume that its domain is an interval $[t_0, +\infty)$; by the previous observations, we know that our object of study is not trivial.

More precisely, we will prove that such kind of motions are completely parabolic, meaning that the velocity of each body goes to zero as $t \to +\infty$. The origin of this name comes from Chazy's classification of the possible final evolution of motions defined for all future time in the three body problem. In fact, we will show that free time minimizers must have zero energy and that its moment of inertia must grow like $I(x(t)) \sim \alpha\, t^{4/3}$ for some positive constant $\alpha > 0$. From these facts we will deduce that the motion must be completely parabolic.

On the other hand, we must recall Marchal's theorem (see [4], [9] and [14]) which will be crucial for our proofs. It asserts that, if $\dim E \geq 2$ then the curves that minimize the action in some $\mathcal{C}(x, y, \tau)$, cannot have collisions in any interior time. In particular we know that, except for the one-dimensional case, every free time minimizers defined in an open interval is a solution of Newton's equation. Therefore, with our notation, we can say that, if the Euclidean space E has dimension at least 2 and $J \subset \mathbb{R}$ is an open interval, then for every free time minimizer $\gamma : J \to E^N$ we have $\gamma(J) \subset \Omega$, where $\Omega = \left\{ x \in E^N \text{ such that } U(x) < +\infty \right\}$ denotes the set of configurations without collisions.

Recently, using Marchal's theorem, the second author has proved in [11] that every free time minimizer defined on an unbounded interval must have fixed center of mass. This result allow us to use a theorem due to Pollard for give a proof of our main theorem:

Theorem 1. *If $x : [t_0, +\infty) \to E^N$ is a free time minimizer of the N-body problem in an Euclidean space E of dimension at least 2, then x corresponds to a completely parabolic motion of the bodies.*

The next result is a consequence of theorem 1 and again of Marchal's theorem. From its discovery, Marchal's theorem was used to prove the existence of special orbits by variational methods. Commonly, the technique consists in minimize the action in some special class of curves, such as periodic curves with topological or symmetry constraints, and then apply the theorem to prove that the minimizer is a true motion. Here we will use Marchal's theorem in the inverse way:

Theorem 2. *If $\dim E \geq 2$ there are no entire free time minimizers for the N-body problem in E, that is to say, an entire motion $x : \mathbb{R} \to E^N$ is never a free time minimizer.*

Another application of theorem 1 can be obtained by means of the weak KAM theory. It was recently established by A. Venturelli and the second author in [12] that for any given configuration $x_0 \in E^N$, and every minimal normalized configuration $c \in E^N$, there is a completely parabolic motion starting at x_0 and asymptotic to a parabolic homothetic motion by c. As usual, a motion $x : [t_0, +\infty) \to E^N$ is said to be completely parabolic in the future when $\lim_{t\to\infty} T(t) = 0$. This is equivalent to say that all the velocities tend to zero when $t \to \infty$. We will easily deduce from theorem 1 that free time minimizers are completely parabolic. On the other hand, as we have say, associated to every critical solution of the Hamilton-Jacobi equation there is a lamination of the space of configurations by calibrating curves which are therefore free time minimizers (see [10] prop. 15). Therefore, we obtain an alternative proof for the abundance of completely parabolic motions:

Theorem 3. *Given N different positions $r_1, r_2, \dots, r_N \in E$, there exist N velocities $v_1, v_2, \dots, v_N \in E$ such that the motion determined by these initial positions and velocities is completely parabolic.*

There is a subtle difference between the first proof of this result given in [12] and the proof given here. Our proof uses the existence of a weak KAM solution, and we lose the possibility of choice for the limit shape of the bodies. On the other hand, we gain a stronger minimization property (the parabolic motion is not only globally minimizing, that is, in every compact subinterval of his domain, but also in free time).

As in [8], the existence of weak KAM solutions for the N-body problem is obtained in [10] by a fixed point argument. We hope that a more refined study of the subject can give the existence of particular weak KAM solutions, in such a way that the limit shape of his calibrating curves can be prescribed in advance. These solutions would be similar to the Busemann functions of a complete non compact manifold.

We do not know as yet if there is a limit configuration for a free time minimizer. In fact, only we can say that, if a free time minimizer has an asymptotic configuration in the sense that the normalized configuration $u(t) = I(x(t))^{-1/2}x(t)$ converges to some configuration $a \in E^N$ with $I(a) = 1$, then the limit configuration a must be a central configuration such that its parabolic homothetic motion is itself a free time minimizer. On the other hand, this last property seems to be the only requirement on the configuration which is needed to define an associated critical Busemann function. We refer the reader to the work of G. Contreras [6] for a construction of the critical Busemann functions of an autonomous Tonelli Lagrangian and his relationship with the weak KAM theory. These considerations show that the set of configurations with such property is playing the role of the Aubry set at infinity.

Another interesting invariant set in the general theory of Tonelli Lagrangians is the Mañé set. We refer the reader to the original paper of Mañé [13] for the definition of a semistatic curve, as well as to the work of G. Contreras and G. Paternain [7]. It is not difficult to see that the semistatic curves in these cited works are precisely the free time minimizers in our context (the critical value is $c(L) = 0$). The Mañé set is defined as the subset of the tangent bundle TM whose elements are the velocity of some entire semistatic curve (the set $\Sigma(L)$ in the cited literature). Therefore, theorem 2 says that the Mañé set of the Newtonian N-body problem is empty.

The paper is organized as follows. The next section is devoted to introduce the main tools and notations we will use. In particular, the lower semicontinuity of the Lagrangian action is showed as well as the homogeneity of the action potential. In the third section, the existence of free time minimizers is proved, and some of its basic properties are discussed. The last section begins recalling a theorem of H. Pollard ([17] theorem 5.1, p. 607) and give the proof of theorem 1 and theorem 2.

2. Preliminaries and notations

As usual, we will use the notation $I(t)$ for $I(x(t))$ when the curve $x(t)$ is understood. In the same way we will write $U(t) = U(x(t))$, $T(t) = I(\dot{x}(t))$. Therefore, if $x(t)$ describes a motion of the system, then the quantity $h = T(t) - U(t)$ is the constant total energy of the motion, and the Lagrange-Jacobi relation (or virial relation) can be written $\ddot{I} = 2U + 4h$.

We will write $x \cdot y$ the mass inner product of two configurations $x, y \in E^N$, thus we have $I(x) = x \cdot x$. It is easy to see that Newton's equations admit the synthetic expression $\ddot{x} = \nabla U(x)$ if the gradient is taken with respect to this inner product. With the obvious identification $TE^N \simeq E^N \times E^N$ we can write $2T(v) = v \cdot v$. If we apply the Cauchy-Schwarz inequality to the product $x(t) \cdot v(t)$, where $v = \dot{x}$, we get the inequality $2IT - \dot{I}^2 \geq 0$, where equality holds if and only if the velocities vector and the configuration vector are collinear. In particular, the equality holds on an open interval of time if and only if the curve is homothetic on this interval.

