CODIMENSION ONE GENERIC HOMOCLINIC CLASSES WITH INTERIOR

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Preliminary draft

Abstract. We study generic diffeomorphisms with a homoclinc class with non empty interior and in particular those admitting a codimension one dominated splitting. We prove that if in the finest dominated splitting the extreme subbundles are one dimensional then the diffeomorphism is partially hyperbolic and from this we deduce that the diffeomorphism is transitive.

1. Introduction

1.1. **Definitions and statement of results.** Let M be a compact connected boundaryless manifold of dimension d and let $\mathcal{D}iff^1(M)$ be the set of diffeomorphisms of M endowed with the C^1 topology. We shall say that a property is generic if and only if there exists a residual set $\mathcal{R}$ of $\mathcal{D}iff^1(M)$ for which for every $f \in \mathcal{R}$ satisfies that property.

For a hyperbolic periodic point $p \in M$ of some diffeomorphism f we denote its homoclinic class by $H(p, f)$ and defined as the clousure of the transversal intersections between the stable and unstable manifolds of p.

The main result of this paper concerns the following conjecture of [ABD]:

Conjecture 1. *Generically, homoclinic classes with interior are the whole manifold.*

Some progress has been made towards the proof of this conjecture (see [ABD] and [ABCD]), in particular, it has been proved in [ABD] that isolated homoclinic classes as well as homoclinic classes admiting a strong partially hyperbolic splitting verify the conjecture. Also, they proved that a homoclinic class with non empty interior must admit a dominated splitting. In [ABCD] the conjecture was proved for surface diffeomorphisms.

In [ABD] the question about whether within the finest dominated splitting the extremes subbundless should be volume hyperbolic was posed. We give a positive answer when the class admits codimension one dominated splitting. This gives also new situations where the above conjecture holds and weren't known.

The main theorem of this paper is the folowing

Theorem 1. *Let f be a generic diffeomorphism with a homoclinic class H with non empty interior and admitting a codimension one dominated splitting $T_H M = E^1 \oplus E^2$ where $dim(E^1) = 1$. Then, the bundle E^1 is uniformly hyperbolic (contracting) for f.*

Recall that Theorem 8 of [ABD] implies that such a homoclinic class must admit dominated splitting.

As a consecuence of our main theorem we get the following easy corolaries:

Corollary 1. *Let H be a homoclinic class with non empty interior for a generic diffeomorphism f such that $T_H M = E^1 \oplus E^2 \oplus E^3$ is a dominated splitting for f and $dim(E^1) = dim(E^3) = 1$. Then, H is partially hyperbolic and $H = M$.*

Proof . The class should be strongly partially hyperbolic because of the previous theorem (applyed to f and to f^{-1}). Corollary 1 of [ABD] (page 185) implies that $H = M$.

□

Corollary 2. *Let H be a homoclinic class with non empty interior for a generic diffeomorphism f of a 3-dimensional manifold M such that H is far from tangencies. Then, $H = M$*

Proof . If the class is not hyperbolic it should have periodic points of different indices. If the class cannot be aproximated by homoclinic tangencies, so (see [Gou2]) it must admit a dominated splitting into 3 subbundles, thus satisfying the hypothesis of our main Theorem.

□

Incidentally, we also give a new proof in the two dimensional case:

Corollary 3. *Let f be a generic surface diffeomorphism having a homoclinic class with nonempty interior. Then f is Anosov.*

Proof . Since the class must admit dominated splitting (Theorem 8 of [ABD]), this should be into 2 one dimensional subbundles. So, the class must be hyperbolic and thus, since the conjecture holds for hyperbolic homoclinic classes f is Anosov.

□

1.2. **Idea of the proof.** The idea of the proof is the following.

First we prove that if the homoclinic class has interior, the periodic points in the class (which are all saddles) should have eigenvalues (in the E_2 direction) exponentially (with the period) far from 1. Otherwise we manage to obtain a sink or a source inside the interior of the class and thus contradicting the fact that the interior of the homoclinic class for generic diffeomorphisms is, roughly speaking, robust (Theorem 4 of [ABD]).

