C^1-robustly Expansive Homoclinic Classes are generically hyperbolic

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Abstract

Let $f : M \to M$ be a diffeomorphism defined in a d-dimensional compact boundary-less manifold M. We prove that generically C^1-robustly expansive homoclinic classes $H(p)$, p an f-hyperbolic periodic point, are hyperbolic.

1 Introduction

In this article we pursue to analyze the influence of robust expansiveness property on the behavior of the tangent map of a homoclinic class associated to a hyperbolic periodic point of a diffeomorphism. In previous papers we use to say that the homoclinic class $H(p, f)$ is robustly expansive when its continuation $H(p_g, g)$ for g close to f are expansive with a common constant of expansiveness $\alpha > 0$ while we say that the class $H(p, f)$ is persistently expansive when the continuations are expansive but the expansivity constant may go to zero. Before stating our result we would like to change the name of these concepts in order to normalize them according to the usual meaning of robustness in the literature.

Let M be a compact connected boundary-less Riemmanian d-dimensional manifold and $f : M \to M$ a homeomorphism. Let K be a compact invariant subset of M and $\text{dist} : M \times M \to \mathbb{R}$ a metric on M and $\alpha > 0$. We say that f restricted to K is α-expansive when $\text{dist}(f^n(x), f^n(y)) \leq \alpha$ for all $x, y \in K$ and all $n \in \mathbb{Z}$ implies $x = y$. The number $\alpha > 0$ is called a constant of expansiveness for f and K, and sometimes we say that f is expansive in K if α is fixed.

Definition 1.1. *We say that $H(p, f)$ is C^r-robustly expansive ($r \geq 1$) iff there exists a C^r-neighbourhood $\mathcal{U}(f)$ of f such that for all $g \in \mathcal{U}(f)$, there exist p_g the continuation of p and $\alpha(g) > 0$ such that g is $\alpha(g)$-expansive on $H(p_g, g)$.*

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Definition 1.2. *Let p be a hyperbolic periodic point of f. We say that the homoclinic class $H(p,f)$ is uniformly C^r-robustly expansive ($r \geq 1$) iff there exist $\alpha > 0$ and a C^r-neighborhood $\mathcal{U}(f)$ of f such that for all $g \in \mathcal{U}(f)$, there exists a continuation p_g of p such that g is α-expansive in $H(p_g)$.*

In [PPV] it is studied the case when $H(p)$ is uniformly C^1-robustly expansive and p has index 2, where M is three dimensional, proving in this case that $H(p)$ has a dominated splitting and for an open dense subset of $\mathcal{U}(f)$ in the C^1-topology, $f|_{H(p)}$ is hyperbolic. Although [PPV] refers only to three dimensional manifolds, these results extend to the case of d-dimensional manifolds with $H(p)$ a codimension one homoclinic class, see Remark 1.2 below. Moreover, in [SV] we studied the case of C^1-robustly expansive homoclinic classes in any codimension (called there persistently expansive homoclinic classes). It is proved in [SV] that for C^1-robustly expansive homoclinic classes $H(p,f)$ there always exists an homogeneous dominated splitting and some sufficient condition is given so that this dominated class is actually hyperbolic. In this paper we study C^1-robustly homoclinic classes under generic conditions. Indeed, our main result is:

Theorem A. *There exists a residual set $\mathcal{R}$ in $\text{Diff}^1(M)$ such that if $f \in \mathcal{R}$ has a hyperbolic periodic point p with its homoclinic class $H(p)$ C^1-robustly expansive, then $H(p)$ is hyperbolic.*

Let us assume that $f/H(p)$ is C^1-robustly expansive. More precisely, let $\mathcal{U}$ a C^1-neighborhood of f such that for all $g \in \mathcal{U}$ $g/H(p_g)$ is expansive. Then we have

Corollary 1.1. *There exists an open and dense subset $\mathcal{V} \subset \mathcal{U}$ such that for all $g \in \mathcal{V}$ we have $g/H(p_g)$ is hyperbolic.*

Remark 1.2. *In fact it was proved in [PPSV] that when $f/H(p)$ is uniformly C^1-robustly expansive and p has index 1 or $d-1$, $\dim(M) = d$, then $H(p)$ is hyperbolic. We now prove that this holds C^1-generically no matter what the index of p is.*

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2 Proof of Theorem A

First we recall a definition and some results from previous papers.

