Regular interval Cantor sets of S^1 and minimality

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Abstract

[1] It is known that not every Cantor set of S^1 is C^1-minimal. In this work we prove that every member of a subfamily of the called regular interval Cantor set is not C^1-minimal. We also prove in general, for a even large class of Cantor sets, that any member of such family can be $C^{1+\epsilon}$-minimal, for any $\epsilon > 0$.

1 Introduction

If $f : S^1 \to S^1$ is a diffeomorphism without periodic points, there exists a unique set $\Omega(f) \subset S^1$ minimal for f (we say that $\Omega(f)$ is C^1-minimal for f). In this case $\Omega(f)$ is a Cantor set or it is S^1. Up to now, the C^1-minimal Cantor sets that are known are the Danjoy examples and its conjugates. However we know that some families are not C^1-minimal. For example, in [2] Mc Duff demonstrates that the usual middle thirds Cantor set is not C^1-minimal and gives some conditions for a Cantor set that imply that it is not C^1-minimal. In [6] we can find other conditions that imply the no C^1-minimality too. In [5] A. Norton demonstrates that the family of the affine Cantor sets is not C^1-minimal too. In this work we construct new families of Cantor sets that are not C^1-minimal and other families of Cantor sets that are not $C^{1+\epsilon}$-minimal (for any $\epsilon > 0$).

1.1 Regular interval Cantor sets

The regular interval Cantor set construction imitates the procedure utilized to obtain the usual middle thirds Cantor set. Given two sequences $\{m_i\}$ and $\{\theta_i\}$ with m_i a positive integer and $0 < \theta_i < 1$, we proceed as follows. In the first step we

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remove m_1 open intervals with the same measure from the circle, distributed in the same way, obtaining the closed set $K_1 = \cup\Delta_{i_1}$ $(i_1 = 1, ..., m_1)$ with Lebesgue measure $|K_1| = \theta_1$, where Δ_{i_1} are the connected components of K_1. In the second step, we remove m_2 open intervals of the same measure from each connected component Δ_{i_1}, distributed in the same way, obtaining the closed set $K_2 = \cup\Delta_{i_1 i_2}$ $(i_2 = 1, ..., m_2 + 1)$ with measure $|K_2| = \theta_2|K_1|$, where $\Delta_{i_1 i_2}$ are the connected components of K_2. Proceeding inductively, we obtain, for each n, a closed set $K_n \subset S^1$, contained in K_{n-1}, with measure $|K_n| = \theta_n|K_{n-1}|$, and $K_n = \cup\Delta_{i_1 \dots i_n}$ $(i_n = 1, ..., m_n + 1\ )$, where $\Delta_{i_1 \dots i_n}$ are connected components of K_n. We define $K = \bigcap K_n$. This set is a Cantor set, and we will call **regular interval Cantor set** to every set K constructed in this way.

1.2 Quasi regular interval Cantor sets

Now we are going to give the construction of a family of Cantor sets that contains the regular interval Cantor sets. Given a sequence $\{n_i\}$ of positive integers with $\sum_{i<j} n_i \leq n_j$, we proceed as follows. In the first step we remove n_1 open intervals of the same measure from S^1, obtaining a closed set $K_1 = \bigcup \Delta_{1i_1}$ $(i_1 = 1, ..., n_1)$, where Δ_{1i_1} are the connected components of K_1. In the second step, we remove n_2 open intervals of the same measure form K_1, removing at least an interval of each connected component of K_1, obtaining the closed set $K_2 = \bigcup \Delta_{2i_2}$ $(i_2 = 1, ..., n_1 + n_2)$, where Δ_{2i_2} are the connected components of K_2. We do not require the intervals removed to be likewise distributed. Proceeding inductively, for each m we obtain a closed set $K_m \subset S^1$ contained in K_{m-1} and we write $K_m = \bigcup \Delta_{mi_m}$ $(i_m = 1, ..., n_1 + ... + n_m)$ where Δ_{mi_m} are the connected components of K_m. Then, we define $K = \bigcap K_m$. The set K is a Cantor set if, and only if, $\nu_m = max\{|\Delta_{mi_m}| : i_m = 1, ..., n_1 + ... + n_m\} \to 0$ when $m \to \infty$. We will call **quasi regular interval Cantor set** to every Cantor set K constructed in this way. Note that with this procedure we do not obtain all Cantor sets of S^1. If $\mu_m = min\{|\Delta_{mi_m}| : i_m = 1, ..., n_1 + ... + n_m\}$, the number $\delta = inf\{\mu_m/\nu_m : m \in \mathbf{N}\}$ gives an idea of the irregularity of the Cantor set K. This number depends on the set K and the procedure to obtain K. Then, we define the regularity of K as the supreme of the set of δ, taking all the possible procedures to obtain K. Note that if the Cantor set K is a regular interval Cantor set, its regularity is 1.

2 Main results

Theorem 1. *If the Cantor set K is C^1-minimal for a diffeomorphism f, and K^c has only one orbit of wandering intervals, then K is not a quasi regular interval*

Cantor set.

Theorem 2. *If K is a quasi regular interval Cantor set of regularity different from 0, then K is not $C^{1+\epsilon}$-minimal for any $\epsilon > 0$.*

As all regular interval Cantor sets have regularity 1 then, from the previous theorem, we have the following result.

Corollary 1. *If K is a regular interval Cantor set, then K is not $C^{1+\epsilon}$-minimal for any $\epsilon > 0$.*

If the regular interval Cantor set K has positive measure and we suppose that it is C^1-minimal for f we obtain several conditions for f'. Let m_i be the quantity of intervals removed in the step i of the construction of K. In this case, we have the following result.

Theorem 3. *If K is a regular interval Cantor set of positive measure and the sequence $\{m_i\}$ is not limited, then K is not C^1-minimal.*

Definition 2.1. *If K is a regular interval Cantor set, for each prime integer we define $A_q = \{i \in \mathbf{N} : m_i + 1 = 0 \ (mod\ q)\}$.*

For the case that A_q is an infinite set we denote its elements by t_n ($n \in \mathbf{N}$), with $t_n < t_{n+1}$. Now we can enunciate de following result.

Theorem 4. *If K is a regular interval Cantor set of positive measure and there exists a prime integer q such that A_q is infinite and $t_{n+1} - t_n \to \infty$, then K is not C^1-minimal.*

3 Generalities

The following lemmas are going to be very useful in the demonstrations of the main results.

Definition 3.1. *If $f : S^1 \to S^1$ is a diffeomorphism, then for each $x \in S^1$ and for each positive integer n we define $F(x, n) = \sum_{i=0}^{n-1} log f'(f^i(x)) = lof(f^n)'(x)$.*

Lemma 3.1. *If the Cantor set K is C^1-minimal for f, then there exists $x \in K$ such that $F(x, n) \geq 0$, for all positive integer n.*

