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On weak KAM theory for N-body problems

Ezequiel Maderna
Instituto de Matemática y Estadístca "Prof. Rafael Laguardia"
Universidad de la República, Montevideo, Uruguay
(e-mail: eze@fing.edu.uy)

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Abstract. We consider N-body problems with $1/r^{2\kappa}$ potential where $\kappa \in (0,1)$, including the Newtonian case ($\kappa = 1/2$). Given $R > 0$ and $T > 0$, we find a uniform upper bound for the minimal action of paths binding in time T any two configurations which are contained in some ball of radius R. Using cluster partitions, we obtain from these estimates Hölder regularity of the critical action potential (i.e. of the minimal action of paths binding in free time two configurations). As an application, we establish the weak KAM theorem for these N-body problems, i.e. we prove the existence of fixed points of the Lax-Oleinik semigroup and we show that they are global viscosity solutions of the corresponding Hamilton-Jacobi equation. We also prove that there are invariant solutions for the action of isometries on the configuration space.

1. *Introduction*

Let E be a finite dimensional Euclidian space, and denote by $x = (r_1, \ldots, r_N) \in E^N$ the configuration vector of N punctual masses $m_1, \ldots, m_N > 0$. By $\| x \|$ we will denote the norm given by $\max\{ \| r_i \|_E \mid 1 \le i \le N \}$, and $|x|$ will denote the norm induced by the mass scalar product

$$< x, y >= \sum_{i=1}^{N} m_i < r_i, s_i >_E$$

for $x = (r_1, \ldots, r_N)$, $y = (s_1, \ldots, s_N) \in E^n$. As usual, we call $I(x) = |x|^2$ the moment of inertia of x regarding the origin of E. The N-body problem is determined once the force function U on E^N (or potential function), negative of the potential energy, is chosen. In this paper, we restrict us to the potential functions which are homogeneous of degree -2κ

$$U_\kappa(x) = \sum_{i<j} m_i m_j \left(r_{ij}\right)^{-2\kappa},$$

where $r_{ij} = \| r_i - r_j \|_E$, and $\kappa \in (0,1)$. The case $\kappa = 1/2$ corresponds to the Newtonian potential. In other words, this means that the laws of motion are given on the open and dense subset $\Omega = \left\{ x \in E^N \mid U_\kappa(x) < +\infty \right\}$ by the differential equation $\ddot{x} = \nabla U_\kappa$, where the gradient is taken with respect to the mass scalar product on E^N. The equivalent variational formulation is given by the Lagrangian defined on $TE^N = E^N \times E^N$,

$$L(x,v) = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 + U_\kappa(x)\,,$$

where $v = (v_1, \ldots, v_n)$. Thus, motions are characterized as critical points of the Lagrangian action $A(\gamma) = \int L(\gamma(s), \dot{\gamma}(s))\, ds$, and the Euler-Lagrange equations define a - non complete - analytical flow on the non compact manifold $T\Omega$.

1.1. *Globally minimizing curves and the action potential.* Let us give a precise definition of the Lagrangian action functional. Recall that a curve $\gamma : [a,b] \to E^N$ is absolutely continuous if it is differentiable almost everywhere, and its derivative $\dot{\gamma}$ satisfies the fundamental theorem of calculus for the Lebesgue integral. Thus the Lagrangian action is well defined on the set of absolutely continuous curves $\mathcal{C}$. More precisely, the action is the function $A : \mathcal{C} \to (0, +\infty]$ given by

$$A(\gamma) = \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds = \frac{1}{2}\int_a^b |\, \dot{\gamma}(s) \,|^2\, ds + \int_a^b U_\kappa(\gamma(s))\, ds\,.$$

where $|\, v \,|$ is the norm in E^N induced by the mass scalar product. It can be seen that absolutely continuous curves with finite action are necessarily 1/2-Hölder continuous, hence they are contained in the Sobolev space $H^1([a,b], E^N)$.

For $T > 0$ and $x, y \in E^N$, denote by $\mathcal{C}(x,y,T)$ the set of all absolutely continuous curves $\gamma : [0,T] \to E^N$ which satisfy $\gamma(0) = x$ and $\gamma(T) = y$. We are interested in the function ϕ defined on $E^N \times E^N \times (0,+\infty)$ by

$$\phi(x,y,T) = \inf \{ \, A(\gamma) \mid \gamma \in \mathcal{C}(x,y,T) \, \} \,.$$

We will say that a curve $\gamma : [a,b] \to E^N$ is globally minimizing, if we have that $A(\gamma) = \phi(\gamma(a), \gamma(b), b-a)$. For a curve defined on a non compact interval, globally minimizing will mean that the property is satisfied for all restrictions of the curve to a compact interval. It is not difficult to see that a globally minimizing curve always exists for any two configurations $x, y \in E^N$ and for all $T > 0$. Essentially, it is a consequence of the lower semi-continuity of the action functional.

In the last years, the global variational methods have been successful to prove the existence of a great variety of particular motions. A typical example is the eight choreography of Chenciner and Montgomery [**4**], among many others closed orbits with topological or symmetry constraints. The main difficulty that raises from these methods for the Newtonian potential, and also for the homogeneous potentials here considered, is the one to assure that global minimizers avoid collisions, that is to say, that they are contained in the open domain Ω. Following an idea of Marchal, Chenciner established a proof of this fact, for the Newtonian N-body problem in the plane or the three-dimensional space, see [**3**], [**14**].

Our first result gives an upper bound for the action of such curves which depends on the size of the configurations. In our opinion, this result is quite fundamental for global variational methods, and it is optimal, in the sense that the bound is reached by homothetic minimizing configurations, as we explain in the following section.

THEOREM 1. *There are positive constants* α, $\beta > 0$ *such that for all* $T > 0$,

$$\phi(x, y, T) \leq \alpha\, T^{-1} R^2 + \beta\, T R^{-2\kappa}\,,$$

whenever x *and* y *are configurations contained in a ball of radius* $R > 0$ *of* E*. The constants* α *and* β *only depend on the degree of homogeneity of the potential* (-2κ)*, the number of bodies* N*, and their masses.*

The next result shall be useful for the study of free time minimizers, that is to say, absolutely continuous curves which minimizes the action in the set of curves $\mathcal{C}(x, y) = \bigcup_{T>0} \mathcal{C}(x, y, T)$. The Mañé's critical action potential (see for instance [**6**]), or the *action potential*, is defined in our setting on $E^N \times E^N$ by

$$\phi(x, y) = \inf \{\, \phi(x, y, T) \mid T > 0 \,\} = \inf \{\, A(\gamma) \mid \gamma \in \mathcal{C}(x, y) \,\}\ .$$

It is clear that $\phi(x, y) = \phi(y, x)$, and that $\phi(x, y) \leq \phi(x, z) + \phi(z, y)$, for any configurations x, y, z in E^N. In fact, proposition 6 shows that the action potential ϕ is a distance function. Notice that as a corollary of theorem 1, we have that $\phi(x, y) \leq (\alpha + \beta) R^{1-\kappa}$ whenever x and y are configurations contained in a ball of radius $R > 0$ of E. With similar arguments as in theorem 1, combined with a cluster decomposition, we obtain the following theorem.

THEOREM 2. *There is a positive constant* $\eta > 0$ *such that for all* $x, y \in E^N$,

$$\phi(x, y) \leq \eta\, \|\, x - y \,\|^{\,1-\kappa}\,.$$

Therefore, the action potential is Hölder continuous respect to the Euclidean norm on $E^N \times E^N$. In other words, for any configurations x, y, z in E^N we have $\phi(x, z) - \phi(y, z) \leq \phi(x, y) \leq \eta\, \|\, x - y \,\|^{\,1-\kappa}$. On the other hand, it is easy to prove that the action potential is locally Lipschitz in the open and dense subset $\Omega \times \Omega \subset E^N \times E^N$.

1.2. *On the weak KAM theory.* In order to give applications, we will show that theorem 2 enables us to prove a weak KAM theorem in the spirit of [**10**], [**11**]. The novelty in this viewpoint, is that we regard the action of the Lax-Oleinik semigroup on a space of Hölder functions.

Let us remember that a function $u : E^N \to \mathbb{R}$ is said *dominated* by L, if it satisfies the condition $u(x) - u(y) \leq \phi(x, y)$ for all $x, y \in E^N$. Since the action potential is symmetric, theorem 2 implies that dominated functions are Hölder continuous. On the other hand, it is not difficult to prove that they are locally Lipschitz in the open subset of total measure $\Omega \subset E^N$, see proposition 7 below. Therefore, dominated functions are differentiable almost everywhere. We shall discuss this in more detail below. Another way to define the set of dominated functions, is using the Lax-Oleinik semigroup: given a function $u : E^N \to [-\infty, +\infty)$ and $t > 0$ we define $T_t^- u : E^N \to [-\infty, +\infty)$ by

$$T_t^- u(x) = \inf \left\{ u(y) + \phi(x, y, t) \mid y \in E^N \right\} .$$

Then, a continuous function u is dominated if and only if $u \leq T_t^- u$ for all $t > 0$. Notice that the set of dominated functions is convex and stable under the Lax-Oleinik semigroup. Setting $T_0^- u = u$ for any function u, we will prove that $(T_t^-)_{t \geq 0}$ is a continuous semigroup on the set of dominated functions equipped with the topology of uniform convergence on compact subsets.

