# $C^1$ STABLE MAPS: EXAMPLES WITHOUT SADDLES.

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ABSTRACT. We give here the first examples of $C^1$ structurally stable maps in manifolds of dimension greater than one that are not diffeomorphisms nor expanding. It is shown that an Axiom A endomorphism all of whose basic pieces are expanding or attracting is $C^1$ stable. A necessary condition for the existence of such examples is also given.

## 1. INTRODUCTION

The problem of stability is central in dynamical systems. As the models appearing in factual sciences cannot reflect exactly the reality, it is important to decide wether a given model can be slightly perturbed without changing its qualitative behaviour. Two nearby systems are said equivalent if one of them is equal to the other if when a change of coordinates is performed. In our context, a discrete dynamical system will be a $C^1$ map of a compact manifold $M$. An equivalence between $f$ and $g$ is a homeomorphism $h$ of $M$ such that $hf = gh$. A map $f \in C^1(M)$ is $C^1$ structurally stable, or simply $C^1$ stable, if there exists a neighborhood $\mathcal{U}$ of $f$ in $C^1(M)$ such that $f$ and $g$ are topologically equivalent for every $g \in \mathcal{U}$.
For invertible self maps of compact manifolds, the characterization of $C^1$ stability was obtained by Robinson ([Rob]) and Mañé ([Ma]). Some years before that, it was shown by Shub ([Sh]) that an expanding map is $C^r$ stable. Since then, no new examples of $C^1$ stable maps in manifolds of dimension greater than one were discovered.

It is already known that if $M$ is a compact manifold then the following conditions are necessary for a map $f \in C^1(M)$ to be $C^1$ structurally stable:

(1) The set of critical points of $f$ is empty.
(2) The map $f$ is Axiom A without cycles.
(3) If the unstable set of a basic piece $\Lambda$ intersects another basic piece, then $\Lambda$ is an expanding basic piece.

Note that this already contrasts with the chronology of discoverings for diffeomorphisms: in this latter case, sufficient conditions (namely, Axiom A + strong transversality) for $C^1$ structural stability were obtained by C.Robinson (1976). It was conjectured by Palis and Smale that these conditions were also necessary for $C^1$ stability. R.Mañé obtained a proof of this central conjecture in 1987. In the case of noninvertible maps, it is already known that the hyperbolicity is necessary for stability, but no set of sufficient conditions were established until now: no nonexpanding examples were known . Looking for a characterization of $C^1$ stable maps, our main interest here is to provide sufficient conditions and analize examples of

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$C^1$ stable maps.
Now we comment briefly the necessary conditions stated above. The first item is obvious since it concerns with $C^1$ maps. It follows that $f$ is locally invertible and so a covering map. There exist examples of maps (in manifolds of dimension greater than one) having critical points that are $C^r$ structurally stable ($r > 1$) and have nontrivial nonwandering sets, see [IPR1] and [IPR2].
The proof of the second item was given by Aoki, Moriyasu and Sumi in [AMS], adapting the proof of the $C^1$ stability conjecture given by Mañé ([Ma]). Adopting the definition of [MP], the meaning of Axiom A is the following: a map $f$ satisfies the Axiom A if the nonwandering set of $f$ is hyperbolic, the set of periodic points of $f$ is dense in the nonwandering set and the restriction of $f$ to $\Lambda$ is injective whenever $\Lambda$ is a basic piece that is not expanding. Actually, the $C^1$ $\Omega$-stable maps without nonwandering critical points were already characterized: Przytycki showed that items (1) and (2) above imply that $f$ is $C^1$ $\Omega$-stable ([Prz]) and the above mentioned theorem of [AMS] implies that the second condition is necessary for $\Omega$-stability when the map is critical points free. When the set of critical points intersects the nonwandering set, some conditions were shown to be sufficient for $C^1$ $\Omega$-stability in [DRRV], but a full characterization was not already established.
The third item was also proved by Przytycki in [Prz].