Definition 2.1. *We say that a compact f-invariant set Λ admits a dominated splitting if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that*

$$\|Df^n_{|E(x)}\| \cdot \|Df^{-n}_{|F(f^n(x))}\| \leq C\lambda^n \;\forall x \in \Lambda,\, n \geq 0.$$

We say that the dominated splitting is homogeneous if the dimension of $E(x)$ is constant for all $x \in \Lambda$.

Definition 2.2. *We say that a compact f-invariant set Λ is hyperbolic if the tangent bundle $T_\Lambda M$ has a continuous f-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that*

$$\|Df^n_{|E(x)}\| \leq C\lambda^n \;\forall x \in \Lambda,\, n \geq 0,$$

and

$$\|Df^{-n}_{|F(x)}\| \leq C\lambda^n \;\forall x \in \Lambda,\, n \geq 0.$$

Theorem 2.1. *Let $f \in Diff^r(M)$, $r \geq 1$, with a hyperbolic periodic point p such that its homoclinic class $H(p)$ is C^1-robustly expansive. Then $H(p)$ has an homogeneous dominated splitting $E \oplus F$.*

Proof. See [SV, Theorem 1]. □

If $q, r \in H(p)$ are hyperbolic periodic points such that $W^s(q) \pitchfork W^u(r) \neq \emptyset$ and $W^u(q) \pitchfork W^s(r) \neq \emptyset$ then we say that r and q are homoclinically related and write $q \sim r$. Clearly if $W^s(q) \pitchfork W^u(r) \neq \emptyset$ and $W^u(q) \pitchfork W^s(r) \neq \emptyset$ then $\dim(W^s(r)) = \dim(W^s(q))$ and $\dim(W^u(r)) = \dim(W^u(q))$. Moreover, by Palis' λ-lemma, "$\sim$" is an equivalence relation and if $q \sim p$ then $H(p) = H(q)$.

Theorem 2.2. *Let $f \in Diff^r(M)$, $r \geq 1$, with a hyperbolic periodic point p such that its homoclinic class $H(p)$ is C^1-robustly expansive. Then there exist $\mathcal{W}$ a C^1-neighbourhood of f, $C > 0$, $0 < \lambda < 1$ and $m > 0$ such that if $g \in \mathcal{W}$, q is a g-hyperbolic periodic point of period $\pi(q)$ and $q \sim p_g$, where p_g is the continuation of p, then*

$$\prod_{i=0}^{k-1} \|Dg^m_{/E^s(g^{im}(q))}\| < C\lambda^k \quad \textit{and} \quad \prod_{i=0}^{k-1} \|Dg^{-m}_{/E^u(g^{-im}(q))}\| < C\lambda^k \tag{1}$$

where $k = [\pi(q)/m]$ (here $[\,.\,]$ represents the integer part.).

Proof. See [SV, Theorem 2], the statement there is slightly different from the given here but both are equivalent. □

The following proposition is a consequence of [HPS], see also [Ma2] and [PS1].

Proposition 2.3. *Let $\Lambda = H(p, f)$ having dominated splitting $E \oplus F$. Then there exist $\phi^s : \Lambda \to Emb_\Lambda(D_1^s, M)$ and $\phi^u : \Lambda \to Emb_\Lambda(D_1^u, M)$ such that defining $W_\epsilon^{cs}(x) = \phi^s(x)D_\epsilon^s$ and $W_\epsilon^{cu}(x) = \phi^u(x)D_\epsilon^u$ the following hold:*

1. *$T_xW^{cs}(x) = E(x)$ and $T_xW^{cu}(x) = F(x)$.*
2. *For every $0 < \epsilon_1 < 1$ there exists $0 < \epsilon_2 < 1$ such that $f(W_{\epsilon_2}^{cs}(x)) \subset W_{\epsilon_1}^{cs}(f(x))$ and $f^{-1}(W_{\epsilon_2}^{cu}(x)) \subset W_{\epsilon_1}^{cu}(f^{-1}(x))$.*

We shall call W_ϵ^{cs} and W_ϵ^{cu} the local center-stable and center-unstable manifolds respectively. Observe that for any $\epsilon > 0$ there exists $\rho(\epsilon) > 0$ such that for all $x \in H(p)$, W_ϵ^{cs} contains a ball of radius $\rho(\epsilon)$ inside the local center stable manifold (with respect to the Riemannian metric inherited from M), the same for $W_\epsilon^{cu}(x)$. For the sake of simplicity we shall assume $\rho(\epsilon) = \epsilon$. Also, for $y \in W_\epsilon^{cs}(x)$ we shall denote $E(y) = T_yW_\epsilon^{cs}(x)$, and for $y \in W_\epsilon^{cu}(x)$ we shall denote $F(y) = T_yW_\epsilon^{cu}(x)$.