Proof. We suppose by contradiction that for all $x \in K$ there exists m_x such that $F(x, m_x) < 0$. By the continuity of f', for each $x \in K$ there exists $\delta_x > 0$ such that for every point y in the interval $(x - \delta_x, x + \delta_x)$, $F(y, m_x) < 0$. As the family of intervals $(x - \delta_x, x + \delta_x)$ with $x \in K$ is a covering of K, and K is a Cantor set, then there exists a finite refinement $\{I_i,\ i = 1, ..., p\}$ of this covering of open intervals, disjoint two to two, that is a covering of K. So, for each I_i there exists $m_i \in \mathbf{N}$ such that for all $y \in I_i$ we have $F(y, m_i) < 0$. Besides, $S^1 \setminus \bigcup_{i=1}^{p} I_i$ is a finite union of closed intervals, each of which is contained in a connected component of K^c that we call J_i, with $i = 1, ..., p$. We consider $m = \max\{m_i : i = 1, ..., p\}$ and $M \geq 1$ the maximum of f'. We consider a wandering interval T of the past of J_1 such that $|T|M^m < \min\{|J_1|, ..., |J_p|\}$. Now we will demonstrate that if j is a positive integer then $|f^j(T)| < |J_1|$, and this is a contradiction. By the choice of T, we know that T is contained in I_i for some i. By the Mean Value Theorem, there exists $\theta \in I_i$ such that

$$|f^{m_i}(T)| = |T|(f^{m_i})'(\theta).$$

As $F(\theta, m_i) < 0$, we have $(f^{m_i})'(\theta) < 1$ and so

$$|f^{m_i}(T)| < |T|.$$

We can repeat this process with $f^{m_i}(T)$ instead of T. Proceeding inductively we conclude that there exists a sequence $\nu_1, \nu_2, ..., \nu_k, ...$ with $\nu_k \in \{m_1, ..., m_p\}$ such that for all positive integer r

$$|f^{\sum_{k=1}^{r} \nu_k}(T)| < |T|.$$

As for all j there exists $r_0 \geq 0$ such that $\sum_{k=1}^{r_0} \nu_k \leq j < \sum_{k=1}^{r_0+1} \nu_k$,we have

$$|f^j(T)| = |f^{j-\sum_{k=1}^{r_0} \nu_k}(f^{\sum_{k=1}^{r_0} \nu_k}(T))| \leq M^m|T| < |J_1|.$$

□

Let K be a Cantor set of the circle and let $K^c = \bigcup I_j$, where I_j are the connected components of K^c . We define the spectrum of K (E_K) as the orderly set $\{\lambda_i\}$ ($\lambda_{i+1} < \lambda_i$), with λ_i the length of I_j, for some j.

Lemma 3.2. *If the Cantor set K is C^1-minimal for f and $\lambda_n/\lambda_{n+1} \not\to 1$, there exists $\eta > 0$ and $x \in K$ such that $F(x, m) \leq -\eta$, for all positive integer m.*

Proof. As $\lambda_n/\lambda_{n+1} \not\to 1$, there exist $\epsilon_0 > 0$ and a sequence $\{n_k\}$ such that $1 + \epsilon_0 \leq \frac{\lambda_{n_k}}{\lambda_{n_k+1}}$. Let I_{n_k} be a connected component of K^c such that $|I_{n_k}| \geq \lambda_{n_k}$ and for all $j > 1$, $|f^j(I_{n_k})| \leq \lambda_{n_k+1}$. By the choice of I_{n_k} we have that $|I_{n_k}| \to 0$ when $k \to \infty$. Let x be a point of accumulation of the set of the intervals I_{n_k} ($x \in K$) and $\{k_i\}$ a sequence such that $d(x, I_{n_{k_i}}) \to 0$ when $i \to \infty$. Therefore, for every $m \geq 1$, there exists i sufficiently large such that

$$1 + \epsilon_0 \leq \frac{\lambda_{n_{k_i}}}{\lambda_{n_{k_i}+1}} \leq \frac{|I_{n_{k_i}}|}{|f^m(I_{n_{k_i}})|}.$$

Then

$$F(x,m) = \log(f^m)'(x) = \log\left(\lim_{i\to\infty} \frac{|f^m(I_{n_{k_i}})|}{|(I_{n_{k_i}})|}\right) \leq -\log(1+\epsilon_0).$$

□

Lemma 3.3. *If the Cantor set K is C^1-minimal for f and $\lambda_n/\lambda_{n+1} \not\to 1$ then for every point $x \in K$, $F(x,m)$ is not limited.*

Proof. By the transitivity of K (for f), it is enough to demonstrate the property for any point of K. Let x and the number η be as in lemma 3.2 and suppose by contradiction that $F(x,m)$ is limited. Therefore if $y = \inf\{F(x,m) : m \in \mathbf{N}\}$, there exists a positive integer p such that $|F(x,p) - y| < \eta/2$. So

$$F(f^p(x),m) = F(x,m+p) - F(x,p) = F(x,m+p) - y - (F(x,p) - y) > \frac{-\eta}{2} \quad (1)$$

for all positive integer m. We consider $\{n_k\}$ such that $f^{p+n_k}(x)$ has limit x when $k \to \infty$. From the uniform continuity of f' we have that

$$|F(f^p(x),p+n_k) - F(x,p+n_k)| \leq \sum_{i=0}^{p-1} |\log f'(f^{p+n_k+i}(x)) - \log f'(f^i(x))| = \delta(n_k) \to 0$$

when $k \to \infty$. Then

$$F(f^p(x),p+n_k) < F(x,p+n_k) + \delta(n_k) < -\eta + \delta(n_k),$$

so utilizing (1) we have a contradiction. □

4 Geometric rigidity

In this section we are going to prove two geometric properties for the quasi regular interval Cantor sets and that if, we suppose that a Cantor set K of this family is C^1-minimal for f, we obtain rigid conditions for f'.

Lemma 4.1. *If K is a quasi regular interval Cantor set, $\mu_n < \frac{2\pi}{2^{n-1}}$, for all integer $n > 1$.*

Proof. We are going to prove that if $\mu_n < \frac{2\pi}{2^{n-1}}$, $\mu_{n+1} < \frac{2\pi}{2^n}$. Proved this, as $\mu_1 < 2\pi$ we have demonstrated the lemma. From the construction of K we know that there exist integers j_1, j_2 and j_3 such that $\Delta_{nj_1} < \frac{2\pi}{2^{n-1}}$ and such that Δ_{n+1,j_2} and Δ_{n+1,j_3} are contained in Δ_{nj_1}. Therefore

$$min\{|\Delta_{n+1,j_2}|, |\Delta_{n+1,j_3}|\} \leq \frac{|\Delta_{n,j_1}|}{2} < \frac{2\pi}{2^n},$$

and from here follows the thesis. □

Lemma 4.2. *If K is a quasi regular interval Cantor set, $\lambda_n/\lambda_{n+1} \not\to 1$, when $n \to \infty$.*

Proof. Let $\{l_i\}$ be the sequence where l_i is the length of the open intervals removed in the step i of the construction of K. From the construction of K we have that the open intervals removed in the step n are contained in K_{n-1} , so from the previous lemma we have that $l_n < 2\pi/2^{n-2}$ for $n > 2$. Then, for $n > 2$ we have

$$\#\left(\{\log \lambda_i\} \cap [-(n-2)\log 2 + \log 2\pi, 0]\right) < n. \tag{2}$$

Suppose by contradiction that $\lambda_n/\lambda_{n+1} \to 1$. Then for all $\epsilon > 0$ there exists $n_0 > 0$ such that for all $n \in \mathbf{N}$

$$0 < \log \lambda_{n_0+n-i} - \log \lambda_{n_0+n+1-i} < \log(1+\varepsilon)$$

with $i = 0, ..., n$, so

$$0 > \log \lambda_{n_0+n} > \log \lambda_{n_0} - n\log(1+\varepsilon).$$

Then

$$\#(\{\log \lambda_i\} \cap [\log \lambda_{n_0} - n\log(1+\varepsilon), 0]) \geq n_0 + n. \tag{3}$$

Utilizing the inequalities (2) e (3) we have

$$\#(\{\log \lambda_i\} \cap [-(n-2)\log 2 + \log 2\pi, 0]) < n < n_0 + n \leq \#(\{\log \lambda_i\} \cap [\log \lambda_{n_0} - n\log(1+\epsilon), 0]).$$

