The real analytic Feigenbaum-Coullet-Tresser attractor in the disk.

Eleonora Catsigeras*, Marcelo Cerminara and Heber Enrich

Instituto de Matemática (IMERL), Facultad de Ingenería
Universidad de la República.
P.O.Box: C.C.30 Montevideo, Uruguay.
Tel-Fax +598-2-711-0621
E-mail addresses:
eleonora@fing.edu.uy
cerminar@fing.edu.uy
enrich@fing.edu.uy

May 2nd., 2008

Abstract

We consider a real analytic diffeomorphism ψ_0 on a n-dimensional disk $\mathcal{D}$, $n \geq 2$, exhibiting a Feigenbaum-Coullet-Trésser (F.C.T.) attractor, being far, in the $C^\omega(\mathcal{D})$ topology, from the standard F.C.T. map ϕ_0 fixed by the double renormalization.

We prove that ψ_0 persists along a codimension-one manifold $\mathcal{M} \subset C^\omega(\mathcal{D})$, and that it is the bifurcating map along any one-parameter family in $C^\omega(\mathcal{D})$ transversal to $\mathcal{M}$, from diffeomorphisms attracted to sinks, to those which exhibit chaos.

The main tool in the proofs is a theorem of Functional Analysis, which we state and prove in this paper, characterizing the existence of codimension one submanifolds in any abstract functional Banach space.

Keywords: bifurcation, Feigenbaum, attractor, manifold of mappings
MSC: 37G35, 58D15, 34C23, 34D45

1 Introduction

In dimension one, the Feigenbaum-Coullet-Tresser (F.C.T.) theory [1, 2, 3] states that the F.C.T. attractor is a codimension one phenomenon when seen locally, in a small neighborhood of the real analytic standard map φ_0. This standard F.C.T. map φ_0, is the real analytic unimodal map in the interval, of quadratic type at its critical point, and such that φ_0 is fixed by the doubling renormalization in the interval.

The existence of a local codimension-one manifold through φ_0 is a consequence of the hyperbolicity of φ_0 as fixed by the doubling renormalization in the space of real analytic maps of the interval. (See the proofs in [4, 5, 11]). This hyperbolic behavior was first proved by Lanford in [4].

The codimension one character is also true for n-dimensional real analytic transformations, as proved by Collet-Eckman and Koch in [17], taking as the fixed point of the doubling renormalization the standard endomorphic F.C.T. map $\phi_0 : \mathcal{D} \mapsto \text{int}\ (\mathcal{D})$. This endomorphism is defined

*Corresponding author: eleonora@fing.edu.uy

from the standard F.C.T. map φ_0 in the interval, by endowing the $2 \leq n$-dimensional disk $\mathcal{D}$ onto its interior, with infinite codimension one contraction, such that $\phi_0(\mathcal{D})$ is the graph of the map φ_0 in the interval. The precise definition of the map ϕ_0 will be reviewed in Definition 2.2 of this paper.

Locally, nearby the standard F.C.T. map φ_0 in the interval, or nearby the standard F.C.T. endomorphism ϕ_0 in the n-dimensional disk $\mathcal{D}$, the codimension one character of infinitely doubling renormalizable maps was proved in the space of C^r transformations, provided that r is large enough. (See [7], [5] and [8]).

Also for C^r maps, far away from the standard endormorphism ϕ_0 in the n-dimensional disk, if $n \geq 2$ and $r \geq 8$, the F.C.T. attractor is a codimension one phenomenon, as proved in [9]). This result is not true in dimension one. For global results for maps in the interval exhibiting a F.C.T. attractor see [5] and [11]).

We address here to the remaining open question about the codimension one character of the F.C.T. attractor in the space $C^\omega(\mathcal{D})$, exhibited by infinitely doubling renormalizable diffeomorhphisms or endomorphisms ψ_0, that are far away from the standard F.C.T. endomorphism ϕ_0 fixed by the renormalization. We prove that, in fact, it is a codimension- one phenomenon.

The key condition to obtain that result, is the dimension two or greater of the manifold. The result is not true in dimension one. In fact, the known proof of the codimension one character of the F.C.T. attractor in the $n-$disk $\mathcal{D}$ (see [9]), requires the use of at least two different spatial directions: a first direction onto which asymptotically the map contracts after successive many doubling renormalizations; and a second direction to perturb the map and to construct a one-parameter family destroying the F.C.T. attractor. This argument was used to prove that this attractor is a one-parameter bifurcating phenomenon in $C^r(\mathcal{D})$ in [9], and we will base on it our new result in $C^\omega(\mathcal{D})$. The one parameter family such constructed shall be transversal to the desired codimension-one manifold of maps exhibiting the continuation of the F.C.T. attractor.

Nevertheless the known proof of the codimension-one character of the F.C.T. attractor in the space $C^r(\mathcal{D})$, does not work for real analytic maps. This is due to the construction, used in [9], of a one-parameter family of non-zero diffeomorphisms or endomorphisms that have null derivative in infinitely many points of the compact disk $\mathcal{D}$. (see Lemma 5.3 of the proof of Theorem 2 in [9]).