Another set which is stable by the Lax-Oleinik semigroup is the set of functions which are invariant by symmetries. If we observe that the group of isometries of E, acts naturally on E^N by symmetries of the potential function, then an obvious question is the existence of invariant fixed points of the semigroup. More precisely, we will say that a function $u : E^N \to \mathbb{R}$ is *invariant* if $u(r_1, \ldots, r_N) = u(Ar_1 + r, \ldots, Ar_N + r)$ for all $x = (r_1, \ldots, r_N) \in E^N$, $r \in E$ and $A \in O(E)$.

THEOREM 3 (INVARIANT WEAK KAM) *There exists an invariant and dominated function $u : E^N \to \mathbb{R}$ such that $u = T_t^- u$ for all $t \geq 0$.*

In section 3, we prove the weak KAM theorem, and we study the relationship with the Hamilton-Jacobi equation. More precisely, we show that weak KAM solutions are global viscosity solutions in Ω.

An important difference with the compact case is that here the Aubry set is empty. In particular the technique used in [**12**] to prove the invariance of all solutions is not available. Moreover, we will show non invariant solutions for the Kepler problem in the plane, which is the subject of the last section.

2. *Hölder regularity of the action potential*

This section is devoted to the study of the action potential, and to give the proofs for theorems 1 and 2.

2.1. *Proof of theorem 1.*

Given $r \in E$ and $R > 0$, we say that a configuration $x = (r_1, \ldots, r_N) \in E^N$ is contained in the ball $B(r, R)$ when we have $\| r_i - r \|_E \leq R$ for all $i = 1, \ldots, N$. Suppose now that we have two configurations x and y such that for some $r \in E$ and some $R > 0$, both x and y are contained in $B(r, R)$. If we tried to bound $\phi(x, y, T)$ with the action of a linear path, then two problems arise. The first one is that the linear path can present collisions in which case the action is infinite. The second one is that, even if the linear path avoid collisions, the distance between two given bodies can be arbitrary small for both configurations, hence the action can be arbitrary large. Both problems are solved in the following way: fix an intermediate configuration p with sufficiently large mutual distances, and take the linear path from x to p defined on $[0, T/2]$ followed by the linear path from p to y defined on $[T/2, T]$. This path has no more than $2N(N-1)$ collisions, and we can determine the values of $t \in [0,T]$ in which these collisions happen. Thus, reparametrizing the path in such a way that in the new times of collisions the action integral converges, we obtain the following proposition, from which we can easily deduce theorem 1.

PROPOSITION 4. *Given two configurations $x, y \in E^N$ contained in a ball $B(r,R)$, $r \in E$, $R > 0$, and given $T > 0$, there is a curve $\gamma \in \mathcal{C}(x,y,T)$, such that $\gamma(t)$ is contained in $B(r, 6NR)$ for all $t \in [0,T]$,*

$$\frac{1}{2}\int_0^T |\,\dot\gamma(t)\,|^{\,2}\,dt \le \alpha\, T^{-1}R^2\,,\ \ \textit{and}\ \ \int_0^T U_\kappa(\gamma(t))\,dt \le \beta\, T R^{-2\kappa}\,,$$

where α and β are positive constants that only depend on the number of bodies, the total mass and the degree of homogeneity of the force function.

Proof. We first observe that it suffices to give the proof for a fixed value of $T > 0$: for $S > 0$, we can define $\sigma : [0,S] \to E^N$ as $\sigma(s) = \gamma(sT/S)$, and we have

$$\int_0^S |\,\dot\sigma(s)\,|^{\,2}\,ds = T^2S^{-2}\int_0^S |\,\dot\gamma(sT/S)\,|^{\,2}\,ds = S^{-1}T\int_0^T |\,\dot\gamma(t)\,|^{\,2}\,dt \le 2\alpha\, S^{-1}R^2\,,$$

$$\int_0^S U_\kappa(\sigma(s))\,ds = \int_0^S U_\kappa(\gamma(sT/S))\,ds = ST^{-1}\int_0^T U_\kappa(\gamma(t))\,dt \le \beta\, SR^{-2\kappa}\,.$$

We will then give the proof for $T = 2$. Take $v \in E$ such that $\|\,v\,\|_E = 6R$, and define $p = (p_1, \ldots, p_N) \in E^N$ by

$$p_i = r + (i-1)\,v\,,\quad i = 1, \ldots, N\,.$$

Notice that the mutual distances $p_{ij} = \|\,p_i - p_j\,\|_E$ of p are greater than $6R$ and smaller than $6NR$. Therefore, the configuration p is contained in $B(r, 6NR)$.

Let now $x = (r_1, \ldots, r_N)$ be a configuration such that $\|\,r_i - r\,\|_E \le R$ for all $i = 1, \ldots, N$. We consider the curve $z_x : [0,1] \to E^N$, defined by $z_x(t) = x + \psi_x(t)(p - x)$, with $\psi_x : [0,1] \to [0,1]$ a function to be determined. Our aim is to choose the function ψ_x conveniently, in order to obtain a bound of $A(z_x)$ which does not depend on x.

Recall that if u and v are two vectors in a Euclidean space, and $v \neq 0$, then we have, for all real number λ,

$$\|\,u + \lambda v\,\|^{\,2} = \left(\lambda\,\|\,v\,\| + \frac{< u, v >}{\|\,v\,\|}\right)^2 + \|\,u\,\|^{\,2} - \frac{< u, v >^2}{\|\,v\,\|^{\,2}}$$

and as a consequence,

$$\|\,u + \lambda v\,\| \ \ge \|\,v\,\| \left|\,\lambda + \frac{< u, v >}{\|\,v\,\|^{\,2}}\,\right|\,.$$

In particular, the minimum of $\|\,u + \lambda v\,\|$, is reached for

$$\lambda = -\frac{< u, v >}{\|\,v\,\|^{\,2}}\,.$$

We will use the notation $u_{ij} = r_i - r_j$ and $v_{ij} = (p_i - p_j) - (r_i - r_j)$ for $i < j$. Thus, the mutual distances of the configuration $z_x(t)$ can be written $d_{ij}(t) = \| u_{ij} + \psi_x(t) v_{ij} \|$. Observe that $\| u_{ij} \|_E \leq 2R$ and $\| v_{ij} \|_E \geq 4R$ for all $i < j$. Therefore, taking $\lambda = \psi_x(t)$, $u = u_{ij}$ and $v = v_{ij}$ in the above considerations, we deduce that each mutual distance $d_{ij}(t)$ verifies

$$d_{ij}(t) \geq \| v_{ij} \|_E \, | \psi_x(t) - t_{ij} | \geq 4R \, | \psi_x(t) - t_{ij} | \, ,$$

where

$$t_{ij} = - \frac{< u_{ij}, v_{ij} >_E}{\| v_{ij} \|_E{}^2} \, .$$

It is clear that $| t_{ij} | < 1/2$ for all $i < j$.

By lemma 5 below, we know that the function ψ_x can be chosen in such a way that, on one side,

$$\int_0^1 \dot{\psi}_x(t)^2 dt \leq 5N^2 \frac{1+\kappa}{1-\kappa} \, ,$$

and on the other side, for each $i < j$ there is a real number s_{ij} for which

$$| \psi_x(t) - t_{ij} | \geq N^{-2} | t - s_{ij} |^{(1/1+\kappa)}$$

for all $t \in [0,1]$. Let us estimate the action $A(z_x)$ for this function ψ_x. We have $\dot{z}_x(t) = \dot{\psi}_x(t) \, (p - x)$, and $\| p_i - r_i \|_E \leq (6N+2)R$ for all $i = 1, \ldots, N$. Hence,

$$\begin{aligned} \frac{1}{2} \int_0^1 | \dot{z}_x(t) |^2 \, dt &= \frac{1}{2} \sum_{i=1}^{N} m_i \| p_i - r_i \|_E^2 \int_0^1 \dot{\psi}_x(t)^2 dt \\ &\leq 160 \, \frac{1+\kappa}{1-\kappa} \, M \, N^4 \, R^2 \, , \end{aligned}$$

and

$$\begin{aligned} \int_0^1 U_\kappa(z_x(t)) \, dt &= \sum_{i<j} \int_0^1 m_i m_j \, d_{ij}(t)^{-2\kappa} \, dt \\ &\leq \sum_{i<j} \int_0^1 m_i m_j \, (4R)^{-2\kappa} \, N^{4\kappa} \, | t - s_{ij} |^{-(2\kappa/1+\kappa)} \, dt \\ &\leq N^6 M^2 R^{-2\kappa} \int_0^1 t^{-(2\kappa/1+\kappa)} \, dt \; = \; \frac{1+\kappa}{1-\kappa} \, N^6 M^2 \, R^{-2\kappa} \, . \end{aligned}$$

To finish the proof, let $y = (s_1, \ldots, s_N)$ be a second configuration contained in $B(r,R)$, and define $\gamma \in \mathcal{C}(x,y,2)$ as follows: $\gamma(t) = z_x(t)$ if $t \leq 1$, and $\gamma(t) = z_y(2-t)$ if $t \geq 1$. We conclude that

$$A(\gamma) = A(z_x) + A(z_y) \leq \, 320 \, \frac{1+\kappa}{1-\kappa} \, M \, N^4 \, R^2 \; + \; 2 \, \frac{1+\kappa}{1-\kappa} \, N^6 M^2 \, R^{-2\kappa} \, .$$

This also proves the proposition for $T = 2$, with

$$\alpha = 640 \, \frac{1+\kappa}{1-\kappa} \, M \, N^4 \quad \text{and} \quad \beta = \frac{1+\kappa}{1-\kappa} \, N^6 M^2 \, .$$

□

We have used the following lemma.