Therefore

$$-(n-2)\log 2 + \log 2\pi \geq \log \lambda_{n_0} - n\log(1+\epsilon).$$

As this inequality is true for all $n \in \mathbf{N}$ and for all $\epsilon > 0$, taking ϵ such that $\log(1+\epsilon) < \log 2$ we have a contradiction. □

Lemma 4.3. *If a quasi regular interval Cantor set K is C^1-minimal for f, there exists $x \in K$ such that $f'(x) > 1$.*

Proof. From the previous lemma, we know that there exists $\epsilon_0 > 0$ and a crescent sequence of positive integers $\{n_j\}$ such that $\lambda_{n_j}/\lambda_{n_j+1} > 1+\epsilon_0$, for all n_j. Let I be a connected component of K^c. Then, the family $\{f^{-n}(I)\}$ with $i \in \mathbf{N}$ is a family of open intervals, disjoint two to two, so $|f^{-n}(I)| \to 0$ when $n \to \infty$. Therefore, if j is sufficiently large there exists $p(j) \in \mathbf{N}$ such that $|f^{-p(j)}(I)| \leq \lambda_{n_j+1}$ and $|f^{-p(j)+1}(I)| \geq \lambda_{n_j}$. Then, we have

$$\frac{|f^{-p(j)+1}(I)|}{|f^{-p(j)}(I)|} \geq \frac{\lambda_{n_j}}{\lambda_{n_j+1}} > 1 + \epsilon_0. \tag{4}$$

Utilizing the Mean Value Theorem, we know that there exists a point $\theta_{p(j)} \in f^{-p(j)}(I)$ such that

$$|f^{-p(j)+1}(I)| = f'(\theta_{p(j)})|f^{-p(j)}(I)|$$

so

$$\frac{|f^{-p(j)+1}(I)|}{|f^{-p(j)}(I)|} = f'(\theta_{p(j)}). \tag{5}$$

From (4) and (5) we have

$$f'(\theta_p) > 1 + \varepsilon_0. \tag{6}$$

If x is an accumulation point of the set $\{f^{-p(j)}(I)\}$, it is an accumulation point of the set $\{\theta_{p(j)}\}$ too and, as $f \in C^1$, we have that $f'(\theta_p) \to f'(x)$ when $j \to \infty$, so from (6) we obtain that $f'(x) > 1$. □

If K is a quasi regular interval Cantor set and $y \in K$ we denote by K_n^y the connected component of K_n that contains y. The following observations will be of use for the demonstrations of the next lemmas.

1. If K is a quasi regular interval Cantor set, C^1-minimal for f, for all $\epsilon > 0$ there exists a positive integer $n(\epsilon)$ such that if $n > n(\epsilon)$ and x_1,x_2 belong to the same connected component of K_n,

$$\frac{1}{1+\varepsilon} < \frac{f'(x_1)}{f'(x_2)} < 1 + \varepsilon.$$

2. For all positive integer n and all point $x \in K$ there exists a positive number υ such that if λ is an element of the spectrum of K, smaller than υ, there exists a connected component of K^c, of length λ, contained in $K_n^{f(x)}$ such that its preimage is contained in K_n^x.

Lemma 4.4. *If the quasi regular interval Cantor set K is C^1-minimal for f and x is any point in K, then for all $\epsilon > 0$ and for all integer m if I is a connected component of K^c of length so small as necessary, there exists a connected component I^* of K^c such that*

$$\frac{(f'(x))^m}{1+\varepsilon} < \frac{|I^*|}{|I|} < (f'(x))^m(1+\varepsilon).$$

Proof. First we suppose that $m \geq 0$. We consider $\epsilon_1 > 0$ sufficiently small and $n = n(\epsilon_1)$ as in observation 1. Let K_n be as in the construction of K. If I is a connected component of K^c of length sufficiently small, there exists I_1, connected component of K^c too, contained in K_n^x such that its length is $|I|$. From the Mean Value Theorem we have that there exists $\theta \in I_1$ such that

$$|f(I_1)| = f'(\theta)|I_1| = f'(\theta)|I|.$$

As $\theta \in K_n^x$, utilizing observation 1 we have

$$\frac{f'(x)}{1+\epsilon_1} < \frac{|f(I_1)|}{|I|} < f'(x)(1+\epsilon_1).$$

If I is sufficiently small we can repeat this procedure with $f(I_1)$ instead of I. Then there exists I_2, connected component of K^c, such that

$$\frac{f'(x)}{1+\epsilon_1} < \frac{|f(I_2)|}{|f(I_1)|} < f'(x)(1+\epsilon_1).$$

Proceeding inductively we conclude that there exist $I_3, ..., I_m$, connected components of K^c, such that

$$\frac{f'(x)}{1+\epsilon_1} < \frac{|f(I_{i+1})|}{|f(I_i)|} < f'(x)(1+\epsilon_1),$$

with $i = 1, ..., m-1$. So

$$\frac{(f'(x))^m}{(1+\epsilon_1)^m} < \frac{|f(I_m)|}{|I|} < (f'(x))^m(1+\epsilon_1)^m. \tag{7}$$

Given $\epsilon > 0$ we choose $\epsilon_1 > 0$ such that $(1+\epsilon_1)^m < 1+\epsilon$. Then, from (7) follows the thesis. In the case $m < 0$ we proceed as follows. If I is a connected component of K^c, sufficiently small, there exists I_1, connected component of K^c too, of length $|I|$, contained in $K_n^{f(x)}$ such that $f^{-1}(I_1)$ is contained in K_n^x. Therefore, there exists $\theta \in I_1$ such that

$$|f^{-1}(I_1)| = (f^{-1})'(\theta)|I_1| = \frac{|I_1|}{f'(f^{-1}(\theta))}.$$

As $f^{-1}(\theta) \in K_n^x$, from observation 1 we have

$$\frac{1}{(1+\epsilon_1)f'(x)} < \frac{|f^{-1}(I_1)|}{|I_1|} = \frac{1}{f'(f^{-1}(\theta))} < \frac{1+\epsilon_1}{f'(x)}.$$

So, proceeding as in the first case we obtain the desired result. □

Lemma 4.5. *If the quasi regular interval Cantor set K is C^1-minimal for f, f' restricted to K is constant by parts. Even more, if the set of values of f' restricted to K is $\{a_1, ..., a_n\}$, then $log a_i / log a_j \in \mathbb{Q}$ ($a_j \neq 1$).*

Proof. Let ϵ_0 and $\{n_j\}$ be as in the proof of lemma 4.3. We need to prove that $A = \{f'(x) : x \in K\}$ is a finite set. We suppose by contradiction that A is a infinite set. As f' is continuous in S^1, the set A has point of accumulation. From here we conclude that there exist $a, b \in K$, $a \neq b$, such that

$$\frac{1}{1+\epsilon_0} < \frac{f'(a)}{f'(b)} < 1. \tag{8}$$

Let ϵ_1 be a positive number such that

$$1+\epsilon_1 < \min\left\{\sqrt{\frac{f'(b)}{f'(a)}}, \sqrt{(1+\epsilon_0)\frac{f'(a)}{f'(b)}}\right\}.$$

From observation 1 we have that there exists $n(\varepsilon_1)$ such that if x_1 and x_2 are in the same connected component of $K_{n(\epsilon_1)}$,

$$\frac{1}{1+\epsilon_1} < \frac{f'(x_1)}{f'(x_2)} < 1+\epsilon_1. \tag{9}$$

Let I_1 be a connected component of K^c contained in the connected component of $K_{n(\epsilon_1)}$ that contains the point a. From the construction of K we have that $K^c_{n(\epsilon_1)}$ only contains a finite quantity of connected components of K^c. By the Mean Value Theorem, there exists $\theta_1 \in I_1$ such that

$$|f(I_1)| = |I_1| f'(\theta_1).$$

Utilizing 9, and that θ_1 and a are in the same connected component of $K_{n(\epsilon_1)}$, we have

$$\frac{|I_1| f'(a)}{1+\epsilon_1} < |f(I_1)| < |I_1|(1+\epsilon_1) f'(a). \tag{10}$$