In Przytycki's above mentioned article, there is an example of an $\Omega$-stable map that satisfies the three items above; it was asked if this example is structurally stable or not. As far as we know, this question remained unsolved since then. It is our pourpose to show that his example is $C^1$ structurally stable in a forthcoming article. The nonwandering set of Przytycki's map is the union of an attracting fixed point, a saddle type basic piece and an expanding set. The main difficulty to prove the stability is that one has to deal with self-intersections of the unstable manifolds of the saddle type basic piece.
In this work, we find examples without basic pieces of the saddle type. More precisely, given an Axiom A map $f$ denote by $\Gamma(f)$ the union of the basic pieces of $f$ that are not expanding nor attracting (repelling sets that are not expanding are contained in $\Gamma$).

**Theorem 1.** *Let $M$ be a compact manifold. If $F \in C^1(M)$ is an Axiom A map without critical points and $\Gamma(F) = \emptyset$, then $F$ is $C^1$ structurally stable.*

We knew how to prove similar assertions some time ago, the ideas are contained in results previously obtained in [IP], [IPR1] and [IPR2]. What we didn't knew were examples of maps verifying its hypothesis; the discovery of simple examples in odd dimensions greater than two motivated us to write the present work.

**Theorem 2.** *Let $M$ be a manifold admitting an expanding map. Assume that there exists an embedding of $M$ into some sphere $S$. Then there exists a noninvertible Axiom A map $F$ in $C^1(M \times S)$ whose nonwandering set is the union of an expanding set and a nonperiodic attractor. It follows that $F$ is $C^1$ structurally stable.*

For example, if $M$ is the torus $T^n$ and $S$ is the sphere $S^{n+1}$, then the hypothesis above are satisfied. However, as our next result asserts, it is not frequent to find examples of maps satisfying the above sufficient conditions of stability. A neighborhood $U$ of an attractor $\Lambda$ of a map $f$ is said *admissible* if it is contained in the basin of attraction of $\Lambda$, the closure of $f(U)$ is contained in $U$ and the restriction of $f$ to $U$ is injective. The attractor $\Lambda$ is said *topologically simple* if there exists an admissible neighborhood $U$ of $\Lambda$ such that every curve in $U$ is homotopic to a curve in $f(U)$ with the homotopy contained in $U$. For example, a periodic attractor and a DA attractor are topologically simple, a solenoid is not.

**Theorem 3.** *Let $M$ be a compact manifold and $f \in C^1(M)$ a nonivertible map. Assume in addition that $f$ is an Axiom A map without critical points, that has a topologically simple attractor. Then $\Gamma(f)$ is not empty.*

The attractors appearing in the examples of Theorem 2 are generalized solenoids. We find relevant to the development of the theory the possible discovery of examples of other type, or the classification of maps satisfying the condition $\Gamma(f) = \emptyset$.
It seems that in dimension two every attractor is topologically simple. Thus there would not exist maps satisfying the hypothesis of theorem 1 in dimension two.
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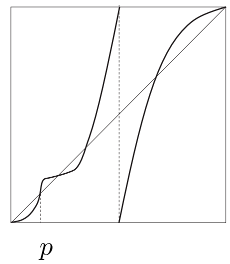
## 2. Sufficient conditions for stability.