LEMMA 5. *Given $\kappa \in (0,1)$ and real numbers $a_1 < \cdots < a_m$, there are real numbers $b_1 < \cdots < b_m$ and an increasing Hölder homeomorphism F of $[0,1]$ such that*

1.
$$|\, F(t) - a_i \,| \geq \frac{1}{m} |\, t - b_i \,|^{\,1/1+\kappa}$$
for all $t \in [0,1]$ and each $i = 1, \ldots, m$, and

2.
$$\int_0^1 F'(t)^2 dt \leq (4 + 2a)(m+1) \frac{1+\kappa}{1-\kappa}\,,$$
where $a = \min\{\, |\, a_1 \,|, \ldots, |\, a_m \,| \,\}$.

Proof. Given $c > 0$, let $g_c : \mathbb{R} \setminus \{\, 0 \,\} \to \mathbb{R}$ defined by $g_c(x) = c\,|\, x \,|^{\,-\kappa/1+\kappa}$. Given $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$ such that $b_1 < \cdots < b_m$, we also define the function $f_{b,c} : \mathbb{R} \setminus \{\, b_1, \ldots, b_m \,\} \to \mathbb{R}$ by

$$f_{b,c}(t) = \max\{\, g_c(t - b_1), \ldots, g_c(t - b_m) \,\}\ .$$

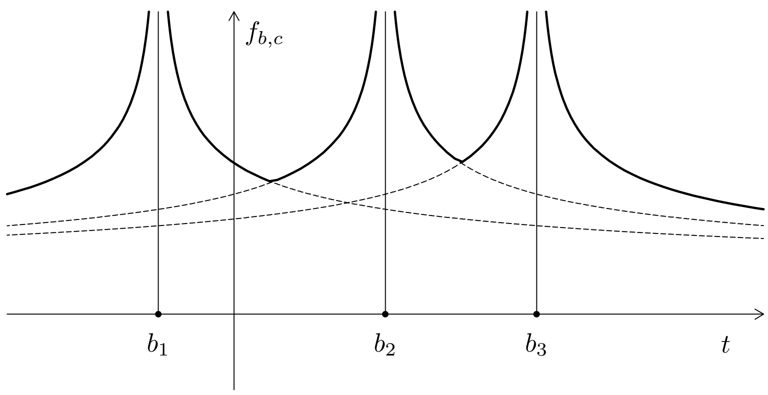

FIGURE 1. Graph of $f_{b,c}$ for $b = (b_1, b_2, b_3)$.

We will define the required function F as a primitive of a function $f_{b,c}$ for a good choice of b and c. More precisely, if we define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(t) = \int_0^t f_{b,c}(s)\, ds$$

then it is no difficult to see that F is an increasing Hölder homeomorphism of $\mathbb{R}$ which satisfy

$$|\, F(t) - F(b_i) \,| \geq c\,(1+\kappa)\,|\, t - b_i \,|^{\,1/1+\kappa}$$

for all $t \in \mathbb{R}$ and each $i = 1, \ldots, m$. Therefore, we must choose $c > 0$ and $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$ such that $F(1) = 1$ and $F(b_i) = a_i$ for all $i = 1, \ldots, m$.

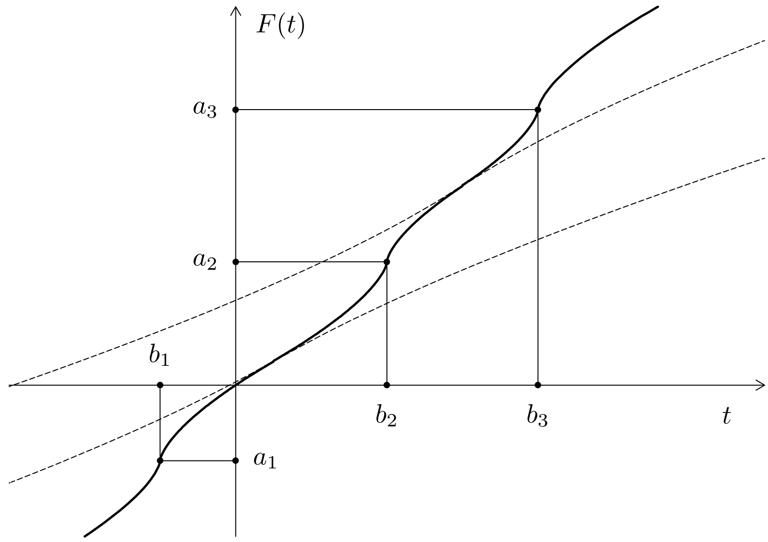

FIGURE 2. Graph of $F(t) = \int_0^t f_{b,c}(s)\, ds$.

If we fix $c > 0$, then we have a unique possible choice for b. To see this, first observe that the $m-1$ distances between the consecutive values of a_i determine the $m-1$ distances between the consecutive values of b_i. If we set $A_i = a_{i+1} - a_i$ and $B_i = b_{i+1} - b_i$ then we must have

$$A_i = \int_{b_i}^{b_{i+1}} f_{b,c}(s)\, ds = 2c \int_0^{B_i/2} s^{-\kappa/1+\kappa} ds = 2^{\kappa/1+\kappa}\, c\,(1+\kappa)\, B_i^{1/1+\kappa}$$

hence $B_i = 2^{-\kappa}[A_i/c\,(1+\kappa)]^{1+\kappa}$. From the condition $F(b_1) = a_1$ we deduce that

$$a_1 = \int_0^{b_1} f_{b,c}(s)\, ds = -\int_0^{-b_1} f_{b,c}(s + b_1)\, ds\,.$$

Therefore b_1 must be the unique solution of the equation $\int_0^{-x} f_{b',c}(s)\, ds = -a_1$ where $b' = (0, B_1, B_1 + B_2, \ldots, B_1 + \cdots + B_{m-1}) = (0, b_2 - b_1, \ldots, b_m - b_1)$. Moreover, we have showed that there is a continuous vector $b(c) \in \mathbb{R}^m$ such that $\int_0^{b_i} f_{b(c),c}(s)\, ds = a_i$ for all $i = 1, \ldots, m$. Therefore, it is clear that

$$\delta(c) = \int_0^1 f_{b(c),c}(s)\, ds$$

also depends continuously on c. We claim that there is $c \in [1/m(1+\kappa), 2+a]$ for which $\delta(c) = 1$. We have

$$\begin{aligned} \delta(c) = \int_0^1 f_{b(c),c}(s)\, ds \quad &\leq \quad \sum_{i=1}^{m} \int_0^1 g_c(s - b_i(c))\, ds \\ &\leq \quad c\,m \int_0^1 s^{-\kappa/1+\kappa} = c\,m\,(1+\kappa)\,. \end{aligned}$$

This shows that $\delta(c) < 1$ when $c < 1/m(1+\kappa)$. In order to prove the claim, it suffices to show that $\delta(c) > 1$ when $c > 2+a$. Since $a = |\,a_j\,|$ for some $j \in \{\,1,\ldots,m\,\}$, we have

$$\begin{aligned} a &= \Big|\int_0^{b_j(c)} f_{b(c),c}(s)\,ds\,\Big| \\ &\geq \Big|\int_0^{b_j(c)} c\,|\,s-b_j(c)\,|^{-\kappa/1+\kappa}\,ds\,\Big| \\ &\geq c\int_0^{|\,b_j(c)\,|} s^{-\kappa/1+\kappa}\,ds = c\,(1+\kappa)|\,b_j(c)\,|^{1/1+\kappa}\,, \end{aligned}$$

which implies $|\,b_j(c)\,| \leq [a/c(1+\kappa)]^{(1+\kappa)}$. On the other hand we have

$$\begin{aligned} \delta(c) &\geq \min\{\,g_c(s-b_j(c)) \mid s\in[0,1]\,\} \\ &\geq c\,(1+|\,b_j(c)\,|)^{-\kappa/1+\kappa}. \end{aligned}$$

Thus, it suffices to prove that $|\,b_j(c)\,| \leq c^{(1+\kappa)/\kappa} - 1$ when $c > 2+a$. By the previous estimation of $|\,b_j(c)\,|$, we only have to prove that $(a/1+\kappa)^{1+\kappa} \leq c^{1+\kappa}(c^{1+\kappa/\kappa}-1)$, but this condition is clearly satisfied if $c>2$ and $c>a$.