If $|I_1|$ is sufficiently small there exists I_2, connected component of $S^1 \setminus K$, of length $|f(I_1)|$, such that $f^{-1}(I_2)$ is in the connected component of $K_{n(\varepsilon_1)}$ that contains b (observation 2). Utilizing the Mean Value Theorem there exists $\theta_2 \in I_2$ such that

$$|f^{-1}(I_2)| = |I_2|(f^{-1})'(\theta_2) = \frac{|I_2|}{f'(f^{-1}(\theta_2))}.$$

From the choice of I_2 we have that $f^{-1}(\theta_2)$ and b are in the same connected component of $K_{n(\varepsilon_1)}$; so applying (9) we obtain

$$\frac{|f(I_1)|}{f'(b)}\frac{1}{1+\epsilon_1} \leq |f^{-1}(I_2)| \leq \frac{|f(I_1)|}{f'(b)}(1+\epsilon_1).$$

From this last inequality and (10) we have

$$\frac{|I_1|}{(1+\epsilon_1)^2}\frac{f'(a)}{f'(b)} \leq |f^{-1}(I_2)| \leq |I_1|(1+\epsilon_1)^2\frac{f'(a)}{f'(b)},$$

and therefore, by the choice of ϵ_1 we have

$$1 < \frac{|I_1|}{|f^{-1}(I_2)|} < 1+\epsilon_0.$$

Summarizing, we have proved that if I is a connected component of $S^1 \setminus K$ with length sufficiently small, there exists another connected component I^* of K^c such that

$$1 < |I|/|I^*| < 1+\epsilon_0.$$

Taking I, of length λ_{n_j}, sufficiently small we have

$$1+\epsilon_0 > \frac{|I|}{|I^*|} \geq \frac{\lambda_{n_j}}{\lambda_{n_j+1}} > 1+\epsilon_0$$

and this is a contradiction. Then, A is a finite set.
Now, we suppose by contradiction that there exist i and j such that $\log a_i/\log a_j \notin \mathbb{Q}$. We are going to prove (as in the previous case) that if I is a connected component of K^c of length sufficiently small, there exists another connected component I^* of K^c such that

$$1 < |I|/|I^*| < 1+\varepsilon_0$$

and we have a contradiction again. As $\log a_i/\log a_j \notin \mathbb{Q}$ then for all $\epsilon_1 > 0$ there exist integers m and n such that

$$-\epsilon_1 < m\log a_i - n\log a_j < 0,$$

so there exist $x, y \in K$ such that

$$e^{-\epsilon_1} < (f'(x))^m (f'(y))^{-n} < 1. \tag{11}$$

From lemma 4.4 we have that given $\epsilon_2 > 0$ and I, connected component of K^c, sufficiently small, there exist I^* and I^{**} such that

$$\frac{(f'(x))^m}{1+\epsilon_2} < \frac{|I^{**}|}{|I|} < (f'(x))^m (1+\epsilon_2) \tag{12}$$

and

$$\frac{(f'(x))^{-n}}{1+\epsilon_2} < \frac{|I^{*}|}{|I^{**}|} < (f'(x))^{-n} (1+\epsilon_2) \tag{13}$$

Utilizing 11, 12 and 13 we have

$$\frac{(f'(x))^{-m}(f'(y))^n}{(1+\epsilon_2)^2} < \frac{|I|}{|I^*|} < \frac{(1+\epsilon_2)^2}{e^{-\epsilon_1}}. \tag{14}$$

We take ϵ_2 such that

$$\frac{(f'(x))^{-m}(f'(y))^n}{(1+\epsilon_2)^2} > 1,$$

and ϵ_1 such that

$$\frac{(1+\epsilon_2)^2}{e^{-\epsilon_1}} < 1 + \epsilon_0.$$

So, from 14 we have proved what we want. □

5 Proof of the theorem 1

For the proof of theorem 1 we need the following two lemmas.

Lemma 5.1. *If $x \in S^1$ and $R_\theta : S^1 \to S^1$ is the rotation of angle θ (irrational in π), for all positive integer m there exists $n > m$ such that the set $A_n = \{R^i_\theta(x) : i = 0, ..., n\}$ determines a division of S^1 in intervals with two possible lengths.*

Proof. We are going to construct a sequence $n_1 < n_2 < ... < n_k < ...$ such that A_{n_k} has the desired properties for all k. We can take $n_1 = 1$. We suppose that n_k is already known. We denote $x_j = R^j_\theta(x)$. Let $T_1, ..., T_p$ (with the same length) and $J_1, ..., J_q$ (with the same length) be the open intervals that determine the partition A_{n_k} in S^1. We can always order the intervals so that $f(T_i) = T_{i+1}$ and $f(J_j) = J_{j+1}$. Now we consider the point x_{n_k+1}. If we assume $|T_i| < |J_j|$, the point x_{n_k+1} belongs to J_1. Even more, this point and the extreme of J_1, different from x, determine an interval of length $|T_1|$. This shows that, in general, the point x_{n_k+j} belongs to J_j ($j = 1, ..., q$), determining, with one of the extremes of J_k, an interval of length $|T_1|$. Therefore, we can take $n_{k+1} = n_k + q$, so that $A_{n_{k+1}}$ has the desire properties. □

Lemma 5.2. *If $f : S^1 \to S^1$ is a continuous function and R_θ is the rotation of irrational angle θ, for all point $x \in S^1$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{0\le i\le n} f(R_\theta^i(x)) = \int_{S^1} f\, dx.$$

Proof. By Birkhoff theorem (see [4]) the affirmation is true for almost every point (with regard to Lebesgue measure in S^1). Therefore, by the uniform continuity of f, for all $x \in S^1$ and $\varepsilon > 0$ there exists y such that

1. $\lim_{n\to\infty} \frac{1}{n} \sum_{0\le i\le n} f(R_\theta^i(y)) = \int_{S^1} f\, dx.$

2. $|f(R_\theta^i(x)) - f(R_\theta^i(y))| < \epsilon.$

Adding, we obtain

$$\left| \frac{1}{n} \sum_{0\le i\le n} f(R_\theta^i(x)) - \frac{1}{n} \sum_{0\le i\le n} f(R_\theta^i(y)) \right| < \epsilon,$$

so the affirmation follows. □

To continue we give the proof of theorem 1.