An excellent presentation of basic geometric constructions for N-body problems with homogeneous potentials is the paper of A. Chenciner [5], to which we refer the reader for other intimately related definitions and properties.

2.1. **The Lagrangian action in polar coordinates.**

When a curve $x : [a, b] \to E^N$ avoid the total collision, we can decompose it as the product of a positive real function by an unitary configuration. In other words, we can write $x(t) = \rho(t)\, u(t)$, with $\rho(t) > 0$ and $I(u(t)) = 1$ for all $t \in [a, b]$. Note that these factors are well defined as $\rho = I(x)^{1/2}$ and $u = \rho^{-1} x$.

Therefore we have $\dot{x} = \dot{\rho}\, u + \rho\, \dot{u}$. Since $u^2 = u \cdot u = I(u)$ is constant, we also have $u \cdot \dot{u} = 0$, from which we deduce that $\dot{x}^2 = \dot{x} \cdot \dot{x} = \dot{\rho}^2 + \rho^2\, \dot{u}^2$. If in addition we consider the homogeneity of the Newtonian potential, we have that $U(x) = U(\rho\, u) = \rho^{-1} U(u)$.

Thus we get the following expression for the action of the curve x, which will be useful to compare it to other paths joining the same endpoints.

$$A(x) = \frac{1}{2} \int_a^b \dot{\rho}(s)^2\, ds + \frac{1}{2} \int_a^b \rho(s)^2 \dot{u}(s)^2\, ds + \int_a^b \rho(s)^{-1} U(u(s))\, ds. \tag{1}$$

Note that if the curve x is homothetic, the second term vanishes. Thus, in this case, the action of x can be viewed as the Lagrangian action of ρ as a curve in $\mathbb{R}^+$ with respect to the Lagrangian associated to a reduced Kepler problem (central force) in this half-line.

2.2. **Lower semicontinuity of the Lagrangian action.** Frequently in the literature, the curves are considered in the Sobolev space H^1, but it is not difficult to see that for this kind of Lagrangian, absolutely continuous curves with finite action must have square-integrable derivative, in particular they are also 1/2-Hölder continuous. If $x : [a, b] \to E^N$ is an absolutely continuous curve such that $A = A(x) < +\infty$, then obviously we have

$$\int_a^b |\,\dot{x}(s)\,|^2 \; ds \leq 2A.$$

On the other hand, it is well know that for any absolutely continuous curve, the distance between its extremities is bounded by the integral of the norm of the speed. Hence, given $a \leq s < t \leq b$, we can apply the Bunyakovsky inequality, and we deduce that

$$|\,x(t) - x(s)\,| \leq \int_s^t |\,\dot{x}(u)\,| \; du \leq (2A)^{1/2} \;\; |\,t - s\,|^{1/2}. \tag{2}$$

Here we use the norm in E^N induced by the mass inner product, which is denoted $|\,x\,|$, but it is clear that the Hölder continuity not depends on the choice of the norm since they are all equivalent. Thus by Ascoli's theorem we obtain the following proposition.

Proposition 4. *Let $x_n : [a, b] \to E^N$ be a sequence of absolutely continuous curves for which there is a positive constant $k < +\infty$ such that $A(x_n) \leq k$ for all $n > 0$. If $x_n(t)$ converges for some $t \in [a, b]$, then there is a subsequence x_{n_k} which converges uniformly.*

In other words, given $x, y \in E^N$, $\tau > 0$ and $k > 0$, we know that the sets of absolutely continuous curves

$$\Sigma(x, y, \tau, k) = \left\{\, \gamma : [0, \tau] \to E^N \mid \gamma(0) = x,\ \gamma(\tau) = y,\ \text{and } A(\gamma) \leq k \,\right\}$$

are relatively compact in $C^0([a, b], E^N)$. As we will see, the compactness of such sets is equivalent to the lower semicontinuity of the Lagrangian action on the subset of absolutely continuous curves. Moreover, we will see that it is a consequence of the well known Tonelli's lemma for strictly convex and superlinear Lagrangians that we state below.

We recall that an autonomous *Tonelli Lagrangian* on a complete Riemannian manifold M is a function $L : TM \to \mathbb{R}$ of class C^2, which is strictly convex on each fiber of TM, and such that for each positive constant $\alpha > 0$ there is $C_\alpha \in \mathbb{R}$ such that $L(x, v) \geq \alpha \,\|\, v \,\| + C_\alpha$ for all $(x, v) \in TM$. We will denote $A_L(x)$ the corresponding Lagrangian action of an absolutely continuous curve x in M.

Lemma 5 (Tonelli's lower semicontinuity)**.** *Let M be a Riemannian manifold, and let $L : TM \to \mathbb{R}$ be a Tonelli Lagrangian on M. Suppose that $x_n : [a, b] \to M$ is a sequence of absolutely continuous curves such that $\sup A_L(x_n) < +\infty$. If x_n converges uniformly to a curve x, then the limit curve x is absolutely continuous, and $A_L(x) \leq \liminf A_L(x_n)$.*

There are several proofs in the literature of the above lemma, see for instance the first appendix in [16], where an equivalent version is given. We can deduce from this lemma the following theorem, also due to Tonelli, which assures the existence of absolutely continuous minimizers.

Theorem 6 (Tonelli)**.** *Let M be a complete connected Riemannian manifold, and let $L : TM \to \mathbb{R}$ be an autonomous Tonelli Lagrangian on M. Given $x, y \in M$ and $\tau > 0$, the action A_L takes a minimum value over the set of all absolutely continuous curves $\gamma : [0, \tau] \to M$ such that $\gamma(0) = x$ and $\gamma(\tau) = y$.*

Proof. For each $k \in \mathbb{R}$, let Σ_k be the set of absolutely continuous curves $\gamma : [0, \tau] \to M$ such that $A_L(\gamma) \leq k$. Since M is connected, the sets Σ_k are nonempty for sufficiently large values of k. From the superlinearity of L we can deduce as before, that each curve in Σ_k is 1/2-Hölder continuous, with a Hölder constant which only depends in k. Therefore, since M is complete, we can apply Ascoli's theorem as in proposition 4, and we get that the sets Σ_k are relatively compact in the C^0-topology. But if we apply Tonelli's lemma to a convergent sequence in some Σ_k we conclude that the limit curve is also in Σ_k. Thus, each Σ_k is actually compact in the C^0-topology.

We note now that the Lagrangian is bounded below, since the superlinearity implies that $A_L(\gamma) \geq \tau\, C_1$. Thus $k_0 = \inf \{\, k \in \mathbb{R} \mid \Sigma_k \neq \emptyset \,\}$ is well defined. Since $\Sigma_{k_0} = \cap_{k > k_0} \Sigma_k$ we can conclude that $\Sigma_{k_0} \neq \emptyset$. But it is clear that each $\gamma \in \Sigma_{k_0}$ is a minimizer of A_L in the required set of curves. □

Using Fatou's lemma we can obtain a Tonelli's theorem which works for the Lagrangian action of the Newtonian N-body problem. As before, first we need to establish the lower semicontinuity of the action.