We take $c \in [1/m(1+\kappa), 2+a]$ such that $\delta(c)=1$ and we define $F : [0,1] \to [0,1]$ by

$$F(t) = \int_0^t f_{b(c),c}(s)\,ds\,.$$

In order to see that this function satisfy all the required conditions, it remains to estimate the L^2 norm of F'. If we observe that $[0,1] \setminus \{\,b_1(c),\ldots,b_m(c)\,\}$ has at most $m+1$ components I_j, and that on each one of these components we have

$$\int_{I_j} f_{b(c),c}(s)^2\,ds \leq 2\int_0^1 c\,s^{-2\kappa/1+\kappa}ds\,,$$

we conclude that

$$\int_0^1 F'(t)^2 dt \leq (4+2a)(m+1)\frac{1+\kappa}{1-\kappa}\,.$$

□

2.2. *Minimizing configurations.* The following observations shows that theorem 1 is optimal in the sense that the bound is reached by some configurations. We shall first recall the notions of *central* and *minimizing* configurations, as well as some properties, see for instance Wintner, [**15**].

We say that a configuration $x \in E^N$ is *minimizing*, if it is a minimum of the potential function U restricted to the sphere $\{\,y\in E^N \mid I(y)=I(x)\,\}$. In particular, minimizing configurations are *central* configurations, that is to say, configurations $x \in E^N$ which are critical points of U restricted to $\{\,y\in E^N \mid I(y)=I(x)\,\}$. Central configurations are characterized as configurations which admit homothetic motions. In other words, a configuration

$x_0 \in E^N$ is central, if and only if $U_\kappa(x_0) < +\infty$ and $x(t) = r(t)x_0$ is a solution for some positive real function $r(t)$.

Suppose that $x_0 \in E^N$ is a central configuration, normalized in the sense that $I(x_0) = 1$. If we look for an homothetic motion through x_0, then we must solve a one dimensional differential equation, which is nothing but the one dimensional Kepler problem when the potential is the Newtonian one. A particular solution, that we shall call parabolic, is given by $x(t) = c\,t^{1/1+\kappa}x_0$ for some value of $c > 0$. A simple computation shows that the action of this solution is

$$\begin{aligned} A(x\,|_{[0,T]}) &= \frac{c^2}{2(1+\kappa)^2}\int_0^T t^{-2\kappa/1+\kappa}\,dt \;+\; c^{-2\kappa}U(x_0)\int_0^T t^{-2\kappa/1+\kappa}\,dt \\ &= \left(\frac{c^2}{2(1-\kappa^2)} + c^{-2\kappa}U(x_0)\frac{1+\kappa}{1-\kappa}\right)T^{(1-\kappa)/(1+\kappa)}\,. \end{aligned}$$

If we set $R_T = \|\,x(T)\,\| = T^{1/1+\kappa}\,\|\,c\,x_0\,\|$, then we can write

$$A(x\,|_{[0,T]}) = \alpha_0\,T^{-1}R_T^2 \;+\; \beta_0\,T\,R_T^{-2\kappa}\,,$$

for a good choice of constants α_0 and β_0.

On the other hand, if x_0 is a minimizing configuration, then the above solution $x(t)$ is globally minimizing. In other words, we have

$$\phi(0, x(T), T) = A(x\,|_{[0,T]})\,,$$

and therefore the bound for $\phi(x, y, T)$ given by theorem 1 can not be improved modulo the choice of the constants. To see this, fix $T > 0$, and take any other curve $\gamma \in \mathcal{C}(0, x(T), T)$. For our purposes, we can suppose $\gamma(t) \neq 0$ for all $t \in (0, T]$. Setting $\gamma(t) = r(t)s(t)$, where $r(t) = |\,\gamma(t)\,| = I(\gamma(t))^{1/2}$, we have that $|\,s(t)\,| = 1$ for all $t > 0$, and the action of γ can be written

$$A(\gamma) = \frac{1}{2}\int_0^T \dot r(t)^2\,dt \;+\; \frac{1}{2}\int_0^T r(t)^2|\,\dot s(t)\,|^{\,2}\,dt \;+\; \int_0^T r(t)^{-2\kappa}U(s(t))\,dt\,.$$

Since x_0 is minimizing, we have $U(s(t)) \geq U(x_0)$ for all $t > 0$. Moreover, we have

$$A(\gamma) \geq A(r\,x_0) = \int_0^T \left(\frac{1}{2}\,\dot r(t)^2\,dt \;+\; U(x_0)r(t)^{-2\kappa}\right)dt\,.$$

But the last integral is minimal for $r(t) = c\,t^{1/1+\kappa}$, because the one dimensional problem has the property that for any two given positions $0 \leq r_0 < r_1$ and $T > 0$, there is one and only one solution $r(t)$ on $[0, T]$ with $r(0) = r_0$ and $r(T) = r_1$. Therefore, we conclude that $A(\gamma) \geq A(x\,|_{[0,T]})$, and that the solution $x(t)$ is globally minimizing.

2.3. *Properties of the action potential and proof of theorem 2.* We start by showing that the action potential is a distance function on E^N.

PROPOSITION 6. *For all $x, y \in E^N$ we have $\phi(x, y) = 0$ if and only if $x = y$.*

Proof. Let $x \in E^N$ be a configuration, and choose a path $\sigma : [0,1] \to E^N$ which satisfies $\sigma(0) = x$ and $A(\sigma) < +\infty$. Then define for $0 < T \leq 2$ the curve $\gamma_T \in \mathcal{C}(x,x,T)$ by $\gamma_T(t) = \sigma(t)$ if $t \leq T/2$, and $\gamma_T(t) = \sigma(T-t)$ if $t \geq T/2$. It is not difficult to see that $A(\gamma_T) \to 0$ as $T \to 0$, from which it follows that $\phi(x,x) = 0$ for all $x \in E^N$.

To see that the condition is necessary, take any two configurations $x = (r_1, \ldots, r_N)$ and $y = (s_1, \ldots, s_N)$ in E^N, and a path $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathcal{C}(x,y)$. If $d = \| y - x \|$, and γ is defined on $[0,T]$, then it must exist $T_0 \in [0,T]$ such that $\| \gamma(T_0) - x \| = d$ and $\| \gamma(t) - x \| \leq d$ for all $t \in [0,T_0]$. Moreover, we must have $d = \| \gamma_i(T_0) - r_i \|_E$ for some $i \in \{1, \ldots, N\}$. If $T_0 \geq 1$, we can write

$$A(\gamma) \geq A(\gamma\,|_{[0,T_0]}) \geq \int_0^{T_0} U_\kappa(\gamma(t))\,dt \geq C > 0\,,$$

where $C = \min \{ U_\kappa(z) \mid \| z - x \| \leq d \}$. If $T_0 \leq 1$ we have

$$A(\gamma) \geq A(\gamma\,|_{[0,T_0]}) \geq \frac{m_i}{2} \int_0^{T_0} \| \dot\gamma_i(t) \|_E^2\,dt \geq \frac{m\,d^2}{2}\,,$$

where $m = \min \{ m_1, \ldots, m_N \}$. The last inequality follows from the fact that γ_i is absolutely continuous and the Cauchy-Schwartz inequality. Therefore, we conclude that if $\phi(x,y) = 0$, then $d = 0$ and $x = y$. □

In the sequel we will denote $\delta(z)$ the minimal distance between the bodies of the configuration z. More precisely, $\delta : E^N \to \mathbb{R}^+$ will be the function defined by $\delta(z) = \min \{ \| z_i - z_j \|_E \mid i < j \}$, where $z = (z_1, \ldots, z_N)$. Thus the set of configurations without collisions is nothing but $\Omega = \left\{ z \in E^N \mid \delta(z) > 0 \right\}$. The next proposition shows that the action potential is locally Lipschitz in $\Omega \times \Omega$.

