New examples of Cantor sets in S^1 that are not C^1-minimal

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February 7, 2007

Abstract

Although every Cantor subset of the circle (S^1) is the minimal set of some homeomorphism of S^1, not every such set is minimal for a C^1 diffeomorphism of S^1. In this work, we construct new examples of Cantor sets in S^1 that are not minimal for any C^1-diffeomorphim of S^1.

[1] [2]

1 Introduction and main results

To study the dynamics of a homeomorphism $f : S^1 \to S^1$ it is important to study the invariant sets for f. We say that a set K is a minimal set for f if it is compact, non empty, invariant and minimal (relative to the inclusion) with regard to the former three properties. Simple examples of minimal sets are the fixed points and the periodic orbits of a homeomorphism, and in general the w-limit (α-limit) of any point. Zorn's lemma implies that every homeomorphism of S^1 has at least a minimal set. If f has periodic points (for example when f does not preserve orientation) then any minimal set is finite. On the other hand, if f does not have periodic points the minimal set is unique, infinite and it is the set of accumulation points of the past orbit and future orbit of any point $x \in S^1$. In the latter case the minimal set is a Cantor set (intransitive case) or all S^1 (transitive case). The following theorem allows us to state that the intransitive case cannot happen when f is a diffeomorphism of class C^2.

Theorem 1.1. *(Denjoy) If f is a diffeomorphism of class C^1 of S^1 without periodic points and with derivate of bounded variation then f is transitive.*

[1]Mathematical subject classification: 37E10, 37C45

[2]Keywords: minimal sets, Cantor sets, C^1 diffeomorphisms.

We can find a proof of this theorem in [1]. In this work, Denjoy also constructs intransitive diffeomorphisms of class C^1 (so called Denjoy's examples). Also there exist examples of intransitive diffeomorphisms of class $C^{1+\alpha}$ for $\alpha < 1$, constructed by Herman in [3]. From the existence of intransitive diffeomorphisms and since any two Cantor sets of S^1 are homeomorphic, it follows that any Cantor set of S^1 is C^0-minimal (i.e. it is minimal for some homeomorphism). This is not true when f is a diffeomorphism of class C^1. It is easy to verify that any finite subset of S^1 is C^1-minimal (i.e. it is minimal for some diffeomorphism of class C^1), but not every Cantor set of S^1 is C^1-minimal. In [2] Mc Duff proved that the usual ternary Cantor set is not C^1-minimal and in [4] Norton proved that the affine Cantor sets are not C^1-minimal.
Let K be a Cantor set of circle and let $K^c = \bigcup I_j$ where I_j are the connected components of K^c. We define the spectrum of K (E_K) as the ordered set $\{\lambda_i\}$ ($\lambda_{i+1} < \lambda_i$), with λ_i the lengths of I_j for some j. We call *covering of the spectrum of* K to every separate family of closed intervals $\{\mathcal{J}_i = [\alpha_i, \beta_i]\}$ such that $E_K \subset \cup \mathcal{J}_i$ and $\alpha_{i+1} \leq \beta_{i+1} < \alpha_i$. In this condition each connected component I_j of K^c is associated to an integer $n(I_j)$ such that $|I_j| \in \mathcal{J}_{n(I_j)}$. In [2] Mc Duff conjectured that if $\lambda_n/\lambda_{n+1} \not\to 1$ the Cantor set K is not C^1-minimal (all known C^1-minimal Cantor sets satisfy $\lambda_n/\lambda_{n+1} \to 1$).

Definition 1.1. *We say that the Cantor set K satisfies the* p**-separation condition** *for a covering $\{\mathcal{J}_i\}$ if there exists a non negative integer p such that for any $N > 0$ there exists $\eta(N) > 0$ such that*

$$\frac{\alpha_{j+n-1}}{\beta_{j+p+n}} \geq (1 + \eta(N)) \frac{\beta_j}{\alpha_{j+p}} \tag{1}$$

for any integer n, $|n| \leq N$, and for all j, sufficiently large.

Adapting the techniques used by Mc Duff in [2], we obtain the following result.

Theorem 1.2. *If the Cantor set K satisfies the p-separation condition then the Cantor set K is not C^1-minimal.*

This theorem is a generalization of the following theorem proved by Mc Duff in [2].