Lemma 7. *Let $x_n : [a, b] \to E^N$ be a sequence of absolutely continuous curves which converges uniformly to a limit curve x, and such that $\sup A(x_n) < +\infty$. Then, x is also an absolutely continuous curve, and $A(x) \leq \liminf A(x_n)$.*

Proof. Let L_0 be the quadratic Lagrangian in E^N given by

$$L_0(x, v) = \frac{1}{2} \,|\, v \,|^2$$

and let A_0 be the associated action. Certainly L_0 is a Tonelli Lagrangian on E^N, therefore Tonelli's lemma can be applied. Thus we have that x is absolutely continuous, and that

$$A_0(x) \leq \liminf A_0(x_n).$$

On the other hand, since U is a positive measurable function, by Fatou's lemma we have

$$\int_a^b U(x(s))\, ds = \int_a^b \liminf U(x_n(s))\, ds \leq \liminf \int_a^b U(x_n(s))\, ds.$$

Since

$$A(x) = A_0(x) + \int_a^b U(x(s))\, ds$$

we get $A(x) \leq \liminf A(x_n)$ what was required to be proved. □

An evident corollary of lemma 7 is the existence of absolutely continuous minimizers on each set of curves $\mathcal{C}(x, y, \tau)$.

Theorem 8 (Tonelli's theorem for the N-body problem)**.** *Given two configurations $x, y \in E^N$ and $\tau > 0$, there is at least one curve $\gamma \in \mathcal{C}(x, y, \tau)$ such that $A(\gamma) = \phi(x, y, \tau)$.*

Proof. By proposition 4 and lemma 7, we already know that given $c \in \mathbb{R}$, the sets of absolutely continuous curves

$$\Sigma(x, y, \tau, c) = \left\{\, \gamma : [0, \tau] \to E^N \mid \gamma(0) = x, \gamma(\tau) = y, \text{ and } A(x) \leq c \,\right\}$$

are compact subsets of $C^0([0,\tau],E^N)$. Moreover, since $L>0$ and E^N is connected, they are empty for $c\leq 0$, and nonempty for sufficiently large values of $c>0$.

We observe now that $\phi(x,y,\tau)=\inf\{\,c>0\mid \Sigma(x,y,\tau,c)\neq\emptyset\,\}$. Hence the intersection for $c>\phi(x,y,\tau)$ of these nonempty compact sets is also nonempty, and each curve γ in the intersection satisfies $A(\gamma)=\phi(x,y,\tau)$. □

2.3. **Regularity of minimizers and Marchal's theorem.** Everything what we said would not be useful for anything, unless we are able to show that absolutely continuous minimizers correspond to true motions or, in other words, to solutions of Newton's equation. We will explain why this happens briefly.

For a Tonelli Lagrangian on a smooth manifold, it is very well known that a C^1 minimizer is in fact of class C^2 and satisfies the Euler-Lagrange equations. Therefore it suffices to show that an absolute continuous minimizer is of class C^1. The proof of this fact, which actually holds even for time dependent Tonelli Lagrangians (see [16]) is a little arduous, but for a mechanical system is not it as much. We must recall first a classical result of the calculus of variations, which tell us that if $\gamma\in\mathcal{C}(x,y,\tau)$ is a critical point of the Lagrangian action, then in local coordinates we can write

$$\mathcal{L}(\gamma(t),\dot\gamma(t))=\left(\gamma(t),\,u+\int_0^t\frac{\partial L}{\partial x}(\gamma(s),\dot\gamma(s))\,ds\right)$$

where $\mathcal{L}:TM\to T^*M$ is the Legendre transform, $u\in(\mathbb{R}^n)^*$, and the equality holds for almost every $t\in[0,\tau]$.

Note that for a mechanical system, i.e. of the form $L(x,v)=g(v,v)+U(x)$, where g is a Riemannian metric, and V a smooth function en M, the right hand of this equality is a continuous function of t, since (in local coordinates) we have

$$\frac{\partial L}{\partial x}(\gamma(s),\dot\gamma(s))=DU(\gamma(s))$$

for all $s\in[0,\tau]$. Since $\mathcal{L}$ is a diffeomorphism of class C^1, we conclude that $\dot\gamma$ is actually a continuous function. We refer the reader to [1] (proposition 3.1) for a more detailed explanation and other basic properties of the Lagrangian action.

Finally, we observe that our Lagrangian is a smooth mechanical system in Ω, the open and dense subset of E^N where $U<+\infty$. Since the above considerations are of a local nature, we conclude that an absolutely continuous minimizer whose image is contained in Ω must be smooth.

Tonelli's theorem results extremely useful combined with Marchal's theorem that we recall now. From such combination and the above considerations we can conclude that, except in the collinear case ($\dim E=1$), Tonelli minimizers are smooth in the interior of its domain.

Theorem 9 (Marchal [14], Chenciner [4], Ferrario-Terracini [9])**.** *Suppose* $\dim E\geq 2$. *If* $\gamma:[a,b]\to E^N$ *is such that* $A(\gamma)=\phi(\gamma(a),\gamma(b),b-a)$*, then* $\gamma(t)\in\Omega$ *for all* $t\in(a,b)$.

Combining Marchal's theorem with 8 and the above considerations we obtain the following corollary.

Corollary 10. *If* $\dim E\geq 2$*, then for every pair of configurations* $x,y\in E^N$*, and every positive time* $\tau>0$*, there is at least one curve* $\gamma\in\mathcal{C}(x,y,\tau)$ *such that*

$$A(\gamma)=\phi(x,y,\tau)$$

and such that

$$\gamma(t)\in\Omega\text{ for every }t\in(0,\tau)\,.$$

In particular the restriction of γ *to* $(0,\tau)$ *satisfies Newton's equations, that is to say, it is a true motion of the* N*-body problem.*

2.4. **Homogeneity of the critical action potential.** We shall introduce now some properties which are consequence of the homogeneity of the Newtonian potential. Given a curve $\gamma \in \mathcal{C}(x, y, \tau)$ and two positive numbers $\lambda, \mu > 0$ we define the curve $\gamma_{\lambda,\mu} \in \mathcal{C}(\lambda x, \lambda y, \mu)$ in the obvious way. If γ is defined for $t \in [a, b]$, with $b - a = \tau$, then $\gamma_{\lambda,\mu}$ can be defined for $s \in [\mu\tau^{-1}a, \mu\tau^{-1}b]$ by

$$\gamma_{\lambda,\mu}(s) = \lambda\, \gamma(\tau\mu^{-1}s)$$

Using these curves we can deduce the following lemma and corollaries.

