STABILITY MODULUS SINGULAR SETS

J. IGLESIAS, A. PORTELA, AND A. ROVELLA

Abstract. A new concept of stability, closely related to that of structural stability, is introduced and applied to the study of C^r endomorphisms with singularities. A map that is stable in this sense will be conjugated to each perturbation that is equivalent to it in a geometric sense. It will be shown that this kind of stability implies Axiom A, Omega-stability and that every critical point is wandering. A partial converse will be shown, providing new examples of C^3 structurally stable maps.

§1. Introduction.

Denote by $C^r(M)$ the space of class C^r self mappings of a manifold M, $1 \leq r \leq \infty$. If M is compact the topology is the usual C^r topology, while for noncompact M the space $C^r(M)$ is endowed with the Whitney (or strong) topology.
Given $f \in C^r(M)$ the set of critical (or singular) points of f (denoted S_f) is the set of points where the differential of f is singular.
Two maps f and g of class C^1 are said *geometrically equivalent* if there exist C^1 diffeomorphisms φ and ψ of M such that $\varphi f = g\psi$. In this case, the image of a critical point of f under ψ is a critical point of g, and the image of a critical value of f under φ is a critical value of g. Moreover, for each neighborhood $\mathcal{Z}$ of the identity of M in C^0 topology, the maps f and g are said $\mathcal{Z}$-geometrically equivalent if the diffeomorphisms φ and ψ are contained in $\mathcal{Z}$. Any pair of C^1 diffeomorphisms are geometrically equivalent, and two endomorphisms without critical points are geometrically equivalent if and only if the absolute values of the degrees are equal. But the concept is purely geometric, it has no dynamical meaning: for example, two quadratic polynomials of one variable are always geometrically equivalent.
In this article, the concept of stability of maps is considered. Two maps f and g are topologically equivalent if there exists a homeomorphism h such that $fh = hg$. A C^r map f is C^r structurally stable if there exists a C^r neighborhood $\mathcal{U}$ of f such that f is topologically equivalent to each $g \in \mathcal{U}$.
In [IPR], a concept of stability of maps was introduced, that generalizes the usual concept and is more adequate to study maps having critical points.

Definition 1. A map $f \in C^r(M)$ is said C^r structurally stable modulus singular sets, denoted $f \in I^r(M)$, if there exist a neighborhood $\mathcal{Z}$ of the identity in $C^0(M)$ and a C^r neighborhood $\mathcal{U}$ of f, such that two $\mathcal{Z}$-geometrically equivalent maps g_1 and g_2 in $\mathcal{U}$ are topologically equivalent.

The need of $\mathcal{Z}$-geometric equivalence (instead of geometric equivalence) will become apparent in theorem C. If f is C^r structurally stable then it belongs to $I^r(M)$. Under generic assumptions on maps g_1 and g_2, topological equivalence implies geometric equivalence.

Date: January 28, 2008.

A map having critical points cannot be C^1 structurally stable, but a map $f \in I^1(M)$ has the following property: given a C^0 neighborhood $\mathcal{Z}$ of the identity, there exists a C^1 neighborhood $\mathcal{U}$ of f such that $\mathcal{Z}$-geometric equivalence is an equivalence relation in $\mathcal{U}$ and coincides with topological equivalence in $\mathcal{U}$. Lemma 3 at the end of section 2 implies that if a map f belongs to $I^1(M)$, then f is topologically equivalent to any C^1 perturbation g that coincides with f in a neighborhood of S_f.
Necessary and sufficient conditions for a diffeomorphism f to be C^1 structurally stable are known some time ago (Robinson,[R], 1976 and Mañé [Ma], 1987). Since then, no new examples of C^r $(r \geq 2)$ structurally stable diffeomorphisms were discovered: it remains open the question if there exist any. Other C^r stable maps are known: expanding maps in compact manifolds were considered by Shub [S] and its stability proved. The case of one dimensional maps, where the situation is easier, will not be specially discussed here.
There exist no examples of noninvertible nonexpanding structurally stable maps without critical points. Allowing critical points, the $I^r(M)$ maps known (in dimension at least two) are those presented in [IPR]. In that article, the concept was introduced and some tools provided the first known examples of C^3 structurally stable maps with critical points, in dimension greater than one, and with nontrivial nonwandering set. It was proved there that a complex polynomial p in the Riemann sphere satisfying the no critical relations property (i.e., no critical point belongs to the future orbit of other critical point and no critical point is periodic) and whose Julia set is hyperbolic and connected, belongs to $I^1(\mathbb{R}^2)$.
Theorems A and B describe some properties of a $I^1(M)$ map.

Theorem A. *Let M be a compact manifold. If $f \in I^1(M)$, then every critical point of f is wandering, and f is Axiom A and C^1 Ω-stable.*

The definitions involved are the following: a point is wandering if it has a neighborhood U such that $f^n(U) \cap U = \emptyset$ for every $n > 0$. The set of nonwandering points of f is denoted by $\Omega(f)$. Two maps f and g are Ω- equivalent if there exists a homeomorphism $h : \Omega(f) \to \Omega(g)$ such that $hf = gh$ in $\Omega(f)$. A map f is C^r Ω-stable whenever all its C^r perturbations are Ω-equivalent to it.
A map f whose nonwandering set has a hyperbolic structure and whose set of periodic points is dense in $\Omega(f)$, has a spectral decomposition: the nonwandering set of f is the union of a finite number of *basic pieces*; these are compact, invariant, transitive sets. Then the map is called Axiom A if, in addition, the restriction of f to a basic piece is either injective or expanding. (This is the definition given in [MP], other authors called this concept *strong Axiom A*, and Przytycki [P] does not require this last condition for a map to be Axiom A). A basic piece Λ is expanding if the stable subspace at x, E^s_x, is $\{0\}$, for $x \in \Lambda$. A basic piece Λ is called a repeller if the stable manifold of each point in Λ is contained in Λ. In this case, the unstable set of Λ, denoted $W^u(\Lambda)$, and defined as the set of points $x \in M$ having a preorbit whose limit set is contained in Λ, is a neighborhood of Λ. Not every repeller is expanding, but if the map is Axiom A, then every repeller that is not expanding must be injective.
Another necessary condition for a map f to belong to $I^1(M)$ is that the critical set must be contained in a particular region of the wandering set:

Theorem B. *If $f \in I^1(M)$ then:*

- *If C is a component of the critical set S_f of f, then there exists a periodic attractor γ such that C is contained in its basin of attraction.*
- *If S_f intersects the unstable set of a basic piece Λ, then Λ is expanding and C is contained in its unstable set.*

It remains as an open problem to know if maps in $I^r(M)$ for $r > 1$ must satisfy any of the conclusions of theorems A and B. There exist no known examples of maps in $I^r(M) \setminus I^1(M)$ if $r > 1$.
There is another necessary condition for a map to belong to $I^1(M)$. This condition was found necessary for C^1 structural stability by Przytycki (theorem C in [P]):
Let $f \in I^1(M)$ and denote by $W^u(\Lambda)$ the unstable set of a basic piece Λ. If Λ_1 and Λ_2 are basic pieces such that $W^u(\Lambda_1) \cap \Lambda_2 \neq \emptyset$, then Λ_1 is expanding.
The proof of this result will be omitted because it is similar to that of the mentioned article of Przytycki.

There exists also a partial converse to the previous theorems.