Proof. We suppose by contradiction, that there exists a quasi regular interval Cantor set K, C^1-minimal for f, and that K^c has only one orbit of wandering intervals. Let $h : S^1 \to S^1$ be the semiconjugate such that $h \circ f = R_\theta \circ h$, with $R_\theta : S^1 \to S^1$ the rotation of angle θ (irrational in π). From lemma 4.5 we have that there exists a covering of K formed by closed intervals $H_1, ..., H_r$, disjoint two to two, such that $f'/H_i \bigcap K = a_i$. It is possible to choose the intervals H_i so that each connected component of the complement of $\bigcup_{i=1}^r H_i$ is a connected component of K^c. If $L_1, ..., L_r$ are the connected components of the complement of $\bigcup_{i=1}^r H_i$, then the image of each L_i by h is a point y_i. As f has only one orbit of wandering intervals, then the points y_i are in the same orbit in the rotation R_θ. Let A_m, $T_1, ..., T_p$, $J_1, ..., J_q$ be as in lemma 5.1 such that $\{y_1, ..., y_r\} \subset A_m$. Now, we define

$$g : \bigcup_1^p T_i \cup \bigcup_1^q J_j \to I\!R$$

such that $g(x) = f'(h^{-1}(x))$ (note that g is well defined even in the case that $h^{-1}(x)$ is an interval). By the choice of the intervals T_i and J_j we have that g is constant in each of them. Even more, if y is a point of S^1 such that $h(y)$ does not belong to $\bigcup_{j\in\mathbf{N}} R_\theta^{-j}(A_m)$ (preorbit of the extremes of the intervals T_i and J_j) then

$$F(y,n) = \sum_{i=0}^{n-1} \log(g(R_\theta^i(h(y)))).$$

Claim:

$$\int_{(\bigcup T_i)\cup(\bigcup J_j)} \log g\, dx = 0.$$

We suppose by contradiction that $\int_{(\bigcup T_i)\cup(\bigcup J_j)} \log g\, dx \neq 0$. Supposing that

$$\int_{(\bigcup T_i)\cup(\bigcup J_j)} \log g\, dx > 0,$$

we have that there exists a continuous function $g_1 : S^1 \to S^1$ such that $g_1 < g$ and $\int_{S^1} \log g_1 dx > 0$. So, by lemma 5.2 we have that given $x \in S^1$ and $k > 0$ there exists $n = n(x,k)$ such that $\sum_{i=0}^{n-1} \log(g_1(R_\theta^i(x))) > k$. Therefore, if $x \in K$ and $h(x) \notin \bigcup_{j\in\mathbf{N}} R_\theta^{-j}(A_m)$ we have that for each $k > 0$ there exists a positive integer n such that

$$F(x,n) = \sum_{i=0}^{n-1} \log(g(f^i(x))) \geq \sum_{i=0}^{n-1} \log(g_1(R_\theta^i(h(x)))) > k. \tag{15}$$

As for each point $x \in K$ there exists a positive integer s such that $h(f^s(x))$ does not belong to $\bigcup_{j\in\mathbf{N}} R_\theta^{-j}(A_m)$, taking k sufficiently large and applying (15) for the point $h(f^s(x))$, we have that there exists a positive integer n such that

$$F(x,n) > 0.$$

Therefore, the result obtained contradicts lemma 3.2. If

$$\int_{S^1} \log g\, dx < 0,$$

working in analogous form we have that for every $x \in K$ there exists a positive integer n such that $F(x,n) < 0$. This result contradicts lemma 3.1. Then we have proved the claim. Now, we are going to prove that

$$\int_{\bigcup T_i} \log g\, dx = \int_{\bigcup J_j} \log g\, dx = 0. \tag{16}$$

We denote $a_i = g/T_i$ e $b_j = g/J_j$. Then

$$\int_{(\bigcup T_i)\cup(\bigcup J_j)} \log g\, dx = \sum |T_i| \log a_i + \sum |J_j| \log b_j = |T_1| \sum \log a_i + |J_1| \sum \log b_j = 0. \tag{17}$$

If $\sum \log a_i \neq 0$, from lemma 4.5 we have $\sum \log b_j / \sum \log a_i \in \mathbb{Q}$. So, by (17) we have that $|T_1|/|J_1| \in \mathbb{Q}$ and this is a contradiction because the extremes of the intervals T_i and J_j are in a same orbit of the irrational rotation R_θ. Then

$$\sum \log b_j = \sum \log a_i = 0.$$

Now, let $y \in K$ be such that $x = h(y) \in T_1$. From the construction of the intervals T_i and J_j we have that $R_\theta^{p+1}(x)$ belongs to T_1 or J_1. If $R_\theta^{p+1}(x)$ belongs to T_1, then $R_\theta^{2p+1}(x)$ belongs to T_1 or J_1. If $R_\theta^{p+1}(x)$ belongs to J_1, then $R_\theta^{p+q+1}(x)$ belongs to T_1 or J_1. Proceeding inductively we have that there exists a crescent sequence n_k such that $n_{k+1} - n_k$ only takes values p and q and $R_\theta^{n_k+1}(x)$ belongs to T_1 or J_1. Therefore, from (16) we have that $F(y, n_k) = 0$, for all k. Finally, given a positive integer n there exists k_0 such that $n_{k_0} \leq n < n_{k_0+1}$ and therefore,

$$F(y, n) = F(y, n_{k_0}) + F(f^{n_{k_0}}(y), n - n_{k_0}) = F(f^{n_{k_0}}(y), n - n_{k_0}).$$

As $n - n_{k_0}$is limited, $F(y, n)$ is limited too and this contradicts lemma 3.3, and the proof is finished. □

6 Covering and levels

Note that if the quasi regular interval Cantor set K is C^1-minimal for f, for each positive integer n we have that if I is a connected component of K^c, so small as necessary, I and $f(I)$ are contained in K_n.

Definition 6.1. *The positive integer s is the level of an interval $I \subset S^1$, if I was removed from the construction of K in step s (we denote $s = \mathcal{L}(I)$).*

Lemma 6.1. *If $\{\mathcal{T}_{ij}\}$, with $j \in \mathbf{N}$ and $i = 1, ..., n$, is a family of closed intervals contained in S^1 such that $\nu_j = \max\{|\mathcal{T}_{ij}|; i = 1, .., n\}$ has limit 0 when $j \to \infty$, there exist a positive integer k and a finite set of intervals $\{\mathcal{J}_t\}$, disjoint two to two, contained in S^1, such that $\mathcal{A} = \bigcup \mathcal{J}_t \supset \bigcup_{i=1}^n \mathcal{T}_{ik}$ and every interval of $\mathcal{A}^c$ has a greater measure than the measure of $\mathcal{A}$.*

Proof. For the demonstration we will use finite induction in n. If $n = 1$ the demonstration is immediate. We suppose that the property is true for $n \geq 1$ and we are going to prove that the property is true for $n+1$. For each $j \in \mathbf{N}$, we denote by $\mathcal{B}_j = \bigcup_{i=1}^{n+1} \mathcal{T}_{ij}$ and by $\mathcal{Y}_{sj}$ $(s = 1, ..., n_j$, with $n_j \leq n+1)$ the connected components of the complement of $\mathcal{B}_j$. We will divide the demonstration in two cases. First, we suppose that $a_j = \min\{|\mathcal{Y}_{kj}|; k = 1, ..., n_j\}$ does not have limit 0 when $j \to \infty$. Then, there exist $\epsilon > 0$ and a crescent sequence $\{j_t\}$ such that $a_{j_t}, > \epsilon$ for all t. By hypothesis we know that $\nu_j \to 0$ when $j \to \infty$, then there exists $r \in \mathbf{N}$ such that $\nu_{j_r} < \epsilon/(n+1)$, so

$$|\mathcal{B}_{j_r}| \leq \sum_{i=1}^{n+1} |\mathcal{T}_{ij_r}| < (n+1)\frac{\epsilon}{n+1} = \epsilon.$$