Then, using the previous fact and some results of [LS] and [PS] we manage to prove the center manifolds integrating a one dimensional extreme subbundle should have nice dynamical properties. For this we also use the connecting lemma for pseudo orbits of [BC].

Finally, in the event that the extreme subbundle is not hyperbolic, we manage to obtain (using dynamical properties and a Lemma of Liao) periodic points near the class with bad contraction or expansion in those extreme subbundles. Using Lyapunov stability of the homoclinic class (which is generic, see [ABD] and [CMP]) we ensure that the periodic points we found belong to the class and thus reach a contradiction.

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2. Preliminary results

In this section we shall state some results we are going to use in the proof of the main theorem. It can be skipped and used as reference when the results are used.

Some generic properties of diffeomorphisms are contained in the following Theorem (see [ABD] and references therein):

Theorem 2. *There exists a residual subset $\mathcal{R}$ of $\mathcal{D}iff^1(M)$ such that if $f \in \mathcal{R}$*

- a1) *f is Kupka Smale (that is, all its periodic points are hyperbolic and their invariant manifolds intersect transversally).*
- a2) *The periodic points of f are dense in the chain recurrent set of f ([1]). Moreover, homoclinic classes coincide with chain recurrent classes.*

[1]The chain recurrent set is the set of points x satisfying that for every $\varepsilon > 0$ there exist an ε-pseudo orbit form x to x, that is, there exist points $x = x_0, x_1, \dots x_k = x$ such that $d(f(x_i), x_{i+1}) < \varepsilon$.

a3) *Every homoclinic class with non empty interior of f is Lyapunov stable for f and f^{-1} ([2]).*
a4) *For every periodic point p of f, $H(p,f) = \overline{W^s(p)} \cap \overline{W^u(p)}$.*
a5) *Homoclinic classes vary continuously with the Haussdorff distance with respect to f.*
a6) *Given a homoclinic class H, if U is an open set such that $\overline{U} \subset int(H)$ then there exists $\mathcal{U}$ neighborhood of f such that for every $g \in \mathcal{U} \cap \mathbb{R}$ $U \subset H_g$ is satisfied (where H_g is the continuation of H for g).*

To obtain dinamical properties of the center manifolds we shall use the following results from [LS] and [PS]. First recall that if $T_H M = E^1 \oplus E^2$ is a dominated splitting then, Theorem 5.5 of [HPS] gives us a local $f-$invariant manifolds W^1_ε tangent to E^1.

Local $f-$invariance means that $\forall \varepsilon > 0$ there exists $\delta > 0$ such that $f(W^1_\delta(x)) \subset W^1_\varepsilon(f(x))$. Taking f^{-1} we have an analog for E^2.

Theorem 3 (Main Theorem of [LS])**.** *Let Λ a compact invariant set of a generic diffeomorphism f admitting a codimension one dominated splitting $T_\Lambda M = E^1 \oplus E^2$ with $dim(E^2) = 1$. Assume that $\overline{Per(f_{/\Lambda})} = \Lambda$. Then, $\forall x \in \Lambda$ and $\forall \varepsilon > 0$ there exists $\delta > 0$ such that*

$$f^{-n}(W^2_\delta(x)) \subset W^2_\varepsilon(f^{-n}(x)) \quad \forall n \geq 0$$

In particular, $W^2_\delta(x) \subset \{y \in M \ : \ d(f^n(x), f^n(y)) \leq \varepsilon\}$.

If there is a dominated splitting for H of the form $T_H M = E^1 \oplus E^2$, then, there exists V neighborhood of H such that if a point z satisfies that $f^n(z) \in V$ $\forall n \in \mathbb{Z}$ then we can define the splitting for z and it will be dominated (see [BDV]).

If I is an interval, we denote by $\omega(I) = \bigcup_{x \in I} \omega(x)$, and by $W^{ss}_\varepsilon(I) = \bigcup_{x\in I} W^{ss}_\varepsilon(x)$ its strong stable manifold. Also $\ell(I)$ denote its length. We shall state the following result which is an inmediate Corollary of Theorem 3.1 of [PS] for generic dynamics.