Theorem 1.3. *If a Cantor set K satisfies the p-separation condition for $p = 0$ then the Cantor set K is not C^1-minimal.*

We say that a covering $\{\mathcal{J}_i\}$ of the spectrum of K is a ϵ-covering (with $\epsilon > 0$) if $\frac{\alpha_j}{\beta_{j+1}} = 1 + \epsilon$, for every j. The other result obtained is the following.

Theorem 1.4. *If $\{\mathcal{J}_i\}$ is a ϵ-covering of the spectrum of a Cantor set K and $\beta_i/\alpha_i = k$ then the Cantor set K is not C^1-minimal.*

Finally, in the last section we give the construction of a Cantor set that satisfies the p-separation condition for $p = 1$, but does not satisfy the condition given by Mc Duff in [2] (this is the p-separation condition for $p = 0$).

2 Proof of the theorems 1.2 and 1.4

The following lemmas will be used in the proof of theorem 1.2.

Lemma 1. *If the Cantor set K is C^1-minimal and $\{\mathcal{J}_i\}$ is a covering of E_K then $\frac{\alpha_i}{\beta_{i+1}}$ is bounded.*

Proof. We can suppose that any interval of the covering of E_K contains some element of E_K. Let f be a diffeomorphism for which K is C^1-minimal. If I is a connected component of K^c and $\{|f^n(I)| : n \in \mathbf{N}\} = \{\gamma_1, ..., \gamma_j, ...\}$ with $\gamma_{j+1} < \gamma_j$, we have

$$\frac{\gamma_j}{\gamma_{j+1}} \leq \max\{M, 1/m\}, \tag{2}$$

where M and m are the maximum and minimum of f' respectively. For every i there exists j_i such that $\gamma_{j_i} \in \mathcal{J}_i$ and $\gamma_{j_i+1} \in \mathcal{J}_{i+1}$. Then

$$\frac{\alpha_i}{\beta_{i+1}} \leq \frac{\gamma_{j_i}}{\gamma_{j_i+1}}. \tag{3}$$

Therefore using (2) and (3) we have

$$\frac{\alpha_i}{\beta_{i+1}} \leq \max\{M, 1/m\}.$$

This ends the proof. □

Lemma 2. *If the Cantor set K is C^1-minimal and satisfies the p-separation condition for $\{\mathcal{J}_i\}$ then $\frac{\beta_j}{\alpha_j}$ is bounded.*

Proof. Taking $N = n = 1$ in (1) we have

$$\frac{\alpha_j}{\beta_{j+p+1}} \geq (1 + \eta(1))\frac{\beta_j}{\alpha_{j+p}}$$

for all j sufficiently large. Then

$$\frac{\beta_j}{\alpha_j} \leq \frac{1}{1 + \eta(1)} \frac{\alpha_{j+p}}{\beta_{j+p+1}}.$$

The result follows from the previous lemma. □

It is simple to verify the following properties.

1. If the Cantor set K is C^1-minimal for f, then for every $r > 1$ there exists a finite covering of K formed by disjoint closed intervals T_i such that if x, y belong to a same T_i,
$$\frac{1}{r} \leq \frac{f'(x)}{f'(y)} \leq r.$$

2. If the Cantor set K satisfies the p-separation condition for $\{\mathcal{J}_j\}$ then
$$\frac{\alpha_j}{\beta_{j+1}} \geq 1 + \eta(1)$$
for all j, sufficiently large.

Lemma 3. *If the Cantor set K is C^1-minimal for f and satisfies the p-separation condition then for every component I of K^c, $|n(I) - n(f(I))|$ is bounded.*

Proof. If m and M are the minimum and maximum of f' respectively then $m|I| \leq |f(I)| \leq M|I|$. If $n(f(I)) \geq n(I)$, using property 2 we have
$$(1+\eta(1))^{n(f(I))-n(I)} \leq \frac{\alpha_{n(I)}}{\beta_{n(I)+1}} \cdot \frac{\alpha_{n(I)+1}}{\beta_{n(I)+2}} \cdots \frac{\alpha_{n(f(I))-1}}{\beta_{n(f(I))}} \leq \frac{\alpha_{n(I)}}{\beta_{n(f(I))}} \leq \frac{|I|}{|f(I)|} \leq \frac{1}{m}.$$
If $n(f(I)) < n(I)$ then
$$(1+\eta(1))^{n(I)-n(f(I))} \leq \frac{\alpha_{n(f(I))}}{\beta_{n(I)}} \leq \frac{|f(I)|}{|I|} \leq M.$$
In both cases we conclude that $|n(I) - n(f(I))|$ is bounded. □