As $a_{j_r} > \epsilon$, we have that every interval of the complement of $\mathcal{B}_{j_r}$ has greater length than $|\mathcal{B}_{j_r}|$. If we define the intervals $\mathcal{J}_t$ as the connected components of $\mathcal{B}_{j_r}$, we have proved the step of the induction in this case. Now, we suppose that $a_j \to 0$ when $j \to \infty$. We denote by $\mathcal{Y}_j^*$ one of the connected components of the complement of $\mathcal{B}_j$ such that its length is a_j. We can suppose, without loss of generality, that $\mathcal{Y}_j^*$ is the interval $Arc(\mathcal{T}_{1j}, \mathcal{T}_{2j}) \setminus (\mathcal{T}_{1j} \cup \mathcal{T}_{2j})$ (considering j sufficiently large and reordering the intervals $\mathcal{T}_{ij}$ as necessary). Now we consider the family of intervals $\mathcal{T}_{ij}^*$ defined as follows. We take

$$\mathcal{T}_{1j}^* = \mathcal{T}_{1j} \cup \mathcal{Y}_j^* \cup \mathcal{T}_{2j}$$

and for $i = 2, ..., n$

$$\mathcal{T}_{i,j}^* = \mathcal{T}_{i+1,j}.$$

Then by the inductive hypothesis there exist a number k and a family of intervals $\mathcal{J}_t$ that satisfy the lemma for the intervals $\mathcal{T}_{ij}^*$. The number k and the family of intervals $\mathcal{J}_t$ obtained for the family of intervals $\mathcal{T}_{ij}^*$ satisfy the conclusion of the lemma for the family of intervals $\mathcal{T}_{ij}$, too. This establishes the step of induction and the proof concludes. □

If the point x is the extreme of a connected component of K^c of level s_0, for each integer $s > s_0$ we denote by I_s the connected component of K^c closest to x. Note that if s is sufficiently large then I_s is unique.

Definition 6.2. *Let x be the extreme of a connected component of K^c of level s_0. For each integer $s > s_0$ we define*

$$\varphi_x(s) = s - \mathcal{L}(f(I_s))$$

.

Lemma 6.2. *If the quasi regular interval Cantor set K, of regularity different from 0, is C^1-minimal for f and x is the extreme of a connected component of K^c of level s_0, then φ_x is upper limited.*

Proof. As the regularity of K is not 0, there exists a procedure that determines K such that $\delta = \inf\{\mu_m/\nu_m : m \in \mathbf{N}\} > 0$. We suppose by contradiction that for each $k > 0$ there exists a positive integer s_k, such that $\varphi(s_k) = s_k - \mathcal{L}(f(I_{s_k})) > k$. We denote $r_k = \mathcal{L}(f(I_{s_k}))$. By the construction of K we have that $\mu_{s_k} \leq 2^{-k}\mu_{r_k}$. If $I_{s_k} = (a_k, b_k)$, with a_k between x and b_k, we have that there exists $\theta_k \in [x, a_k]$ such that $d(f(x), f(a_k)) = f'(\theta_k)d(x, a_k)$. So

$$d(f(x), f(a_k)) \leq f'(\theta_k)\nu_{s_k} \leq f'(\theta_k)\frac{\mu_{s_k}}{\delta} \leq \frac{f'(\theta_k)}{\delta}2^{-k}\mu_{r_k} \leq \frac{f'(\theta_k)}{\delta}2^{-k}d(f(x), f(a_k)).$$

From here it follows that $f'(\theta_k) \to \infty$ when $k \to +\infty$, and this is a contradiction. □

7 Proof of the theorem 2

Proof. We suppose by contradiction that there exists $\epsilon > 0$ and a diffeomorphism f, of class $C^{1+\epsilon}$ such that K is minimal for f. By lemmas 4.5 and 4.3 we have that there exist a positive integer n_0 and a point x, extreme of a connected component of K^c, such that:

1. the restriction of f' to K is constant in each connected component of K_{n_0}.
2. $f'(x) = \nu > 1$.
3. by the continuity of f' we have that if n_0 is sufficiently large, for every connected component I of K^c, contained in $K^x_{n_0}$ (connected component of K_{n_0} that contains x), we have that $|f(I)| > |I|$, so $f(I)$ and I have different level.

Given a positive integer n we denote by $I_n = (a_n, b_n)$ the interval of level $n + n_0$ contained in $K^x_{n_0}$ nearest to x. We fix m and for each integer $n > m$ we consider the family of intervals $\{I^j_n\}_{j\in\mathbf{N}}$ with the following properties:

1. the interval $I^0_n = I_n$.
2. the interval I^j_n is the connected component of K^c with the same level that the level of $f(I^{j-1}_n)$ nearest to x (in the proof we are going to work with a finite quantity of these).

Let $q = \max\{\mathcal{L}(I) - \mathcal{L}(f(I))\}$ be the integer given by lemma 6.2. We define $p_n = \min\{j : \mathcal{L}(I_n^j) \leq \mathcal{L}(I_{m+q-1}) = n_0 + m + q - 1\}$. We need to prove that the set $D_n = \{j : \mathcal{L}(I_n^j) \leq \mathcal{L}(I_{m+q-1})\}$ is not empty. We suppose by contradiction that D_n is empty. Then, for all j we have that $|I_n^{j-1}| < |I_n^j|$ and that I_n^j is between x and I_{m+q-1} and this is a contradiction. So D_n is not empty. Now, we consider the finite family $\{I_n^j\}$ with $j = 1, ..., p_n$. By lemma 6.2 follows that $n_0 + m + q > \mathcal{L}(I_n^{p_n}) \geq n_0 + m$. By the Mean Value Theorem we know that there exist $\theta_j \in I_n^j$, $j = 0, ..., p_n - 1$ such that

$|f(I_n^j)| = f'(\theta_j)|I_n^j| = |I_n^{j+1}|$. Therefore,

$$|I_n| = \frac{|I_n^{p_n}|}{f'(\theta_0)...f'(\theta_{p_n-1})}. \tag{18}$$

We denote $r_j = \mathcal{L}(I_n^j)$, with $j = 0, ..., p_n - 1$. Note that as $i \neq j$, $r_i \neq r_j$ and $r_j \geq m + n_0$, for every j. For every j, we have that θ_j and x are in the same connected component of K_{r_j-1}, so from lemma 4.1 and if r_j is sufficiently large we have

$$|\theta_j - x| < \frac{2}{\delta 2^{r_j-2}}.$$

Therefore, as f is the class $C^{1+\varepsilon}$ (this is $|f'(x) - f'(y)| \leq \widetilde{k}|x-y|^\epsilon$) we have

$$1 - \frac{k}{\nu}\frac{1}{2^{(r_j-2)\epsilon}} < \frac{f'(\theta_j)}{\nu} < 1 + \frac{k}{\nu}\frac{1}{2^{(r_j-2)\epsilon}}, \tag{19}$$

where $k = \widetilde{k}(\frac{2}{\delta})^\epsilon$. From (18) e (19) we have

$$\frac{|I_n^{p_n}|}{\nu^{p_n}} \prod_{i=0}^{p_n-1} \left\{1 + \frac{k}{\nu}\left(\frac{1}{2^{r_i-2}}\right)^\epsilon\right\}^{-1} \leq |I_n| \leq \frac{|I_n^{p_n}|}{\nu^{p_n}} \prod_{i=0}^{p_n-1} \left\{1 - \frac{k}{\nu}\left(\frac{1}{2^{r_i-2}}\right)^\epsilon\right\}^{-1}.$$

Therefore,

$$\log|I_n^{p_n}| - p_n \log\nu - P_2(m) \leq \log|I_n| \leq \log|I_n^{p_n}| - p_n \log\nu - P_1(m) \tag{20}$$

where

$$P_1(m) = \sum_{j=m+n_0}^{\infty} \log\left\{1 - \frac{k}{\nu}\left(\frac{1}{2^{j-2}}\right)^\epsilon\right\} \leq \log\prod_{i=0}^{p_n-1}\left\{1 - \frac{k}{\nu}\left(\frac{1}{2^{r_i-2}}\right)^\epsilon\right\} < 0$$

and

$$P_2(m) = \sum_{j=m+n_0}^{\infty} \log\left\{1 + \frac{k}{\nu}\left(\frac{1}{2^{j-2}}\right)^\epsilon\right\} \geq \log\prod_{i=0}^{p_n-1}\left\{1 + \frac{k}{\nu}\left(\frac{1}{2^{r_i-2}}\right)^\epsilon\right\} > 0.$$