Theorem 4 ([PS])**.** *Let $f \in \mathcal{D}iff^1(M)$ a generic diffeomorphism and Λ compact invariant set admiting a codimension one dominated splitting $T_\Lambda M = E^1 \oplus E^2$ (where $dim(E^2) = 1$). Then, there exists δ_0 such that if I is an interval integrating the subbundle E^2 satisfying $\ell(f^n(I)) < \delta < \delta_0$ $\forall n \geq 0$ and that its orbit remains in an addapted neighborhood of Λ, then, only one of the following holds:*

[2]Lyapunov stability of Λ means that $\forall U$ neighborhood of Λ there is $V \subset U$ neighbourhood of Λ such that $f^n(V) \subset U$ $\forall n \geq 0$.

(1) *$\omega(I)$ is contained in the set of periodic points of f restricted to the addapted neighborhood of Λ and also, some of them is an attractor.*
(2) *I is wandering (that is, $W^{ss}_\varepsilon(f^n(I)) \cap W^{ss}_\varepsilon(f^m(I)) = \emptyset$ for all $n \neq m$). This implies that $\ell(f^n(I)) \to 0$ as $|n| \to \infty$.*

Other result we shall use is the following well known Lemma of Franks:

Theorem 5 (Frank's Lemma [F]). *Let $f \in Diff^1(M)$. Given $\mathcal{U}(f)$ C^1 neighborhood of f, $\exists \mathcal{U}_0(f)$ and $\varepsilon > 0$ such that if $g \in \mathcal{U}_0(f)$, $\theta = \{x_1, \dots, x_m\}$ and*

$$L : \bigoplus_{x_i \in \theta} T_{x_i}M \to \bigoplus_{x_i \in \theta} T_{g(x_i)}M \quad \textit{such that} \quad \left\|L - Dg|_{\bigoplus T_{x_i}M}\right\| < \varepsilon$$

Then, $\tilde{g} \in \mathcal{U}(f)$ exists such that $D\tilde{g}_{x_i} = L|_{T_{x_i}M}$ and if R is a compact set disjoint from θ we can consider $\tilde{g} = g$ in R.

Finally we state the following Lemma of Liao. A proof can be found (with the same notation) in [W]. We shall state the Theorem in the particular case of index one dominated splitting with an addapted metric (which always exist because of [Gou1], recall also that for one dimensional spaces $\prod_i \|A_i\| = \|\prod_i A_i\|$). The theorem holds in a wider context.

Lemma 1 (Liao [L]). *Let Λ be a compact invariant set of f with dominated splitting $T_HM = E^1 \oplus E^2$ such that $\|Df|_{E^1(x)}\|\|Df^{-1}|_{E^2(x)}\| < \gamma$ $\forall x \in \Lambda$ and $dim(E^1) = 1$. Assume that*

(1) *There is a point $b \in \Lambda$ such that $\|Df^n|_{E^1(b)}\| \geq 1$ $\forall n \geq 0$.*
(2) *There exists $\gamma < \gamma_1 < \gamma_2 < 1$ such that given $x \in \Lambda$ satisfying $\|Df^n|_{E^1(x)}\| \geq \gamma_2^n$ $\forall n \geq 0$ we have that there is $y \in \omega(x)$ satisfying $\|Df^n|_{E^1(y)}\| \leq \gamma_1^n$ $\forall n \geq 0$.*

Then, for any $\gamma_2 < \gamma_3 < \gamma_4 < 1$ and any neighborhood U of Λ there exists a periodic point p of f whose orbit lies in U, is of the same index as the dominated splitting and satisfies $\|Df^n|_{E^1}(p)\| < \gamma_4^n$ $\forall n \geq 0$ and $\|Df^n|_{E^1(p)}\| \geq \gamma_3^n$ $\forall n \geq 0$.

3. Proof of the main theorem

For $p \in Per(f)$, $\pi(p)$ denotes the period of p.