2.1 Proof of theorem 1.2

Proof. Suppose by contradiction that the Cantor set K is C^1-minimal for f and satisfies the p-separation condition for the covering $\{\mathcal{J}_i\}$. From lemma 3 there exists a non negative integer N_0 such that $|n(I) - n(f(I))| < N_0$ for any connected component I of K^c. Consider a covering of K formed by disjoint open intervals $T_1, ..., T_s$, such that if x and y belong to a same T_i, then
$$\frac{f'(x)}{f'(y)} < 1 + \frac{\eta(N_0)}{3}. \tag{4}$$
From property 1 we know that such covering exists. Let I and J be two intervals of K^c contained in a same T_i, such that $n(I) - n(J) \leq p$ (p is the integer given

by the condition of p-separation). We will prove now that $n(f(I)) - n(f(J)) \leq p$. Suppose by contradiction that $n(f(J)) < n(f(I)) - p$. Then

$$\frac{|f(J)|}{|f(I)|} \geq \frac{\alpha_{n(f(J))}}{\beta_{n(f(I))}} \geq \frac{\alpha_{n(f(J))}}{\beta_{n(f(J))+p+1}}.$$

Using the p-separation condition and that $|n(J) - n(f(J))| < N_0$, we obtain

$$\frac{|f(J)|}{|f(I)|} \geq (1 + \eta(N_0))\frac{\beta_{n(J)}}{\alpha_{n(J)+p}}.$$

On the other hand, using (4) we obtain

$$\frac{|f(J)|}{|f(I)|} \leq \frac{|J|}{|I|}\left(1 + \frac{\eta(N_0)}{3}\right) \leq \left(1 + \frac{\eta(N_0)}{3}\right)\frac{\beta_{n(J)}}{\alpha_{n(I)}} \leq \left(1 + \frac{\eta(N_0)}{3}\right)\frac{\beta_{n(J)}}{\alpha_{n(J)+p}}$$

and this is a contradiction. Therefore, if I and J are in the same component T_i such that $n(I) - n(J) \leq p$ then $n(f(I)) - n(f(J)) \leq p$. For each component of the complement of $\cup T_i$ there exists a component of K^c that contains it. Let us denote such components by $L_1, ..., L_s$. Let I be a component of K^c. As $|f^j(I)| \to 0$ when $j \to \infty$ then there exists j_0 such that for all $j > j_0$,

$$n(f^j(I)) > p + \max\{n(L_i) : i = 1, ..., s\}.$$

In these conditions there exists i_0 such that $f^{j_0}(I) = (a_{j_0}, b_{j_0})$ is contained in T_{i_0}. Let c_{j_0} be a point of K contained in T_{i_0} such that $|(c_{j_0}, a_{j_0})| < |f^{j_0}(I)|$. From here, if J is a connected component of K^c contained in (c_{j_0}, a_{j_0}) then $n(f^{j_0}(I)) - n(J) \leq p$, and $n(f^{j_0+1}(I)) - n(f(J)) \leq p$. From the choice of j_0 we have that $n(f(J)) > \max\{n(L_i) : i = 1, ..., s\}$ so $f(J) \neq L_i$ for $i = 1, ..., s$. This shows that $f^{j_0+1}(I)$ and $f((c_{j_0}, a_{j_0}))$ are in the same T_i. Proceeding inductively we have that for any interval J of K^c contained in (c_{j_0}, a_{j_0}), $f^n(J) \neq L_i$, for all positive integer n and $i = 1, ..., s$. This is a contradiction because for any interval L_i there exist infinite $n > 0$ such that $f^{-n}(L_i) \subset (c_{j_0}, a_{j_0})$. □

2.2 Proof of theorem 1.4

Proof. Suppose by contradiction that the Cantor set K is C^1-minimal for a diffeomorphism f.

Claim: There exist connected components, T and I, of K^c such that $|T|$ and $|I|$ belong to the same interval $\mathcal{J}_i$, but $|f(T)|$ and $|f(I)|$ belong to different ones.