PROPOSITION 7. *Given a configuration $z \in E^N$ without collisions, there is $k > 0$ and $\epsilon > 0$ such that, if $x \in E^N$ satisfies $\| x \| < \epsilon$, then $\phi(z, z + x) \leq k \| x \|$.*

Proof. We give the proof for $\epsilon = \delta(z)/4$. Since z is without collisions, we have $\epsilon > 0$. For $T > 0$ we define the curve $\gamma : [0,T] \to E^N$, by $\gamma(t) = z + (t/T)x$. If $z = (z_1, \ldots, z_N)$ and $x = (r_1, \ldots, r_N)$, then $\gamma(t) = (\gamma_1(t), \ldots, \gamma_N(t))$, where $\gamma_i(t) = z_i + (t/T)r_i$. Hence, for $i < j$ and $t \in [0,T]$ we can write

$$\gamma_{ij}(t) = \| \gamma_i(t) - \gamma_j(t) \|_E \geq \| z_i - z_j \|_E - (t/T) \| r_i - r_j \|_E \geq \delta(z)/2\,,$$

and

$$U_\kappa(\gamma(t)) = \sum_{i<j} m_i m_j\, \gamma_{ij}(t)^{-2\kappa} \leq M^2 N^2 \left[\delta(z)/2\right]^{-2\kappa}\,.$$

Therefore, using that $|\, x \,|^2 = I(x) \leq MN \| x \|^2$, we deduce that

$$\begin{aligned} A(\gamma) &= \frac{1}{2} \int_0^T |\, x/T \,|^2\,dt + \int_0^T U(\gamma(t))\,dt \\ &\leq MN \| x \|^2/2T + M^2 N^2 \left[\delta(z)/2\right]^{-2\kappa} T\,. \end{aligned}$$

If $x = 0$ there is nothing to prove, since we already know that $\phi(z,z) = 0$. If $x \neq 0$, we can take $T = \| x \|$, and the above estimation gives $A(\gamma) \leq k \| x \|$ for $k = MN/2 + M^2 N^2 \left[\delta(z)/2\right]^{-2\kappa}$. □

We introduce now a notion of *cluster partition* of a subset $A \subset E$ adapted to our purposes. Given $\lambda > 1$, we will say that the set $\{ r_1, \ldots, r_K \} \subset E$ defines a λ-cluster partition of size $R > 0$ of A, if the following two conditions are satisfied:

1. $\| r_i - r_j \|_E \geq 2\lambda R$ for all $1 \leq i < j \leq K$,
2. A is contained in the union $\bigcup_{i=1}^{K} B(r_i, R)$.

It is clear that if A is finite and R is small enough, then A defines itself a cluster partition of size R of A. It is also clear that if A is bounded, then any $r \in A$ defines a trivial cluster partition of size R for any $R > \mathrm{diam}(A)$.

We will need the following lemma.

LEMMA 8. *Given $\lambda > 1$, $A = \{ r_1, \ldots, r_N \} \subset E$ and $\epsilon > 0$, there is a subset $A' \subset A$, and $R(\epsilon) > 0$ such that: (i) $\epsilon \leq R(\epsilon) < (2\lambda)^N \epsilon$, (ii) A' defines a λ-cluster partition of size $R(\epsilon)$ of A.*

Proof. We reason recursively. We begin setting $A'_1 = A$. If A'_1 does not define a λ-cluster partition of size ϵ, then there are $r, s \in A'_1$ such that $\| r - s \|_E < 2\lambda\epsilon$. If that is the case, we define $A'_2 = A'_1 \setminus \{ s \}$. Then we reason as before: if A'_2 does not defines a λ-cluster partition of size $2\lambda\epsilon$ then we have $r, s \in A'_2$ such that $\| r - s \|_E < (2\lambda)^2 \epsilon$, and we set $A'_3 = A'_2 \setminus \{ s \}$. It is clear that the process finish at the most in N steps. □

Proof of theorem 2. Fix a configuration $x = (r_1, \ldots, r_N) \in E^N$, and denote by A_x the set $\{ r_1, \ldots, r_N \} \subset E$. Let $y = (s_1, \ldots, s_N) \in E^N$ such that $\epsilon = \| y - x \| > 0$. If we apply lemma 8 to A_x with $\epsilon = \| y - x \|$ and $\lambda = 24N$, we conclude that there are $r_{i_1}, \ldots, r_{i_K} \in A_x$, and $R(\epsilon) > 0$ with the following properties.

1. $\epsilon \leq R(\epsilon) < (48N)^N \epsilon$,
2. for all $1 \leq j < k \leq K$, we have $\| r_{i_j} - r_{i_k} \|_E \geq 48N\, R(\epsilon)$, and
3. $A_x \cup A_y$ is contained in the disjoint union $\bigcup_{j=1}^{K} B_j$ where $B_j = B(r_{i_j}, 2R(\epsilon))$.

Therefore, both configurations x and y are decomposed in K clusters, each one contained in a ball B_j. More precisely, we have a partition $\{ 1, \ldots, N \} = I_1 \cup \cdots \cup I_K$ such that $i \in I_j$ if and only if both r_i and s_i are in B_j. Denote by $N_j = \mathrm{card}(I_j)$ the number of bodies in cluster j, and by M_j the total mass of this cluster, that is $M_j = \sum_{i \in I_j} m_i$. Thus we have $N = N_1 + \cdots + N_K$ and $M = M_1 + \cdots + M_K$.

We consider now the N_j-body problem composed by the bodies in the ball B_j. Given $T > 0$, we apply proposition 4 in each ball B_j, $j = 1, \ldots, K$, with initial and final condition conformed by the N_j bodies of x and y contained in B_j. Therefore we obtain, a path $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathcal{C}(x, y, T)$ such that for all $j = 1, \ldots, K$ we have,

1. If $i \in I_j$, then $\gamma_i(t) \in B(r_{i_j}, 12N\, R(\epsilon))$ for all $t \in [0, T]$,
2. $$T_j = \frac{1}{2} \int_0^T \sum_{i \in I_j} m_i \| \dot{\gamma}_i(t) \|_E^2 \, dt \leq \; 10^4 \, \frac{1+\kappa}{1-\kappa} \, M_j \, N_j^4 \, R(\epsilon)^2 / T \, , \text{ and}$$

3.

$$W_j = \int_0^T \sum_{i,\,k\,\in I_j}^{i<k} m_i m_k \| \, \gamma_i(t) - \gamma_k(t) \, \|_E^{-2\kappa} \, dt \le \frac{1+\kappa}{1-\kappa} \, N_j^6 M_j^2 \, 2^{-2\kappa} \, R(\epsilon)^{-2\kappa} \, T \, .$$

Notice that the action of the curve $\gamma = (\gamma_1, \dots, \gamma_N) \in \mathcal{C}(x, y, T)$ is

$$A(\gamma) = \sum_{j=1}^{K} T_j \, + \sum_{j=1}^{K} W_j \, + W_0$$

where W_0 is the integral of the terms of the potential function U_κ corresponding to pairs of bodies in different clusters. More precisely,

$$W_0 = \int_0^T \sum_{1 \le j < l \le K} \; \sum_{i \in I_j, \, k \in I_l} m_i m_k \, \| \, \gamma_i(t) - \gamma_k(t) \, \|_E^{-2\kappa} \, dt \, .$$

Since the balls $B(r_{i_j}, 24NR(\epsilon))$ are disjoint, we deduce that

$$W_0 \le N^2 M^2 \, (24N)^{-2\kappa} \, R(\epsilon)^{-2\kappa} \, T \, .$$

Finally, taking $T = R(\epsilon)^{1+\kappa}$, we obtain $A(\gamma) \le k \, R(\epsilon)^{1-\kappa}$ for

$$k = 10^4 \, \frac{1+\kappa}{1-\kappa} \, M \, N^4 + 2^{-2\kappa} \, \frac{1+\kappa}{1-\kappa} \, M \, N^6 + (24N)^{-2\kappa} \, M^2 N^2 \, ,$$

and we conclude the proof using that $R(\epsilon) < (24N)^N \, \epsilon = (24N)^N \, \| \, y - x \, \| \, .$ □

2.4. *Homogeneity of the action potential.* We include at the end of this section a property of homogeneity of the action potential due to the homogeneity of the potential function U_κ. We does not have used this property in the preceding proofs, but we think that it is useful for complete the picture of the action potential. The proof can be done reparametrizing conveniently homothetic paths of a given path.

PROPOSITION 9. *If $\lambda > 0$, then $\phi(\lambda \, x, \lambda \, y) = \lambda^{1-\kappa} \phi(x, y)$ for all $x, y \in E^N$.*

3. *Weak KAM theory*

It is well know the relationship between global solutions of the Hamilton-Jacobi equation and globally minimizing solutions of the corresponding Lagrangian flow. Let us recall that the *Hamiltonian*, defined on $T^*E^N = E^N \times (E^*)^N$ is the function

$$H(x, p) = \frac{1}{2} \, | \, p \, |^{\,2} - U_\kappa(x) \, ,$$

where $| \, p \, |$ denotes the dual norm of $p \in (E^*)^N$ with respect to the norm on E^N induced by the mass scalar product. More precisely, if we identify canonically the space E with its dual E^*, and $p = (p_1, \dots, p_N) \in (E^*)^N$, then

$$| \, p \, |^{\,2} = \sum_{i=1}^{N} m_i^{-1} \, \| \, p_i \, \|_E^2 \, .$$

A closely related function is the *total energy*, defined on TE^N as $\mathcal{E} = H \circ \mathcal{L}$, where $\mathcal{L} : TE^N \to T^*E^N$ is the Legendre transform $\mathcal{L}(\,x\,;v_1,\dots,v_N) = (\,x\,;p_1,\dots,p_N)$, $p_i = m_i v_i$. It is easy to see that $\mathcal{E}$ is a first integral of the motion.