Lemma 2. *Let H be a homoclinic class with interior of a generic diffeomorphism f admiting a dominated splitting $E^1 \oplus E^2$. . Then, there exists $\lambda < 1$ such that for all $p \in Per(f|_H)$ the following hold:*

(1) *If* $dimE^1 = 1$ *then* $\|Df^{\pi(p)}_{/E^1(p)}\| \leq \lambda^{\pi(p)}$
(2) *If* $dimE^2 = 1$ *then* $\|Df^{-\pi(p)}_{/E^2(p)}\| \leq \lambda^{\pi(p)}$

Proof . We shall prove just item 2). The first one is analogous and also follows by applying the result to f^{-1}.

Arguing by contradiction assume that does not hold, that is, for every $\lambda < 1$ there exists $p \in Per(f|H)$ such that $\|Df^{-\pi(p)}_{/E^2(p)}\| \geq \lambda^{\pi(p)}$ which is equivalent to $\|Df^{\pi(p)}_{/E^2(p)}\| \leq \lambda^{-\pi(p)}$ since E^2 is one dimensional.

Let U be an open set such that $\overline{U} \subset int(H)$. Since f is generic, property $a6$) of Theorem 2 ensure us the existence of a neighbourhood $\mathcal{U}$ of f such that for every g in a residual subset of $\mathcal{U}$ we have $U \subset H_g$ (H_g is the continuation of H for g).

Frank's Lemma implies the existence of $\varepsilon > 0$ such that if we fix an arbitrary finite set of points, we can perturb the difeomorphism as near as we want of those points obtaining a new diffeomorphism with arbitrary derivatives ($\varepsilon-$close to the originals) inside $\mathcal{U}$.

Let us fix $1 > \lambda > 1 - \varepsilon/2$ and let $p \in Per(f|_H)$ as before. Since f is generic, the periodic points of the same index as p are dense in H so, we can choose $q \in U \cap Per(f)$ homoclinically related to p.

Let $x \in W^s(p) \cap W^u(q)$ y $y \in W^s(q) \cap W^u(p)$, we get that the set $\Lambda = \mathcal{O}(p) \cup \mathcal{O}(q) \cup \mathcal{O}(x) \cup \mathcal{O}(y)$ hyperbolic.

Consider the following periodic pseudo orbit contained in Λ,

$$\{..., p, f(p), ..., f^{N\pi(p)-1}(p), f^{-n_0}(y), \dots, f^{n_0}(y), f^{-n_0}(x), \dots, f^{n_0}(x), p, ...\}$$

which we shall denote as $\wp^N$. Clearly, given $\beta > 0$ there exists n_0 such that $\wp^N$ is a β-pseudo orbit. At the same time, if we choose N large enough we obtain a pseudo orbit which stays near p much longer than of q and then inherit the behaviour of the derivative of p rather than that of q.

The shadowing lemma for hyperbolic sets (see [Sh]) implies that for every $\alpha > 0$ there exists β such that every closed β-pseudo orbit is α-shadowed by a periodic point. So, let us choose α in such a way that the following conditions are satisfied:

(a) $B_{2\alpha}(q) \subset U$.
(b) If $d(z, w) < \alpha$ and x, y are in an adapted neighbourhood of H then,

$$\frac{\|Df_{/E^2(z)}\|}{\|Df_{/E^2(w)}\|} < 1 + c$$

(c verifies $(1 + c)(1 - \frac{\varepsilon}{2})^{-1} < 1 + \varepsilon$).

Let $\beta < \alpha$ be given from the Shadowing Lemma for that α and let n_0 be such that $\wp^N$ is a β-pseudo orbit. Therefore there exists a periodic orbit r of period $\pi(r) = N\pi(p)+4n_0$ such that α- shadows $\wp^N$. Therefore, setting $k = \sup_{x\in M}\|Df_x\|$, we have

$$\begin{aligned}\|Df^{N\pi(p)+4n_0}_{/E^2(r)}\| &\leq k^{4n_0}(1+c)^{N\pi(p)}\|Df^{\pi(p)}_{/E^2(p)}\|^N \\ &\leq k^{4n_0}\left((1+c)(1-\frac{\varepsilon}{2})^{-1}\right)^{N\pi(p)} < (1+\varepsilon)^{\pi(r)}\end{aligned}$$

where the last inequality holds provided N is large enough. Notice that the orbit of r passes through U. On the other hand, by domination, we have that $\|Df^{\pi(r)}_{/E^1(r)}\| < \|Df^{\pi(r)}_{/E^2(r)}\|$. Since E^1 and E^2 are invariant we conclude that any eigenvalue of $Df^{\pi(r)}_r$ is less than $(1+\varepsilon)^{\pi(r)}$.