In this section we prove theorem 1. The proof of this theorem is inspired in that of theorem C given in [IPR2]. In that result, there were critical points in the basin of the attractor but it was assumed that the nonwandering set of the map was completely invariant, that is not the case here.
The map $F$ is $C^1$ $\Omega$-stable: as was explained in the introduction, it is sufficient to prove that $F$ is critical points free and Axiom A without cycles. The no cycles condition is a trivial consequence of the nature of the basic pieces; indeed, cycles between expanding sets are transverse because there are no critical points and so it must be contained in the nonwandering set.
There exists a $C^1$ neighborhood $\mathcal{U}$ of $F$ such that for every $G \in \mathcal{U}$ one can define $A(G)$ as the union of the attracting basic pieces of $G$, $B(G)$ as the union of the basins of elements of $A(G)$ and $J(G)$ as the union of the expanding basic pieces of $G$ with all its preimages. It follows that $J(G) \cup B(G) = M$. Moreover, the restrictions $F_{|\Omega(F)}$ and $G_{|\Omega(G)}$ are conjugated by a conjugacy close to the identity.
For a map $F$ satisfying the hypothesis of theorem 1, it is not necessarily true that $F^{-1}(\Omega(F))$ is equal to $\Omega(F)$ (see figure 1 below), and so $J(F)$ is not necessarily contained in the nonwandering set of $F$. However, it holds that $J(F)$ is always an expanding set: there exist constants $C > 0$ and $\lambda > 1$ such that $||DF^n_x(v)|| \geq C\lambda^n||v||$ for every vector $v$, $n > 0$ and $x \in J(F)$.

**Lemma 1.** *$J(F)$ is an expanding set.*

*Proof.* Note that $J(F)$ is compact because it is the complement of $B(F)$. The future orbit of any point $x \in J(F)$ eventually lands on an expanding basic piece; this implies that $\lim_{n\to+\infty} ||DF^n_x(v)|| = +\infty$. A standard argument, that we repeat here, implies the assertion of the lemma. Denote by $SJ$ the set of pairs $(x, v)$, where $x \in J(F)$ and $v$ is a unit vector tangent to $M$ at $x$.
**Claim.** There exists $N > 0$ such that, for any $(x, v) \in SJ$, there exists an $n \leq N$ such that $||DF^n_x(v)|| \geq 2$.
We prove the claim by contradiction: assume that for every $N$ there exists $(x_N, v_N)$ such that $||DF^j_{x_N}(v_N)|| < 2$ for every $0 \leq j \leq N$. Passing to a subsequence, suppose

FIGURE 1.




*$F$ is an Axiom A map of $S^1$. The basic pieces are: two attracting fixed points, a fixed repeller $p$ and a expanding Cantor set. $F^{-1}(p)$ is not contained in $\Omega(F)$ .*

that $(x_N, v_N)$ converges to an $(x, v) \in SJ$. Given any $m > 0$, note that

$$||DF_x^m(v)|| = \lim_N ||DF_{x_N}^m(v_N)|| \leq 2,$$

which contradicts the fact that the limit of $||DF_x^m(v)||$ is $\infty$.

Given $(x, v) \in SJ$, define $n(x, v)$ as the maximum $n \leq N$ such that the assertion of the claim holds, that is, $||DF_x^n(v)|| \geq 2$.
Now let $\lambda = 2^{1/N}$ and let $C > 0$ be the minimum of the norms $||DF_x^j(v)||$ for $(x, v) \in SJ$ and $0 \leq j \leq N$. Let $x_n = F^n(x)$ and $v_n = DF_x^n(v)$. By the claim, there exists numbers $n_i$, $0 \leq i \leq k$ such that $0 = n_0 < n_1 < \cdots < n_k \leq n$ and $n_{i+1}$ is defined by induction by $n_{i+1} - n_i = n(x_{n_i}, \frac{v_{n_i}}{||v_{n_i}||})$. Clearly, $n_{i+1} - n_i \leq N$, $n - n_k \leq N$ and

$$\frac{||v_{n_{i+1}}||}{||v_{n_i}||} = ||DF_{x_i}^{n_{i+1}-n_i}(v_{n_i}/||v_{n_i}||)|| \geq 2,$$

by definition of $n_i$. Finally, note that

$$||v_n|| = \left[\prod_{i=0}^{k-1} \frac{||v_{n_{i+1}}||}{||v_{n_i}||}\right] . \frac{||v_n||}{||v_{n_k}||} \geq C.2^k \geq (C/2).\lambda^n,$$

because $k \geq \frac{n}{N} - 1$. □

It is a well known fact that hyperbolic attractors are stable. As the restriction of $F$ to a neighborhood of $A(F)$ is a diffeomorphism onto its image, there exists a conjugacy $h$ between the restrictions of $F$ and $G$ to neighborhoods $U(F)$ of $A(F)$ and $U(G)$ of $A(G)$. Moreover, the conjugacy can be taken as close to the identity as wished by diminishing the neighborhood $\mathcal{U}$ of $F$.