For each m we define the set $A_m = \{\log|I_r|; r > m\}$ (the difference between this set and the set $\{\log \lambda_i\}$ is a finite quantity of elements). Now, we consider the quotient $A_m/\log\nu.\mathbb{R} = \mathcal{A}_m$ as a subset of the affine manifold $\mathcal{S} = \mathbb{R}/\log\nu.\mathbb{R}$ that is isomorphic to S^1. From the inequality (20) we have that for each m there exists a finite quantity of closed intervals $\mathcal{T}_{mj}$, $j = 1, ..., q$, contained in $\mathcal{S}$ such that $\bigcup_{j=1}^{q} \mathcal{T}_{mj} \supset \mathcal{A}_m$ and $a_m = \max\{|\mathcal{T}_{mj}|; j = 1, ..., q\} = P_2(m) - P_1(m)$. From the definitions of $P_1(m)$ and $P_2(m)$ follows that a_m has limit 0 when $m \to \infty$. From lemma 6.1 we know that there exist m_0 and a family of intervals $\mathcal{J}_k$ contained in $\mathcal{S}$, with $k = 1, ..., h$, such that

$$\mathcal{A}_{m_0} \subset \bigcup_{j=1}^{q} \mathcal{T}_{m_0 j} \subset \bigcup \mathcal{J}_k = \mathcal{M}$$

and every connected component of the complement of $\mathcal{M}$ has greater length than $|\mathcal{M}|$. If we consider the lifting of the previous sets we have that there exist a number $\delta > 0$ and a family of intervals $[\alpha_s, \beta_s]$, with $\alpha_s \le \beta_s$ e $\beta_{s+1} < \alpha_s$, $s = 1, ..., \infty$ (they are the lifting of the intervals $\mathcal{J}_t$) such that $A_{m_0} \subset \bigcup_{s=1}^{\infty}[\alpha_s, \beta_s]$ and $\alpha_s - \beta_{s+1} < \beta_s - \alpha_s + \delta$. It is easy to see that this condition implies the Mc Duff condition and this is a contradiction (see Proposition 4.2 in [2]) .

□

8 Proof of the theorems 3 and 4

We will begin proving certain lemmas that will be of utility in the demonstrations of theorems 3 and 4. If I and J are sets contained in $S^1 \backslash K$, we denote by $Arc(I, J)$ the smaller arch that contains I and J.

Lemma 8.1. *Let K be a regular interval Cantor set and let I_1, I_2, I_3 and I_4 be connected components of $S^1 \backslash K$, disjoint two to two, removed in steps n_1, n_2, n_3 and n_4 of the construction of K, respectively. If $n_4 \ge \max\{n_1, n_2, n_3\}$ and $Arc(I_3, I_4) \backslash (I_3 \cup I_4)$ is a connected component of K_{n_4}, there exists a positive integer m such that $|K \cap Arc(I_1, I_2)| = m|K \cap Arc(I_3, I_4)|$.*

Proof. From the construction of K, we know that $I_1, I_2, I_3, I_4 \subset S^1 \setminus K_{n_4}$, so $Arc(I_1, I_2) \cap K_{n_4}$ is a union of m connected components of K_{n_4}, that we denote by $K^1_{n_4}, ..., K^m_{n_4}$. Then

$$Arc(I_1, I_2) \cap K = (Arc(I_1, I_2) \cap K_{n_4}) \cap K = (\bigcup_{i=1}^{m} K^i_{n_4}) \cap K.$$

Therefore, $|Arc(I_1, I_2) \cap K| = \sum_{i=1}^{m} |K_{n_4}^i \cap K|$. So, by the construction of K, we have

$$|Arc(I_1, I_2) \cap K| = m|K_{n_4}^1 \cap K|. \tag{21}$$

As $Arc(I_3, I_4) \setminus (I_3 \cup I_4)$ is a connected component of K_{n_4} then

$$|K_{n_4}^1 \cap K| = |(Arc(I_3, I_4) \setminus (I_3 \cup I_4)) \cap K| = |Arc(I_3, I_4) \cap K|. \tag{22}$$

Then from (21) e (22) we have

$$|K \cap Arc(I_1, I_2)| = m|K \cap Arc(I_3, I_4)|.$$

□

Lemma 8.2. *If the regular interval Cantor set K, of positive measure, is C^1-minimal for f and $f'(x) > 1$ for $x \in K$, $f'(x)$ is a positive integer.*

Proof. Let ϵ_0, $\{n_j\}$ and $\{\lambda_{n_j}\}$ be as in the proof of lemma 3.2, and we consider $\epsilon_1 = \min\{\epsilon_0, f'(x) - 1\}$. By lemma 4.5 and the construction of K we know that there exists a positive integer n such that f' is constant in the intersection of K with each connected component of K_n and if n is sufficiently large, by the continuity of f' we have

$$\frac{1}{1+\epsilon_1} < \frac{f'(x_1)}{f'(x_2)} < 1 + \epsilon_1$$

with x_1 and x_2 in the same connected component of K_n. Without loss of generality, we can suppose that x is an extreme of a connected component I of K^c such that I and $f(I)$ are contained in $S^1 \setminus K_n$. We consider j_0 such that $\lambda_{n_{j_0}}$ is smaller than the length of some connected components of K^c contained in K_n. For each $j > j_0$ we consider I_j as the connected component K^c contained in K_n^x (connected component of K_n that contains x) nearest to x and $|I_j| \geq \lambda_{n_j}$. Then, we have that $|I_j| \to 0$ and $d(x, I_j) \to 0$ when $j \to \infty$. This implies that there exists a positive integer j_1 such that if $j \geq j_1$ then $f(I_j)$ is contained in $K_n^{f(x)}$. By the choice of ϵ_1 we have that

$$d(f(x), f(I_j)) > \frac{f'(x)}{1+\epsilon_1} d(x, I_j) \geq d(x, I_j). \tag{23}$$

Now, we will demonstrate that if $j \geq j_1$ there does not exist another connected component of K^c with length $|f(I_j)|$, contained in $K_n^{f(x)}$ and within $f(x)$ and $f(I_j)$. By contradiction we suppose that there exists I^* in the previous conditions. Then $f^{-1}(I^*)$ is between x and I_j. By the Mean Value Theorem we know that there exists $\theta^* \in f^{-1}(I^*)$ and $\theta_j \in I_j$ such that $|f^{-1}(I^*)| = \frac{|I^*|}{f'(\theta^*)}$ and $|f(I_j)| = f'(\theta_j)|I_j|$ so

$|f^{-1}(I^*)| = \frac{f'(\theta_j)}{f'(\theta^*)}|I_j|$. As θ^* and θ_j are in the same connected component of K_n, we have

$$\frac{|I_j|}{1+\epsilon_1} < |f^{-1}(I^*)| < |I_j|(1+\epsilon_1)$$

so

$$|f^{-1}(I^*)| > \frac{|I_j|}{1+\epsilon_1} > \frac{|I_j|}{1+\epsilon_0} \geq \frac{\lambda_{n_j}}{1+\epsilon_0} > \lambda_{n_j+1}.$$

From here we conclude that $|f^{-1}(I^*)| \geq \lambda_{n_j}$ and this contradicts the definition of I_j. More over, utilizing (23) we have that if $f(I_j)$ was removed in the step n_1 and I_j was removed in the step n_2, $n < n_1 < n_2$. This observation allows us to apply lemma 8.1, so there exists $p \in \mathbf{N}$ such that

$$|K \cap Arc(f(x), f(I_j))| = p|K \cap Arc(x, I_j)|. \tag{24}$$

As f' restrict to $K \cap Arc(x, I_j)$ is constant, then

$$|f(K \cap Arc(x, I_j))| = f'(x)|K \cap Arc(x, I_j)| = |K \cap Arc(f(x), f(I_j)|. \tag{25}$$

Therefore, from (24) e (25) and utilizing that $|K| > 0$ we have that $1 < f'(x) = p \in \mathbf{N}$ and this concludes the proof. □

To continue we will give the proof of theorem 3.