Now, if we compose in the orbit of r its derivatives with homoteties of value $(1+\varepsilon)^{-1}$ we obtain, by using Frank's Lemma, a diffeomorphism g so that all the eigenvalues associated to the periodic orbit r are less than 1, that is, r is a periodic attractor (sink). This contradicts the generic assumption, since the sink is persistent, so every residual $\mathcal{R} \in \mathcal{U}$ will have diffeomorphisms with a sink near r, thus contained in U, and thus contradicting that the interior is persistent.

□

Lemma 3. *Let H be a homoclinic class with non empty interior for a generic diffeomorphism f such that $T_HM = E^1 \oplus E^2$ is a dominated splitting for f and $dim(E^2) = 1$. Then, there exists ε_0 such that for all $\varepsilon \leq \varepsilon_0$ there exists δ such that $\forall x \in H$,*

$$W^2_\delta(x) \subset W^{uu}_\varepsilon(x) := \{y \in M : \ d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon \ ; \ d(f^{-n}(x), f^{-n}(y)) \to 0\}$$

.

Proof . First we shall prove the Lemma for periodic points and then, using this fact prove the general statement. Let $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(H)$ is contained in the adapted neighborhood of H and such that if $d(x,y) < \varepsilon_0$ then

$$\frac{\|Df^{-1}_{/E^2(x)}\|}{\|Df^{-1}_{/E^2(y)}\|} < \lambda^{-1}$$

where λ is given by Lemma 2. Let $\varepsilon \leq \varepsilon_0$ and let $\delta > 0$ from Theorem 3 correspoding to this ε.

Let $p \in Per(f|_H)$ for which there is $y \in W^2_\delta(p)$ such that $d(f^{-n}(y), f^{-n}(p)) \nrightarrow 0$. Since $W^2_\delta(p)$ is one dimensional, $W^2_\delta(p)\backslash\{p\}$ is a disjoint union of two intervals. Denote I_δ the connected component of $W^2_\delta(p)\backslash\{p\}$ that contains y. By Theorem 3 we have either $f^{2\pi(p)}(I_\delta) \subset I_\delta$ or $f^{2\pi(p)}(I_\delta) \supset I_\delta$. In any event, since $y \in I_\delta$ we conclude that there exits a point $z_0 \in W^2_\varepsilon(p)$ fixed under $f^{2\pi(p)}$ and such that $\|Df^{2pi(p)}_{/E^2(z_0)}\| \leq 1$.

This contradicts the previous Lemma, since by the way ε was chosen we get (since we know that $d(f^i(p), f^i(z_0)) < \varepsilon$ for all i) that

$$\|Df^{2\pi(p)}_{/E^2(p)}\| = \prod_{i=0}^{2\pi p-1} \|Df_{/E^2(f^i(p))}\| < \lambda^{-2\pi(p)} \prod_{i=0}^{2\pi p-1} \|Df_{/E^2(f^i(z_0))}\| =$$
$$= \lambda^{-2\pi(p)} \|Df^{2\pi(p)}|_{E^2(z_0)}\| < \lambda^{-2\pi(p)}$$

Now, lets prove the general statement. Let us suppose that for every $\varepsilon > 0$ there exist $x \in H$ and a small arc $I \subset W^2_\delta(x)$ containing x such that $\ell(f^{-n}(I)) \nrightarrow 0$. We know, because of Theorem 3 that $\ell(f^{-n}(I)) \leq \varepsilon$, then, takeing $n_j \to +\infty$ such that $\gamma \leq \ell(f^{-n_j}(I)) \leq \varepsilon$ and takeing limits, we obtain an arc J integrateing E^2 such that $\ell(f^n(J)) \leq \varepsilon$ $\forall n \in \mathbb{Z}$ and containing a point $z \in J \cap H$ (a limit point of $f^{-n_j}(x)$).