We will prove the existence of critical global (weak) solutions for the Hamilton-Jacobi equation $H(x, d_x u) = c$. The critical value of this Hamiltonian can be defined as the infimum of the values of $c \in \mathbb{R}$ such that the Hamilton-Jacobi equation admits global subsolutions. Since $\inf_{E^N} U_\kappa(x) = 0$, and constants functions are global subsolutions for $c = 0$, it follows that the critical value is $c = 0$. Therefore, we are interested in global solutions of

$$|\,d_x u\,|^2 = 2\,U_\kappa(x)\,. \qquad \text{(HJ)}$$

We will obtain global solutions as fixed points of a continuous semigroup acting on the set of weak subsolutions, namely the Lax-Oleinik semigroup. There are no new ideas in the method that we apply here. In fact, we will follow the scheme introduced by Fathi in [**10**], with some adaptations to our setting. As we have said in the introduction, the difference is that we consider a space of Hölder functions on which the semigroup acts, and theorem 2 will assure that the method works with this space.

3.1. *The Lax-Oleinik semigroup.* Given a continuous function $u : E^N \to \mathbb{R}$ and $t > 0$, we define $T_t^- u : E^N \to [-\infty, +\infty)$ by

$$T_t^- u(x) = \inf \left\{\, u(y) + \phi(x,y,t) \;\mid\; y \in E^N \,\right\}\,.$$

We also define $T_0^- u = u$ for all function u. The semigroup property follows from the definition . In other words, for any function u we have that $T_t^-(T_s^- u) = T_{t+s}^- u$ for all $t, s \geq 0$. We will restrict the semigroup to the set $\mathcal{H}$ of dominated functions. More precisely, we define

$$\mathcal{H} = \left\{\, u : E^N \to \mathbb{R} \;\mid\; u(x) - u(y) \leq \phi(x,y) \text{ for all } x, y \in E^N \,\right\}\,.$$

Notice that $u : E^N \to \mathbb{R}$ is in $\mathcal{H}$ if and only if $u \leq T_t^- u$ for all $t \geq 0$. On the other hand, $u \leq v$ implies that $T_t^- u \leq T_t^- v$ for all $t \geq 0$. Therefore, the semigroup property implies that $T_t^- u \in \mathcal{H}$ for all $u \in \mathcal{H}$. Also notice that $\mathcal{H}$ is convex, and nonempty since it contains all constant functions.

In that follows, the set $\mathcal{H}$ will be endowed the compact open topology, that is to say, the topology generated by the sets

$$U_K(u, \epsilon) = \{\, v \in \mathcal{H} \;\mid\; |\,v(x) - u(x)\,| < \epsilon \text{ for all } x \in K \,\}\,,$$

with $u \in \mathcal{H}$, $K \subset E^N$ compact, and $\epsilon > 0$.

Proposition 10. *The map $T^- : \mathcal{H} \times [0, +\infty) \to \mathcal{H}$, $(u,t) \mapsto T_t^- u$ is continuous.*

We will use the following lemma.

Lemma 11. *For all $x, y \in E^N$ and $T > 0$ we have $\phi(x,y,T) \geq (m/2T)\|\,x - y\,\|^2$, where $m = \min\{\,m_1, \dots, m_N\,\}$.*

Proof. Let $r, s \in E$ and $\sigma : [0, T] \to E$ an absolutely continuous curve such that $\sigma(0) = r$ and $\sigma(T) = s$. We observe that

$$\| r - s \|_E \leq \int_0^T \| \dot\sigma(t) \|_E \, dt \leq T^{1/2} \left(\int_0^T \| \dot\sigma(t) \|_E^2 \, dt \right)^{1/2} ,$$

hence

$$\| r - s \|_E^2 \leq T \int_0^T \| \dot\sigma(t) \|_E^2 \, dt \, .$$

If $x = (r_1, \ldots, r_N)$ and $y = (s_1, \ldots, s_N)$ are two configurations, then we can choice $i \in \{ 1, \ldots, N \}$ such that $\| r_i - s_i \|_E = \| x - y \|$. Take now $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathcal{C}(x, y, T)$. By the previous observation we have,

$$A(\gamma) \geq (m_i/2) \int_0^T \| \dot\gamma_i(t) \|_E^2 \, dt \geq (m_i/2T) \| r_i - s_i \|_E^2 \geq (m/2T) \| x - y \|^2 ,$$

which proves the lemma since $\phi(x, y, T) = \inf \{ A(\gamma) \mid \gamma \in \mathcal{C}(x, y, T) \}$. □

Proof of proposition 10. As a first step, we show that given $R > 0$ and $t > 0$, there is a constant $k(R, t) > 0$ such that

$$T_t^- u(x) = \inf \{ u(y) + \phi(x, y, t) \mid \| y - x \| \leq k(R, t) \}$$

for all $u \in \mathcal{H}$ and all $x \in E^N$ with $\| x \| \leq R$. To see this, fix $R > 0$, $t > 0$, $u \in \mathcal{H}$ and $x \in E^N$ such that $\| x \| \leq R$. Suppose that $y \in E^N$ is such that $\| y - x \| > 1$ and $u(y) + \phi(x, y, t) \leq u(x) + \phi(x, x, t)$. Then, by lemma 11 and theorem 2 we have

$$\frac{m}{2t} \| y - x \|^2 \leq \eta \| y - x \|^{1-\kappa} + \phi(x, x, t) \, .$$

Therefore, using that $\| y - x \| > 1$ and theorem 1 we deduce

$$m \| y - x \|^2 \leq 2\eta \, t \| y - x \| + 2\alpha R^2 + 2\beta \, t^2 R^{-2\kappa} ,$$

hence that $\| y - x \| \leq k_0(R, t)$ where

$$k_0(R, t) = \eta \, t/m + \left(\eta^2 t^2/m^2 + 2\alpha \, R^2/m + 2\beta \, t^2 R^{-2\kappa}/m \right)^{1/2} .$$

Setting $k(R, t) = \max \{ 1, k_0(R, t) \}$, it follows that $u(y) + \phi(x, y, t) > u(x) + \phi(x, x, t)$ for all $y \in E^N$ such that $\| y - x \| > k(R, t)$, and we conclude that

$$\begin{aligned} T_t^- u(x) &= \inf \left\{ u(y) + \phi(x, y, t) \mid y \in E^N \right\} \\ &= \inf \{ u(y) + \phi(x, y, t) \mid \| y - x \| \leq k(R, t) \} \, . \end{aligned}$$

Let now $u, v \in \mathcal{H}$ and $t > 0$. Let $K \subset E^N$ be a compact subset, and $R > 0$ such that $\| x \| \leq R$ for all $x \in K$. If we set

$$K_t = \bigcup_{x \in K} \left\{ y \in E^N \mid \| y - x \| \leq k(R, t) \right\} ,$$

then for all $x \in K$ we have $T_t^- v(x) = \inf \{ v(y) + \phi(x, y, t) \mid y \in K_t \}$. On the other hand, since $v(y) \leq u(y) + \sup \{ | u(y) - v(y) | \mid y \in K_t \}$ for all $y \in K_t$, we deduce that $T_t^- v(x) \leq \inf\{\, u(y) + \phi(x,y,t) \mid y \in K_t \,\} + \sup\{\, |\, u(y) - v(y)\,| \mid y \in K_t \,\}$. Thus we have proved that $T_t^- v(x) - T_t^- u(x) \leq \sup\{\, |\, u(y) - v(y)\,| \mid y \in K_t \,\}$ for all $x \in K$. Moreover, since $k(R,t)$ is non decreasing in t, given $b \geq 0$ we have that

$$|\, T_t^- v(x) - T_t^- u(x)\,| \leq \sup\{\, |\, u(y) - v(y)\,| \mid y \in K_b \,\}$$

for all $t \leq b$ and all $x \in K$. Since the subset $K_b \subset E^N$ is compact, this implies the continuity of the map T^-. □

3.2. *Proof of theorem 3.* Let $\widehat{\mathcal{H}}$ be the quotient space of $\mathcal{H}$ by the subspace of constants functions. Thus, $\widehat{\mathcal{H}}$ is homeomorphic to $\mathcal{H}_0 = \{\, u \in \mathcal{H} \mid u(0) = 0 \,\}$. By theorem 2 we have that dominated functions are uniformly equicontinuous. It follows that $\mathcal{H}_0$ is compact by Ascoli's theorem. Therefore, $\widehat{\mathcal{H}}$ is a compact, convex, and nonempty subset of $\widehat{C^0}(E^N, \mathbb{R})$, the quotient of the vector space $C^0(E^N, \mathbb{R})$ by the subspace of constant functions. Notice that $\widehat{C^0}(M, \mathbb{R})$ is endowed with the quotient topology of the compact open topology on $C^0(M, \mathbb{R})$. In particular, $\widehat{C^0}(M, \mathbb{R})$ is a locally convex topological vector space.