Lemma 2.4. *Let C, λ be as in Theorem 2.2 and let $\delta > 0$ be such that $\lambda' = \lambda(1+\delta) < 1$ and let $q \sim p$. Then, there exists $0 < \epsilon_1 < \epsilon$ such that if for all $0 \le n \le \pi(q)$ it holds that for some $\epsilon_2 > 0$, $f^n(W_{\epsilon_2}^{cs}(q)) \subset W_{\epsilon_1}^{cs}(f^n(q))$ then*

$$f^{\pi(q)}(W_{\epsilon_2}^{cs}(q)) \subset W_{C\lambda'^{\pi(q)}\epsilon_2}^{cs}(q)\,.$$

Similarly, if $f^{-n}(W_{\epsilon_2}^{cu}(q)) \subset W_{\epsilon_1}^{cu}(f^{-n}(q))$ then

$$f^{-\pi(q)}(W_{\epsilon_2}^{cu}(q)) \subset W_{C\lambda'^{\pi(q)}\epsilon_2}^{cu}(q)\,.$$

Proof. See [SV, Lemma 4]. □

Let $N > 0$ be such that $C\lambda'^N \le 1/2$ and let us define $S = \{q \sim p \,/\, \pi(q) \ge N\}$. Clearly we have $\mathrm{clos}(S) = H(p)$.

2.1 Generic assumptions.

Now we define the residual set $\mathcal{R}$ required in the statement of Theorem A: it is defined as the set that satisfy the generic properties listed below. We give references that do not intend to be exhaustive, see also [ABCDW].

There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that if $f : M \to M$ is a diffeomorphisms belonging to $\mathcal{R}$ then

1. f is Kupka-Smale, (i.e.: all periodic points are hyperbolic and their stable and unstable manifolds intersect transversally) (see [PM]).

2. for any pair of saddles p, q, either $H(p,f) = H(q,f)$ or $H(p,f) \cap H(q,f) = \emptyset$

3. for any saddle p of f, $H(p,f)$ depends continuously on $g \in \mathcal{G}$ (see [CMP]).

4. every chain recurrent class containing a periodic point p is the homoclinic class associated to that point (see [BC]).

Lemma 2.5. *Let $f \in \mathcal{R}$ and $H(p)$ be C^1-robustly expansive. Let α be an expansivity constant of $H(p,f)$. Then, given $\epsilon_1 < \alpha$ there exists ϵ_2 such that the following statements hold:*
(a) For all $q \in S$, $f^n(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q)) \; \forall \; n \geq 0$. Similarly for the center-unstable manifolds.
(b) for all $y \in W^{cs}_{\epsilon_2}(q)$, $q \in S$, $\lim_{n\to+\infty} \operatorname{dist}(f^n(q), f^n(y)) = 0$. Similarly for the center-unstable manifolds.

Proof. To prove (a) let us begin defining

$$\varepsilon(q) = \sup\{\epsilon > 0 \,/\, f^n(W^{cs}_{\epsilon}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q)), \, \forall n \geq 0\}.$$

On account of the periodicity of $q \sim p$ and by Proposition 2.3 and Lemma 2.4 $\varepsilon(q) > 0$. If we prove that the infimum ϵ_2 in $q \in S$ of $\varepsilon(q)$ is positive then we are done. Suppose on the contrary that for some sequence $\{q_n\} \subset S$ we have that $\varepsilon(q_n) \to 0$ when $n \to +\infty$. Let $m_n > 0$ and $y_n \in W^{cs}_{\varepsilon(q_n)}(q_n)$ be such that $\operatorname{dist}(f^{m_n}(q_n), f^{m_n}(y_n)) = \epsilon_1$. By Lemma 2.4

$$f^{\pi(q_n)} W^{cs}_{\varepsilon(q_n)}(q_n) \subset W^{cs}_{C' \lambda^{\pi(q_n)} \varepsilon(q_n)}(q_n).$$