Denote by $h$ the conjugacy between $F$ and $G$ referred above. The first step of the proof consists in extending $h$ to $B(F)$. This would be trivial if $f$ were a diffeomorphism. For let $x$ be a point in $F^{-k}(U(F))$; to define $h(x)$ one has to choose a $G^k$-preimage of $h(F(x))$, and there are a lot of them. However, our arguments will imply that there exists one of these preimages that is closest to $x$. This is easy to prove for a finite number of preimages, but at each step, one should be forced to diminish the neighborhood of $F$. Hence a different argument must be applied when the preimages taken are sufficiently close to an expanding set. Next

lemma 2 will provide the precise estimates, and corollary 1 will explain the order of choices of neighborhoods and constants. The second step of the proof consists in extending $h$ to the complement $J(F)$ of $B(F)$.
Some definitions and notations are in order before proceeding to the statement of the main lemma. Define $U_k(F) = F^{-k}(U(F))$. Let $d$ denote the distance in $M$, and $B(x;r)$ be the ball of center $x$ and radius $r$. As the maps have no critical points and the manifold $M$ is compact, there exists $\epsilon_0 > 0$ such that $G(x) = G(y)$ implies $x = y$ or $d(x,y) > \epsilon_0$ whenever $G \in \mathcal{U}$. If $W$ is a subset of $M$ and $\delta$ is a positive real number, denote by $\mathcal{N}_\delta(W)$ the set of homeomorphisms $h : W \to h(W) \subset M$ that are $\delta$ close to the identity of $W$. If, moreover, $h$ conjugates corresponding restrictions of $F$ and $G$, then we write $h \in \mathcal{N}_\delta(W;G)$.

**Lemma 2.** (1) *Let $\rho$ be a positive constant less than or equal to $\epsilon_0/2$. Then there exist $\delta = \delta(\rho) > 0$ and a neighborhood $\mathcal{U} = \mathcal{U}(\rho)$ of $F$ such that, if $W$ is any subset of $M$ and $h$ belongs to $\mathcal{N}_\delta(W;G)$ for some $G \in \mathcal{U}$, then there exists a unique extension $h'$ of $h$ in $\mathcal{N}_\rho(W \cup F^{-1}(W);G)$.*
(2) *There exist a positive number $\delta_0$, and a neighborhood $V$ of $J(F)$ such that the following property holds:*
*Given any $\delta < \delta_0$ there exists a neighborhood $\mathcal{U}$ of $F$ such that, given any $W \subset V$, any $G \in \mathcal{U}$ and any $h \in \mathcal{N}_\delta(W;G)$, there exists a unique extension $h'$ of $h$ in $\mathcal{N}_\delta(W \cup F^{-1}(W);G)$.*

*Proof.* (1) Let $x \in F^{-1}(W)$, one has to prove that there exists a unique $x' \in B(x;\rho)$ such that $G(x') = h(F(x))$.
Note that given any $\rho > 0$ there exists a neighborhood $\mathcal{U}$ of $F$ and a positive number $\delta$ such that, for every $G$ in $\mathcal{U}$ and $x \in M$ it holds that:

$$G(B(x;\rho)) \supset B(G(x);2\delta) \tag{1}$$

Note also that if $\rho > 0$ is less than $\epsilon_0/2$, then $G_{|B(x;\rho)}$ is a homeomorphism onto its image.
To prove part (1) it suffices to show that $h(F(x)) \in B(G(x);2\delta)$. But