Since $T_t^-(u + c) = T_t^- u + c$ for all $c \in \mathbb{R}$, it is clear that the semigroup T^- defines canonically a continuous semigroup $\widehat{T}_t^- : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$. If we apply the Schauder-Tykhonov theorem, see [**8**] pages 414–415, we conclude that $\widehat{T}_t^-$ has a fixed point in $\widehat{\mathcal{H}}$. That is to say, there is a function $u \in \mathcal{H}$ such that $T_t^- u = u + c(t)$ for some function $c : [0, +\infty) \to \mathbb{R}$. The semigroup property and the continuity of T^- imply that $c(t) = c(1)t$. Since $u \in \mathcal{H}$, we have that $u \leq T_t^- u$ for all $t \geq 0$, hence we must have $c(1) \geq 0$. We will prove that $c(1) = 0$. Notice that $T_t^- u = u + c(1)t$ implies $u(x) - u(y) \leq \phi(x,y,t) - c(1)t$ for all $x, y \in E^N$. Hence, by theorem 1 we have that

$$u(x) - u(y) \leq \alpha \, \frac{R^2}{t} + \left(\frac{\beta}{R^{2\kappa}} - c(1) \right) t$$

whenever x and y are contained in a ball of E of radius $R > 0$. Since this must be true for R and t arbitrary large, we conclude that $c(1) = 0$. Therefore $T_t^- u = u$ for all $t \geq 0$.

It remains to prove that there are fixed points of T^- which are invariant by the group of symmetries. This can be done as in [**12**] as follows. We define the $\mathcal{H}_{inv}$ as the set of functions in $\mathcal{H}$ which are invariant by symmetries. Thus $\mathcal{H}_{inv}$ is also convex, closed and nonempty since constant functions are invariant. Moreover, $\mathcal{H}_{inv}$ is stable by the Lax-Oleinik semigroup. Therefore, the quotient of this set by the subspace of constants functions is also compact, convex, nonempty and stable by the induced semigroup $\widehat{T}^-$. With the same arguments as above we obtain an invariant fixed point. □

3.3. *Viscosity solutions and subsolutions.* It is well known that the notion of dominated function is related to a notion of subsolution of the Hamilton-Jacobi equation, namely the notion of viscosity subsolution. On the other hand, viscosity solutions (see below) can be detected as fixed points, modulo constants, of the Lax-Oleinik semigroup. An introduction to the subject of viscosity solutions can be found for instance in the books [**1**], [**2**] or [**9**]. However, our setting presents some technical differences, essentially due to the fact that the potential function is infinite in the set of configurations with collisions. The following is a little adaptation of some results in section 5 of [**11**].

Recall that $u : E^N \to \mathbb{R}$ is a *viscosity subsolution* at $x_0 \in E^N$ of (HJ), if for each C^1 function $\psi : E^N \to \mathbb{R}$ such that x_0 is a maximum of $u - \psi$ we have $|\, d_{x_0}\psi \,|^2 \leq 2\, U_\kappa(x_0)$. Given $V \subset E^N$, we say that u is a viscosity subsolution in V if it is viscosity subsolution at each $x \in V$. We remark that any function is trivially a viscosity subsolution in Ω^c, where $\Omega \subset E^N$ denotes the set of configurations without collisions.

Analogously, a function $u : E^N \to \mathbb{R}$ is said to be a *viscosity supersolution* at $x_0 \in E^N$ of (HJ), if for each C^1 function $\psi : E^N \to \mathbb{R}$ such that x_0 is a minimum of $u - \psi$ we have $|\, d_{x_0}\psi \,|^2 \geq 2\, U_\kappa(x_0)$. If $x_0 \in \Omega^c$, then u is a viscosity supersolution at x_0 if and only if there are no C^1 functions ψ such that x_0 is a minimum of $u - \psi$. As for subsolutions, given $V \subset E^N$, we say that u is a viscosity supersolution in V if it is viscosity supersolution at each $x \in V$.

We say that a continuous function $u : E^N \to \mathbb{R}$ is a *viscosity solution* of (HJ) in $V \subset E^N$ if it is both a subsolution and a supersolution in V. It is not difficult to see that a viscosity solution u satisfies (HJ) at each point $x \in V$ where the derivative $d_x u$ exists. We will prove the following.

PROPOSITION 12. *(1) Any $u \in \mathcal{H}$ is almost everywhere differentiable and a viscosity subsolution of Hamilton-Jacobi in E^N. (2) If $u \in \mathcal{H}$ is a fixed point of the Lax-Oleinik semigroup, then u is a viscosity supersolution of Hamilton-Jacobi in $\Omega = \left\{ x \in E^N \mid U_\kappa(x) < +\infty \right\}$.*

Proof. The fact that dominated functions are differentiable almost everywhere follows from proposition 7 and the Rademacher's theorem. In order to prove that they are viscosity subsolutions, take $u \in \mathcal{H}$ and $\psi : E^N \to \mathbb{R}$ of class C^1 such that $u - \psi$ admits a maximum at some $x_0 \in E^N$. Let $v \in E^N$. For all $t > 0$ we have

$$\psi(x_0) - \psi(x_0 - tv) \leq u(x_0) - u(x_0 - tv) \leq \frac{1}{2}\int_{-t}^{0} |\, v \,|^2 ds + \int_{-t}^{0} U_\kappa(x_0 + sv)\, ds\,.$$

Dividing by t and taking the limit for $t \to 0$ we obtain

$$d_{x_0}\psi(v) \leq \frac{1}{2}\,|\, v \,|^2 + U_\kappa(x_0)\,.$$

If we define $p_1, \ldots, p_N \in E$ by the condition $d_{x_0}\psi(v) = \sum_{i=1}^N < p_i, v_i >_E$ for all $v = (v_1, \ldots, v_N) \in E^N$, then we can write

$$|\, d_{x_0}\psi \,|^2 = \sum_{i=1}^{N} m_i^{-1} \|\, p_i \,\|_E^2 = d_{x_0}\psi(w)$$

where $w = (w_1, \ldots, w_N)$ and $m_i w_i = p_i$ for all $i = 1, \ldots, N$. Therefore, using the last inequality with $v = w$ we obtain $|\, d_{x_0}\psi \,|^2 \leq 2\, U_\kappa(x_0)$.

It remains to prove (2). Suppose that $u \in \mathcal{H}$ is such that $T_t^- u = u$ for all $t > 0$. Let $\psi : E^N \to \mathbb{R}$ be a C^1 function such that $u - \psi$ has a minimum at some $x_0 \in \Omega$.

With the same arguments as in the proof of proposition 10, we deduce that there is a constant $k > 0$ such that $T_1^- u(x_0) = \inf \{ u(y) + \phi(x_0, y, 1) \mid \| y - x_0 \| \leq k \}$. Therefore, using theorem 1 and the lower semi-continuity of the Lagrangian action we can choose $y_0 \in E^N$ such that $\| y_0 - x_0 \| \leq k$ and a curve $\gamma \in \mathcal{C}(x_0, y_0, 1)$ such that

$$u(x_0) = T_1^- u(x_0) = u(y_0) + \frac{1}{2} \int_0^1 | \dot\gamma(t) |^2 dt + \int_0^1 U_\kappa(\gamma(t))\, dt \,.$$

In particular, since u is a dominated function we must have

$$u(x_0) - u(\gamma(t)) = \frac{1}{2} \int_0^t | \dot\gamma(s) |^2 ds + \int_0^t U_\kappa(\gamma(s))\, ds$$

for all $t \in [0, 1]$, which says that γ is a calibrated curve for u. We use now the hypothesis that x_0 is a configuration without collisions. Since $x_0 \in \Omega$, there is $\delta > 0$ such that $\gamma([0, \delta]) \subset \Omega$. On the other hand, γ is globally minimizing, hence a solution of the Euler-Lagrange flow in $[0, \delta]$. Therefore γ is differentiable at $t = 0$.

We have that

$$\psi(x_0) - \psi(\gamma(t)) \geq u(x_0) - u(\gamma(t)) = \frac{1}{2} \int_0^t | \dot\gamma(s) |^2 ds + \int_0^t U_\kappa(\gamma(s))\, ds \,.$$

Dividing by t and taking the limit for $t \to 0$ we obtain

$$d_{x_0}\psi(v) \geq \frac{1}{2} | v |^2 + U_\kappa(x_0) \,,$$

where $v = -\dot\gamma(0)$. On the other hand, always we have $2\, p(v) \leq | p |^2 + | v |^2$ for $p \in (E^*)^N$ and $v \in E^N$, which is nothing but the Fenchel's inequality. Thus we conclude that $| d_{x_0}\psi |^2 \geq 2\, U_\kappa(x_0)$. We have proved that u is a viscosity supersolution at x_0. □

3.4. *Lax-Oleinik and weak KAM solutions.* Following the analogy with the weak KAM theory for Tonelli Lagrangians on compact manifolds, we show that the fixed points of the Lax-Oleinik semigroup are the weak KAM solutions defined by Fathi in [**10**]. More precisely, we show that the fixed points of Lax-Oleinik semigroup are characterized by the following property: given any configuration $x \in E^N$, always we have a calibrated curve $\gamma_x : (-\infty, 0] \to E^N$ such that $\gamma_x(0) = x$.

Recall that if $u : E^N \to \mathbb{R}$ is a dominated function, then a curve $\gamma : I \to E^N$ is said to be calibrated when satisfies

$$u(\gamma(b)) - u(\gamma(a)) = A(\gamma|_{[a,b]})$$

for all compact interval $[a, b] \subset I$. In particular, the calibrated curves of a dominated function are free time minimizers, meaning that

$$A(\gamma|_{[a,b]}) = \phi(\gamma(a), \gamma(b))$$

for all compact interval $[a, b] \subset I$.