Therefore, without loss of generality, we may assume that $0 < m_n < \pi(q_n)$. It follows that $m_n \to +\infty$ and $\pi(q_n) - m_n \to +\infty$ when $n \to +\infty$. Without loss of generality we may assume (taking subsequences if necessary) that $f^{m_n}(q_n)$ and $f^{m_n}(y_n)$ converges to x and y respectively. It follows that for all $k \in \mathbb{Z}$ we have $\operatorname{dist}(f^k(x), f^k(y)) \leq \epsilon_1$. We know that $x \in H(p)$. We claim that y also belongs to $H(p)$ which implies a contradiction on the account that ϵ_1 is less than a expansivity constant α. To prove our claim is enough to show that y is in the same recurrent class of p since for our generic assumptions the recurrent class of p and its homoclinic class coincide. Now, let ϵ be any positive number. Let n be such that $\epsilon(q_n) < \epsilon$. Then,

$$q_n, f(y_n), \dots f^{m_n - 1}(y_n), y, f^{m_n+1}(y_n), \dots, f^{\pi(q_n)-1}(y_n), q_n$$

is, for n large enough, a periodic ϵ-chain through y and having a point in $H(p)$. This implies that y belongs to the chain recurrent class of p and hence, since we are assuming $f \in \mathcal{R}$, it is in $H(p)$. This is a contradiction as we said above and item a) is proved. Therefore $x = y$ by the expansivity inside the class, and this is a contradiction since $\text{dist}(x, y) = \epsilon$ as we said above. The item (a) is proved.

The proof of item (b) is a consequence of the fact that there is a finite number of f-periodic points of bounded period on $H(p)$ and that for $q \in S$, $\pi(q) > N$ and therefore we have $C\lambda'^{\pi(q)}\epsilon_2 < \frac{1}{2}\epsilon_2$.

As we have said above, similar results hold for the center-unstable manifolds. □

Taking into account Lemma 2.5 we see that for $x \in S$ we have that the local center stable and unstable manifolds are true stable and unstable ones. For any point in the class we have the following.

Corollary 2.6. *Let $f \in \mathcal{R}$ and $H(p)$ be C^1-robustly expansive. For all $x \in H(p)$, $\epsilon_1 > 0$ there exists $\epsilon_2 > 0$ such that for all $n \geq 0$: $f^n(W^{cs}_{\epsilon_2}(x)) \subset W^{cs}_{\epsilon_1}(f^n(x))$. Moreover, if $y \in W^{cs}_{\epsilon_2}(x) \cap H(p)$ then $\text{dist}(f^n(x), f^n(y)) \to 0$ as $n \to \infty$. Similarly for the center-unstable manifolds.*

Proof. The first part follows since the statement holds for periodic points homoclinically related to p and they are dense in $H(p)$ and the continuity of $W^{cs}_\epsilon(x)$ with respect to x. The second part follows from the first one and using the expansivity inside the class. For, if it were not the case that $\text{dist}(f^n(x), f^n(y)) \to 0$ when $n \to \infty$, then there exist $\rho > 0$ and a subsequence $n_k \to \infty$ when $k \to \infty$, such that $\text{dist}(f^{n_k}(x), f^{n_k}(y)) \geq \rho$. Since M is compact we may assume that $f^{n_k}(x) \to z$ and $f^{n_k}(y) \to w$ and from $\text{dist}(f^{n_k}(x), f^{n_k}(y)) \geq \rho$ we obtain that $\text{dist}(z, w) \geq \rho$. Since $H(p)$ is closed and f-invariant we have from $x, y \in H(p)$ that $z, w \in H(p)$. On account of the first part of the Corollary if $y \in W^{cs}_{\epsilon_2}(x)$ we have that $\text{dist}(f^n(x), f^n(y)) \leq \epsilon_1$ for all $n \geq 0$. Therefore $\text{dist}(f^j(z), f^j(w)) \leq \epsilon_1$ for all $j \in \mathbb{Z}$ contradicting expansiveness of $f/H(p)$. □

Remark 2.7. *Corollary 2.6 says that $W^{cs}_{\epsilon_2}(x) \subset W^{s}_{\epsilon_1}(x)$, that is, the local center stable manifolds are stable ones in the sense that $\text{dist}(f^n(x), f^n(y) \leq \epsilon_1$ for all $x \in H(p)$, $y \in W^{cs}_{\epsilon_2}(x)$, $n \geq 0$. Moreover, by Lemma 2.5, if x is a periodic point then $\text{dist}(f^n(x), f^n(y)) \to 0$ when $n \to \infty$. On the other hand, if x is not periodic then we have $\text{dist}(f^n(x), f^n(y)) \to 0$ when $n \to \infty$ only if $y \in H(p) \cap W^{cs}_{\epsilon_2}(x)$. In order to simplify notation we will assume that $\epsilon_1 = \epsilon_2 = \epsilon$. This assumption implies no loss of generality.*

Lemma 2.8. *Let $f \in \mathcal{R}$ and $H(p)$ C^1-robustly expansive. There exist positive ϵ and δ such that if $x, y \in H(p)$ and $\text{dist}(x, y) < \delta$ then $\emptyset \neq W^{cs}_\epsilon(x) \cap W^{cu}_\epsilon(y) \in H(p)$.*

Proof. Notice that given $\epsilon > 0$ there exists $\delta > 0$ such that $W^{cs}_\epsilon(x) \cap W^{cu}_\epsilon(y) \neq \emptyset$ (and consists of a single point) whenever $\mathrm{dist}(x, y) < \delta$ (this follows by transversality of E and F).