Now, we shall use Theorem 4 to reach a contradiction. It is not dificult to discard the first posibility in the Theorem because it will contradict what we have proved for periodic points.

On the other hand, if J is wandering, we know that it can not be acumulated by periodic points. Since f is generic, we reach a contradiction if we prove that the points in J are chain recurrent (see property $a2$) of Theorem 2). Theorem 4, implies that, $\ell(f^n(J)) \to 0$ ($|n| \to +\infty$), then, since $z \in H \cap J$, if we fix ϵ, and $y \in J$, then, for some future iterate k_1 and a past one $-k_2$, we know that $f^{k_1}(y)$ is ϵ-near of $f^{k_1}(z)$ and $f^{-k_2}(y)$ is ϵ-near $f^{-k_2}(z)$. Since homoclinic classes are chain recurrent classes, there is an ϵ pseudo orbit from $f^{k_1}(z)$ to $f^{-k_2}(z)$ and then, y is chain recurrent, a contradiction.

□

Corollary 4. *Let H be a homoclinic class with non empty interior for a generic diffeomorphism f such that $T_H M = E^1 \oplus E^2$ is a dominated splitting for f and $dim(E^2) = 1$. Then, E^2 is uniquely integrable.*

Proof . It follows from the fact that the center stable manifold is dynamically defined (see [PS] and [HPS]).

□

Corollary 5. *Let $H = H(p, f)$ be a homoclinic class with non empty interior for a generic diffeomorphism f such that $T_H M = E^1 \oplus E^2$ is a dominated splitting for f and dim$(E^2) = 1$. Then, for all $L > 0$ and $l > 0$ there exists n_0 such that if I is a compact arc integrating E^2 whose lenght is smaller than L, then $\ell(f^{-n}(I)) < l$ $\forall n > n_0$.*

Proof . It is easy to see that every compact arc integrating E^2 should have its iterates of lenght going to zero in the past because of Theorem 3 (it is enough to consider a finite covering of I where the Theorem applies).

Lets suppose then that there exists L and l such that for every $j > 0$ there is an arc I_j integrating E^2 of lenght smaller than L and $n_j > j$ such that $\ell(f^{-n_j}(I_j) \geq l$. We can suppose without loss of generality that $\ell(I_j) \in (L/2, L)$.

Also, we can assume (maybe considering subsequences) that I_j converges uniformly to an arc J integrating E^2 and verifying $L/2 \leq \ell(J) \leq L$.

Since the lenght of J is finite and it integrates E^2 we know that $\ell(f^{-n}(J)) \to 0$ with $n \to +\infty$.

Let $\varepsilon = l/2$ and δ given by Theorem [LS] which ensures that $W^2_\delta(x) \subset W^u_\varepsilon(x)$ $\forall x$.

Let n_0 such that $\forall n \geq n_0$ we have $\ell(f^{-n}(J)) < \delta/4$. Let also be γ small enough such that if $x \in B_\gamma(J)$ then $d(f^{-k}(x), f^{-k}(J)) < \delta/4$ $\forall 0 \leq k \leq n_0$.

Now, if we consider j large enough (in particular $j > n_0$) such that $I_j \subset B_\gamma(J)$ we obtain $\ell(f^{-n_0}(I_j)) < \delta$ and so $\ell(f^{-n}(I_j)) < \varepsilon < l$ $\forall n \geq n_0$, so, $n_j < n_0$ which is a contradiction.

□

We are ready to give the proof of our main theorem:

Theorem 6. *Let H be a homoclinic class with non empty interior for a generic diffeomorphism f such that $T_H M = E^1 \oplus E^2$ is a dominated splitting for f and dim$(E^1) = 1$. Then, E^1 is uniformly contracting (i.e. $\|Df^n|_{E^1(x)}\| \to 0$ with $n \to +\infty$).*

Proof . Because of the existence of an addapted norm for the dominated splitting (see [Gou1]) we can assume that $\|Df|_{E^1(x)}\|\|Df^{-1}|_{E^2(f(x))}\| < \gamma$ (for the sake of simplicity).