Theorem C. *Let M be a compact manifold and assume that a map $f \in C^1(M)$ satisfies the following conditions:*

1. *f is Axiom A.*
2. *Every critical point of f is wandering.*
3. *Every basic piece is expanding or a periodic attractor.*
4. *$f^{-1}(\Omega(f)) = \Omega(f)$.*
5. *f satisfies the non critical relations property.*

Then f belongs to $I^1(M)$.

This will be proved in the last section. The non critical relations property will be defined later. This condition, and the first two conditions in the statement above, are also necessary for a map to belong to $I^1(M)$. The assumption that constitutes the great gap to obtain an equivalence to this type of stability is the third one. It may happen that a $I^1(M)$ map has a saddle type basic piece whose stable and unstable sets do not intersect the critical set. There are no examples known (apart of diffeomorphisms) of $I^1(M)$ maps having saddle type basic pieces.

§2. Critical sets.

The objective of this section is to prove that if $f \in I^1(M)$, then every critical point is wandering. The use of the C^1 topology in the assumption $f \in I^1(M)$ is essential in the proof. The question wether $f \in I^r(M)$ for $r > 1$ implies the same conclusion is still open. An affirmative answer would imply also a conjecture stated in [MP]: *If f is a C^r structurally stable map, then every critical point of f is wandering.*
The section begins with some definitions and known results.

Definition 2. Let f belong to $C^r(M)$.

1. $S_k(f)$ is the set of points $z \in S_f$ such that the dimension of the kernel of Df_z is equal to k.

(2) If $r \geq 2$, called a point $z \in S_k(f)$ *generic* for f, if there exist local charts (U, τ_1) at z and (V, τ_2) at $f(z)$ such that the map $x \in \mathbb{R}^m \to D(\tau_2^{-1} f \tau_1)_x \in L(\mathbb{R}^m)$ is transverse to L_k (where $L(\mathbb{R}^m)$ is the set of linear maps from $\mathbb{R}^m$ to $\mathbb{R}^m$, L_k denotes the possibly non closed submanifold of $L(\mathbb{R}^m)$ of transformations having kernel of dimension k).
(3) If $x \in S_k(f)$ is generic, then S_k is (locally at x) a codimension k^2 submanifold of M (see next theorem). A generic critical point $x \in S_1(f)$ is fold type if the kernel of Df_x is transverse to $S_1(f)$ at x.

Consider the simpler case of a one-dimensional map f: if x is a critical point, then x is generic if and only if he second derivative of f does not vanish at x. Going a little bit ahead, let $f \in C^2(M)$, M an m-dimensional manifold. Suppose that $x \in S_1(f)$. Put coordinates in a neighborhood of x such that the first $m-1$ rows of the differential Df_x form a linearly independent set and the last one is equal to $0 \in \mathbb{R}^m$. If $f = (f_1, \ldots, f_m)$ and H_m is the Hessian matrix of f_m (the matrix of second derivatives), then x is a generic critical point of f if and only if the $(2m-1) \times m$ matrix obtained adding to H_m the first $m-1$ rows of Df_0 has rank m.
It was proved by Whitney that around a critical point of fold type a map $f \in C^\infty(M)$ is locally geometrically equivalent to the map

$$q(x_1, x_2, \ldots, x_m) = (x_1^2, x_2, \ldots, x_m),$$

acting in $\mathbb{R}^m$. We beg the next theorem from differential topology, its proof, as well as the assertions above, can be found sparse through the text [GG].

Theorem 1. *Given any manifold M, there exists an open and dense set $\mathcal{R}(M) \subset C^\infty(M)$ such that, for every $f \in \mathcal{R}(M)$ the following conditions hold:*
(1) *Each critical point of f is generic.*
(2) *$S_1(f)$ is a codimension one submanifold of M, and its closure equals S_f.*
(3) *The set of fold type points of f is open and dense in $S_1(f)$.*

It is not true that the genericity of each critical point of a map f implies that the map belongs to $\mathcal{R}$. However, if x is a fold point of $f \in C^\infty(M)$, then there exists a neighborhood U of x such that $f \in \mathcal{R}(U)$ and every critical point of f in U is fold type. The following semicontinuity holds in general: given a neighborhood V of $S_f \cap U$, it holds that $S_g \cap U \subset V$ for every map g in a C^1 neighborhood of f. The first idea for the proof of theorem A is quite obvious: the set of critical points can be locally modified in an arbitrary way in C^1 topology. If one wants to preserve the C^∞ genericity of the maps considered, the following can be said.

Lemma 1. *Let $x \in S_f$ be a critical point of a map $f \in C^1(M)$. Given a C^1 neighborhood $\mathcal{U}$ of f and a codimension one submanifold $N \subset M$ containing x, there exists a map $g \in \mathcal{U} \cap \mathcal{R}(M)$ such that $S_1(g)$ contains a neighborhood of x in N.*
In addition, the map g can be C^∞ approximated by a map $h \in \mathcal{R}(M)$ that is geometrically equivalent to g and such that $S_h \cap N$ has empty interior in the relative topology of N.

This statement contains the perturbation mechanism that will be needed to obtain a contradiction from the assumption: $f \in I^1(M)$ and $S_f \cap f^n(S_f) \neq \emptyset$. The submanifold N will be $f^n(S_f)$ $(n > 1)$.

Some definitions and remarks concerning the structure of critical sets of perturbed maps are in order before proceeding with the proof of this lemma.
Remark 1 : If x is a critical point of fold type of a map $f \in \mathcal{R}$, then the local geometrical equivalence with q implies that there exist neighborhoods U of x and V of $f(x)$ such that $f(S_f)$ separates V in two components V^- and V^+; f is two to one from $U \setminus S_f$ onto V^+ and no point in V^- has preimages in U. If two maps f and g in $\mathcal{R}$ are topologically conjugated ($hf = gh$) then the homeomorphism h must carry critical points of f to critical points of g, because the local forms imply that this is true for fold type points, and the fact that fold type points are dense in S_f implies the assertion for the other critical points. Analogously it comes that critical values of g are carried by the conjugacy to critical values of f.
This proves an assertion of the introduction: if two topologically equivalent maps f and g belong to $\mathcal{R}$, then f and g are geometrically equivalent.

The basic idea to prove that no critical point is periodic is the following. It is well known that maps without critical periodic points constitute a residual set. However this is not enough for our purposes, we have to find two geometrically equivalent maps close to f such that one of them still has a periodic critical point and the other does not. This will give a contradiction.
It seems intuitive the fact that a nongeneric map $f \in C^1(M)$ having a nongeneric critical point x can be C^1 perturbed to a generic map $g \in \mathcal{R}$ for which the same point is still critical. To prove this we make a sequence of perturbations within a given C^1 neighborhood $\mathcal{U}$ of f.

Proposition 1. *Let f be a C^1 map, $\mathcal{U}$ a C^1 neighborhood of f, and $x \in S_f$. Then there exists $g \in \mathcal{R}(M) \cap \mathcal{U}$ such that x is a fold type point of g.*
Moreover, if f satisfies one of the following conditions:

1. *$f^n(x) \in S_f$ for some positive n.*
2. *$f^k(x)$ is periodic for some $k \geq 0$.*
3. *x belongs to the stable manifold of a periodic point p of f,*

then the map $g \in \mathcal{R}(M)$ can be chosen satisfying the corresponding property.