$$d(h(F(x)),G(x)) \leq d(h(F(x)),F(x)) + d(F(x),G(x)) \leq 2\delta,$$

if the $C^0$ distance between $F$ and $G$ is less than $\delta$. This defines $h$ in $F^{-1}(W)$; it is a homeomorphism since it is open by definition (locally $h = G^{-1}hF$). Moreover $h(x) = h(y)$ implies $h(F(x)) = h(F(y))$, hence $F(x) = F(y)$ and so $x = y$ because $h$ is close to the identity.
(2) If $\mathcal{U}$ is a small neighborhood of $F$ and $V$ is a small neighborhood of $J(F)$, then lemma 1 implies that there exists a number $\lambda > 1$ and an adapted metric in $M$ such that $DG_x$ $\lambda$-expands any direction, for any $x \in V$ and $G \in \mathcal{U}$. Using part (1), let $\delta_0 = \delta(\epsilon_0/2)$. Now, if $\delta < \delta_0$, and if for some $G \in \mathcal{U}(\rho)$ one has an $h \in \mathcal{N}_\delta(W;G)$, then there is an extension $h'$ of $h$ to $F^{-1}(W)$ that still conjugates $F$ and $G$. It must be shown that the extension remains in the $\delta_0$ neighborhood of the identity. Indeed, if $F(x) \in W$, and $h(x) = x'$, then $d(F(x),F(x')) \geq \lambda d(x,x')$. Moreover,

$$d(F(x),F(x')) \leq d(F(x),G(x')) + d(G(x'),F(x')) \leq \delta_0 + d_0,$$

where $d_0$ is the $C^0$ distance between $F$ and $G$. Taking $\mathcal{U}$ small so that $d_0 < (\lambda-1)\delta_0$, it follows that $d(x,x') < \delta$. □

By part one of the lemma, it follows that one can extend $h$ to $U_1(F)$ if the neighborhood $\mathcal{U}$ is diminished once. Therefore this proceeding can be repeated a finite number of times, which is not enough to cover $B(F)$. The second part of the previous lemma then implies that the proceeding of taking preimages will provide an extension of $h$ to the whole $B(F)$, because the complement of $B(F)$ is an expanding set.

It follows that there exist neighborhoods $V$ of $J(F)$ and $\mathcal{U}$ of $F$, and a positive constant $\lambda > 1$, such that for an adapted metric, it holds that $DG_x$ expands vectors at a rate at least $\lambda$ for any $x \in V$ and every $G \in \mathcal{U}$. Note that the neighborhood $V$ of $J(F)$ can be taken backward invariant for every $G \in \mathcal{U}$.

**Corollary 1.** *Given any $\delta > 0$ there exists a neighborhood $\mathcal{U}$ of $F$ such that for every $G \in \mathcal{U}$ the set $\mathcal{N}_\delta(B(F); G)$ is not empty.*

*Proof.* Fix an admissible neighborhood $U(F)$ of $A(F)$. Then choose neighborhoods $V$ of $J(F)$ and $\mathcal{U}$ of $F$ such that every $G \in \mathcal{U}$ is $\lambda$-expanding in $V$. By lemma 1 there exists a positive integer $k$ such that $V \cup U_k(F) = M$. Diminishing $\mathcal{U}$ one can obtain, for some fixed $G \in \mathcal{U}$, an $h \in \mathcal{N}_\rho(U(F), G)$ in such a way that repeatedly applying part (1) of lemma 2, there exists an extension of $h$ in $\mathcal{N}_\delta(U_k(F); G)$, again denoted by $h$. Now, as $V$ is backward invariant, the fundamental neighborhood $U_{k+1}(F) \setminus U_k(F)$ is contained in $V$, so part (2) of lemma 2 gives an extension of $h$ to $U_{k+1}(F)$, and this extension remains in the $\delta$ neighborhood of the identity. By induction the homeomorphism $h$ is extended to $\cup_{n>0} U_n(F) = B(F)$ to a conjugacy between $F_{|B(F)}$ and $G_{|B(G)}$ that is $\delta$ close to the identity in $B(F)$. □

It remains to prove the second part of the theorem, that consists in extending $h$ to the whole manifold.