Lemma 11. *If $\mu = \lambda^{3/2}\tau$, then $A(\gamma_{\lambda,\mu}) = \lambda^{1/2}A(\gamma)$.*

Proof. A simple computation shows that the action of $\gamma_{\lambda,\mu}$ is

$$\begin{aligned} A(\gamma_{\lambda,\mu}) &= \frac{\lambda^2\tau^2}{\mu^2}\frac{1}{2}\int_{\mu\tau^{-1}a}^{\mu\tau^{-1}b} \left|\, \dot\gamma(\tau\mu^{-1}s)\,\right|^2 \, ds + \frac{1}{\lambda}\int_{\mu\tau^{-1}a}^{\mu\tau^{-1}b} U(\gamma(\tau\mu^{-1}s))\, ds \\ &= \frac{\lambda^2\tau}{\mu}\frac{1}{2}\int_a^b |\,\dot\gamma(t)\,|^2 \, dt + \frac{\mu}{\lambda\tau}\int_a^b U(\gamma(t))\, dt. \end{aligned}$$

It suffices now to make the substitution $\mu = \lambda^{3/2}\tau$. □

Corollary 12. *For all $x, y \in E^N$ and for every $\tau, \lambda > 0$ we have*

$$\phi(\lambda x, \lambda y, \lambda^{3/2}\tau) = \lambda^{1/2}\phi(x, y, \tau).$$

Proof. Let $\epsilon > 0$ and $\gamma \in \mathcal{C}(x, y, \tau)$ such that $A(\gamma) \leq \phi(x, y, \tau) + \epsilon$. Setting $\mu = \lambda^{3/2}\tau$, we write γ_λ instead of $\gamma_{\lambda,\mu}$. Thus we can apply lemma 11, and we get that

$$\begin{aligned} A(\gamma_\lambda) &= \lambda^{1/2}A(\gamma) \\ &\leq \lambda^{1/2}\phi(x, y, \tau) + \lambda^{1/2}\epsilon. \end{aligned}$$

Since $\gamma_\lambda \in \mathcal{C}(\lambda x, \lambda y, \lambda^{3/2}\tau)$, and $\epsilon > 0$ is arbitrary, we deduce that

$$\phi(\lambda x, \lambda y, \lambda^{3/2}\tau) \leq \lambda^{1/2}\phi(x, y, \tau).$$

Therefore we also have

$$\begin{aligned} \phi(x, y, \tau) &= \phi(\lambda^{-1}\lambda x, \lambda^{-1}\lambda y, \lambda^{-3/2}\lambda^{3/2}\tau) \\ &\leq \lambda^{-1/2}\phi(\lambda x, \lambda y, \lambda^{3/2}\tau), \end{aligned}$$

which proves the reverse inequality. □

Corollary 13. *For all $x, y \in E^N$, and every $\lambda > 0$, we have $\phi(\lambda x, \lambda y) = \lambda^{1/2}\phi(x, y)$.*

Proof. Take the infimum over $\tau > 0$ in the equality given by corollary 12. □

Corollary 14. *Given a free time minimizer $\gamma : [a, b] \to E^N$, and $\lambda > 0$, the curve*

$$\begin{aligned} \gamma_\lambda : [\lambda^{3/2}a, \lambda^{3/2}b] &\to E^N \\ t &\mapsto \gamma_\lambda(t) = \lambda\gamma(\lambda^{-3/2}t) \end{aligned}$$

is also a free time minimizer.

Proof. If we denote $x = \gamma(a)$ and $y = \gamma(b)$, we have

$$\begin{aligned} A(\gamma_\lambda) &= \lambda^{1/2}A(\gamma) \\ &= \lambda^{1/2}\phi(x, y) \\ &= \phi(\lambda x, \lambda y). \end{aligned}$$

On the other hand, it is clear that $\gamma_\lambda \in \mathcal{C}(\lambda x, \lambda y)$, thus γ_λ is a free time minimizer.
□

2.5. **The Mañé critical energy level.** In the Mañé works, the critical energy level of a Tonelli Lagrangian $L : TM \to \mathbb{R}$ on a connected compact manifold M is defined as

$$c(L) = \inf \{ \, c \in \mathbb{R} \mid A_{L+c}(\gamma) \geq 0 \text{ for every closed curve } \gamma \, \} \, .$$

It is easy to show that, if for some $c \in \mathbb{R}$ there is a closed curve γ such that $A_{L+c}(\gamma) < 0$, then for any given pair of points $x, y \in M$ the Lagrangian action of $L + c$ has no lower bound in the set of all absolutely continuous curves from x to y. On the other hand, it can be proved that the Lagrangian $L + c(L)$ admits free time minimizers for any pair of prescribed endpoints $x, y \in E^N$. Moreover, the energy constant of this curves (also called semistatics) is exactly $c(L)$, and using them and the compactness of the manifold it can be proved the existence of invariant measures supported in the critical energy level, and the existence of several compact invariant sets with interesting dynamical properties. We did not wish to develop here this theory more than necessary to show the existing analogy, even if our Lagrangian flow is not complete. We will only show that in the Newtonian N-body problem we have $c(L) = 0$, and that free time minimizers have zero energy.

For a natural mechanical system with bounded potential energy $V : M \to \mathbb{R}$, it is very easy to see that $c(L) = \sup V$. Suppose first that $c < \sup V$. Then there is some open subset $A \subset M$ in which $c < V$. Any constant curve $\gamma(t) = p \in A$ defined in some time interval $[0, \tau]$ is a closed curve, and clearly we have $A_{L+c}(\gamma) = \tau(c - V(p)) < 0$. This proves that $c(L) \geq \sup V$. On the other hand, if $c = \sup V$, then $L + c = T - V + c \geq 0$, thus $A_{L+c}(\gamma) \geq 0$ whatever the curve γ is. In our setting the potential energy is the function $V = -U$, hence the critical energy level is zero.

Suppose that $\gamma : [a, b] \to M$ is an absolutely continuous curve, with finite action $A_L(\gamma)$, and binding the points $x = \gamma(a)$ and $y = \gamma(b)$ in time $\tau = b - a$. A particular kind of variation of the curve γ can be obtained by reparametrization. Consider for instance, for $\alpha > 0$, the linear reparametrization $\gamma_\alpha : [\alpha a, \alpha b] \to E^N$, $\gamma_\alpha(s) = \gamma(\alpha^{-1}s)$. Thus we have $\gamma_\alpha \in \mathcal{C}(x, y, \alpha\tau)$ and $\gamma_1 = \gamma$. The action of γ_α is

$$\begin{aligned} A(\gamma_\alpha) &= \alpha^{-2} \, \frac{1}{2} \int_{\alpha a}^{\alpha b} \left| \, \dot\gamma(\alpha^{-1}s) \, \right|^2 \, ds + \int_{\alpha a}^{\alpha b} U(\gamma(\alpha^{-1}s)) \, ds \\ &= \alpha^{-1} \, \frac{1}{2} \int_a^b | \, \dot\gamma(t) \, |^2 \, dt + \alpha \int_a^b U(\gamma(t)) \, dt. \end{aligned}$$

Therefore

$$\frac{d}{d\alpha} A(\gamma_\alpha) \mid_{\alpha=1} = -\int_a^b \left(\frac{1}{2} \, | \, \dot\gamma(t) \, |^2 - U(\gamma(t)) \right) \, dt,$$

from which the following lemma can be deduced.