Proof. Take a one dimensional subspace V contained in the kernel of Df_x and let H be a complementary hyperplane. Now let h be a map in $\mathcal{U}$ such that $Dh_x(V) = 0$ and Dh_x is injective in H. This $h \in \mathcal{U}$ is arbitrarily C^1 close to f and $x \in S_1(h)$. Now it will be constructed a map ℓ in $\mathcal{U}$ such that $x \in S_1(\ell)$ is a generic critical point. Without loss of generality, one can assume that h acts on $\mathbb{R}^m$ and that the subspace H is $\mathbb{R}^{m-1} \times \{0\}$ in the product $\mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R}$; hence, if $h = (h_1, h_2)$ then the derivative of h_1 with respect to $x_1 \in \mathbb{R}^{m-1}$ is injective for every (x_1, x_2) in the ball centered at x and radius ρ in $\mathbb{R}^m$. There exists a map r such that $h(y) = h(x) + A(y - x) + r(y)$ and $\frac{||r(y)||}{||y-x||} \to 0$ when $y \to x$, where $A = Dh_x$. Choose a symmetric linear map B such that the critical point 0 of the quadratic map $Q(X) = AX + \langle BX, X\rangle$ is generic (of course an open and dense set of possible choices for that B exist). Finally define $\ell(y) = h(x) + Q(y - x)$. It is clear that ℓ is C^1 close to h in a neighborhood of x, so an adequate bump function should be used to construct ℓ in the whole manifold M.
The final step consists in producing the map g. To do this, first perturb ℓ to a map g_0 in $\mathcal{R}$ that is C^∞ close to ℓ in a neighborhood U of x. Because the critical point x was generic for ℓ, there exists a generic critical point $y \in S_1(g_0) \cap U$. Let τ be

a C^∞ map close to the identity, carrying y to x and define $g = g_0 \circ \tau$. This map satisfies the property required in the first statement. The proof that the others properties are preserved by adequate small perturbations is quite similar and relies on the fact that $g(x) = f(x)$ holds by construction. □

The first application of this result is the following:

Proof of lemma 1: First perturb f to a map (still called f) in $\mathcal{R}(M) \subset C^\infty(M)$ for which x is a critical point, as in the last proposition. As the assertion is local, one can assume without loss of generality that $M = \mathbb{R}^m$ and that 0 is a critical point of f. It can also be assumed that $N = \{x = (x_1, \cdots, x_m) \in \mathbb{R}^m \quad x_m = 0\}$, because there exists a diffeomorphism T carrying N to this set and the origin to itself. In addition, if L is an adequate linear isomorphism, then $L \circ Df_0 \circ T^{-1}$ is a linear map whose associated matrix in canonical coordinates is upper triangular, has the first $m-1$ rows linearly independent and the last one null. Maintain the notation f for the map in these coordinates, so that it satisfies all the conditions. These assumptions imply that the last coordinate of f satisfies $f_m(x) = f_m(0)+r(x)$ for some C^1 function r such that $\frac{|r(x)|}{||x||} \to 0$ when $x \to 0$. Given any $\epsilon > 0$ there exist a number $\rho \in (0, \epsilon/2)$ and a function $\varphi = \varphi_\epsilon : \mathbb{R} \to [0,1]$, of class C^∞, that is equal to 0 in $|x| \le \rho$, equal to 1 outside $|x| < \epsilon$, and such that $|\varphi'(x)| < 2/\epsilon$. Let

$$g_m(x_1, \cdots, x_m) = f_m(0) + \varphi(||x||)r(x) + (1 - \varphi(||x||))\epsilon x_m^2$$

and define $g = (f_1, \cdots, f_{m-1}, g_m)$. Note that

$$\nabla g_m(x) = \varphi'(||x||)\frac{r(x) - \epsilon x_m^2}{||x||}x + \varphi(||x||)\nabla r(x) + 2(1 - \varphi(||x||))\epsilon(0, \dots, 0, x_m).$$

Note that for $||x|| \le \rho$, the determinant of Dg_x is equal to 0 if and only if $x_m = 0$, (because the assumptions on f imply that the $(m-1) \times (m-1)$ block above and at the left in the matrix of Df_x is nonsingular). Hence $S_g \cap B(0;\rho) \subset N$, where $B(0;\rho)$ is the ball centered at the origin and with radius ρ. It remains to prove that ϵ can be chosen so that $g \in \mathcal{U}$. Note that $f(x) = g(x)$ for $x \notin B(0;\epsilon)$ and that within this ball it holds that:

$$|g_m(x) - f_m(x)| \le |\varphi(||x||) - 1| \left(||r(x)|| + \epsilon||x||^2\right)$$

Hence g is C^0 close to f if ϵ is small. Finally note that

$$||\nabla g_m(x) - \nabla f_m(x)|| \le (2/\epsilon).|r(x) - \epsilon x_m^2| + |\varphi(||x||) - 1| \left(||\nabla r(x)|| + 2\epsilon||x||\right)$$

Observe that given any $\delta > 0$, one can choose ϵ such that $|r(x)| \le \delta||x||$ and $||\nabla r(x)|| \le \delta$ for every $x \in B(0;\epsilon)$. It follows that the C^1 distance between f and g is at most $2(\delta + \epsilon^2) + \delta + 2\epsilon^2$. It remains to show that $g \in \mathcal{R}$; observe that by construction, the critical point at the origin is generic: indeed, the Hessian matrix of g_m at the origin has entries a_{ij}, where $a_{ij} = 0$ for every $(i,j) \ne (m,m)$, and $a_{mm} = 2$, but the set of vectors $\{\nabla f_1(0), \dots, \nabla f_{m-1}(0), (0, \dots, 0, 2)\}$ is linearly independent, by the choice of local coordinates. This implies that $g \in \mathcal{R}(U)$ for some neighborhood U of the origin. On the other hand, as f was taken in $\mathcal{R}(M)$, it follows that $g \in \mathcal{R}(B^c)$, where B^c is the complement of $B(0;\epsilon)$ (there, f and g coincide). Take a bump function θ that is equal to 0 in the complement of $U \cap B(0;\rho)$ and is equal to 1 in a neighborhood V of x. Let d be the C^∞ norm of θ. As $\mathcal{R}(M)$ is dense in $C^1(M)$, choose any $g_1 \in \mathcal{R}(M)$ such that the C^∞ distance between g and g_1 is less than τ/d, (where every map in a τ-neighborhood of g belongs to

$\mathcal{R}(U)$. Finally define $g_0 = \theta g + (1-\theta)g_1$; this g_0 satisfies all conditions: it belongs to $\mathcal{R}(U)$, because it is τ-close to g there; it belongs to $\mathcal{R}(U^c)$ because it is equal to g_1 there; it satisfies $S_{g_1} \cap V = N \cap V$, because in that neighborhood it coincides with g.

It remains to prove the second assertion of the lemma. Let φ be a C^∞ diffeomorphism of M such that $\varphi(x) = x$, $\varphi(V) = V$ and $\varphi^{-1}(N) \cap N$ does not contain a neighborhood of x in N. It is clear that such a map can be obtained arbitrarily C^∞ close to the identity (for example $D\varphi_0$ can take T_0N to any other hyperplane containd in T_0M). Observe that $h = g\varphi$ is C^∞ close to g, so that it belongs to $\mathcal{U} \cap \mathcal{R}(M)$; h is geometrically equivalent to g (because $ih = g\varphi$, where i is the identity map) and $S_h \cap V = \varphi^{-1}(S_g \cap V)$, so that S_h cannot contain a neighborhood of x in N.