Following the remark of Tresser ([10]), and using the result of Theorem 2 in [9] combined with the density of the real analytic maps in the space $C^r(\mathcal{D})$ maps, we obtain the following Theorem, whose proof is the first purpose of this paper:

Theorem 1 *If $\psi_0 : \mathcal{D} \mapsto int(\mathcal{D})$ is a real analytic map of the $n-$dimensional compact disk $\mathcal{D}$ to its interior, where $n \geq 2$, and if ψ_0 exhibits a F.C.T. attractor (see Definition 2.8), then there exist a local codimension-one C^1 manifold $\mathcal{M}$ in the Banach space $C^\omega(\mathcal{D})$ of real analytic maps from $\mathcal{D}$ to $\mathbb{R}^n$, such that $\psi_0 \in \mathcal{M}$ and for all $\psi \in \mathcal{M}$ the map $\psi : \mathcal{D} \mapsto int(\mathcal{D})$ also exhibits a F.C.T. attractor.*

We prove Theorem 1 in section 4.

We base our arguments on a result stated in Theorem 3.3 of this paper, which applies the classical tools on Functional Analysis, dealing with non linear infinite dimensional submanifolds of an abstract Banach space H. More precisely, in Theorem 3.3 we give necessary and sufficient conditions to obtain codimension-one submanifolds of a functional Banach space H, where some phenomenon appears, in terms of the persitence of the bifurcating quality of this phenomenon, along one-parameter families in H.

The same arguments also work to prove the following second result:

Theorem 2 *If ψ_0 verifies the hypothesis of Theorem 1 then ψ_0 belongs to a one-parameter family of real analytic maps in $\mathcal{D}$ such that at ψ_0 there exists a global bifurcation from maps that pass through a cascade of period doubling bifurcations to maps that pass through a sequence of homoclinic tangencies bifurcations.*

Even more, any one-parameter family that is transversal at ψ_0 to the manifold $\mathcal{M} \subset C^\omega(\mathcal{D})$ of the thesis of Theorem 1, has the property above.

We prove Theorem 2 in the last part of Section 4.

The result of Theorem 2, restricted to one-dimensional quadratic unimodal maps in the interval that are near the standard F.C.T. map φ_0 (see Definition 2.1), was first obtained in [6]. Afterwards, the result was generalized to the disk in dimension $n \geq 2$ in [12], but also, only for maps that are in a small neighborhood of the standard F.C.T. map ϕ_0 (see Definition 2.2).

As a consequence of Theorem 2, any ψ_0 showing a F.C.T. attractor, even far from the standard F.C.T. attractor, is the bifurcating map along any one-parameter family in $C^\omega(\mathcal{D})$ transversal to $\mathcal{M}$. At one side of ψ_0 the maps of the family exhibit sinks, in a cascade of period doubling bifurcations, while at the other side they exhibit chaos (hyperbolic horseshoes, and also Hénon-like attractors), due to the sequence of homoclinic bifurcations that accumulate on ϕ_0 (see [13]).

The conclusion of the F.C.T as a generic route to chaos, is widely known and applied in other sciences to physical autonomous dynamical systems, but no mathematical proof of it was known before, in open sets of $C^\omega(\mathcal{D})$, far away from the standard F.C.T. map.

Let us suggest that a similar result to that in the thesis of Theorem 1, can be obtained for other kind of infinitely renormalizable diffeomorphisms or endomorphisms in $n \geq 2$ dimensions. In fact, instead of considering the classical standard F.C.T. map, we can look at other fixed map by the doubling renormalization in the interval, with a non quadratic critical point or in the critical circle ([14], [15]). We would define other classes of Cantor set attractors that are not the F.C.T. attractor.

Provided that the unidimensional map fixed by the renormalization also has a local hyperbolic behavior in the functional space ([14], [16]), it defines the corresponding endomorphisms in $n \geq 2$ dimensions, fixed and locally hyperbolic by the renormalization, with the same arguments used in [17].

Finally, the technique tools we use in this paper can be applied to a diffeomorphism ψ in n dimensions, exhibiting the Cantor set attractor, but which is initially far away from that fixed and locally hyperbolic endomorphism. We shall assume that its sequence of renormalized maps converges to that fixed endomorphism, in the functional space. We note that the renormalization is neither a linear operator nor a Fréchet differentiable transformation in the functional space. Nevertheless, to construct a manifold $\mathcal{M}$ of the thesis of Theorem 1, where the infinitely renormalizable Cantor set attractor persists, the main arguments in [9] and in this paper can be applied. We are aware that the theory does not hold in dimension one.

2 Definitions and previous results.

A *analytic n-disk $\mathcal{D}$,* (or simply a *disk*), is the image by a real analytic diffeomorphism of the unit closed ball of R^n. ($n \geq 2$). Note that we call *real analytic diffeomorphism* to a real analytic transformation that is invertible and whose inverse is also a real analytic transformation.

Analogously a $C^r - n - disk$ is the image by a C^r diffeomorphism of the unit closed ball of R^n. All analytic disk is a C^r-disk.

2.1 The standard Feigenbaum-Coullet-Tresser map in the interval.

We call *the standard F.C.T. map in the interval,* to the unique real analytic unimodal map $\varphi_0 : [-1,1] \mapsto [-1,1]$ such that $\varphi_0(0) = 1$, $\varphi'(0) = 0$, $\varphi_0''(0) < 0$ and

$$\varphi_0(1)^{-1} \cdot \varphi_0 \circ \varphi_0(\varphi_0(1) \cdot x) = \varphi_0(x) \quad \forall x \in [-1,1]$$

The existence and unicity of φ_0 was the central conjecture of the F.C.T. theory [1, 2, 3] and was proved in [4, 5]. We denote λ to the number $-\varphi_0(1) = 0.3995\ldots$. The map φ_0 has a single fixed point in $[-1,1]$, which is larger than λ. The analytic map φ_0 is symmetric: $\varphi_0(x) = g_0(x^2)$ where g_0 is an analytic diffeomorphism from $[0,1]$ to $[-\lambda, 1]$. It can be analytically uniquely extended to an open interval.