Lemma 15. *Let $I \subset \mathbb{R}$ be an open interval. If $\gamma : I \to E^N$ is a free time minimizer, then γ is a trajectory with zero energy.*

Proof. It is clear that for every compact subinterval $[a, b] \subset I$, we have that $\gamma \mid_{[a,b]}$ is a Tonelli minimizer, meaning that $A(\gamma \mid_{[a,b]}) = \phi(\gamma(a), \gamma(b), b - a)$. By Marchal's theorem we know that $\gamma(t) \in \Omega$ for every $t \in (a, b)$. Since $[a, b] \subset I$ is arbitrary, we conclude that $\gamma(I) \subset \Omega$. This implies that γ is a smooth curve which corresponds to a true motion. Therefore γ must have constant energy $h = T(t) - U(t)$.

We fix now some interval $[a, b] \subset I$ and we define the variation γ_α as above given by linear reparametrization of time. Since $\gamma \mid_{[a,b]}$ is also a free time minimizer, we must have

$$\frac{d}{d\alpha} A(\gamma_\alpha) \mid_{\alpha=1} = \; -h(b - a) \; = \; 0$$

which proves that $h = 0$. □

3. Existence of free time minimizers

3.1. **Free time minimizers between two given configurations.** We know that for each configuration $x \in E^N$ and each $\tau > 0$ there is a Tonelli minimizer $\gamma \in \mathcal{C}(x, x, \tau)$ defined in $[0, \tau]$. It was proved ([10] corollary 10) that we must have $A(\gamma) = \phi(x, x, \tau) \leq \mu\tau^{1/3}$ for a constant $\mu > 0$ which not depends on x. Therefore we have $\phi(x, x) = 0$. In other words, given $x \in E^N$, we can leave x and return to x with small displacements, in small times, and in such a way that the action becomes arbitrarily small. However, this does not happen when the extremal configurations are different. If $x \neq y$ and we try to minimize the action from x to y, we can see that a curve defined on a short interval of time has a too expensive action because the average kinetic energy must be large. On the other hand, also we will see that once two configurations $x, y \in E^N$ are fixed, the minimal action $\phi(x, y, \tau)$ becomes arbitrarily large for $\tau > 0$ large enough. These arguments enable us to prove the following result.

Theorem 16. *Given any two different configurations $x \neq y$ in E^N, there is $\tau > 0$ and $\gamma \in \mathcal{C}(x, y, \tau)$ such that $A(\gamma) = \phi(x, y)$.*

Of course, in case $x = y$ we can define the free time minimizer as the constant curve on a trivial interval of zero length, but this convention will be useless for us. We will need the following lemma.

Lemma 17. *Let $d = \| x - y \|$, $m_0 = \min\{ m_1, \ldots, m_N \}$ and $\tau > 0$. If $\gamma \in \mathcal{C}(x, y, \tau)$ is such that $A(\gamma) \leq A$, then $2A\,\tau \geq m_0\, d^2$.*

Proof. Let $x = (r_1, \ldots, r_N)$ and $y = (s_1, \ldots, s_N)$. Since $\| x - y \| = \max \| r_i - s_i \|_E$, we can choose $i_0 \in \{ 1, \ldots, N \}$ for which $d = \| r_{i_0} - s_{i_0} \|_E$. Using now the inequality (2) of the precedent section, we get $2A(\gamma)\,\tau \geq |\, x - y\,|^2$. Since $|\, x - y\,|^2 \geq m_{i_0} \| r_{i_0} - s_{i_0} \|_E^2$ we conclude that $2A\,\tau \geq m_0\, d^2$. □

Proof of theorem 16. Let $x \neq y$ be two given configurations. A sequence of curves $\gamma_n \in \mathcal{C}(x, y, \tau_n)$, $n \geq 0$, will be called minimizing if it satisfies $\phi(x, y) = \lim A(\gamma_n)$. Of course, the existence of such sequences of curves follows from the definition of $\phi(x, y)$.

As a first step of the proof we will show that, given a sequence $\gamma_n \in \mathcal{C}(x, y, \tau_n)$ for which $A = \sup\{ A(\gamma_n) \} < +\infty$, there are positive constants $0 < T_0 < T_1$ such that for all $n \geq 0$ we have $T_0 \leq \tau_n \leq T_1$. We observe first that by lemma 17, we know that the lower bound $\tau_n \geq T_0 = m_0\, d^2/2A$, where $d = \| x - y \| > 0$ and $m_0 = \min\{ m_1, \ldots, m_N \}$, holds for all $n \geq 0$. Moreover, if we fix any $n \geq 0$, and we restrict the curve γ_n to an interval $[0, t]$ with $t \leq \tau_n$, once again by application of lemma 17 we deduce that

$$d_n(t) = \| \gamma_n(t) - x \| \leq (2A\,\tau_n/m_0)^{1/2}$$

hence we have

$$\| \gamma_n(t) \| \leq \| x \| + (2A\,\tau_n/m_0)^{1/2}. \tag{3}$$

Once we know the positions are bounded, we get a lower bound for the Newtonian potential throughout the curve γ_n. Indeed, if $\| z \| \leq K$ then $U(z) \geq m_0^2/2K$. Thus we have

$$U(\gamma_n(t)) \geq \frac{m_0^2}{2(\| x \| + (2A\,\tau_n/m_0)^{1/2})}$$

for all $t \in [0, \tau_n]$, and we conclude that the inequality

$$A \geq A(\gamma_n) \geq \frac{\tau_n\, m_0^2}{2(\| x \| + (2A\,\tau_n/m_0)^{1/2})}$$

holds for all $n \geq 0$. But the right hand of the last inequality is upper bounded if and only if the sequence τ_n also is it. Thus we have proved the existence of the positive constants $T_0 < T_1$ as required.

The second step of the proof consists in applying the first step to some minimizing sequence of curves, which allow us to deduce the existence of a new minimizing sequence of curves, but contained in a fixed set $\mathcal{C}(x, y, \tau)$.