Remark 2 : A point x is preperiodic for f if there exist $k \geq 0$ and $p > 0$ such that $f^{k+p}(x) = f^k(x)$. If $k = 0$ then x is periodic. The period of a preperiodic point is the minimum p satisfying the above equation. Denote $P_{k,p}(f)$ the set of points x such that the above holds and k is minimum, and by $Per(f)$ the union of the $P_{0,p}(f)$ for positive p.

A periodic point x of period p is hyperbolic if the differential of f^p at x has no eigenvalue null or of modulus one. It is well known that given any $p \geq 1$, the set of maps $\mathcal{G}_p$ for which every periodic point of period at most p is hyperbolic, is open and dense in every $C^r(M)$. For every map in this open and dense set, the number of periodic points of period at most p is finite and locally constant. It is clear that if $f \in \mathcal{G}_p$ then $P_{0,p}(f) \cap S_f = \emptyset$. This argument can be completed to obtain an open and dense set $\mathcal{G}_{k,p}$ of maps such that $P_{k,p}(f) \cap S_f$ is empty.

On the other hand, assume that a map $f \in C^1(M)$ has a critical point $x \in P_{k,p}(f)$. The second assertion of proposition 1 gives a map $g \in \mathcal{R}(M) \cap \mathcal{G}_p$, C^1 close to f, such that x still belongs to $P_{k,p}(g) \cap S_g$.

Lemma 2. *If $f \in I^1(M)$, then no preperiodic point of f is critical.*

Proof. Note that intersections between $P_{k,p}(f)$ and S_f can be avoided by small perturbations; what must be proved now is that this can be done within the same class of geometric equivalence. Therefore one can assume that $x \in S_1(f)$ is an isolated point of $P_{k,p}(f)$, that $f^k(x)$ is hyperbolic, and that $f \in \mathcal{R}(M)$. We will arrive to a contradiction if we can find a map g that is C^1 close to f, geometrically equivalent to f and such that no critical point of g belongs to $P_{k,p}(g)$.

This will be done in local charts. Let (U, τ_1) and (V, τ_2) be local charts at x and $f(x)$ respectively. The local coordinates can be chosen so that $\tilde{f}(x_1, \ldots, x_m) = \tau_2 f \tau_1^{-1}(x_1, \ldots, x_m) = (x_1^2, \ldots, x_m)$. Let $h(x_1, \ldots, x_m) = (h_1(x), x_2, \ldots, x_m)$, where $h_1(x) = x_1^2 - \epsilon\rho(||x||)x_1$, ϵ is an arbitrary positive number and ρ satisfies the following conditions: The function ρ is C^∞, $\rho(0) = 1$, $\rho(x) = 0$ for every $|x| > 1$ and the C^2 norm of ρ is less than a constant k. The map g will be $\tau_2^{-1} h \tau_1$. To see that g satisfies the above condition we note that h is C^1 close to $\tilde{f}$ if ϵ is small enough, that $g(x) = f(x)$ and that x is not a critical point of g: hence $P_{k,p}(g) \cap S_g = \emptyset$. It remains to show that f and g are geometrically equivalent and for this it is enough to prove that $\tilde{f}$ and h are geometrically equivalent.

The equation of the critical points of h is $\partial_1 h_1(x) = 0$; as $\partial_{11} h_1$ is close to 2 if ϵ is small it follows by the implicit function theorem that there exists a C^∞ function

$c(x_2,\ldots,x_m)$ whose graph is the set of critical points of h. Define

$$\psi(y) = (y_1 - h_1(c(y_2,\ldots,y_m),y_2,\ldots,y_m),y_2,\ldots,y_m),$$

so that

$$\psi(h_1(x),x_2,\ldots,x_m) = (h_1(x) - h_1(c(x_2,\ldots,x_m),x_2,\ldots,x_m),x_2,\ldots,x_m).$$

If $\varphi_1(x)$ is a function such that $\varphi_1^2(x) = h_1(x) - h_1(c(x_2,\ldots,x_m),x_2,\ldots,x_m)$ then $\tilde{f}\varphi = \psi h$, where $\varphi(x) = (\varphi_1(x),x_2,\ldots,x_m)$. It remains to prove that φ is a C^∞ diffeomorphism. Note that

$$h_1(x) - h_1(c(x_2,\ldots,x_m),x_2,\ldots,x_m) = \alpha(x_1,\ldots,x_m)(x_1 - c(x_2,\ldots,x_m))^2$$

with α a positive C^∞ function. □

Proposition 2. *If $f \in I^1(M)$, then there exist neighborhoods $\mathcal{U}$ of f and U of S_f such that $U \cap h^n(U) = \emptyset$ for every $n \geq 1$ and every $h \in \mathcal{U}$.*

Proof. It is first claimed that $f^n(S_f)$ does not intersect S_f if $n \geq 1$.
Assume by contradiction that $f \in I^1(M)$, and that there exists a point $x \in S_f$ such that $f^n(x) \in S_f$ for some (minimum) $n > 0$. By proposition 1, one can assume that $f \in \mathcal{R}$. Let $\{U_j \ : 0 \leq j \leq n\}$ be a disjoint sequence of open sets such that each U_j is a neighborhood of $f^j(x)$ and $f(U_j) \subset U_{j+1}$. This is possible since x is not preperiodic.
Note then that $f^n(S_f)$ is a codimension one submanifold of M containing $f^n(x)$; indeed, $f(S_f \cap U_0)$ is a submanifold since the restriction of f to $S_1(f)$ is an immersion whenever x is a fold point. Then the fact that $Df_{f^j(x)}$ is an immersion for every $j \geq 1$ implies the asserted.
Now apply lemma 1. The first assertion there, gives a map $g \in \mathcal{R}$ for which S_g contains a neighborhood of $g^n(x)$ in $N = f^n(S_f \cap U_0)$ (note that the support of this perturbation is contained in U_n, so $f^n(S_f \cap U_0) \cap U_n = g^n(S_g \cap U_0) \cap U_n$. The second perturbation gives a map h for which $S_h \cap h^n(S_h)$ has empty interior in the submanifold $h^n(S_h \cap U_n)$. The support of this last perturbation is also contained in U_n, hence the set of critical points in U_0 and their images until n are the same for f, g and h. A contradiction follows, because, on one hand, g and h must be topologically conjugate since lemma 1 says that g and h are geometrically equivalent C^1 perturbations of f, and on other hand, g and h cannot be topologically conjugate since a such a conjugacy must carry points is the interior of $S_g \cap g^n(S_g)$ (a nonempty set) to points in the interior of $S_h \cap h^n(S_h)$ (empty). This proves the claim.
Now assume that the conclusion of the proposition is false. Then one can find a map g arbitrarily C^1 close to f, a point x and an integer j such that x and $g^j(x)$ are arbitrarily close to S_f. By an argument similar to that of Franks' lemma, ([F]) one can find a small C^1 perturbation g_1 of f such that x and $g_1^j(x) = g^j(x)$ are both contained in S_{g_1}. This contradicts the first claim. □

We have the desired conclusion:

Corollary 1. *If $f \in I^1(M)$, then $\Omega(f) \cap S_f = \emptyset$.*

Finishing this section, the main perturbation result is presented:

Lemma 3. *Let M be a compact manifold, f be a map in $I^1(M)$ and W a neighborhood of S_f. There exists a C^1 neighborhood $\mathcal{U}$ of f such that if a map $g \in \mathcal{U}$ is equal to f in W, then f and g are geometrically equivalent.*