There exists one single periodic orbit of φ_0 of period 2^N for each natural $N \geq 0$, and this orbit is a hyperbolic repeller. The orbit of countably many points eventually fall on one of these repellers. All the other orbits of φ_0 are attracted to a Cantor set K in the interval which we call *the standard F.C.T. attractor in the interval.*

Let $n \geq 2$. Let $\mathcal{D}$ be n-dimensional compact disk containing the segment $[-\lambda, 1] \times \{0\}^{n-2} \times [-\lambda, 1]$.

Definition 2.2 The standard Feigenbaum-Coullet-Tresser map in n-dimensions.

The map $\phi_0 : \mathcal{D} \mapsto \text{int}(\mathcal{D})$ defined as

$$\phi_0(x_1, x_2, \ldots, x_{n-1}, x_n) = (x_n, 0, \ldots, 0, \varphi_0(x_n)) = (x_n, 0, \ldots, 0, g_0(x_n^2))$$

for all $x \in \mathcal{D}$, will be called *the standard F.C.T. map in n dimensions.* It inherits the Cantor set attractor of the map φ_0, that we call the *standard F.C.T. attractor in n dimensions.*

Remark 2.3 Observe that the standard F.C.T. map in n dimensions has a one dimensional character: it is an endomorphism of $\mathcal{D}$ endowing it to a one-dimensional image, contained in its interior, and following the graph of φ_0.

The repellers of φ_0 are transformed into periodic hyperbolic saddles of ϕ_0 with infinite contraction along their stable manifolds. There exist such a periodic orbit with period 2^N for each natural $N \geq 0$. The unstable manifolds of the saddles have dimension one and are contained in $\phi_0(\mathcal{D})$. The stable manifold of each saddle is the union of their pre-images by ϕ_0, formed by the intersection with $\mathcal{D}$, of the horizontal $(n-1)$-dimensional hyperplanes for which x_n is constant.

All the orbits of ϕ_0, except those in the stable manifolds of the saddles, are attracted to its standard F.C.T. attractor.

We are interested in studying some Cantor set attractors for other n-dimensional maps, particularly for diffeomorphisms that might be far away from the standard F.C.T. map.

2.4 Functional spaces.

Given a analytic n-disk $\mathcal{D}$, the space $C^\omega(\mathcal{D})$ is the open set of the Banach space of the real analytic maps $\psi : \mathcal{D} \mapsto R^n$, such that $\psi(\mathcal{D}) \subset \text{int}(\mathcal{D})$. The topology in $C^\omega(\mathcal{D})$ is that given by the supreme norm $\parallel \psi \parallel = max\{\|\psi(x)\|_{R^n} : x \in \mathcal{D}\}$.

Analogously the space $C^r(\mathcal{D})$ is the open set of the Banach space of all the C^r maps $\psi : \mathcal{D} \mapsto R^n$, such that $\psi(\mathcal{D}) \subset \text{int}(\mathcal{D})$. The topology in $C^r(\mathcal{D})$ is that given by the supreme norm $\parallel \psi \parallel_r = max\{\|\psi(x)\|_{R^n}, \|D\psi(x)\|, \|D^2\psi(x)\|, \ldots, \|D^r\psi(x)\| : x \in \mathcal{D}\}$.

In some parts of this paper we will need to work with the whole Banach space of real analytic maps, or of C^r maps, from $\mathcal{D}$ to R^n although their images are not contained in the interior of $\mathcal{D}$. We will still denote them as $C^\omega(\mathcal{D})$ and $C^r(\mathcal{D})$, if there were no risk of confusion.

Definition 2.5 Doubling renormalization.

A map $\psi \in C^\omega(\mathcal{D})$ ($\psi \in C^r(\mathcal{D})$) is *doubling renormalizable* if there exists a analytic (resp. C^r) $n-$disk $\mathcal{D}_1 \subset \text{int}\mathcal{D}$ such that:

$$\psi(\mathcal{D}_1) \cap \mathcal{D}_1 = \emptyset$$

$$\psi^2(\mathcal{D}_1) \subset \text{int}(\mathcal{D}_1)$$

If ψ is doubling renormalizable and $\xi : \mathcal{D} \mapsto \mathcal{D}_1$ is a real analytic (resp C^r) diffeomorphism (called *change of variables*), the map $\mathcal{R}\psi$ defined as $\mathcal{R}\psi = \xi^{-1} \circ \psi \circ \psi \circ \xi$ is a *renormalized map* of ψ.

Note that doubling renormalizability is an open condition in $C^\omega(\mathcal{D})$ (resp. $C^r(\mathcal{D})$). Also note that $\mathcal{R}\psi$ is not uniquely defined: small perturbations of the change of variables ξ give other renormalized map of ψ. When referring to the properties of $\mathcal{R}\psi$ we understand that there exists some renormalized map of ψ having these properties.

By induction we define:

m-times doubling renormalizable maps.