Assume first that x, y are periodic points and $x \sim p, y \sim p$, then choosing $\epsilon = \epsilon_2$ as in Lemma 2.5 we know that $W^{cs}_\epsilon(x) \subset W^s_\epsilon(x)$ and analogously $W^{cu}_\epsilon(x) \subset W^u_\epsilon(x)$, and similarly for y. As x, y are homoclinically related with p we also have, by the λ-lemma, that $W^s_\epsilon(x)$ is accumulated by $W^s(p)$ and $W^u_\epsilon(x)$ is accumulated by $W^u(p)$. Thus if $z \in W^s_\epsilon(x) \cap W^u_\epsilon(y)$ then $z \in H(p)$.

In the general case let $x, y \in H(p)$ and take ϵ and $\delta > 0$ as above. Take sequences $\{x_n\}$ and $\{y_n\}$ of periodic points homoclinically related to p converging to x and y respectively. As periodic points homoclinically related to p are dense in $H(p)$ (see [Sm1]) such sequences exist. Then $W^{cs}_\epsilon(x_n)$ converges to $W^{cs}_\epsilon(x)$ and $W^{cu}_\epsilon(y_n)$ converges to $W^{cu}_\epsilon(x)$. As $W^{cs}_\epsilon(x_n) \cap W^{cu}_\epsilon(y_n) = \{z_n\}$ we have that $z_n \to z \in W^{cs}_\epsilon(x) \cap W^{cu}_\epsilon(y)$. As $z_n \in H(p)$ and $H(p)$ is closed we conclude that $z \in H(p)$. □

Definition 2.3. *We say that a compact f-invariant set Λ has a local product structure if given $\epsilon > 0$ there exists $\delta > 0$ such that if $\mathrm{dist}(x, y) < \delta$ and $x, y \in \Lambda$ then*

$$\emptyset \neq W^s_\epsilon(x) \cap W^u_\epsilon(y) \subset \Lambda$$

where $W^s_\epsilon(x) := \{y \in M : dist(f^n(x), f^n(y)) < \epsilon \text{ for } n \geq 0\}$ (analogously for W^u_ϵ).

From Corollary 2.6 (see also Remark 2.7) and Lemma 2.8 we have

Corollary 2.9. *Let $f \in \mathcal{R}$ and $H(p)$ be C^1-robustly expansive. Then $H(p)$ has a local product structure.*

Theorem 2.10. *If $f : \Lambda \to \Lambda$ is an expansive homeomorphism of the compact metric space (Λ, dist), then there exists a metric D on Λ defining the same topology as dist in Λ, and numbers $r > 0$ and $k > 1$ such that*

$$\forall\, x,\, y \in \Lambda : \max\{D(f(x), f(y)), D(f^{-1}(x), f^{-1}(y))\} \geq \min\{kD(x, y), r\}\,.$$

Proof. See [Ft, Theorem 5.1]. □

Lemma 2.11. *Let $f \in \mathcal{R}$ and $H(p)$ C^1-robustly expansive. Given $\eta > 0$ there exists $\theta > 0$ such that any θ-pseudo-orbit $\{x_n\} \subset H(p)$ is η-shadowed by an orbit in $H(p)$. Moreover, if θ, η are less than half the expansivity constant then the orbit is unique (and periodic if the pseudo orbit is periodic).*