Proof. Note first that there exist $\rho > 0$, $\lambda_0 > 0$, and a C^1 neighborhood $\mathcal{U}_0$ of f such that, for every $x \in W^c$ (W^c is the complement of W), $\lambda \leq \lambda_0$ and $g \in \mathcal{U}_0$, it holds that the restriction of g to $B(x;\lambda)$ (the ball of center x and radius λ) is one to one, and that

$$g(B(x;\lambda)) \supset B(g(x);\rho\lambda). \tag{1}$$

This assertion is clear, since for every g in a whole C^1 neighborhood of f, the norm of the inverse of Dg_x at a point $x \in W^c$ is uniformly bounded.
Let $\delta = \inf\{d(x,y) \ : \ f(x) = f(y),\ x \in W^c \text{ and } x \neq y\}$. If $\delta = 0$, then there exist sequences $\{x_n\} \subset W^c$ and $\{y_n\}$ such that $f(x_n) = f(y_n)$ and $0 < d(x_n, y_n) \to 0$; if w is a limit point of both sequences, then w is a critical point of f, this contradicts $w \notin W$, and proves $\delta > 0$. Moreover, diminishing the neighborhood $\mathcal{U}_0$, one can be obtain that $\inf\{d(x,y) \ : \ g(x) = g(y),\ x \in W^c \text{ and } x \neq y\} \geq \delta/2$
To prove the assertion, C^1 diffeomorphisms φ and ψ must be found such that $f\varphi = \psi g$. Take ψ equal to the identity map. Let λ be a positive number less than half the distance from S_f to W^c and less than $\min\{\lambda_0, \delta/4\}$ and take $g \in \mathcal{U}_0$ such that the C^0 distance between f and g is less than $\min\{\delta, \rho\lambda\}$. It is claimed now that if $x \notin W$ then there exists $y \in M$ such that:

(1) $f(y) = g(x)$.
(2) $d(x,y) \leq \delta/4$.
(3) $d(z,x) > \delta/2$ if $f(z) = g(x)$ and $z \neq y$.

The first statement follows from equation 1 and the fact that $g(x) \in B(f(x);\rho\lambda)$; hence the second item holds by the choice of λ. Note that $y \notin W$, otherwise $g(y) = f(y) = g(x)$ and $d(x,y) < \delta/4$, contradicting the definition of δ. Therefore $d(y,z) \geq \delta$ whenever $f(z) = f(y)$ and $z \neq y$. The third item follows. Then, given $x \notin W$ there exists a unique point $y \in f^{-1}(g(x))$ that minimizes the distance to x. The same assertion holds for points $x \in W$, because there one can take $y = x$. Define $\varphi(x) = y$. It follows that with this φ the equation $f\varphi = g$ holds. If f_x denotes the restriction of f to the ball $B(x;\lambda)$, then $\varphi(x) = f_x^{-1}g(x)$ in W^c, from which the required smoothness of φ is obtained. Finally, the C^1 distance between φ and the identity can be made arbitrarily small by diminishing δ. Therefore φ is a diffeomorphism. □

§3. Hyperbolicity

Lemma 4. *If $f \in I^1(M)$ then every periodic point of f is hyperbolic.*

Proof. Suppose that f has a nonhyperbolic periodic point x with period n. Let g be a map in $\mathcal{R}$ such that x is periodic nonhyperbolic for g, has period n and every other periodic point of period less than or equal to n of g is hyperbolic. To do this, first perturb f to a map such that the periodic point x is nonhyperbolic but is isolated within the set of periodic points of period n of f. Then apply the usual mechanisms to make the other periodic points of period at most n are hyperbolic. Now we construct two C^∞ maps, arbitrarily C^1 close to f, such that the periodic point x is hyperbolic for both maps but has different character (the dimension of the stable space changes) and such that the perturbation has support outside the set of critical points of f. By lemma 3 these maps are geometrically equivalent, which contradicts the fact that $f \in I^1(M)$. □

Proof of theorem A

By corollary 1 no critical point is wandering. By lemma 4, every periodic point is hyperbolic. Now we use theorem A of Aoki, Moriyazu and Sumi in [AMS], that implies the following:

If a map f with $S_f \cap \Omega(f) = \emptyset$ has a C^1 neighborhood contained in the set of mappings having every periodic point hyperbolic, then the nonwandering set of f has a hyperbolic structure and the set of periodic points of f are dense in the nonwandering set of f.

As shown by Przytycki ([P]), this is not enough to obtain the C^1 Ω-stability of f: for this it will be necessary to show first that each basic piece is either expanding or injective. We will prove the Ω-stability of f directly from the definition.

Let $\mathcal{U}$ be the neighborhood of f given by the definition of $I^1(M)$ and U such that proposition 2 holds for $\mathcal{U}$ and U. Let W be a neighborhood of S_f whose closure is contained in U. There exists a C^1 neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of f such that:

(1) The conclusions of lemma 3 hold for the neighborhoods W of S_f and $\mathcal{U}_0$ of f.
(2) If f_1 and f_2 belong to $\mathcal{U}_0$ then there exists a map $F = F(f_1, f_2) \in \mathcal{U}$ such that F is equal to f_1 in W and equal to f_2 in the complement of U.

Let $\mathcal{U}_1 \subset \mathcal{U}_0$ be a neighborhood of f such that $F(f_1, f_2) \in \mathcal{U}_0$ whenever f_1 and f_2 belong to $\mathcal{U}_1$.

Let $g \in \mathcal{U}_1$, and $h = F(g, f) \in \mathcal{U}_0$. By lemma 3, g and h are topologically equivalent. By proposition 2, the periodic points of f and h are contained in U^c, where the maps coincide. It follows that $Per(f) = Per(h)$, which implies that $f = h$ in $\Omega(f) = \Omega(h)$, and we conclude that f and h are Ω- equivalent.

§4. Location of critical sets.

In this section we prove theorem B. It was already shown that every critical point of an $f \in I^1(M)$ is wandering, and that f is an Axiom A map. It follows that every point is contained in the stable set of some basic piece and in the unstable set of a basic piece. It will be shown that a basic piece whose stable (resp. unstable) set intersects S_f must be a periodic attractor (resp. an expanding set). Indeed, if this is not the case, and a stable or unstable manifold of a basic piece of another type contains a critical point, then this critical point can be perturbed in the same class of geometric equivalence in order to produce some nonequivalent dynamical consequences.

We refer the reader to the article of Przytycki ([P]) for the definitions of stable and unstable sets and properties of Axiom a maps. It is clear, and will be used in the sequel, that $P_{k,p}(f)$ is an invariant of topological conjugacy, as well as the union of the stable (resp. unstable) sets of its points.

The idea is the following: Let z be a generic critical point of fold type of f and assume that it belongs to the stable manifold of a basic piece that is not a periodic attractor. By the density of periodic points in each basic piece, it can be assumed without loss of generality, that there exists a periodic point x of f whose stable manifold contains z.