A map $\psi \in C^\omega(\mathcal{D})$ (resp. $\psi \in C^r(\mathcal{D})$) is *$m$-times (doubling) renormalizable* if it is $m-1$-times (doubling) renormalizable and its $m-1$-renormalized $\mathcal{R}^{m-1}\psi$ is doubling renormalizable. It is defined a *m-renormalized map of* ψ as $\mathcal{R}^m\psi = \mathcal{R}\mathcal{R}^{m-1}\psi$

Infinitely doubling renormalizable maps.

A map $\psi \in C^\omega(\mathcal{D})$ (resp. $\psi \in C^r(\mathcal{D})$) is *infinitely (doubling) renormalizable* if it is m-times (doubling) renormalizable for all natural m.

The main example of infinite doubling renormalizable maps in the n-disk is the fixed map standard F.C.T. endormorphism ϕ_0, defined in 2.2.

Remark 2.6 An infinitely (doubling) renormalizable diffeomorphism or endormorphism $\psi \in C^\omega(\mathcal{D})$ (resp. $C^r(\mathcal{D})$) in the $n-$disk $\mathcal{D}$, may in general be far away from the stardard F.C.T. endormorphism ϕ_0. Nevertheless its sequence, or some subsequence, of some renormalized maps $\mathcal{R}^n\psi$ may converge to ϕ_0 in the $C^\omega(\mathcal{D})$ (resp. $C^r(\mathcal{D})$) topology. If this happens, roughly speaking ψ exhibits a Cantor set attractor, that assympthotically inside, looks like the standard F.C.T. attractor. In more precise words, this is the statement of the following theorem:

Theorem 2.7 *If a map $\psi \in C^\omega(\mathcal{D})$ (resp. $\psi \in C^r(\mathcal{D})$) is infinitely doubling renormalizable and if there is a sequence $\mathcal{R}^j\psi$ of $j-$times renormalizations of ψ such that*

$$\lim_{j\to\infty} \mathcal{R}^j\psi = \phi_0$$

in $C^\omega(\mathcal{D})$ (resp. in $C^r(\mathcal{D})$), where ϕ_0 is the F.C.T. map in $n-$ dimensions, then there exists a minimal Cantor set $K \in int(\mathcal{D})$ such that $\psi(K) = K$, there exits a nighborhood $U \subset int(\mathcal{D})$ of K such that K attracts almost all the orbits in U and there exists a single periodic saddle orbit in U of period 2^N for all natural number N large enough.

Proof: See Theorem 2.12 of [9] for the proof in $C^r(\mathcal{D})$. Exactly the same arguments of the proof in $C^r(\mathcal{R})$ work in $C^\omega(\mathcal{D})$, due that the topology of $C^\omega(\mathcal{D})$ is induced from the $C^r(\mathcal{D})$ topology.

Definition 2.8 The F.C.T. attractor.

A map $\psi \in C^\omega(\mathcal{D})$ (resp. $\psi \in C^r(\mathcal{D})$) from the $n-$ dimensional disk $\mathcal{D}$ to its interior has a *real analytic F.C.T. attractor* if ψ is infinitely doubling renormalizable and there is a sequence $\mathcal{R}^j\psi$ of $j-$times renormalizations of ψ, such that

$$\mathcal{R}^j\psi \ {}_{j\to\infty} \to \phi_0$$

in $C^\omega(\mathcal{D})$ (resp. in $C^r(\mathcal{D})$), where ϕ_0 is the standard F.C.T. map in $n-$ dimensions.

Remark 2.9 The Cantor set attractor K of Theorem 2.7 and Definition 2.8 has bounded geometry in the sense that the diameter of the connected compact atoms that asymptotically define K, decrease with an asymptotic rate below 1. When looking microscopically the decreasing rate, it tends to the number $\lambda = 0.3995\ldots$, that is a spatial universal constant defined for the standard F.C.T map ϕ_0. In fact, λ is the contraction rate of the change of variables to renormalize ϕ_0, and it is also the asymptotic contraction rate of the change of variables to pass from the $\mathcal{R}^j\psi$ to $\mathcal{R}^{j+1}\psi$, if ψ verifies the Definition 2.8.

We note that Gambaudo and Tresser in [18] give an example of a n dimensional infinitely renormalizable map whose renormalized maps do not converge to the F.C.T. map ϕ_0. In spite of that, this example has a Cantor set attractor that verifies the thesis of the theorem 2.7. Its geometry is also bounded, but the bounds are different from λ. We do not call that Cantor set a F.C.T. attractor.

Let us recall some results from [9] in which we will found part of the proofs of Theorems 1 and 2:

Theorem 2.10 *For $r \geq 8$, if $\psi_0 \in C^r(\mathcal{D})$ has a F.C.T. attractor, then there exists a local codimension-one C^1 manifold $\mathcal{M}$ in $C^r(\mathcal{D})$ such that $\psi_0 \in \mathcal{M}$ and χ has a F.C.T. attractor for all $\chi \in \mathcal{M}$.*

Proof: See Theorem 2 of [9].

Theorem 2.11 *For $r \geq 8$, if $\psi_0 \in C^r(\mathcal{D})$ has a F.C.T. attractor, then any C^1 one-parameter family $\Psi = \{\psi_t\}_{t\in[-1,1]} \subset C^r(\mathcal{D})$ transversal to the manifold $\mathcal{M}$ of Theorem 2.10 at ψ_0, exhibits, at one side of $t = 0$ a sequence of period doubling bifurcations from sinks of period 2^N (for any sufficiently large natural number N) to saddles of the same period and sinks of double period; and exhibits, at the other side of $t = 0$, a sequence of homoclinic tangency bifurcations of saddles of period 2^N (for any sufficiently large natural number N).*

Proof: See Corollary 4 of [9].