More precisely, we start by choosing a minimizing sequence $\gamma_n \in \mathcal{C}(x, y, \tau_n)$, $n \geq 0$. Since we know that $0 < T_0 \leq \tau_n \leq T_1$ for all $n \geq 0$, we can assume without loss of generality that $\tau_n \to \tau > 0$. Then we define $\gamma_n^* : [0, \tau] \to E^N$ by $\gamma_n^*(t) = \gamma_n(\tau_n\, t/\tau)$, and we can write

$$A(\gamma_n^*) = \frac{\tau_n}{\tau}\frac{1}{2}\int_0^{\tau_n} |\,\dot\gamma_n(t)\,|^2\; dt \;+\; \frac{\tau}{\tau_n}\int_0^{\tau_n} U(\gamma_n(t))\, dt$$

from which we deduce that $\lim A(\gamma_n^*) = \lim A(\gamma_n) = \phi(x, y)$. Since each curve γ_n^* is in $\mathcal{C}(x, y, \tau)$, we conclude that $\phi(x, y, \tau) \leq \phi(x, y)$. Thus we must have $\phi(x, y, \tau) = \phi(x, y)$, which reduces the proof to the application of the Tonelli's theorem 8. □

3.2. **Homothetic free time minimizers.** Now we prove, as we announced in the introduction, that every parabolic homothetic motion by a minimal configuration $a \in E^N$ is a free time minimizer. Recall that a minimal configuration is nothing but a global minimum of the homogeneous function $I^{1/2}U$.

We start assuming that the minimal configuration a_0 is also a normal configuration, in the sense that $I(a_0) = 1$. Thus we have $U(a_0) = U_0$ where $U_0 = \min\{\, U(x) \mid I(x) = 1\,\}$. The corresponding parabolic homothetic ejection is the curve $\gamma_0(t) = \mu_0\, t^{2/3} a_0$, where μ_0 is the only positive constant such that the curve γ_0 defines a motion for $t > 0$. A simple computation shows that the value of μ_0 must be $(9\, U_0/2)^{1/3}$. Note that γ_0 passes through a_0 in time $t_0 = \mu_0^{-3/2} > 0$.

Lemma 18. *If $t_1 > t_0$ then we have $A(\gamma) \geq A(\gamma_0\,|_{[t_0,t_1]})$ for every $\gamma \in \mathcal{C}(a_0, \gamma_0(t_1))$. Moreover, the equality holds if and only if $\gamma = \gamma_0\,|_{[t_0,t_1]}$ (modulo translation in time).*

Proof. Let γ be a curve in $\mathcal{C}(a_0, \gamma_0(t_1))$. Let us suppose that γ is defined in a given interval of time $[s_o, s_1]$. Thus we have $\gamma(s_0) = \gamma_0(t_0) = a_0$ and $\gamma(s_1) = \gamma_0(t_1)$. It is clear that there is a unique number $s' \in [s_0, s_1)$ such that $I(\gamma(s')) = 1$ and such that $I(\gamma(s)) > 1$ for every $s \in (s', s_1]$. Thus we can define the curve γ_1 as $\gamma\,|_{[s',s_1]}$, and obviously we have $A(\gamma) \geq A(\gamma_1)$ with equality holding if and only if $s' = s_0$.

Since $\gamma_1(s) \neq 0$ we can now write γ_1 in polar coordinates $\gamma_1(s) = \rho_1(s)u_1(s)$, where $\rho(s) \geq 1$ and $I(u_1(s)) = 1$ for all $s \in [s', s_1]$. Let now $\gamma_2 \in \mathcal{C}(a_0, \gamma_0(t_1))$ be a second curve, that we define as $\gamma_2(s) = \rho_1(s)a_0$ for $s \in [s', s_1]$. Using the expression of the Lagrangian action in polar coordinates deduced in section (2.1) we conclude that $A(\gamma_2) \leq A(\gamma_1)$ and that the equality holds if and only if $\gamma_2 = \gamma_1$.

More precisely the action of γ_2 in polar coordinates is

$$A(\gamma_2) = \frac{1}{2}\int_{s'}^{s_1} \dot\rho_1(s)^2\, ds + U_0 \int_{s'}^{s_1} \rho_1(s)^{-1}\, ds. \tag{4}$$

Note that this quantity is exactly the Lagrangian action of $\rho_1(s)$ for the Kepler problem in the line with Lagrangian

$$L_\kappa(\rho, \dot\rho) = \frac{1}{2}\dot\rho^{\,2} + \frac{U_0}{\rho}\,.$$

On the other hand, we know that for this Keplerian Lagrangian there is a free time minimizer curve from $\rho_1(s') = 1$ and $\rho_1(s_1) = \mu_0 t_1^{2/3}$, and must have zero energy. This assertion can be proved by direct computations, or using the same arguments given in the proof of theorem 16. It is very easy to see that there is only one extremal curve of the Lagrangian L_κ (modulo translation of the time interval), with zero energy and the required extremities, namely $\rho(s) = \mu_0 s^{2/3}$ for $s \in [t_0, t_1]$.

We conclude that $A(\gamma_2) \geq A(\gamma_0 \mid_{[t_0,t_1]})$. Moreover, the equality holds if and only if $\rho_1(s) = \mu_0(t_0 + s - s')^{2/3}$ for all $s \in [s', s_1]$, in which case we must also have $s_1 - s' = t_1 - t_0$.

The above considerations finish the proof of the lemma, since we have showed that

$$A(\gamma) \geq A(\gamma_1) \geq A(\gamma_2) \geq A(\gamma_0 \mid_{[t_0,t_1]})$$

and that we have equality if and only if $\gamma(s) = \gamma_0(t_0 + (s - s_0))$. □

Proposition 19. *Let $a \in E^N$ be a minimal configuration, and $\mu > 0$ such that the curve defined for $t > 0$ as $\gamma(t) = \mu\, t^{2/3} a$ is an homothetic (parabolic) motion. The continuous extension of γ to $[0, +\infty)$ is a free time minimizer with total collision at $t = 0$.*

Proof. In order to apply the previous lemma, we write γ in the form $\gamma(t) = \mu_0\, t^{2/3} a_0$ with $I(a_0) = 1$. Therefore we know that, if $\gamma(t_0) = a_0$ and $t_1 > t_0$, then $\gamma \mid_{[t_0,t_1]}$ is a free time minimizer.

Let us fix $T > 0$ and $\epsilon \in (0, T)$. Taking $\lambda > 0$ in such a way that $\epsilon = \lambda^{3/2} t_0$ and using corollary 14, we can deduce that $\gamma \mid_{[\epsilon,T]}$ is also a free time minimizer. This means that

$$A(\gamma \mid_{[\epsilon,T]}) = \phi(\gamma(\epsilon), \gamma(T))\,.$$

But

$$\lim_{\epsilon \to 0} A(\gamma \mid_{[\epsilon,T]}) = A(\gamma \mid_{[0,T]})$$

and ϕ is continuous, so we conclude that $\gamma \mid_{[0,T]}$ is a free time minimizer. Since $T > 0$ is arbitrary, the proof is complete. □

3.3. **Calibrating curves of weak KAM solutions.** As we said, thanks to the weak KAM theorem we know that there are a lot of free time minimizers defined over unbounded intervals. Moreover, this theory allows to establish that for any configuration of bodies $x \in E^N$, there is at least one free time minimizer $\gamma_x(t)$ defined for all time $t \geq 0$ and such that $\gamma_x(0) = x$ (see [10], proposition 15).