A first lemma will be needed show that by means of a C^1 perturbation, one can create a map g having a segment L close to z such that the image of L is a unique point. If maps g_1 and g_2 are geometrically equivalent to g, then there exists a

segment L_1 (resp. L_2) where g_1 (resp. g_2) is constant. Then g_1 and g_2 are topologically equivalent, and the conjugacy must send L_1 to L_2 and $g_1(L_1)$ to $g_2(L_2)$. A contradiction will be found if one can put the point $g_1(L_1)$ in the stable manifold of the g_1-periodic point x and the point $g_2(L_2)$ outside the stable manifold of a periodic point of period equal to that of x. This will be possible since, by assumption, the periodic point x of f was not an attractor. The proof for the repelling case uses a similar argument.
Note that by proposition 1, one can choose the map f in $\mathcal{R}$, preserving its other properties.

Lemma 5. *Let z be a fold point of a map f in $\mathcal{R}$, $\mathcal{U}$ a C^1 neighborhood of f and U a neighborhood of x. Then there exists a map $g \in \mathcal{U}$ and a segment $L \subset U$ such that $S_g = S_f \cup L$ and $L \cap S_f = \emptyset$.*
Moreover, g is constant in L.

A segment is the image of a smooth injective curve $\alpha : [0,1] \to M$.

Proof. One can assume that $f(x_1, \ldots, x_m) = (x_1^2, \ldots, x_m)$ and the point z is the origin.
Given $\epsilon_0 > 0$ and $r \in (0, a)$, there exist an interval $I \subset \mathbb{R}$, and a C^2 function q such that

- $q(x) = x^2$ for $|x| \geq r/2$
- $q(x)$ is C^1 ϵ_0-close to $x \to x^2$.
- For every x, $q''(x) \geq 0$ and $q(x) = 0$ if and only if $x \in I$.
- The interval I is contained in $(0, r/2)$

Define a function g_1 as follows:

$$g_1(x_1, \ldots, x_m) = h_\rho(x_1, \ldots, x_m) + x_1 q'(x_1) - q(x_1),$$

where, given any $\rho > 0$, h_ρ is a function such that $h_\rho(x) = 0$ for every $|x| > r$, while for every $|x| \leq r/2$ it holds that

$$h_\rho(x_1, \ldots, x_m) = \rho \int_0^{x_1} t(x_2^2 + \cdots + x_m^2) dt.$$

depending just on r, one can choose ρ small in order to obtain that h_ρ is C^∞ arbitrarily close to the null function.
Finally define

$$g(x_1, \ldots, x_m) = (g_1(x_1, \ldots, x_m), x_2, \ldots, x_m).$$

Note that the intersection of S_g with the ball centered at the origin and of radius $r/2$ is the set of points x such that

$$\partial_1 g_1(x) = \partial_1 h_\rho(x) + x_1 q''(x_1) = x_1 \left(q''(x_1) + \rho \sum_{j=2}^{m} x_j^2 \right) = 0,$$

that equals the union of $x_1 = 0$ with the set of points $\{(x_1, \ldots, x_m)\}$ such that $x_1 \in I$ and $x_2 = \cdots = x_m = 0$. On the other hand, $g = f$ in $|x| > r$. Finally, in the annulus $|x| \in (r/2, r)$, $g(x)$ is C^∞ close to f if one makes ρ small. From the general theory of singularities it follows that the intersection of S_g and the annulus is a submanifold arbitrarily close to S_f, so one can perturb in a small neighborhood of this annulus just to make coincide S_g with S_f there without changing the map in the $r/2$ neighborhood of the origin. □

Note that $g(L) = f(z)$, but $g(L) \notin g(S_g \setminus L)$.

Proof of theorem B

Part 1. The map f belongs to $I^1(M)$, and has a critical point z contained in the stable manifold $W^s(x, f)$ of a fixed point x that is not an attractor. It can also be assumed that f is generic and that z is a fold type point.

To produce a perturbation g_1 of f such that $L \cap W^s(x; g_1) \neq \emptyset$, just take the map g of the previous lemma. If the neighborhood U was taken such that its future iterates under f do not intersect U, then $g(L) = f(z)$ belongs to the stable manifold of x.

The construction of the map g_2 is not so easy, because the intersection of the stable manifold of x with the neighborhood U can have infinitely many components. Consider first the case where the basic piece that contains the periodic point x is not an attractor. The same proof made for Axiom A diffeomorphisms can be adapted to show that the union of the basins of the attracting basic pieces is open and dense. Before applying the lemma, perturb the mp f in U so that z belongs to the basin of an attractor, without changing the class of geometric equivalence of f nor the condition of critical point of z. Indeed, let τ be a translation supported in a small neighborhood W of $f(U)$ (that is not necessarily open) such that $\tau(f(z)) \in B$ where B is equal to the intersection of the basin of an attractor with $f(U)$. Define the new map f' as follows: if $y \in U$, then $fy) = \tau(f(y))$, and if $y \notin U$, then $f'(y) = f(y)$. Note that if W is sufficiently small, then the preimages of W are disjoint open sets, so f' is well defined, smooth, and close to f. Moreover, by the choice of U (disjoint of its future iterates) it holds that the set B is still contained in the basin of an attractor of f'. Now one can apply the lemma, with L contained in the basin ot the referred attractor, giving a map g_2 that is geometrically but not topologically equivalent to g_1.

To treat the remaining case, assume Λ is an attracting basic piece and that $x \in \Lambda$. Let W be a neighborhood of Λ such that $f(W) \subset W$ and f is injective in W (f is injective in Λ because f is Ω-stable). Let g be a map like in the lemma so that L is contained in the basin of Λ. Let $k > 0$ be the first positive integer such that $y = g^k(L) \in W$; as periodic points are dense in Λ one can perturb the map g in a neighborhood of Λ to a map g_2 such that y belongs to the stable manifold of a periodic point of period greater than that of x. This last perturbation must be geometrically equivalent ot g_1, but cannot be topologically equivalent.

Part 2. The map f belongs to $I^1(M)$ and there exists a nonrepelling fixed point x whose unstable set contains a critical point z. The unstable set is defined as the union of the images of a local unstable manifold. Now there exists a neighborhood U of z such that the future images of U do not intersect U. This implies that a perturbation of f in U does not produce any change in $W^u(x, f) \cap U$. So one can find a perturbation satisfying that both extreme points of L are contained in the complement of W^u and another perturbation satisfying that at least one extreme point of L belongs to W^u.

§5. Sufficient conditions and examples

This section contains the proof of the partial converse, theorem C of the introduction. In the mentioned article [IPR], the existence of C^3 structurally maps was shown. These were perturbations of complex polynomials, so the components of the set of critical points were arbitrarily small, which provide some simplifications on the proof presented here, that follows the same ideas.

Definition 3. A C^1 map f satisfies the noncritical relations property if there exist open connected sets $\{U_1, U_2, \dots, U_n\}$ such that:

(1) $S_f \subset \cup_i U_i$.
(2) The closures of the sets U_i are disjoint.
(3) Given nonnegative integers j and l such that $f^j(U_k) \cap f^l(U_i) \neq \emptyset$, then $j = l$ and $k = i$.
(4) The restriction of f to the closure of $f^j(U_i)$ is injective for every $j > 0$ and $1 \leq i \leq n$.

It is important to note that items 3 and 4 do not represent an infinite number of conditions, since, under the hypothesis of theorem C, each component of S_f is entirely contained in the basin of a unique periodic attractor and hence is eventually contained in an open set where the map is injective. Therefore, each U_i is contained in a component of the basin of an attractor.

Denote by B_f the union of the basins of the periodic attractors of f. Define also the Julia set of f as the set of nonwandering points of f that are not periodic attractors. This set was denoted by $\Omega'(f)$. The first assertion describes global aspects of the dynamics of every g in a neighborhood of f.