Let $\delta > 0$ be as small as necessary. Let $T_1, ..., T_s$ be as in the proof of theorem 1.2 such that if x and y belong to a same T_i, then

$$\frac{1}{1+\delta} \leq \frac{f'(x)}{f'(y)} \leq 1 + \delta. \tag{5}$$

Let I, i_0, a_{j_0} and c_{j_0} be as in the proof of theorem 1.2. Recall that $f^{j_0}(I) = (a_{j_0}, b_{j_0}) \subset T_{i_0}$. Denote $R = f^{j_0}(I)$. If L is any connected component of K^c contained in (c_{j_0}, a_{j_0}), then

$$\frac{1}{(1+\delta)^q}\frac{|L|}{|R|} \leq \frac{|f^q(L)|}{|f^q(R)|} \leq (1+\delta)^q \frac{|L|}{|R|}$$

while $f^{\widetilde{q}}((c_{j_0}, b_{j_0}))$ is contained in $\cup T_i$ for $0 \leq \widetilde{q} \leq q$. As $\{\mathcal{J}_i\}$ is a ϵ-covering with $\beta_i/\alpha_i = k$, if δ is taken sufficiently small, it follows that

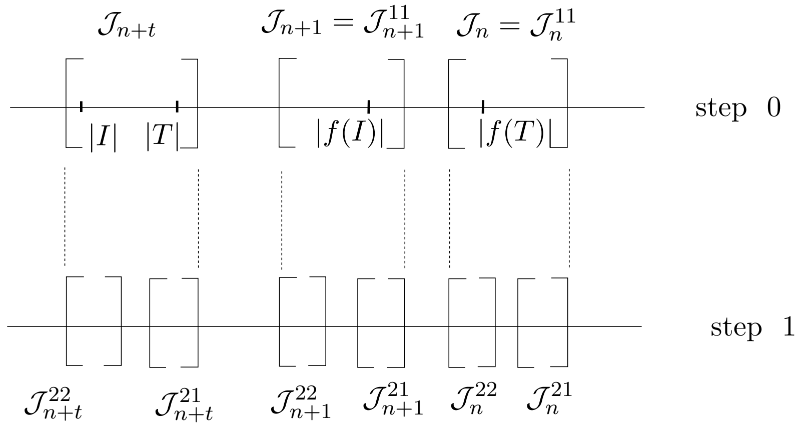
$$|(n(f^{q_1}(L)) - n(f^{q_1}(R))) - (n(f^{q_1+1}(L)) - n(f^{q_1+1}(R)))| \leq 1 \tag{6}$$

for $0 \leq q_1 \leq q$. As remarked at the end of the proof of theorem 1.2, we can take $L = f^{-q_2}(L_1)$ for an adequate $q_2 > 0$. Then $n(f^{-q_2}(L_1)) - n(R) \geq 0$ and $n(f^{q_2}(f^{-q_2}(L_1))) - n(f^{q_2}(R)) < 1$. Then (6) implies that there exist $q_3, q_4 > 0$ and L_j such that $n(f^{q_3}(f^{-q_4}(L_j))) - n(f^{q_3}(R)) = 0$ and $n(f^{q_3+1}(f^{-q_4}(L_j))) - n(f^{q_3+1}(R)) = -1$. Taking $T = f^{q_3-q_4}(L_j)$ and $I = f^{q_3}(R)$ the proof of the claim is finished.
Note, from the proof of the claim, that the intervals T and I are so close as necessary. Also note that given $\delta' > 0$ there exists $\eta > 0$ such that, if $x, y \in E(z, \eta)$ we have

$$\frac{1}{1+\delta'} < \frac{f'(x)}{f'(y)} < 1 + \delta' \tag{7}$$

for any $z \in K$. Then, given $\delta' > 0$, there exist $\eta > 0$, $z \in K$ and $T, I \subset E(z, \eta)$ as in the claim, such that, if $x, y \in E(z, \eta)$ then x, y satisfy (7). As $|f(T)|$ and $|f(I)|$ do not belong to the same $\mathcal{J}_i$, there exists a 'gap' between $|f(T)|$ and $|f(I)|$. Therefore, as by hypothesis $\frac{\beta_i}{\alpha_i} = k$, this 'gap' produces a new 'gap' for the spectrum of the Cantor set $K \cap E(z, \eta)$ in between each one of the original 'gaps'. Formally, we have that there exists a covering $\{\mathcal{J}_i^{21} = [\alpha_i^{21}, \beta_i^{21}]\} \cup \{\mathcal{J}_i^{22} = [\alpha_i^{22}, \beta_i^{22}]\}$ of the spectrum of $K_2 = E(z, \eta) \cap K$ such that $\mathcal{J}_i^{21} \cup \mathcal{J}_i^{22} \subset \mathcal{J}_i$ and $\frac{\beta_i^{2r}}{\alpha_i^{2r}} < k\frac{1+\delta'}{1+\epsilon}$ with $r = 1, 2$ (see figure 1).
As any C^1-minimal Cantor set is locally C^1-minimal (see [2]), there exists $K_2' \subset K_2$, C^1-minimal with $\{\mathcal{J}_i^{21}\} \cup \{\mathcal{J}_i^{22}\}$ as a covering of its spectrum. Proceeding inductively we obtain a Cantor set K_n', C^1-minimal with $\{\mathcal{J}_i^{n1}\} \cup \{\mathcal{J}_i^{n2}\} \cup ... \cup \{\mathcal{J}_i^{nn}\}$ as a covering of its spectrum and such that $1 \leq \frac{\beta_i^{nr}}{\alpha_i^{nr}} < k(\frac{1+\delta'}{1+\epsilon})^{n-1}$. As ϵ is fixed and δ' is as small as we want, taking n sufficiently large we obtain a contradiction, and the proof is finished. □