Proof. Let Λ be the homoclinic class $H(p)$. Since $f/H(p)$ is expansive there are $k > 1$, $r > 0$ and D given by Theorem 2.10, where D is a metric on $H(p)$ defining the same topology as dist. Set $\lambda = 1/k$ and shrink $\epsilon > 0$ given by Lemma 2.8, if it were necessary, in order to have $\mathrm{dist}(x, y) < \epsilon \Longrightarrow D(x, y) < \lambda^2 r = r/k^2$. Since $H(p)$ is compact and both metrics, D and dist, define the same topology such an ϵ exists. We have that $\forall\, x,\, y \in H(p) : \max\{D(f(x), f(y)), D(f^{-1}(x), f^{-1}(y))\} \geq \min\{kD(x, y), r\}$. If $y \in W^s_\epsilon(x) \cap H(p)$ then $\min\{kD(x, y), r\} = kD(x, y)$ by the choice of ϵ. If for $y \neq x$ it were the case that $D(f(x), f(y)) \geq kD(x, y)$ then we have $D(f^2(x), f^2(y)) \geq kD(f(x), f(y)) \geq k^2 D(x, y)$, $D(f^3(x), f^3(y)) \geq k^3 D(x, y)$ and so on till we have $k^h D(x, y) \geq r$ which would imply that $\mathrm{dist}(f^h(x), f^h(y)) > \epsilon$ contradicting that $y \in W^s_\epsilon(x)$. Hence we have $D(f^{-1}(x), f^{-1}(y)) \geq kD(x, y)$ and $D(x, y) \geq kD(f(x), f(y))$ which is the same as $D(f(x), f(y)) \leq \lambda D(x, y)$. By induction we obtain $D(f^n(x), f^n(y)) \leq \lambda^n D(x, y)$ for all $n \geq 0$. Similarly if $y \in W^u_\epsilon(x)$ then $D(f^{-n}(x), f^{-n}(y)) \leq \lambda^n D(x, y)$ for all $n \geq 0$. Using these inequalities the proof of the lemma is similar to that given in [Bo, Proposition 3.6] taking into account that by Lemma 2.9 there is a local product structure in $H(p)$. □

Remark 2.12. *Perhaps it is worthwhile to note that the local stable manifold* $W^s_\epsilon(x) = \{y \in M : \mathrm{dist}(f^n(x), f^n(y)) < \epsilon,\, n \geq 0\}$ *is defined with respect to the original Riemannian distance* dist. *On the other hand it is not difficult to see that there are* $c > 0$ *and* $\rho > 0$ *such that* $W^s_c(x) \cap H(p) \subset W^s_{\rho,D}(x) = \{y \in H(p) : D(f^n(x), f^n(y)) < \rho,\, n \geq 0\} \subset W^s_\epsilon(x) \cap H(p)$*, and similarly with respect to the unstable manifolds.*

Proposition 2.13. *Let* $f \in \mathcal{R}$ *and* $H(p)$ *be* C^1*-robustly expansive and let* $q \in H(p)$ *be a periodic point. Then we have that*

1. $W^s(q) \cap W^u(p) \neq \emptyset$, $W^s(p) \cap W^u(q) \neq \emptyset$,

2. q *is hyperbolic and index*$(p) =$ *index*(q),

Proof. By Lemma 2.9 we have that if x is sufficiently close to y then $W^s_\epsilon(x)$ cuts $W^u_\epsilon(y)$ and $W^u_\epsilon(x)$ cuts $W^s_\epsilon(y)$. Since homoclinically related periodic points are dense in $H(p)$ we have a hyperbolic periodic point q_1, homoclinically related to p as close as we wish to q. Therefore $W^s_\epsilon(q_1)$ cuts $W^u_\epsilon(q)$ and $W^u_\epsilon(q_1)$ cuts $W^s_\epsilon(q)$. It follows that $\dim(W^s(q)) = \dim(W^s(q_1)) = \dim(W^s(p))$. By the λ-lemma, $W^u(p)$ accumulates in $W^u_\epsilon(q_1)$ and $W^s(p)$ accumulates in $W^s_\epsilon(q_1)$. Thus, by domination, $W^u(p)$ cuts $W^s_\epsilon(q)$ and $W^s(p)$ cuts $W^u_\epsilon(q)$.

By the generic assumptions given in 2.1 we have that q is hyperbolic. Hence $\mathrm{index}(p) = \mathrm{index}(q)$. □

As a consequence of the previous proposition and Theorem 2.2 we have:

Corollary 2.14. *Let $f \in \mathcal{R}$ and $H(p)$ be C^1-robustly expansive. Then there exists $0 < \mu < 1$, $L > 0$ and $m > 0$ such that if q is a periodic point in $H(p)$ of period $\pi(q)$ then*

$$\prod_{i=0}^{k-1} \|Df^m_{/E^s(f^{im}(q))}\| < \mu^k \quad \textit{and} \quad \prod_{i=0}^{k-1} \|Df^{-m}_{/E^u(f^{-im}(q))}\| < \mu^k$$

where $k = [\pi(q)/m] > L$.