The interesting fact here, is that we have a lamination of E^N by such curves associated to each weak KAM solution of the Hamilton-Jacobi equation. They are called calibrating curves of the weak KAM solution. One of the main reasons for studying the dynamics of these curves is precisely the link with weak KAM theory. We hope that the results presented here will be useful to characterize the set of these weak solutions, either in the general case or for generic values of the masses.

4. Proof of theorems 1 and 2

The following two results established by Pollard in [17] will be used in the proof of theorem 1. In both cases it is assumed that the center of mass of the motion is fixed at $0 \in E$.

Theorem 20 (Pollard [17], theorem 1.2)**.** *If the energy of a motion $x : [t_0, +\infty) \to \Omega$ is zero, and the center of mass satisfies $G(x(t)) = 0$ for all $t \geq t_0$, then there is a positive constant $\alpha_0 > 0$ such that $I(t) \geq \alpha_0 (t - t_0)^{4/3}$ for all $t \geq t_0$.*

Theorem 21 (Pollard [17], theorem 5.1)**.** *Let $x : [t_0, +\infty) \to \Omega$ be a motion of zero energy such that the center of mass satisfies $G(x(t)) = 0$ for all $t \geq t_0$. Then either*

$$U(t) \sim \alpha\, t^{-2/3} \quad \textit{and} \quad I(t) \sim (9/4)\, \alpha\, t^{4/3}\,,$$

for some positive constant $\alpha > 0$*, or*

$$\lim r(t)\, t^{-2/3} = 0 \quad \textit{and} \quad \lim I(t)\, t^{4/3} = +\infty\,,$$

where $r(t) = \min\{\, \| r_{ij}(t) \| \mid 1 \leq i < j \leq N \,\}$.

Recall now that the center of mass of a given configuration $x = (r_1, \ldots, r_N) \in E^N$ is barycenter of the weighted positions r_i. We can define it as the linear map $G : E^N \to E$ given by $G(x) = M^{-1} \sum m_i r_i$, where $M = \sum m_i$ is the total mass of the system.

The important property of the center of mass is that he has an affine motion when it is computed along every motion. This fundamental property which correspond to the conservation of the linear moment, allow us to reduce the study of a given motion $x(t)$ to the corresponding *internal motion*, $y(t) = (r_1(t) - G(x(t)), \ldots, r_N(t) - G(x(t))$, since $y(t)$ must be also a solution of Newton's equations. Therefore we can say that every motion is the composition of an uniform translation in space with a particular motion contained in $\ker G$ which is nothing but the orthogonal space of the diagonal $\Delta \subset E^N$ with respect to the mass inner product.

For the above reason it is usual in almost all the literature on the subject, to assume that the motions have the center of mass fixed at $0 \in E$.

In order to prove our main results, we shall combine these theorems of Pollard with the following lemma recently proved by the second author, which exclude the collinear case in theorem 1, since his proof uses Marchal's theorem.

Lemma 22 ([11], lemma 3)**.** *If* $\dim E \geq 2$ *and* $x : [0, +\infty) \to E^N$ *is a free time minimizer then the center of mass* $G(x(t))$ *is constant.*

However, we expect that this lemma remains true even in case $\dim E = 1$, as well as theorem 1. This would be true for example if we could prove the following conjecture.

Conjecture 23. *If* $\dim E = 1$ *and* $x : [0, +\infty) \to E^N$ *is a free time minimizer then there is a finite set* $\mathcal{T}_x \subset [0, +\infty)$ *such that* $x(t)$ *is a configuration with collisions if and only if* $t \in \mathcal{T}_x$.

The collinear case seems to be more approachable, since we know that there are exactly $n!$ central configurations and it can be proved that for generic values of the masses only two (symmetric) of such configurations are minimal.

We start the proof with the analysis of the inertia of a free time minimizer.

Proposition 24. *Let* $x : [t_0, +\infty) \to \Omega$ *be a free time minimizer. Then the function* $g : [t_0, +\infty) \to \mathbb{R}$ *defined by*

$$g(t) = \dot{I}(t)\, I(t)^{-1/4}$$

is increasing and bounded.

Proof. Since $h = 0$, we have that $T(t) = U(t)$ for all $t \geq t_0$, and the Lagrange-Jacobi relation gives $\ddot{I}(t) = 2U(t) = 2T(t)$. Thus the derivative of g is

$$\begin{aligned} \dot{g} &= \ddot{I}\, I^{-1/4} - \frac{1}{4}\dot{I}^2 I^{-5/4} \\ &= \frac{1}{4}\left(8\, IT - \dot{I}^2\right) I^{-5/4}. \end{aligned}$$

But on the other hand, we know that $2\, IT - \dot{I}^2 \geq 0$. Therefore we conclude that

$$\dot{g} \geq \frac{3}{2}\, U I^{-1/4} > 0.$$

Thus we have proved that the function g is increasing. We must use now the minimization property in order to prove that g is bounded.

Fix $t > t_0$ and any normal configuration $a \in E^N$, that is, such that $I(a) = 1$. We will compare the Lagrangian action of the free time minimizer x restricted to the interval $[t_0, t]$ with the action of the homothetic curve $\hat{x} : [t_0, t] \to E^N$ given by $\hat{x}(s) = \rho(s)a$ where $\rho(s) = I(x(s))^{1/2}$. Here we will use the polar notation $x = \rho\, u$ where $u(s)$ is the normalized configuration of $x(s)$. Also we will write ρ_0 and ρ_t for denote $\rho(t_0)$ and $\rho(t)$ respectively, as well as u_0 and u_t for $u(t_0)$ and $u(t)$.

By the triangular inequality we have

$$A(x\mid_{[t_0,t]}) = \phi(\rho_0 u_0, \rho_t u_t) \leq A(\hat{x}) + \phi(\rho_0 u_0, \rho_0 a) + \phi(\rho_t a, \rho_t u_t).$$

Moreover, since $S = \left\{ u \in E^N \mid I(u) = 1 \right\}$ is compact, using corollary 13 we can write

$$A(x\mid_{[t_0,t]}) \leq A(\hat{x}) + \Lambda(\rho_0^{1/2} + \rho_t^{1/2}),$$

where $\Lambda = \max\{\, \phi(x, y) \mid x, y \in S \,\}$.