Let $\epsilon$ be a constant of expansivity of the restriction of $F$ to $J(F)$, that is, for every $z \neq w$ in $J(F)$, there exists $N \geq 0$ such that $d(F^N(z), F^N(w)) > \epsilon$. For every $G$ in a neighborhood of $F$ the same $\epsilon$ is a constant of expansivity for the restriction of $G$ to $J(G)$.

By corollary 1, one can choose $\mathcal{U}$ such that the distance between the identity and $h$ is less than $\epsilon/2$, where $h : B(F) \to B(G)$ is a conjugacy between $F$ and some fixed $G \in \mathcal{U}$. Let $x \in \partial B(F)$ and $\{x_n\}$ a sequence in $B(F)$ that converges to $x$. We claim that the sequence $\{h(x_n)\}$ converges. Otherwise, one can choose accumulation points $z \neq y$ of the set $\{h(x_n)\}$. By the choice of $\epsilon$ there exists $N \geq 0$ such that $d(G^N(y), G^N(z)) > \epsilon$. Then the sequence $\{hF^N(x_n) : n > 0\}$ accumulates at $G^N(y)$ and $G^N(z)$, but as $\{F^N(x_n)\}$ converges to $F^N(x)$, a contradiction appears because $h$ is $\epsilon/2$ close to the identity. This proves the claim. Define $h$ in the boundary of $B(F)$ as the limit of $\{h(x_n)\}$. The claim implies that $h$ is continuous and surjective. Finally $h$ is injective because two points $z$ and $w$ with the same image would verify that $d(F^n(z), F^n(w))$ eventually becomes greater than $\epsilon$, while $h(F^n(z)) = h(F^n(w))$ for every $n > 0$. This extends $h$ to the closure of $B(F)$ that equals the whole manifold unless $J(F)$ has nonempty interior, in which case the map is expanding and the stability already established by Shub.

## 3. Existence of examples

This section is devoted to the proof of Theorem 2. Let $T$ be an expanding map of degree greater than one on a manifold $M$ and assume that there exists an embedding $J$ from $M$ into $S^n$. Consider $S^n$ as the one point compactification of $\mathbb{R}^n$ and assume that $JM$ is contained in the ball $B(0; 1)$. To simplify notation we will also assume that $J$ is the inclusion. Let $\alpha > 0$ be such that $T(x) = T(y)$ implies $x = y$ or $|x - y| > \alpha$, where $|.|$ is the Euclidean norm in $\mathbb{R}^n$. For each $z \in M$ let $f_z : \mathbb{R}^n \to \mathbb{R}^n$ given by $f_z(w) = aw + (1-a)z$, where $a < 1$ is a positive real number to be chosen. Note that $f_z$ can be extended to a diffeomorphism of $S^n$ having an attracting fixed point at $z$ and a repelling fixed point at $\infty$. Define $F : M \times S^n \to M \times S^n$ by $F(z, w) = (T(z), f_z(w))$. Note that $F$ is a locally invertible map with the same degree and class of differentiability as $T$.

We make the following choices: if $a < \frac{\alpha}{\alpha+2}$, then $\frac{(1-a)\alpha}{2a} > 1$; take a number $r \in (1, \frac{(1-a)\alpha}{2a})$, and define $U = M \times B(0; r)$.

**Claim 1.** The closure of $F(U)$ is contained in $U$.

Note that $f_z(B(0; r))$ is equal to the ball $B((1-a)z; ar)$, whose closure is contained in $B(0; r)$, because $r > 1$ implies that $(1 - a)|z| + ar \le (1 - a) + ar < r$. This implies the claim.

**Claim 2.** The restriction of $F$ to $U$ is injective.

Assume that $F(z, w) = F(z_1, w_1)$ with $z \neq z_1$, $|w| < r$ and $|w_1| < r$. This implies that $Tz = Tz_1$, so $|z - z_1| > \alpha$. Moreover, $f_z(w) = f_{z_1}(w_1)$ implies that $aw + (1 - a)z = aw_1 + (1 - a)z_1$. But this is impossible because

$$|a(w - w_1) + (1 - a)(z - z_1)| \ge (1 - a)\alpha - 2ar > 0$$

by the choice of $r$.