Proof. By Proposition 2.13 all periodic points in $H(p)$ are hyperbolic and homoclinically related to p. Thus, by Theorem 2.2 we have $\prod_{i=0}^{k-1} \|Df^m_{/E^s(f^{im}(q))}\| < C\lambda^k$. Choose $0 < \mu < 1$ such that $\frac{\mu}{\lambda} > 1$ and find $L > 0$ such that for all $k \geq L$ it holds $C\lambda^k < \mu^k \iff C < \left(\frac{\mu}{\lambda}\right)^k$. Thus $\prod_{i=0}^{k-1} \|Df^m_{/E^s(f^{im}(q))}\| < \mu^k$. Similarly $\prod_{i=0}^{k-1} \|Df^{-m}_{/E^u(f^{-im}(q))}\| < \mu^k$. $\square$

Now, we shall conclude the proof of Theorem A, that is, let $f \in \mathcal{R}$ and $H(p)$ C^1-robustly expansive and we will show that $H(p)$ is hyperbolic. Let μ, L and m as in Corollary 2.14, and take $\gamma > 0$ such that $\mu(1+\gamma) < 1$. It is not difficult to prove that there exists $\nu > 0$ such that if $dist(x, y) \leq \nu, x, y \in H(p)$ then

$$1 - \gamma \leq \frac{\|Df^m_{/E(x)}\|}{\|Df^m_{/E(y)}\|} \leq 1 + \gamma.$$

For any $0 < \eta \leq \nu$ consider $\theta = \theta(\eta)$ from lemma 2.11. We may assume that η and θ are smaller than the expansivity constant of f.

To prove that $H(p)$ is hyperbolic it is enough to prove that $\|Df^n_{/E(x)}\| \to 0$ as $n \to \infty$ and $\|Df^{-n}_{/F(x)}\| \to 0$ as $n \to \infty$ for any $x \in H(p)$. Let us show only that $\|Df^n_{/E(x)}\| \to 0$ as $n \to \infty$, the other one being similar. For this, it is enough to show that for some m and any $x \in H(p)$ there exists $\tilde{k}$ such that for any x there exists $0 < k = k(x) \leq \tilde{k}$ such that

$$\prod_{i=0}^{k} \|Df^m_{/E(f^{im}(x))}\| < \frac{1}{2}.$$

Arguing by contradiction, assume this does not hold. Then, there exist sequences $x_n \in H(p)$ and $k_n \to \infty$ such that

$$\prod_{i=0}^{k} \|Df^m_{/E(f^{im}(x_n))}\| \geq \frac{1}{2}, \; 0 \leq k \leq k_n.$$

Let z be an accumulation point of x_n. It follows that

$$\prod_{i=0}^{k} \|Df^m_{/E(f^{im}(z))}\| \geq \frac{1}{2} \ \ \forall k \geq 0.$$

Observe that the above property implies that z cannot be a periodic point since all of them are hyperbolic. Let w be an accumulation point of the sequence $f^{im}(z)$, i. e., there exists $i_k \to \infty$ such that $f^{i_k m}(z) \to w$. Since $f^{\pi(p)}/H(p)$ is topologically mixing, see [Me], there exist $z_1 \in H(p)$ and n_0 such that $dist(z_1, w) < \theta/2$ and $dist(f^{n_0 m}(z_1), z) < \theta$, where n_0 is a multiple of $\pi(p)$.

Let $K = inf\{\|Df^m_{/E(x)}\| : x \in H(p)\}$ and take j_0 such that

$$((1+\gamma)\mu)^{j+n_0} < \frac{1}{2}K^{n_0} \ \ \forall j \geq j_0.$$

Finally, choose $i_k > j_0$ such that $dist(f^{i_k m}(z), w) < \theta/2$. Consider now the θ-periodic-pseudo-orbit defined by $\{z, ..., f^{i_k m}(z), z_1, ..., f^{n_0 m}(z_1), z\}$. By lemma 2.11 there exists a periodic point q that η-shadows the pseudo orbit and $(i_k + n_0)m$ is (a multiple of) $\pi(q)$. Moreover $\pi(q)$ is larger than L if η is small enough: otherwise, we would get, by taking a sequence $\eta_n \to 0$ (and hence $\theta(\eta_n) \to 0$), that z is periodic. Therefore

$$\begin{aligned}
\frac{1}{2}K^{n_0} &\leq \prod_{i=0}^{i_k-1} \|Df^m_{/E(f^{im}(z))}\| \prod_{i=0}^{n_0-1} \|Df^m_{/E(f^{im}(z_1))}\| \\
&\leq \prod_{i=0}^{i_k+n_0-1} (1+\gamma)\|Df^m_{/E^s(f^{im}(q))}\| \\
&\leq ((1+\gamma)\mu)^{i_k+n_0} < \frac{1}{2}K^{n_0},
\end{aligned}$$

a contradiction. This completes the proof of Theorem A.