3 Characterization of codimension-one submanifolds in abstract Banach spaces.

We recall the following definitions, and the we state a theorem dealing with the differentiable submanifolds of codimension one, in abstract Banach spaces:

Definition 3.1 (The Banach space of C^1 one-parameter families.) Let H be a Banach space and let $\Psi = \{\psi_t\}_{t\in[-1,1]}$ be a C^1 one-parameter family in H, i.e. Ψ is a C^1 application taking $t \in [-1, 1]$ to $\psi_t \in H$. We denote as $\partial\psi_t/\partial t \in H$ to the derivative respect to t of the application $t \mapsto \psi_t \in H$. The set $F = C^1([-1,1], H)$ of all C^1 one-parameter families in H is a Banach space with the C^1 topology derived from the following C^1 norm

$$\|\Psi\|_F = max\{\|\psi_t\|_H, \|\partial\psi_t/\partial t\|_H : t \in [-1,1]\}$$

We denote $B_\epsilon(\Psi)$ to the open ball in F centered at Ψ and with radius equal to $\epsilon > 0$.

Given a C^1 one parameter family $\Psi = \{\psi_t\}_{t\in[-1,1]]} \in F$ and given a fixed real number t_0 such that $|t_0| \leq 1$ we construct new families (many) in F denoted as $(t_0)^*\Psi$ defined as:

$$(t_0)^*\Psi = \{\widehat{\psi_t}\}_{t\in[-1,1]} \in F, \text{ where } \widehat{\psi_t} = \psi_{t+t_0} \text{ if } t \in [-1,1], t+t_0 \in [-1,1]$$

Note that, to define $(t_0)^*\Psi = \{\widehat{\psi_t}\}_{t\in[-1,1]} = \{\psi_{t+t_0}\}_{t\in[-1,1]}$, it is required to choose any C^1-extension of ψ_{t+t_0}, for the values of $t \in [-1,1]$ such that $t + t_0 \notin [-1,1]$.

Definition 3.2 (Persistent phenomena in C^1 one-parameter families.)
Let H be a Banach space, let $\mathcal{P}$ be any non empty subset of H, and let $\Psi = \{\psi_t\}_{t\in[-1,1]} \in F$ be a C^1 one-parameter family in H such that $\psi_0 \in \mathcal{P}$.

We say that *the set $\mathcal{P}$ (or the phenomenon $\mathcal{P}$) is persistent in C^1 one-parameter families near* Ψ if there exist $\epsilon > 0$ and a C^1 real function $a : B_\epsilon(\Psi) \subset F \mapsto [-1,1]$ such that for all $\Gamma = \{\gamma_t\}_{t\in[-1,1]} \in B_\epsilon(\Psi) \subset F$:

a) $\gamma_{a(\Gamma)} \in \mathcal{P}$

b) If $\gamma_0 = \psi_0$ then $a(\Gamma) = 0$. (In particular $a(\Psi) = 0$.)

c) If $|t_0|$ is small enough then $a((t_0)^*\Gamma) = a(\Gamma) - t_0$. (In particular $a((t_0)^*\Psi) = -t_0$.)

To explicit the value of ϵ in this definition we will refer to the set $\mathcal{P}$ as being $\epsilon-$*persistet in* C^1 *one-parameter families near* Ψ.

Theorem 3.3 *Let H be a Banach space, let $\mathcal{P}$ be any non empty subset of H and let $\psi_0 \in \mathcal{P}$. The following assertions are equivalent:*

i) *There exists a C^1 local manifold $\mathcal{M}$ in H with codimension one such that $\psi_0 \in \mathcal{M}$ and $\chi \in \mathcal{P}$ for all $\chi \in \mathcal{M}$.*

ii) *There exists a C^1 one-parameter family $\Psi = \{\psi_t\}_{t\in[-1,1]}$ passing through ψ_0 for $t = 0$ and such that the set $\mathcal{P}$ is persistent in C^1 one-parameter families near* Ψ (according with definition 3.2).

iii) *There exists $v_0 \in H$ such that the set $\mathcal{P}$ is persistent in C^1 one-parameter families near* $\Psi = \{\psi_0 + tv_0\}_{t\in[-1,1]}$ (according with definition 3.2).

Proof:

We first prove that i) implies ii):

We apply the C^1 persistence of the transversal intersection between C^1 manifolds in H (see [19]): Being $\mathcal{M}$ a codimension one, C^1 manifold o H passing through ψ_0, it is locally characterized by a real equation $b(\chi) = 0$ in a neighborhood of radius $\delta > 0$ of $\psi_0 \in H$:

$$\mathcal{M} = \{\chi \in H : \|\chi - \psi_0\|_H < \delta, \ b(\chi) = 0\}$$

where $b : \{\chi \in H : \|\chi - \psi_0\|_H < \delta\} \mapsto \mathbb{R}$ is some real function of C^1 class and with surjective derivative.

Let $v_0 \in H$ be transversal to $\mathcal{M}$ at ψ_0. That is

$$Db_{\psi_0} \cdot v_0 \neq 0$$

If $\|v_0\|_H > 0$ is small enough, then the family $\Psi = \{\psi_t\}_{t\in[-1,1]} = \{\psi_0 + tv_0\}_{t\in[-1,1]} \in F$ verifies $\|\psi_t - \psi_0\|_H < \delta$ for all $t \in [-1, 1]$.