Lemma 6. *If f satisfies the hypothesis of theorem C, then the following assertions hold:*

(1) $M = B_f \cup \Omega'(f)$.
(2) *Either B_f is empty (and f is an expanding map) or $M = \overline{B_f}$, ($\overline{A}$ denotes the closure of the set A).*
(3) *f is C^1 Ω-stable, and hence the same conclusions hold for every map g in a C^1 neighborhood of f.*

Proof. Assume that there exists a point $x \in M \setminus B_f$ and let U be a neighborhood of x. As the sequence $\{f^n(x)\}$ converges to $\Omega(f)$, there exists an $m > 0$ such that $f^m(x) \in \Omega'(f)$ and hence $x \in \Omega'(f)$ by assumption (4). This proves the first item. If $U \subset \Omega'(f)$ then $\Omega'(f) = M$, hence B_f is empty or $U \cap B_f \neq \emptyset$.
As f is Axiom A, has no cycles by hypothesis (3) of theorem C, and every critical point is wandering, then the theorem of Przytycki implies that f is C^1 Ω-stable. □

There exists a uniform constant of expansivity for the restrictions of the maps g in a neighborhood of f to the respective Julia sets. Let ϵ be this constant. Let $\alpha > 0$ be less than ϵ and less than the the distance between different U_i's. Let $\mathcal{U}$ be a C^1 neighborhood of f such that every $g \in \mathcal{U}$ is Ω-equivalent to f and such that $S_g \subset \cup U_i$. The number α and neighborhood $\mathcal{U}$ will be diminished later. Take g_1 and g_2 in $\mathcal{U}$ that are $\mathcal{Z}$-geometrically equivalent, where $\mathcal{Z}$ is the C^0 neighborhood of the identity of size α. One has:

$$\varphi g_1 = g_2 \psi, \tag{2}$$

where the distance from $\psi(x)$ and $\varphi(x)$ to x is less than α for every $x \in M$.

Proof of theorem C.

The idea is to construct a conjugacy h from B_{g_1} to B_{g_2} that is ϵ-C^0 close to the identity in a neighborhood of $\Omega'(g_1)$, and then continuously extend it to the closure of $B_{g_1} = M$, using that $\Omega'(g_1)$ is expanding.

Construction of a fundamental domain.

Let x be an attracting periodic point of f; assume that x is fixed to simplify notation. Let V be a neighborhood of x such that the closure of $f(V)$ is contained in V and f restricted to V is injective. The first step consists in construct an open set $V' \subset V$ such that the same properties of V hold and such that for every U_i contained in the basin of x there exists a positive integer n_i such that $f^{n_i}(U_i)$ is contained in the interior of the fundamental domain $V' \setminus f(V')$. Assume first that there exists only one of the sets U_i contained in the basin of x. Let n be the minimum positive integer such that the closure of $f^n(U_i)$ is contained in V and denote by W the closure of $f^n(U_i)$. As the point x cannot belong to W, there exists a finite number of future iterates of V that intersect W, say that $f^p(V) \cap W$ is empty for every $p > N$. Define a sequence of compact sets $\{W_0, W_1, \dots, W_N\}$ such that the following holds:

- $W_0 = W$.
- W_k is contained in the interior of W_{k+1} for every $k = 0, \dots N-1$.
- W_N is contained in V and does not intersect $f^p(V)$ for every $p > N$.
- $f(W_N) \cap W_N = \emptyset$.

Then define

$$V' = V \setminus \bigcup_{k=1}^{k=N} f^{-k}(W_k),$$

and prove that V' satisfies the above claim. Indeed, if $x \in V$, then $f(x)$ belongs to the interior of V; if, in adition, $x \notin f^{-k}(W_k)$, then $f(x)$ does not belong to $f^{1-k}(W_k)$ whose interior contains $f^{1-k}(W_{k-1})$, because f is a diffeomorphism in V. This proves that $V' \setminus f(V')$ is a fundamental domain for f. Finally, if $y \in V'$, then $y \notin f^{-1}(W_1)$, that contains $f^{-1}(W)$ in its interior; this implies that W is contained in the interior of the fundamental domain.

Assume now that $U_1, \dots, U_L$ are contained in the basin of x, and let n_i such that $f^{n_i}(U_i)$ is contained in V for the first time. The proof is equal if one defines now $W = \cup f^{n_i}(U_i)$, because the preimage of one of the sets $f^{n_i}(U_i)$ cannot intersect an image of an U_j.

The open set $\mathcal{U}$ can be diminished again in order to assume that $V' \setminus g(V')$ is a fundamental domain whose intersection with $f^{n_i}(U_i)$ contains $g^{n_i}(S_g \cap U_i)$ whenever U_i is contained in the basin of x and $g \in \mathcal{U}$. For $i = 1, 2$, denote by x_i the fixed point that the map g_i has in V'.

Defintion of the conjugacy h in the neighborhood V' of x_1.

It is easy to construct a local conjugacy between g_1 and g_2 that is close to the identity, but this local homeomorphism may not preserve critical images. We refer the reader to [IPR] where a similar construction was done (there, the critical components U_i were arbitrarily small and the manifold was two dimensional).

Let α be diminished again in order that α is less than the distance between different $f^{n_i}(U_i)$'s. Let $Z_i(g_1)$ be the closure of $g_1^{n_i}(U_i)$ and $Z_i(g_2) = \varphi(Z_i(g_1))$. As φ is α-C^0 close to the identity, it follows that $Z_i(g_2)$ contains $g_2^{n_i}(S_{g_2} \cap U_i)$. For each

$g = g_1, g_2$ let $Z'_i(g)$ be a small neighborhood of $Z_i(g)$. The construction begins with a homeomorphism h:

$$h : V' \setminus \bigcup_i (\cup_{n\geq 0} g_1^n(Z'_i(g_1))) \to V' \setminus \bigcup_i (\cup_{n\geq 0} g_2^n(Z'_i(g_2))),$$

such that $hg_1 = g_2 h$, and such that the C^0 distance between h and the identity is less than ρ, an arbitrary positive constant to be determined later. Next define $h = \varphi$ in $Z_i(g_1)$ and finally extend h to $Z'_i(g_1) \setminus Z_i(g_1)$, such that h is a homeomorphism α-C^0 close to the identity. To prove that this last extension is possible, note that the boundary of $Z_i(g)$ can be taken smooth and that $Z'_i(g)$ may be taken as the union of a tubular neighborhood of the boundary of $Z_i(g)$ with $Z_i(g)$. Note also that the boundary of $f^{n_i}(U_i)$ has a finite number of components, so the positive number α can be taken small in order that φ identifies components of the boundaries of $Z_i(g_1)$ and $Z_i(g_2)$ in the same way that h identifies components of the boundary of $Z'_i(g_1)$ and $Z'_i(g_2)$. It follows that the problem of constructing this last extension is reduced to show that a C^1 map that is C^0 close to the identity on an embedded manifold, can be extended to a homeomorphism that coincides with the identity outside a tubular neighborhood of it. Once h was defined in the fundamental domain, one can extend it dinamically to the whole V'.

Definition of h in the basin B_{g_1}.