**Claim 3.** The intersection $\Lambda$ of the future iterates of $U$ is a transitive hyperbolic attractor.

This part of the construction is a trivial generalization of the solenoid attractor: the solenoid is obtained when $M$ is the circle $S^1$, $T(z) = z^2$ and $n = 2$. Consider the inverse limit of $T$, that is, the set $\Sigma$ of sequences $\underline{z} = \{z(m) \ : \ m \ge 0\}$ such that $T(z(m)) = z(m-1)$ for every $m > 1$, and endow it with the product topology. Given $z \in M$ let $U_z = \{z\} \times B(0; r)$. If $\underline{z} = \{z_m\} \in \Sigma$, note that the sequence $F^n(U_{z(m)})$ is a decreasing sequence of relatively compact sets whose diameters converge to 0, which implies that its closures intersect in a unique point, denoted $i(z(0))$. It is then easily seen that $i : \Sigma \to \Lambda$ is a homeomorphism realizing a conjugacy between the restriction of $F$ to $\Lambda$ and the shift $\sigma$ given by $\sigma(\underline{z})(m) = T(z(m))$.

**Claim 4.** The basin of attraction of $\Lambda$ is equal to $M \times S^n \setminus M \times \{\infty\}$.

Note that $|f_z(w)| \le a|w| + (1 - a)$, but the function $x \in \mathbb{R} \to ax + (1 - a) \in \mathbb{R}$ has a fixed attractor at $x = 1$ that attracts every $x > 1$. It follows that for any $w \in S^n \setminus \{\infty\}$ such that $|w| > 1$, there exists a positive $k$ such that $F^k(z, w) \in U$.

**Claim 5.** $F$ is Axiom A with $\Gamma(F) = \emptyset$.

Note that $M \times \{\infty\}$ is an expanding basic piece. It follows that the nonwandering set of $F$ is the union of $\Lambda$ with this expanding set. By claim 2, the restriction of $F$ to $\Lambda$ is injective. The claim and the theorem are proved.

## 4. Proof of Theorem 3

We give first a short description of the proof. The hypothesis on the attractor implies that the restriction of $f$ to the immediate basin is injective. Next it is assumed that $\Gamma(f)$ is empty to obtain, applying lemma 1, that the boundary of the immediate basin is contained in an expanding set. There exists an admissible neighborhood $N$ of the attractor $\Lambda$ having a smooth boundary. If $\partial N_k$ denotes the boundary of the intersection of $f^{-k}N$ with the immediate basin, then $\partial N_k$ must converge to the boundary of the immediate basin. On the other hand, the

Lebesgue measure of $\partial N_k$ converges exponentially to zero since it is contained in a neighborhood of an expanding set. This is a contradiction: it implies that the boundary of the immediate basin was a finite set of points, and hence the map was a diffeomorphism.

Let $\Lambda$ be a topologically simple attractor of a noninvertible Axiom A map $f$. The immediate basin of $\Lambda$, denoted by $B^0 = B^0(\Lambda)$, is the union of the connected components of the basin that intersect $\Lambda$. Taking an iterate of $f$ one can assume that the immediate basin is connected and satisfies $f(B^0) = B^0$.

Let $U$ be an admissible neighborhood of $\Lambda$ such that every closed curve in $U$ is homotopic to a closed curve in $f(U)$, with the homotopy contained in $U$. Define by induction an increasing sequence of open sets as follows: let $U_0 = U$ and $U_n$ be the connected component of $f^{-1}(U_{n-1})$ that contains $U_{n-1}$. The first four claims give the proof that $f$ is injective in $B^0$.