Taking a smaller positive value for δ, let us define the transformation $G : B_\delta(\Psi) \times [-1, 1]$ such that, if $\Gamma = \{\gamma_t\}_{t\in[-1,1]} \in B_\delta(\Psi) \subset F$ and if $t \in [-1, 1]$, then:

$$G(\Gamma, t) = b(\gamma_t)$$

The transformation G is C^1 because it is the composition of the C^1 real function b with the parameter evaluation γ_t of the C^1 parameter family Γ.

As $\psi_0 \in \mathcal{M}$ we have

$$G(\Psi, 0) = 0, \quad \left.\frac{\partial G}{\partial t}\right|_{\Gamma=\Psi,t=0} = Db|_{\psi_0} \cdot v_0 \neq 0$$

Then, by the implicit function theorem there exists $\epsilon > 0$ and $a : B_\epsilon(\Psi) \subset F \mapsto [-1, 1]$ of C^1 class, such that

$$G(\Gamma, a(\Gamma)) = 0$$

Thus $b(\gamma_{a(\Gamma)}) = 0$. Therefore $\gamma_{a(\Gamma)} \in \mathcal{M} \subset \mathcal{P}$. We conclude that the C^1 real function a verifies condition a) of Definition 3.2.

The Implicit Function Theorem also asserts that if $G(\Gamma, a) = 0$ for some $\Gamma \in B_\epsilon(\Psi)$ and some a with $|a|$ small enough, then $a = a(\Gamma)$. This last assertion proves that the real function a verifies also conditions b) and c) of Definition 3.2, as wanted.

Let us now prove that ii) implies iii):

Let $\Psi = \{\psi_t\}_{t\in[-1,1]} \in F$ be the C^1 one-parameter family given in ii) and $\epsilon > 0$ the real number given in the definition 3.2 of persistence of $\mathcal{P}$ in one-parameter families near Ψ. Let us call $v_0 = (\partial\psi/\partial t)|_{t=0}$. The C^1 condition of Ψ implies that there exists $0 < \delta < 1$ such that for $|t| \leq \delta$:

$$\|\psi_0 + tv_0 - \psi_t\|_H < \epsilon/2, \quad \|\frac{\partial\psi_t}{\partial t} - v_0\|_H < \epsilon/2$$

Take any C^1 extension $\rho = \{\rho_t\}_{t\in[-1,1]} \in F$ of $\{\psi_0 + tv_0\}_{t\in[-\delta,\delta]}$ such that $\rho \in B_{\epsilon/2}(\Psi) \subset F$.

As the phenomenon $\mathcal{P}$ is $\epsilon-$persistent in C^1 one parameter families near Ψ, and ρ is $\epsilon/2-$near Ψ, we obtain that $\mathcal{P}$ is $\epsilon/2$ persistent in C^1 one parameter families near ρ.

Now we shall construct a family $\Lambda \in F$ to be linear on the parameter $t \in [-1, 1]$ as in the thesis iii), from $\rho \in F$. (The one parameter family ρ is linear only in the small δ- neighborhood of the parameter value $t = 0$.)

Consider $\Lambda = \{\lambda_t\}_{t\in[-1,1]} \subset F$ defined as $\lambda_t = \psi_0 + \delta tv_0$ for all $t \in [-1, 1]$.

As $\rho_0 = \psi_0$ then $a(\rho) = 0$. By continuity of the function a there exists ϵ' such that $0 < \epsilon' < \epsilon/2$ and $|a(\tilde{\Gamma})| < \delta/2$ if $\|\tilde{\Gamma} - \rho\|_F < \epsilon'$.

Take $\epsilon'' = \epsilon'\delta$. It is enough to prove that the phenomenon $\mathcal{P}$ is ϵ'' persistent in C^1 one-parameter families near Λ.

In fact, for all $\Gamma = \{\gamma_t\}_{t\in[-1,1]} \in B_{\epsilon''}(\Lambda) \subset F$ we have

$$\|\gamma_t - (\psi_0 + \delta tv_0)\|_H < \epsilon'' = \epsilon'\delta < \epsilon' \quad (1)$$

$$\left\| \frac{\partial \gamma_t}{\partial t} - \delta v_0 \right\|_H < \epsilon'' = \epsilon' \delta, \quad \left\| \frac{1}{\delta} \frac{\partial \gamma_t}{\partial t} - v_0 \right\|_H < \epsilon' \quad (2)$$

Observe that $\rho_t = \lambda_{t/\delta} = \psi_0 + tv_0$ for $|t| \leq \delta$. Analogously, for $\Gamma = \{\gamma_t\}_{t\in[-1,1]} \in B_{\epsilon''}(\Lambda) \subset F$ define $\tilde{\gamma}_t = \gamma_{t/\delta}$ for $|t| \leq \delta$, and consider any C^1 extension $\tilde{\Gamma} = \{\tilde{\gamma}_t\}_{t\in[-1,1]} \in F$.

Using inequalities (1) and (2) we check that $\tilde{\gamma}_t$ is $C^1\epsilon'-$near ρ for parameter values $|t| \leq \delta$, and so the extension $\tilde{\Gamma} \in F$ to the whole parameter domain $t \in [-1, 1]$ could be chosen such that $\tilde{\Gamma} \in B_{\epsilon'}(\rho) \subset F$.