Figure 1:



3 Examples of Cantor sets that satisfy the p-separation condition

In this section we will construct a family of Cantor sets that satisfy the p-separation condition for $p = 1$ but does not satisfy the McDuff's condition [2].

3.1 Construction of the Cantor set

First we determine a set of real numbers that will be the spectrum of the Cantor set (here we are not considering the order). Let γ be a positive number such that $\gamma < 3$ and $\gamma^{3/2} > 3$. For each positive integer n we consider the set

$$A(n) = \{\eta_{nj} = \frac{\gamma^{\frac{j}{2n}}}{3^{4n+2}} : j = -n, ..., n\}.$$

If $S(n)$ is the sum of the elements of $A(n)$ we have

$$S(n) = \sum_{j=-n}^{n} \eta_{nj} \leq \frac{2n+1}{3^{4n+2}} \gamma^{1/2} \leq \frac{\gamma^{1/2}}{3^{2n}}.$$

Then $\sum_{n=1}^{\infty} S(n)$ is finite, so the sum of the elements of

$$B = \left\{\eta_i = \frac{1}{3^i} : i \in \mathbf{N}\right\} \cup \bigcup_{i=1}^{\infty} A(i)$$

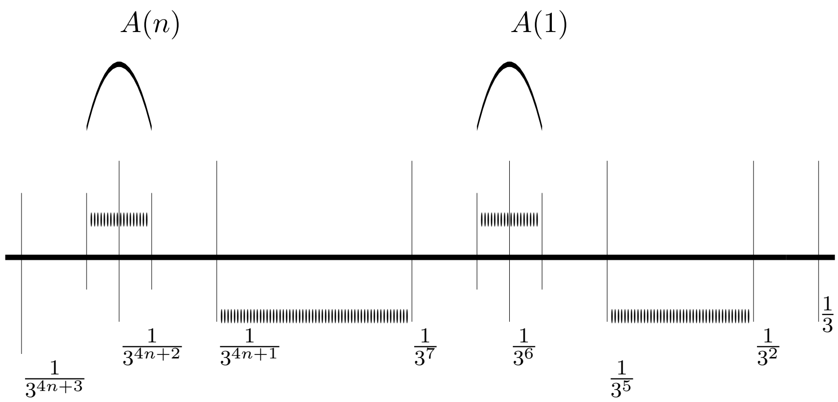


Figure 2:

is finite too. We denote this sum by μ. For the set B we have the figure 2 . Consider the set

$$C = \left\{ \frac{2\pi x}{\mu} : x \in B \right\}.$$

The sum of the elements of C is 2π. Let R_θ be a rotation of irrational angle θ in S^1 and x a point in S^1. Let $m : Z \to C$ be a bijection. We define a family of open intervals (a_j, b_j), $j \in Z$ as follows.

$$a_0 = 0, \qquad b_0 = m(0)$$

and for any positive integer j

$$a_j = b_0 + \sum_{R_\theta^k(x) \in (x, R_\theta^j(x))} m(k), \qquad b_j = a_j + m(j).$$

We define $K = S^1 \setminus (\bigcup_{j \in Z} (e^{ia_j}, e^{ib_j}))$. Then K is a Cantor set and C is its spectrum.