Using the formula for the action in polar coordinates 1 we have that

$$A(x\mid_{[t_0,t]}) = \frac{1}{2}\int_{t_0}^{t} \dot{\rho}(s)^2\, ds + \frac{1}{2}\int_{t_0}^{t} \rho(s)^2 \dot{u}(s)^2\, ds + \int_{t_0}^{t} \rho(s)^{-1} U(u(s))\, ds\,,$$

and that

$$A(\hat{x}) = \frac{1}{2}\int_{t_0}^{t} \dot{\rho}(s)^2\, ds + U(a)\int_{t_0}^{t} \rho(s)^{-1}\, ds\,.$$

We note that both expressions have the same first term, that the second term in the expression of $A(x\mid_{[t_0,t]})$ is positive, and that using the Lagrange-Jacobi relation (which gives $\ddot{I} = 2\,U$ in this case) we can write

$$\int_{t_0}^{t} \rho(s)^{-1} U(u(s))\, ds = \int_{t_0}^{t} U(x(s))\, ds = \frac{1}{2}\int_{t_0}^{t} \ddot{I}(s)\, ds = \frac{1}{2}(\dot{I}(t) - \dot{I}(t_0))\,.$$

Therefore, from the above considerations and the previous inequality we deduce that

$$\frac{1}{2}(\dot{I}(t) - \dot{I}(t_0)) \leq U(a)\int_{t_0}^{t} \rho(s)^{-1}\, ds + \Lambda(\rho_0^{1/2} + \rho_t^{1/2})\,.$$

But $\rho(s) = I(s)^{1/2}$, thus by theorem 20 we have

$$\int_{t_0}^{t} \rho(s)^{-1}\, ds \leq \alpha_0^{-1/2}\int_{t_0}^{t} (s - t_0)^{-2/3}\, ds = 3\alpha_0^{-1/2}\,(t - t_0)^{1/3}\,,$$

and we get the inequality

$$\dot{I}(t) \leq \alpha_1 (t - t_0)^{1/3} + \alpha_2\, I(t)^{1/4} + \alpha_3$$

for some positive constants α_1, α_2 and α_3. Finally, dividing by $I(t)^{1/4}$ and using again theorem 20 we get

$$g(t) = \dot{I}(t)\, I(t)^{-1/4} \leq \alpha_1\alpha_0^{-1/4} + \alpha_2 + \alpha_3\alpha_0^{-1/4}(t - t_0)^{-1/3}$$

from which we conclude that the function g is bounded. □

We can now deduce the following two corollaries

Corollary 25. *If $x : [t_0, +\infty) \to \Omega$ is a free time minimizer then there is a constant $\beta > 0$ such that $I(t) \leq \beta\, t^{4/3}$.*

Proof. By the previous proposition we know that there is a positive constant β_0 such that $\dot{I}\, I^{-1/4} < \beta_0$. Integrating between t_0 and $t > t_0$ we get

$$\frac{4}{3}\,(I(t)^{3/4} - I(t_0)^{3/4}) \leq \beta_0(t - t_0)$$

hence

$$I(t) \leq (\beta_1(t - t_0) + \beta_2)^{4/3}$$

for some positive constants β_1 and β_2. Therefore $I(t)\,t^{-4/3}$ must be bounded. $\square$

Corollary 26. *If* $\dim E \geq 2$*, and* $x : [t_0, +\infty) \to E^N$ *is a free time minimizer, then*

$$U(t) \sim \alpha\, t^{-2/3} \quad \textit{and} \quad I(t) \sim (9/4)\,\alpha\, t^{4/3}\,,$$

for some positive constant $\alpha > 0$*.*

Proof. By lemma 22 we know that $G(x(t))$ is constant. If we call G this constant vector of E, $\delta_G = (G, \ldots, G) \in E^N$ the configuration of total collision at G, and we write $y(t) = x(t) - \delta_G$ for the *internal motion* of x, then it is well known or easy to check that:

(1) $y : [t_0, +\infty) \to E^N$ is also a free time minimizer,
(2) $G(y(t)) = 0$ for all $t \geq t_0$,
(3) $I(x(t)) = M \parallel G \parallel^2 + I(y(t))$ where $M = m_1 + \cdots + m_N$ is the total mass, and
(4) $U(x(t)) = U(y(t))$ for all $t \geq t_0$.

In particular, Marchal's theorem implies that $y(t) \in \Omega$ for every $t > t_0$ and we can apply corollary 25 to the curve $y(t)$. The proof follows then from Pollard's theorem 21. $\square$

Proof of theorem 1. Suppose that $x : [t_0, +\infty) \to E^N$ is a free time minimizer and that $\dim E \geq 2$. By corollary 26 we have that $U(t) \to 0$. This implies that all mutual distances $r_{ij}(t)$ tend to infinity. Moreover, since by lemma 15 we know that the energy of the motion is zero, we also have that $T(t) \to 0$, which is equivalent to say that $\dot{r}_i(t) \to 0$ for all $i = 1, \ldots, N$. $\square$

Proof of theorem 2. Suppose that $x : (-\infty, +\infty) \to E^N$ is a free time minimizer and that $\dim E \geq 2$. Since $I(t) > 0$ the normalized configuration $u(t) = I(t)^{-1/2}x(t)$ is well defined for all $t \in \mathbb{R}$. The set of normal configurations $S = \left\{\, x \in E^N \mid I(x) = 1 \,\right\}$ is compact, therefore there should be an increasing sequence of positive integers $(n_k)_{k\geq 0}$ and normal configurations $a, b \in S$ such that $\lim u(-n_k) = a$ and $\lim u(n_k) = b$. Note that application of corollary 26 we know that there are positive constants $\alpha, \beta > 0$ such that $I(t) \sim \alpha^2\, t^{4/3}$ for $t \to -\infty$ and $I(t) \sim \beta^2\, t^{4/3}$ for $t \to +\infty$.

To each $k \geq 0$ we will associate a free time minimizer defined on the interval $[-1, 1]$ using corollary 14 and the restriction of x to the interval $[-n_k, n_k]$. Thus the sequence of free time minimizers is given by

$$\gamma_k : [-1, 1] \to E^N \quad \gamma_k(t) = n_k^{-2/3} x(n_k\, t)\,.$$

Hence we have

$$\lim \gamma_k(-1) = \lim(n_k^{-2/3} I(-n_k)^{1/2}).\, \lim u(-n_k) = \alpha\, a\,,$$

and

$$\lim \gamma_k(1) = \lim(n_k^{-2/3} I(n_k)^{1/2}).\, \lim u(n_k) = \beta\, b\,.$$

Since for each $k \geq 0$ the curve γ_k is a free time minimizer, we have that

$$\lim A(\gamma_k) = \lim \phi(\gamma_k(-1), \gamma_k(1)) = \phi(\alpha\, a, \beta\, b)\,.$$

Therefore we can apply proposition 4, and we deduce that there is a subsequence of $(\gamma_k)_{k\geq 0}$ which converges uniformly to some free time minimizer $\gamma : [-1, 1] \to E^N$. In particular, we must have $A(\gamma) = \phi(\alpha\, a, \beta\, b)$ and $\gamma(0) = 0$, but this is impossible because it contradicts Marchal's theorem. $\square$

Acknowledgments. The first author would like to thank Soledad Villar for her infinite patience, support and for her useful advise. The second author would like to express its gratitude to Alain Chenciner, Albert Fathi, Christian Marchal, Susanna Terracini and Andrea Venturelli for very helpful discussions and comments on the subject. Finally, we thank the referee for providing constructive comments and help in improving the contents of this paper.

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