The $\epsilon'-$persistence of the phenomenon $\mathcal{P}$ in one-parameter families near ρ, for the chosen value of $\epsilon' < \epsilon/2$, allows the existence of a C^1 function $\tilde{a} : B_{\epsilon'} \mapsto [-\delta/2, \delta/2]$ verifying the conditions of the definition 3.2.

As the values of $\tilde{a}$ are contained in the $\delta/2$-neighborhood of 0 in the parameter domain, we have that $\tilde{\gamma}_{\tilde{a}(\tilde{\Gamma})} = \gamma_{\tilde{a}(\tilde{\Gamma})/\delta} \in \mathcal{P}$. The map $\tilde{a}$ depends on the given $\Gamma \in B_{\epsilon''}(\Lambda)$ and is independent on the choice of the extension $\tilde{\Gamma}$ for parameter values outside $[-\delta, \delta]$.

Define $a(\Gamma) = \tilde{a}(\tilde{\Gamma})/\delta$. It is straightforward to check that the real function a verifies the conditions of the definition 3.2

Finally let us prove that iii) implies i):

Let $\Psi = \{\psi_0 + tv_0\}_{t\in[-1,1]} \in F$ be the one-parameter family given in the hypothesis (iii). Let $\epsilon > 0$ be the radius of the ball centered at Ψ in F, where the C^1-real function a is defined, according to Definition 3.2 of persistence of the phenomenon $\mathcal{P}$.

Let us choose $\delta > 0$ such that if $\|\chi - \psi_0\|_H < \delta$ then

$$\Gamma(\chi) = \{\chi + tv_0\}_{t\in[-1,1]} \in B_\epsilon(\Psi) \subset F \quad (3)$$

Let us define $b : B_\delta(\psi_0) \subset H \mapsto [-1, 1]$ as

$$b(\chi) = a(\Gamma(\chi))$$

By construction we have $b(\psi_0) = 0$ and

$$b(\chi) = 0 \;\Rightarrow\; \chi \in \mathcal{P}$$

Our aim is to prove that the set $\mathcal{M} \subset H$, defined as

$$\mathcal{M} = \{\chi \in B_\delta(\psi_0) \subset H : b(\chi) = 0\},$$

is an embedded C^1 local manifold. It is immediate that $\psi_0 \in \mathcal{M}$. It is enough to prove that the real function b is of C^1- class and that its derivative at $\chi = \psi_0$ ($Db(\psi_0) : H \mapsto R$) is surjective.

First, the function b is the composition $b(\chi) = a \circ \Gamma(\chi)$. As $a : B_\epsilon(\Psi) \subset F \mapsto R$ is of C^1 class by assumption, it is left to prove that $\Gamma : B_\delta(\psi_0) \subset H \mapsto B_\epsilon(\Psi) \subset F$ defined in (3) is differentiable with continuous derivative.

Let us take $\Delta\chi \in H$ such that $\chi + \Delta\chi \in B_\delta(\psi_0)$. From (3) the increment of the function Γ in χ is

$$\Gamma(\chi + \Delta\chi) - \Gamma(\chi) = \{\Delta\chi\}_{t\in[-1,1]} \in F$$

In other words, the increment of Γ is $\Delta\Gamma = i(\Delta\chi)$, where i is the inclusion defined as follows:

$$i : H \hookrightarrow F \;\text{ such that }\; \forall\gamma \in H : i(\gamma) = \{\gamma_t\}_{t\in[-1,1]} \in F \quad \text{where } \gamma_t = \gamma \;\forall t \in [-1, 1]$$

As the inclusion $i : H \hookrightarrow F$ is linear and continuous, then the map $\Gamma : B_\delta(\psi_0) \subset H \mapsto B_\epsilon(\Psi) \subset F$ defined in (3) is of C^1 class as wanted.

Finally, it is left to prove that $Db(\Psi_0) : H \mapsto \mathbb{R}$ is surjective. It is enough to prove that it is not null the directional derivative of the real function $b : B_\delta(\psi_0) \subset H \mapsto R$ along the direction $v_0 \in H$ at ψ_0. (Note that although it was not asked $v_0 \neq 0$ in the hypothesis iii), the assertion above also proves that iii) implies $v_0 \neq 0$.)

In fact, for any real number λ sufficiently small in absolute value so that $\psi_0+\lambda v_0 \in B_\delta(\psi_0) \subset H$, we have:

$$b(\psi_0 + \lambda v_0) = a(\{\psi_0 + \lambda v_0 + tv_0\}_{t\in[-1,1]}) = a(\lambda^*\Psi) = a(\Psi) - \lambda = -\lambda$$

Therefore

$$\frac{d}{d\lambda} b(\psi_0 + \lambda v_0) = \frac{d}{d\lambda}(-\lambda) = -1 \neq 0$$

■

4 Conclusion of the main results: Theorems 1 and 2.

Proof of Theorem 1: Let $\mathcal{P}$ be the set of transformations exhibiting a F.C.T. attractor in $C^r(\mathcal{D}) \subset C^\omega(\mathcal{D})$. Let $\psi_0 \in \mathcal{P} \cap C^\omega(\mathcal{D})$.

Let F^r be the space of C^1 one-parameter families of maps in $C^r(\mathcal{D})$ and let F^ω be the space of C^1 one-parameter families of maps in $C^\omega(\mathcal{D})$. We have that $F^\omega \subset F^r$ and the topology in F^ω defined in 3.1 is the induced topology from F^r.