Proof. We suppose, by contradiction, that K is C^1-minimal for f and $\{m_i\}$ is not limited. By lemmas 4.3 and 8.2 we know that there exists an extreme of a wandering interval I, that we call x, such that $f'(x) = p \in \mathbf{N}$ with $p > 1$. Therefore, by the uniform continuity of f' and by lemma 4.1 we know that there exists $n_0 \in \mathbf{N}$ such that $f'/(K \cap K^x_{n_0}) = p$, where $K^x_{n_0}$ is the connected component of K_{n_0} that contains x. As $\{m_i\}$ is not limited, there exists i_0 sufficiently large such that $m_{i_0} > p+2$. Let J_{i_0} be the interval of level i_0 nearest to x and $K^x_{i_0} = [x, y_{i_0}]$ (connected component of K_{i_0} that contains x). As f' restricted to $K \cap K^x_{n_0}$ is p, then

$$|f(K \cap K^x_{i_0})| = |K \cap [f(x), f(y_{i_0})]| = p|K \cap K^x_{i_0}|.$$

Utilizing that K has positive measure we have that the interval $[f(x), f(y_{i_0})]$ contains exactly p connected components of K_{i_0}. As $f(x)$ is an extreme of $f(I)$ (its level is greater than i_0, if i_0 is sufficiently large) and in step i_0 we removed more than $p+2$ intervals, the level of $f(J_{i_0})$ is i_0. Therefore $|J_{i_0}| = |f(J_{i_0})|$. Besides, we have that $J_{i_0} \subset K_{i_0-1}$ and $|K_{i_0-1}| \to 0$ when $i_0 \to \infty$. But then, utilizing the continuity of f', we know that if i_0 is sufficiently large $|J_{i_0}| < |f(J_{i_0})|$, and this is a contradiction. □

The following lemmas will be of utility for the demonstration of theorem 4.

Lemma 8.3. *If the regular interval Cantor set K, of positive measure, is C^1-minimal for f, and there exists $x \in K$ and a positive integer p ($p > 1$) such that $f'(x) = p$, then p is multiple of $m_i + 1$ for an i sufficiently large.*

Proof. From lemma 4.5 we can suppose that x is an extreme of a connected component of K^c. We denote by $I_i = (a_i, b_i)$ the connected component of K^c of level i nearest to x (if i is sufficiently large, I_i is determined). Then, $f([x, a_i])$ contains exactly p connected components of K_i, so the level of $f(I_i)$ is less than or equal to i. If i is sufficiently large we have that $|f(I_i)| > |I_i|$, so the level of $f(I_i)$ is less than i. Therefore, the quantity of connected components of K_i that contains $f([x, a_i])$ is multiple of $m_i + 1$. □

Lemma 8.4. *If K is a regular interval Cantor set of positive measure, $\frac{l_n}{\sigma_n} \to 0$ when $n \to \infty$, where σ_n is the length of the connected components of K_n and l_n is the length of the open intervals removed in step n of the construction of K.*

Proof. From the construction of K we have that $|K| = \lim_{n\to\infty} \theta_1....\theta_n > 0$, so $\theta_n \to 1$. If x is an extreme of some open interval that was removed in step j, then for all $n > j + 1$ we have

$$\theta_n = \frac{|K_n|}{|K_{n-1}|} = \frac{|K_n^x|(m_n + 1)}{|K_{n-1}^x|} = \frac{|K_n^x|(m_n + 1)}{|K_n^x|(m_n + 1) + m_n l_n},$$

so $\frac{l_n}{|K_n^x|} \to 0$ when $n \to +\infty$.

□

To continue we will give the proof of theorem 4.

Proof. We suppose by contradiction that K is C^1-minimal for f. Let x, I, p and n_0 be as in the proof of theorem 3. For each $i > n_0$, we denote by $J_i = (y_i, z_i)$ the wandering interval of level i nearest to $f(x)$. By hypothesis, there exists a positive integer n_0 such that if $n \geq n_0$, $t_{n+1} - t_n > 3p$.
Claim 1: For all $i > t_{n_0}$, if $f^{-1}(J_i)$ is the interval of level j nearest to x then $f^{-1}(J_j)$ is not the interval of level $k = \mathcal{L}(f^{-1}(J_j))$ nearest to x. We suppose by contradiction that $f^{-1}(J_j)$ is not in the desired conditions. Therefore $[x, f^{-1}(y_i)]$ is a connected component of K_j and $[x, f^{-1}(y_j)]$ is a connected component of K_k. Then $(m_{i+1}+1)...(m_j+1) = p$ and $(m_{j+1}+1)...(m_k+1) = p$. Utilizing lemma 8.3 and that q is a prime number we have that there exist less than two elements of the set $\{(m_{i+1}+1), ..., (m_j+1), ..., (m_k+1)\}$ that are multiple of q. As this set doest not have more than $2p$ elements, if i is sufficiently large we have a contradiction.

Then we have demonstrated claim 1.
Claim 2: If i is sufficiently large there exists $k > i$ such that

$$\frac{|J_k|}{|K_k^{f(x)}|} > \frac{3}{2}\frac{|J_i|}{|K_i^{f(x)}|}.$$

By the Mean Value Theorem, for all i, there exist θ_1 and θ_2 (they depend on i) contained in $[x, f^{-1}(z_i)]$ such that $|J_i| = |f^{-1}(J_i)|f'(\theta_1)$ and $|(f(x), y_i)| = |(x, f^{-1}(y_i))|f'(\theta_2)$. Then

$$\frac{|J_i|}{|K_i^{f(x)}|} = \frac{|J_i|}{|(f(x), y_i)|} = \frac{f'(\theta_1)}{f'(\theta_2)}\frac{|f^{-1}(J_i)|}{|(x, f^{-1}(y_i))|} \to \frac{|f^{-1}(J_i)|}{|(x, f^{-1}(y_i))|}, \tag{26}$$

when $i \to \infty$. We have two possibilities.

1. If $f^{-1}(J_i)$ is the interval nearest to x of level $j = \mathcal{L}(f^{-1}(J_i))$, from claim 1, we have that $f^{-1}(J_j)$ is not the interval of level $k = \mathcal{L}(f^{-1}(J_j))$ nearest to x, therefore $|(x, f^{-1}(y_j))| > 2.|K_k^x|$. Then, utilizing (26),

   $$\frac{|J_i|}{|K_i^{f(x)}|} \to \frac{|J_j|}{|K_j^{f(x)}|} \to \frac{|J_k|}{|(x, f^{-1}(y_j))|} < \frac{|J_k|}{2|K_k^{f(x)}|},$$

   when $i \to \infty$. So, it follows claim 2.

2. If $f^{-1}(J_i)$ is not the interval nearest to x of level $k = \mathcal{L}(f^{-1}(J_i))$, $|(x, f^{-1}(y_i))| > 2.|K_k^x|$. So the demonstration follows in analogous form to the previous item.

From claim 2 we have that $\frac{|J_n|}{|K_n^{f(x)}|} \not\to 0$ when $n \to \infty$ and this contradicts lemma 8.4. □

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