Therefore, the fixed points of the Lax-Oleinik semigroup can be characterized in terms of calibrated curves as follows (recall that our Lagrangian is symmetric).

PROPOSITION 13. *Let $u \in \mathcal{H}$ be a dominated function. Then $u = T_t^- u$ for all $t > 0$ if and only if, for each $x \in E^N$ there is a curve $\gamma_x : [0, +\infty) \to E^N$ with $\gamma_x(0) = x$ and such that $u(x) = u(\gamma_x(t)) + A(\gamma|_{[0,t]})$ for all $t > 0$.*

Proof. Suppose first that the condition is satisfied. Take $x \in E^N$ and the corresponding calibrated curve $\gamma_x : [0, +\infty) \to E^N$ with $\gamma_x(0) = x$. Since $u \in \mathcal{H}$ already we known that $u \leq T_t^- u$ for all $t > 0$. On the other hand, if we fix $t > 0$ we have $T_t^- u(x) \leq u(\gamma_x(t)) + A(\gamma|_{[0,t]}) = u(x)$. Therefore u is a fixed point.

Suppose now that $u \in \mathcal{H}$ is a fixed point of (T_t^-). Given a configuration $x \in E^N$ and $t > 0$ we have

$$u(x) = T_t^- u(x) = \inf \left\{ u(y) + \phi(x, y, t) \mid y \in E^N \right\} .$$

Using lemma 11 and theorem 2 (as in the proof of proposition 10) we deduce that there is a constant $k > 0$ (depending on x and t) such that

$$u(x) = T_t^- u(x) = \inf \left\{ u(y) + \phi(x, y, t) \mid y \in E^N \text{ and } \| y - x \| \leq k \right\} .$$

Therefore, using theorem 1 and the lower semi-continuity of the Lagrangian action we can choose $y(x, t) \in E^N$ such that $\| y(x, t) - x \| \leq k$ and a curve $\gamma_{x,t} \in \mathcal{C}(x, y(x, t), t)$ such that

$$u(x) = T_t^- u(x) = u(y(x, t)) + A(\gamma_{x,t}).$$

For each positive integer $n > 0$ we define the curve $\gamma_n : [0, n] \to E^N$ as the curve $\gamma_{x,n}$. Observe that if $m > n$ then $\gamma_m |_{[0,n]}$ minimizes the action in $\mathcal{C}(x, \gamma_m(n), n)$. Now we apply theorem 1 and once again lemma 11 and we deduce that for a fixed positive integer $n > 0$, the sequence $(A(\gamma_m |_{[0,n]}))_{m>n}$ is bounded. It is not difficult to see (using the Cauchy-Schwartz inequality) that an absolutely continuous curve $\gamma : I \to E^N$ with finite Lagrangian action must satisfies $| \gamma(t) - \gamma(s) | \leq 2A(\gamma) \, | t - s |^{1/2}$ for all $t, s \in I$. Then we can apply Ascoli's theorem and deduce the existence of a convergent subsequence of $(\gamma_m |_{[0,n]})_{m>n}$. By a diagonal process we can extract an increasing sequence of indexes $m_k \in \mathbb{N}$ such that, for each positive integer $n > 0$, the sequence $(\gamma_{m_k} |_{[0,n]})_{m_k>n}$ converges uniformly, when $k \to \infty$. Observe now that by construction, each curve $\gamma_{m_k} |_{[0,n]})_{m_k>n}$ calibrates the function u, hence by continuity the curve $\gamma_x : [0, +\infty) \to E^N$ defined by $\gamma_x(t) = \lim_{k\to\infty} \gamma_{m_k}(t)$ is also calibrated. □

We remark that for the Newtonian potential ($\kappa = 1/2$), Marchal's theorem implies – except of course in the collinear case ($\dim E = 1$) – that the calibrated curves of weak KAM solutions are true motions for $t > 0$ since they must be contained in Ω, the set of configurations without collisions. The dynamics of the free time minimizers of the Newtonian N-body problem is described in [**7**].

4. *The Kepler problem*

Unfortunately, the proof of the weak KAM theorem do not give any explicit solution. Nevertheless, we can give explicit solutions for the Kepler problem. We will find here translation invariant solutions.

Suppose first that we have two bodies of mass 1 in a line ($N = 2$, $k = 1$). Then an invariant solution $u : \mathbb{R}^2 \to \mathbb{R}$ must be of the form $u(x, y) = f(|\, x - y \,|)$, and the Hamilton-Jacobi equation reads

$$u_x^2 + u_y^2 = 2 \,|\, x - y \,|^{\,-1} \,.$$

Replacing $u(x, y)$ by $f(|\, x - y \,|)$ and solving the differential equation in f we conclude that the unique global solutions (up to an additive constant) are the functions

$$u_\pm(x, y) = \pm\, 2|\, x - y \,|^{\,1/2} \,.$$

In fact, the positive solution is the unique fixed point of the *forward* Lax-Oleinik semi-group

$$T_t^+ u(x) = \inf \left\{\, u(y) - \phi(x, y, t) \mid y \in E^N \,\right\}$$

and therefore, the negative one is the unique fixed point of the backward semigroup T_t^-. Of course, since the lagrangian is symmetric, we have that $u \in \mathcal{H}$ is a backward solution if and only if $-u$ is a forward solution.

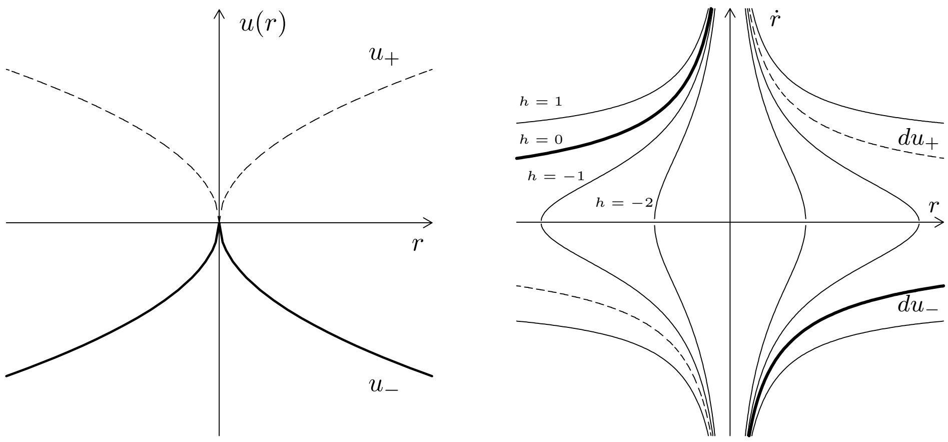

FIGURE 3. The two solutions and their derivatives for the 1-dimensional Kepler problem.

For the planar Kepler problem there are many solutions. It is convenient to reduce first the problem by fixing the center of mass at the origin, or equivalently, to look for translation invariant solutions. Since the configuration is then determined by the position of the first body $x \in \mathbb{R}^2$, the problem reduces as usual to the center fix problem. If we denote $x = (x_1, x_2)$ the position of the body, then the Hamilton-Jacobi equation reads

$$u_{x_1}^2(x) + u_{x_2}^2(x) = 2\|\, x \,\|^{\,-1} \,.$$

Doubtlessly, the most simplest solution that we can give is the rotation invariant solution

$$u(x_1, x_2) = -(x_1^2 + x_2^2)^{1/4} \,.$$

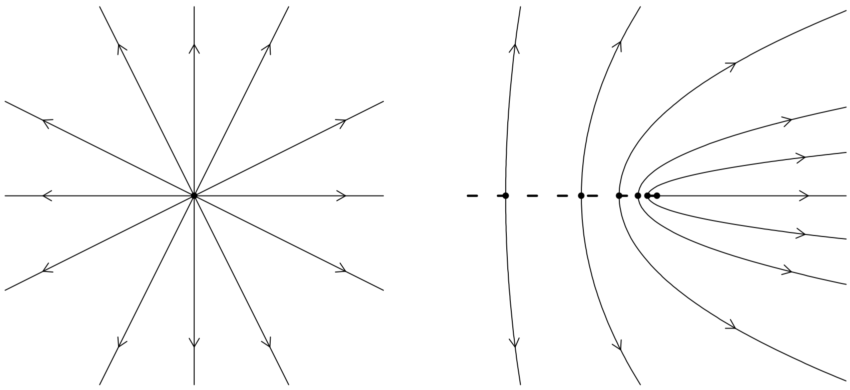

FIGURE 4. Calibrated curves of solutions for the planar Kepler problem.

His calibrated curves are all the parabolic homothetic motions, represented in the left side of figure 4. The half parabolas at the right side are the calibrated curves of a Buseman type solution, which is constant and not differentiable over the dashed line. A computation made by A. Venturelli shows that this last solution can be explicitly defined by the formula

$$u(x_1, x_2) = -((x_1^2 + x_2^2)^{1/2} + x_1)^{1/2}.$$

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