3.2 p-separation condition for K

We will show that the Cantor set K satisfies the p-separation condition for $p = 1$. The elements of C are of the form

$$\omega_i = \frac{2\pi}{\mu 3^i}, \qquad \omega_{ij} = \frac{2\pi \gamma^{\frac{j}{2i}}}{\mu 3^{4i+2}}$$

with $i \in \mathbf{N}$ and $j = -i, ..., i$. Therefore

$$\frac{2\pi\gamma^{-\frac{1}{2}}}{\mu 3^{4i+2}} \leq \omega_{ij} = \frac{2\pi\gamma^{\frac{j}{2i}}}{\mu 3^{4i+2}} \leq \frac{2\pi\gamma^{\frac{1}{2}}}{\mu 3^{4i+2}}.$$

Now we construct a covering $\{\mathcal{J}_j\}$ of C, $\mathcal{J}_j = [\alpha_j, \beta_j]$, $j > 0$. If $j = 4k+2$ for some $k > 0$ then we define

$$\alpha_j = \frac{2\pi\gamma^{-\frac{1}{2}}}{\mu 3^j}, \qquad \beta_j = \frac{2\pi\gamma^{\frac{1}{2}}}{\mu 3^j},$$

if not

$$\alpha_j = \beta_j = \frac{2\pi}{\mu 3^j}.$$

So, for all integer n we have

$$\frac{\alpha_{j+n-1}}{\beta_{j+n+1}} \geq \frac{9}{\gamma^{\frac{1}{2}}}$$

and

$$\frac{\beta_j}{\alpha_{j+1}} \leq 3\gamma^{\frac{1}{2}}.$$

As $\gamma < 3$, then K satisfies a p-separation condition for $p = 1$. Note that from theorem 1.2 we know that the Cantor set K is not C^1-minimal.

3.3 The Cantor set K does not satisfy the McDuff's condition

Suppose that K satisfies the McDuff's condition (the 0-separation condition) for a covering $\{L_i\}$, $L_i = [\alpha_i, \beta_i]$. Note that the McDuff's condition implies that every 'gap' $\frac{\alpha_i}{\beta_{i+1}}$ is greater than every 'non gap' $\frac{\beta_i}{\alpha_i}$. For a fixed k we have

$$\frac{\omega_{kj}}{\omega_{k,j-1}} = \frac{\gamma^{\frac{j}{2k}}}{\gamma^{\frac{j-1}{2k}}} = \gamma^{\frac{1}{2k}}$$

and it limits is 1 when $i \to \infty$. Then, for a sufficiently large k, every ω_{kj} belongs to the same interval $L_{i_k} = [\alpha_{i_k}, \beta_{i_k}]$, so $\frac{\beta_{i_k}}{\alpha_{i_k}} \geq \gamma$.

1. If $\beta_{i_k} < \frac{2\pi}{\mu 3^{4k+1}}$ then there exists α_r with $r < i_k$ such that

$$\beta_{i_k} < \alpha_r \leq \frac{2\pi}{\mu 3^{4k+1}},$$

so

$$\frac{\alpha_r}{\beta_{i_k}} \leq \frac{\frac{2\pi}{\mu 3^{4k+1}}}{\frac{2\pi\gamma^{1/2}}{\mu 3^{4k+2}}} = \frac{3}{\gamma^{1/2}} < \gamma \leq \frac{\beta_{i_k}}{\alpha_{i_k}}.$$

Then there exists a ‘gap‘ smaller than α_r/β_{i_k}, which is smaller than a ‘non gap‘ $\frac{\beta_{i_k}}{\alpha_{i_k}}$, and this is a contradiction.

2. If $\alpha_{i_k} > \frac{2\pi}{\mu 3^{4k+3}}$ a contradiction is proved in a similar way.

3. If $\beta_{i_k} \geq \frac{2\pi}{\mu 3^{4k+1}}$ and $\alpha_{i_k} \leq \frac{2\pi}{\mu 3^{4k+3}}$ then $\frac{\beta_{i_k}}{\alpha_{i_k}} \geq 9$. Then we have that the ‘gap‘ β_{i_k}/α_{i_k} is greater than every ‘non gap‘ (every non ‘gap‘ is equal or smaller than 3) and this is a contradiction.

Then the Cantor set K do not satisfy the McDuff’s condition.

Acknowledgment: This work is part of my PHD thesis. I would like to thank my Advisor, Edson de Faria, not only for our helpful discussions and his many useful remarks on mathematical structure and style, but also for his constant encouragement. I would like to thank J. Paulo Almeida, Jorge Iglesias, Roberto Markarián and Alvaro Rovella too for our helpful discussions. This work was partially supported by CNPq.

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