Suppose the theorem is not true. Thus, for every $0 < \nu < 1$ there exists some $x \in H$ such that $\|Df^n|_{E^1(x)}\| \geq \nu$, $\forall n \geq 0$ (otherwise for every x there would be some $n_0(x)$ which would be the first one for which $\|Df^n|_{E^1(x)}\| < \nu$ and by compactness $n_0(x)$ are uniformly bounded, then E^1 would be hyperbolic). If we choose points x_m satisfying $\|Df^n|_{E^1(x)}\| \geq 1 - 1/m \ \forall n \geq 0$, so a limit point x will satisfy $\|Df^n|_{E(x)}\| \geq 1 \ \forall n \geq 0$.

First of all, we consider the case where we cannot use the Shifting Lemma of Liao (Lemma 1), that is, $\forall \gamma < \gamma_1 < \gamma_2 < 1$, there exists $x \in H$ such that

$$\|Df^n|_{E^1(x)}\| \geq \gamma_2^n \qquad \forall n \geq 0$$

but, $\forall y \in \omega(x)$ we have that

$$\|Df^n|_{E^1(y)}\| \geq \gamma_1^n \qquad \forall n \geq 0$$

So, if we work in $\omega(x)$ which is a closed invariant set, we have that the subbundle E^2 will be hyperbolic since the dominated splitting implies that $\forall z \in \omega(x)$

$$\|Df^{-1}|_{E^2(z)}\| < \frac{\gamma}{\|Df|_{E^1(f^{-1}(z))}} < \frac{\gamma}{\gamma_1} < 1$$

This implies that, since we have dynamical properties for the manifolds integrating the subbundle E^1, that we can shadow recurrent orbits. Indeed, if we have a recurrent point $y \in \omega(x)$, for every small ε (in particular, such that the stable and unstable manifolds of y are well defined) we can consider n large enough so that $d(f^n(y), y) \leq \varepsilon/3$, $f^n(W^1_\varepsilon(y)) \subset W^1_{\varepsilon/3}(f^n(y))$ and $f^{-n}(W^2_\varepsilon(f^n(y))) \subset W^2_{\varepsilon/3}(y)$ which gives us (using clasical arguments) a periodic point p of f which verifies that has period n and remains ε-close to the first n iterates of y. It is not dificult to see that we can consider this periodic point to be of index 1 and such that its unstable manifold intersects the stable manifold of y. This implies that $\overline{W^u(p)} \cap H \neq \emptyset$ which also implies (from the Lyapunov stability of H) that $p \in H$.

Since γ_1 was arbitrary, we can choose it to satisfy $\gamma_1 > \lambda$ where λ is as in Lemma 2. Also, we can choose ε small so that $\|Df^n|_{E^1(p)}\| > \lambda^n$ contradicting Lemma 2.

Now, we shall study what happens if Liao's shifting Lemma can be applied. That is, there exists $\gamma < \gamma_1 < \gamma_2 < 1$ such that for all $x \in H$ satisfying

$$\|Df^n|_{E^1(x)}\| \geq \gamma_2^n \qquad \forall n \geq 0$$

there exists $y \in \omega(x)$ such that

$$\|Df^n|_{E^1(x)}\| \leq \gamma_1^n \qquad \forall n \geq 0$$

So, using the Shifting Lemma we have that for every $\gamma_2 < \gamma_3 < \gamma_4 < 1$ we have a periodic orbit p_U of f contained in any neighborhood U of Λ and satisfying that

$$\|Df^n|_{E^1(p)}\| \leq \gamma_4^n$$
$$\|Df^n|_{E^1(f^i(p))}\| \geq \gamma_3^n$$

for some $i \in 0, \ldots, \pi(p)$ (remember that E^1 is one dimensional, so the product of norms is the norm of the product). But since this periodic points are not very contracting in the direction E^1, if we choose $\gamma_3 > \lambda$ (as before) and U suficiently small to ensure that the unstable manifold of some periodic point will intersect the stable one of a point in H we reach the same contradiction as before.

□

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