References

[ABCDW] Abdenur F., Bonatti Ch., Crovisier S., Díaz L.J., Wen L., *Periodic points and homoclinic classes,* Ergodic Theory Dynam. Systems, **27** (2007), p. 1-22.

[Ao] N. Aoki, *The set of Axiom A diffeomorphisms with no cycles.,* Bol. Soc. Bras. Mat., Nova Sér., **23, No.1-2,** (1992), p. 21-65.

[BC] Bonatti,Ch, Crovisier, S, *Recurrence and genericity,* Invent. Math., **Vol. 158 No. 1** (2004), p. 33-104.

[Bo] Bowen, R., *Equilibrium states and the ergodic theory of Anosov diffeomorphisms,* Springer Lecture Notes in Math, **470,** 1975.

[CMP] Carballo, C. M.; Morales C. A.; Pacifico M. J., *Homoclinic classes for C^1-generic vector fields,* Erg Th & Dyn Sys, **23** (2003), p. 403-415.

[Fr] Franks, J., *Necessary conditions for stability of diffeomorphisms,* Trans. Amer. Math. Soc., **158** (1971), p. 301-308.

[Ft] Fathi A., *Expansiveness, Hyperbolicity and Hausdorff Dimension,* Communications in Mathematical Physics, **126** (1989), p. 249-262.

[Ha] Hayashi, S., *Diffeomorphisms in $\mathcal{F}^\infty(\mathcal{M})$ satisfy Axiom A.,* Ergodic Theory Dyn. Syst., **12, No.2** (1992), p. 233-253.

[HK] Hale J., Koçak H., *Dynamics and Bifurcation,* Springer-Verlag, **Texts in applied Mathematics 3** 1991.

[HPS] Hirsch, M.W.; Pugh, C.C.; Shub, M., *Invariant manifolds,* Lecture Notes in Mathematics. 583, **Springer-Verlag** 1977.

[Ma1] R. Mañé, *Expansive diffeomorphisms,* Lectures Notes in Mathematics, Springer, Berlin, **468** (1975), p. 162-174.

[Ma2] R. Mañé, *Contributions to the Stability Conjecture,* Topology, **17** (1978), p. 383-396.

[Ma3] R. Mañé, *An ergodic closing lemma,* Annals of Mathematics, **116** (1982), p. 503-540.

[Me] L. Mendoza, *Topological entropy of homoclinic closures,* Trans. AMS , **311** (1989), p. 255-266.

[Nh] S. E. Newhouse, *Hyperbolic Limit Sets,* Trans. of the Amer. Math. Soc., **Vol. 167** (May 1972), p. 125-150.

[Pu] Pugh, C., *The closing lemma,* Amer. Jour. of Math., **89** (1967), p. 956-1009.

[PM] Palis, Jacob jun.; de Melo, Welington, *Geometric theory of dynamical systems. An introduction,* Transl. from the Portuguese by A. K. Manning. (English), **Springer-Verlag** (1982).

[PPSV] M. J. Pacifico, E. Pujals, M. Sambarino, J. Vieitez, *Robustly expansive codimension-one homoclinic classes are hyperbolic,* To appear in Ergodic Theory and Dynam. Systems, (), p. -.

[PPV] M. J. Pacifico, E. R. Pujals, J. L. Vieitez, *Robustly expansive homoclinic classes,* Erg Th & Dyn Sys, **25** (2005), p. 271-300.

[PS1] E. Pujals, M. Sambarino, *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms,* Annals of Mathematics, **151** (2000), p. 961-1023.

[PS2] E. Pujals, M. Sambarino, *On the Dynamics of Dominated Splitting,* Preprint, (2002), p. -.

[PT] Palis, J; Takens, F., *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations,* Cambridge Studies in Advanced Mathematics. 35, **Cambridge University Press** 1993.

[Sm1] S. Smale, *Diffeomorphisms with many periodic points,* Differential and Combinatorial Topology, **Princeton University Press** 1965.

[Sm2] S. Smale, *The* Ω*-stability theorem,* Proc. Sympos. Pure Math. AMS, **vol. 14** (1970), p. 289-297.

[SV] M. Sambarino, J. Vieitez, *On* C^1*-persistently Expansive Homoclinic Classes,* Discrete and Continuous Dynamical Systems, **14, No.3** (2006), p. 465-481.

[We] L. Wen, *Homoclinic Tangencies and Dominated Splittings,* Nonlinearity, **15** (No.5), p. 2002-1445.1469

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