**Claim 1.** The restriction of $f$ to $U_n$ is a covering map.

The restriction is locally injective because $f$ has no critical points. To prove that it is a covering map it suffices to show that it is proper. Let $\{x_k\}$ be a sequence in $U_n$ converging to a point $x \notin U_n$. The sequence $\{f(x_k)\}$ converges to a point $y$ in the closure of $U_{n-1}$. We have to prove that $y \notin U_{n-1}$. If $y \in U_{n-1}$, then there exists a ball $B$ centered at $x$ such that $f(B) \subset U_{n-1}$ which is absurd since $U_n \cup B$ is a connected set whose image is contained in $U_{n-1}$ and strictly contains $U_n$.

**Claim 2.** Every closed curve in $U_n$ is homotopic to a closed curve contained in $U_{n-1}$.

Indeed, given a closed curve $\gamma$ contained in $U_n$ let $\gamma'$ be a closed curve in $f(U_0)$ that is homotopic to $f^n(\gamma)$. This implies that the $f^n$-lift of $\gamma'$ is a closed curve contained in $U_{n-1}$ and homotopic to $\gamma$.

**Claim 3.** There exists a map $g$ defined in $\tilde{U} = \cup U_n$ such that $g(f(x)) = x$.

Define $g : f(U_0) \to U_0$ as the inverse of $f_{|U_0}$. Assume $g$ was extended until $U_{n-1}$ and take any $x \in U_n$. If $\gamma_i$, $i = 1, 2$ are curves in $U_n$ joining a point in $\Lambda$ with $x$, then $\gamma_1\gamma_2^{-1}$ is a closed curve in $U_n$ that has a homotopic curve $\gamma'$ in $U_{n-1}$. As $\gamma'$ has a closed lift under $f$, namely $g(\gamma')$, it follows that the $f$-lift of $\gamma_1\gamma_2^{-1}$ is closed. Therefore the $f$-lifts of $\gamma_1$ and $\gamma_2$ have the same final point $x'$, which must be sent to $x$ by $f$. This allows us to define $g(x) = x'$, thus extending $g$ to a diffeomorphism from $U_n$ with the property $f(g(x)) = x$. Note also that $g(U_n) = U_{n+1}$.

**Claim 4.** The restriction of $f$ to $B^0(\Lambda)$ is injective.

The above claim implies that $f$ is injective on $\tilde{U}$. It remains to show that $\tilde{U} = B_0(\Lambda)$. Indeed, let $x \in B_0(\Lambda)$ and let $\alpha$ be a curve in $B_0(\Lambda)$ joining $x$ with a point in $\Lambda$. There exists $K > 0$ such that $f^K(\alpha) \subset U_0$, but as $U_K$ is the connected component of $f^{-K}(U_0)$ that contains $U_0$, we conclude that $\alpha \subset U_K$, whence $x \in U_K$.

From now on it is assumed, by contradiction, that $\Gamma(f) = \emptyset$. This implies by lemma 1 that the complement of $B(F)$, the union of the basins of the attractors, is an expanding set.

**Claim 5.** There exists a neighborhood $N$ of $\Lambda$ such that the following properties hold:

(1) $N$ is an admissible nieghborhood of $\Lambda$.

(2) The boundary of $N$ is a finite union of connected submanifolds of codimension one.

Given an admissible neighborhood $U$, one can obtain a new admissible neighborhood contained in $U$ which consists of a finite union of balls. A small perturbation of this neighborhood would be also an admissible neighborhood with smooth boundary. This proves the claim.

For $k > 0$, let $N_k$ denote the preimage of $N$ under $(f_{|B^0})^k$. Given a neighborhood $V$ of the boundary of $B^0$, there exists $k_0$ such that the boundary of $N_k$ is contained in $V$ for every $k > k_0$. If the neighborhood $V$ of the boundary of $B^0$ is small, then $f$ is expanding in $V$. It follows that the boundary of $N_k$ converges with $n$ to a finite union of single points. But this implies that the map $f$ is a diffeomorphism, a contradiction.

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