Due to Theorem 2.10 and Lemma 3.3 there exists $v_0 \in C^r(\mathcal{D})$ such that the phenomenon $\mathcal{P}$ is persistent in one-parameter families in F^r near $\Psi = \{\psi_0 + tv_0\}_{t\in[-1,1]}$. Let $\epsilon > 0$ be the number of the definition 3.2.

As $C^\omega(\mathcal{D})$ is dense in $C^r(\mathcal{D})$ there exists $w_0 \in C^\omega(\mathcal{D})$ such that $\|w_0 - v_0\|_{C^r(\mathcal{D})} < \epsilon/2$. Define the one-parameter family $\widetilde{\Psi} \in F^\omega$ as follows: $\widetilde{\Psi} = \{\psi_0 + tw_0\}_{t\in[-1,1]}$.

We obtain now that the phenomenon $\mathcal{P}$ is persistent in one parameter families of F^r near $\widetilde{\Psi}$, taking $\epsilon/2$ instead of ϵ in the definition 3.2. Restricting to the subspace F^ω we conclude that the phenomenon $\mathcal{P} \cap C^\omega(\mathcal{D})$ is persistent in one parameter families of F^ω near $\widetilde{\Psi}$. Finally, applying Lemma 3.3 again, we obtain that there exists in $C^\omega(\mathcal{D})$ the local codimension one manifold $\mathcal{M}$ of maps in $\mathcal{P} \cap C^\omega(\mathcal{D})$ as wanted.

■

Proof of Theorem 2: We denote F^r and F^ω as in the last proof. Applying Theorem 2.11 all one-parameter family $\Psi \in F^r$ that is transversal to the local codimension one manifold $\mathcal{M}$ of maps exhibiting a F.C.T. attractor in $C^r(\mathcal{D})$ verifies the thesis of Theorem 2. As in the last proof, the density of $C^\omega(\mathcal{D})$ in $C^r(\mathcal{D})$ implies that there exists a one-parameter family $\widetilde{\Psi} \in F^\omega$ transversal to $\mathcal{M}$ in $C^r(\mathcal{D})$, and thus, verifying the thesis.

■

References

[1] P. Coullet, C. Tresser: *Itérations d'endomorphismes et groupe de renormalisation.* C.R. Acad. Sci. Paris 287 (1978) pp. 577–588

[2] M. J. Feigenbaum: *Quantitative universality for a class of nonlinear transformations.* Journ. Stat. Physics, vol 19 (1978) pp. 25–52

[3] M. J. Feigenbaum: *The universal metric properties of non-linear transformations.* Journ. Stat. Physics, vol 21 (1979) pp. 669–706

[4] O. Lanford III: *A computer assisted proof of the Feigenbaum's conjectures.* Bulletin of the A.M.S. 6 (1982) pp. 427–434

[5] D. Sullivan: *Bounds, quadratic differentials and renormalization conjectures.* Mathematics into the Twenty First Century vol. 2 Am. Math. Soc., Providence, RI (1991)

[6] J. P. Eckmann, P. Wittwer: *A complete proof of the Feigenbaum conjectures.* Journ. of Stat. Physics 46, N.3/4 (1987) pp. 455–475

[7] A. M. Davie: *Period doubling for $C^{2+\epsilon}$ mappings.* Comm. Math. Phys. 176 No. 2 (1996) pp. 261–272

[8] E. Catsigeras, H. Enrich: *Homoclinic tangencies near cascades of period doubling bifurcations.* Ann. de l'IHP. An. non lin., vol.15, No.3 (1998) pp. 255–299

[9] E. Catsigeras, H. Enrich: *Persistence of the Feigenbaum attractor in one-parameter families.* Commun. Math. Phys. 207. (1999) pp. 621–640.

[10] C. Trésser: Personal communication. (2000)

[11] M. Lyubich: *Feigenbaum-Coullet-Tresser universality and Milnor's hairiness conjecture.* Ann. of Math. 149, no. 2 (1999), pp. 319–420.

[12] E. Catsigeras: *Cascades of period doubling bifurcations in n dimensions.* Nonlinearity 9 (1996) pp.1061–1070

[13] J. Palis, F. Takens: *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations.* Cambridge University Press (1993)

[14] M. Yampolsky: *Hyperbolicity of Renormalization of Critical Circle.* Publ. Math. Inst. Hautes Études Sci., Vol. 96 (2002) pp. 1–41

[15] M. Yampolsky: *Siegel Disks and Renormalization Fixed Points.* ArXiv. Math. 0602 No. 678v. Math. D.S. (2006) Cornell Univ.

Available at http://arxiv.org/PS_cache/math/pdf/0602/0602678v1.pdf

[16] E. de Farias, W. de Melo: *Rigidity of critical circle mappings II.* J.Amer. Math. Soc. 19 No. 2 (2000) pp. 343–370

[17] P. Collet, J. P. Eckmann, H. Koch: *Period doubling bifurcations for families of maps on R^n.* Journ. of Stat. Phys., 25, N.1 (1981) pp.1–14

[18] J. M. Gambaudo, C. Tresser: *Self similar constructions in smooth dynamics: rigidity, smoothness and dimension.* Comm. Math. Phys. 150 (1992) pp.45–58

[19] S. Lang: *Differentiable Manifolds.* Addison- Wesley. (1972)