This part is subdivided into two steps. The first one consists in extending h to the complement in B_{g_1} of the union of the preimages of the $\cup_i Z_i(g_1)$. First extend h to the first preimage of V'. For $g = g_1, g_2$ let

$$V^1(g) = g^{-1}(V' \setminus \cup Z_i(g)).$$

Note that hg_1 is a finite to one covering map from each component of $V^1(g_1)$ to a component of $V' \setminus \cup Z'_i(g_2)$. The map g_2 is a covering map from each component of $V^1(g_2)$ to a component of $V' \setminus \cup Z_i(g_2)$. The domains of these covering maps are homeomorphic and there exists an obvious isomorphism between the first homotopy groups associated. The actions of corresponding coverings on homotopy groups are equal modulus that isomorphism. From this it follows that there exists a unique lift $\tilde{h} : V^1(g_1) \to V^1(g_2)$ such that $hg_1 = g_2\tilde{h}$ and $\tilde{h}(x_1) = x_2$. By construction the map $\tilde{h}$ is a homeomorphism that extends h.

This proceeding can be repeated to further preimages, thus giving an extension of the conjugacy h to a homeomorphism

$$h : B_{g_1} \setminus \cup_{n\geq 0} g_1^{-n}(\cup_i(Z_i(g_1))) \to B_{g_2} \setminus \cup_{n\geq 0} g_2^{-n}(\cup_i(Z_i(g_2))). \tag{3}$$

The second step consists in the extension of h to the preimages of Z_i. The homeomorphism h can be extended in an unique way to a conjugacy defined in the preimages of $g_1^{-j}(Z_i(g_1))$ for every i and j such that $j < n_i$, because g_1 was injective there. To define it in $g_1^{-n_i}(Z_i(g_1)) = U_i$ one must use the map ψ of equation 2, to take care of the critical set contained there. It is claimed now that h (given by equation 3) must coincide with ψ in the boundary of U_i. Note that f is an immersion if restricted to the (smooth) boundary of U_i. It follows that there exists a number $c > 0$ such that two points in the boundary of U_i that have the same image under f must be at a distance at least $2c$. Let α be diminished again in such a way that $\alpha < c$. If also the neighborhood $\mathcal{U}$ of f is diminished, then the same property holds for every g there. Note also that as the restrictions of both ψ and h to the boundary of U_i satisfy the functional equation $\Phi g_1 = g_2\varphi$ (where

Φ is the unknown variable of the equation), it comes that ψ and h coincide in a relative open subset of the boundary of U_i. But as both maps are α-C^0 close to the identity, then the claim follows. therefore one can extend h to U_i as equal to ψ. the remaining extension to the preiamges of the sets U_i is know obvious.

Extension to the boundary.

This part is similar to the proof given in [IPR]. Fix a neighborhood U of the Julia set of f where f is expanding, say that with an adapted metric one has that the differential of f expands in U at a rate $\lambda > 1$. By lemma 6 it holds that there exists some positive constant N such that $f^{-n}(V' \setminus f(V'))$ is contained in U for every $n \geq N$. This also holds for every $g \in \mathcal{U}$. Moreover the constant ρ given in the definition of h in the fundamental domain can be taken small so that h is ϵ-C^0 close to the identity in $g_1^{-N}(V' \setminus g_1(V'))$. Using the expansion of g_1 in U one can show, as in [IPR], corollary 3, that h is ϵ-C^0 close to the identity. Then,taking sequences and using the expansivity of g_1 in $\Omega'(g_1)$ the fact that h can be extended to the boundary of B_{g_1} can be made as in the above reference. Trivially the extended h is a conjugacy between g_1 and g_2. This proves the theorem.

An example. In [IPR] examples of perturbations of complex polynomials were shown to be C^3 structurally stable. In that case, each component of the set of critical points was a small Jordan curve whose image was disjoint from its interior. We show now how to construct a stable map in the sphere such that S_f is a circle whose image is contained in the component of its complement that contains the fixed attracting point. Let $f(x,y) = \rho(x^2 - y^2 + \lambda y, 2xy + \mu x)$ defined in a ball B_r of center the origin radius $r = 1/2$. If ρ, λ and μ are small positive numbers, then the origin is an attractor and the set of critical points is a circle contained in B_r. Moreover, if ρ is diminished again, then $f(S_f)$ is contained in the bounded component of the complement of S_f. It can also be seen that the restriction of f to S_f is injective, from which it follows that $f(S_f)$ is also a topological circle and that the origin is contained in the bounded component of its complement. This makes f a C^3 geometrically stable map, in the sense that it is geometrically equivalent to each C^3 perturbation (to prove this assertion and the injectivity of $f|S_f$ consult [DGRRV]).

To define f in the whole sphere, let it coincide with $z \to z^2$ in the annulus $\{z \ : \ |z| \in [3/4, 5/4]\}$ and with a map g that in the complement of the ball of radius 2 verifies that $1/g(1/z) = f(z)$. Then extend f to the whole sphere. It holds that the set of critical points of f has two components each contained in the basin of an attracting fixed point, the origin and ∞. The nonwandering set contains an expanding basic piece $\{z \ : \ |z| = 1\}$. Moreover the extended map f is still C^3 geometrically stable. By theorem C it follows that f is C^3 structurally stable.

Final comment.

We do not know examples of maps satisfying the hypothesis of theorem C in dimension greater than two. To find other examples of structurally stable maps, one would have to admit saddle type basic pieces, which represents an additional difficulty, since their unstable manifolds have a wild behaviour, as they can have infinitely many intesection points. Przytycki has presented the simplest possible example in the last section of [P]. As far as we know, nobody has ever answered his question about the C^1 structural stability of this example.

References

[AMS] N. Aoki, K. Moriyasu, N. Sumi. C^1 maps having hyperbolic periodic points. Fundamenta Mathematicae. 169, 1-49, 2001.

[DGRRV] J.Delgado, A.Garrido, N.Romero, A.Rovella, F.Vilamajó. Critical points of quadratic maps of the plane. premat.fing.edu.uy 2005/85

[F] J. Franks. Necessary conditions for stability of diffeomorphisms. Trans. Amer. Math. Soc. 158, 301-308, 1971.

[GG] M. Golubitsky, V. Guillemin. Stable mappings and their singularities. Graduate texts in mathematics 14. Springer, New York, 1973.

[IPR] J. Iglesias, A. Portela, A. Rovella. Structurally stable perturbations of polynomials in the Riemann sphere. To appear in IHP .

[Ma] R. Mañé. A proof of the C^1 stability conjecture. (English) Publ. Math., Inst. Hautes Étud. Sci. 66, 161-210, 1987.

[MP] R. Mañé and C. Pugh. Stability of endomorphisms. Lecture Notes in Math. Vol 468. Springer Verlag. 175-184, 1975.

[P] F. Przytycki. On Ω-stability and structural stability of endomorphisms satisfying Axiom A.Studia Mathematica 60, 61-77, 1977.

[R] C. Robinson. Structural stability of C^1 diffeomorphisms. J. Diff. Eq., 22, 28-73, 1976.

[S] M. Shub. Endomorphisms of compact differentiable manifols. Amer. J. Math., 91, 175-199, 1969.

Universidad de La República. Facultad de Ingenieria. IMERL. Julio Herrera y Reissig 565.

E-mail address: jorgei@fing.edu.uy

Universidad de La República. Facultad de Ingenieria. IMERL. Julio Herrera y Reissig 565.

E-mail address: aldo@fing.edu.uy

Universidad de La República. Facultad de Ciencias. Centro de Matemática. Iguá 4225. C.P. 11400. Montevideo, Uruguay

E-mail address: leva@cmat.edu.uy