Optimal stopping of Hunt and Lévy processes

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Abstract

The optimal stopping problem for a Hunt processes on $\mathbf{R}$ is considered via the representation theory of excessive functions. In particular, we focus on infinite horizon (or perpetual) problems with one-sided structure, that is, there exists a point x^* such that the stopping region is of the form $[x^*, +\infty)$. Corresponding results for two-sided problems are also indicated. The main result is a spectral representation of the value function in terms of the Green kernel of the process. Specializing in Lévy processes, we obtain, by applying the Wiener-Hopf factorization, a general representation of the value function in terms of the maximum of the Lévy process. To illustrate the results, an explicit expression for the Green kernel of Brownian motion with exponential jumps is computed and some optimal stopping problems for Poisson process with positive exponential jumps and negative drift are solved.

Keywords: optimal stopping problem, Markov processes, Hunt processes, Lévy processes, Green kernel, diffusions with jumps, Riesz decomposition.

AMS Classification: 60G40, 60J25, 60J30, 60J60, 60J75.

1 Introduction

Consider an optimal stopping problem for a real-valued Markov process $X = \{X_t \colon t \geq 0\}$ with reward function g and discount rate $r \geq 0$. Denote by V the value function of the problem, and by τ^* the optimal stopping time. In this paper we analyze this optimal stopping problem departing from three main sources: (i) the characterization of the value function V as the least excessive majorant of the reward function g, due to Snell [23] for discrete martingales and to Dynkin [6] for continuous time Markov processes; (ii) the representation of excessive functions as integrals of the Green kernel of the process, as exposed in Kunita and Watanabe [11] and Dynkin [7], and exploited by Salminen [20] in the framework of optimal stopping for diffusions; and (iii) recent results expressing the solution of some optimal stopping problems for Lévy processes and random walks in terms of the maximum of the process, see Darling et. al.[5], Mordecki [15], Boyarchenko and Levendorskij [3], Novikov and Shiryaev [18] and Kyprianou and Surya [12]. For papers on optimal stopping of Lévy processes using other methods, see, e.g., McKean [13], Gerber and Shiu [9], Chan [4] and Kou and Wang [10].

We then try to understand the structure of the solution of the optimal stopping problem in a regular enough framework of Markov processes, precisely the class of Hunt processes, concluding that finding the solution of such a problem is equivalent to finding the representation of the value function in terms of the Green kernel. The Radon measure that appears in this representation is called the *spectral measure* corresponding to the excessive function V, and furthermore, it results that the support of this spectral measure is the stopping region for the problem. This is our main result, presented in section 3.

Let us specialize to Lévy processes. Firstly, observe that in the case $r > 0$ the Green kernel is proportional to the distribution of the process stopped at an exponential time with parameter r, independent of the process. Secondly, relying on the Wiener-Hopf factorization for the Lévy process, we express this random variable in the distributational sense as the sum of two independent random variables, the first one having the distribution of the supremum of the process up to the exponential time and the second one with the distribution of the infimum of the process in the same random interval. Simple calculations taking into account this fact, and the one-sided structure of the solution of the optimal stopping problem, gives a representation of the solution of an optimal stopping problem in terms of the maximum of the Lévy process – a result that has been obtained earlier in several particular cases. This analysis is carried out in section 4.

The rest of the paper is as follows. In section 2 the framework of Hunt processes in which we are working is described. Section 5 consists of two subsections. In the first one we illustrate the made assumptions concerning the Hunt processes and Lévy processes by studying Browian motion with exponential jumps. In the second one an optimal stopping problem for a compound Poisson process with negative drift and positive exponential jumps and the reward functon $g(x) = (x^+)^\gamma,\ \gamma \geq 1$, is analyzed. Our interest in this particular reward function was arised by Alexander Novikov's talk in the Symposium on Optimal Stopping with Applications held in Manchester 22.– 27.1.2006 [17] where the optimal stopping problem for the same reward functions and general Lévy processes were considered.

2 Preliminaries on Hunt processes

Let $X = \{X_t\}$ be a transient Hunt process taking values in $\mathbf{R}$, where we omit $t \geq 0$ in the notation, as all the considered processes are indexed in the same set. In particular, X is a strong Markov process, quasi left continuous on $[0, +\infty)$ and the sample paths of X are right continuous with left limits (see Kunita and Watanabe [11] and Blumenthal and Getoor [2] p. 45). The notations $\mathbf{P}_x$ and $\mathbf{E}_x$ are used for the probabilty measure and the expectation operator, respectively, associated with X when $X_0 = x$. The resolvent $\{G_r : r \geq 0\}$ of X is defined via

$$G_r(x, A) := \int_0^\infty \mathrm{e}^{-rt}\, \mathbf{P}_x(X_t \in A)\, dt, \tag{2.1}$$

where $x \in \mathbf{R}$ and A is a Borel subset of $\mathbf{R}$. We assume also that there exists a dual resolvent $\{\widehat{G}_r : r \geq 0\}$ with respect to some σ-finite (duality) measure m, that is, for all $f, g \in \mathcal{B}_o$ and $r \geq 0$ it holds

$$\int m(dx)\, f(x)\, G_r g(x) = \int m(dx)\, \widehat{G}_r f(x)\, g(x),$$

where $\mathcal{B}_o$ denotes the set of measurable bounded functions with compact support. Moreover, it is assumed that $\{\widehat{G}_r : r \geq 0\}$ is a resolvent of a transient Hunt process $\widehat{X}$ taking values in $\mathbf{R}$. Finally, we impose Hypothesis (B) from [11] p. 498:

(h$_1$) $G_r(x,dy) = G_r(x,y)\,m(dy)$,

(h$_2$) $\sup_{x\in A} G_0(x,B) < \infty$ for all compact A and B,

(h$_3$) $x \mapsto \widehat{G}_r f(x)$, $f \in \mathcal{B}_o$, is continuous and finite.

The assumption that the dual process is a Hunt process implies that $\widehat{G}$ is regular (see [11] p. 494).

We remark that when X is a Lévy process a dual resolvent always exists as the resolvent of the dual process $\widehat{X} = \{-X_t\}$. Hereby the Lebesgue measure serves as the duality measure (see section II.1 in [1]).

Under these assumptions it can be proved that the function G_r given in (h$_1$) constitutes a potential kernel (often called a Green kernel) of exponent r associated with the pair $(X, \widehat{X})$. This means that for each given $r \geq 0$ the function $(x,y) \mapsto G_r(x,y)$ is jointly measurable and has the properties

(p$_1$) $\widehat{G}_r(y,dx) = G_r(x,y)\,m(dx)$,

(p$_2$) $x \mapsto G_r(x,y)$ is r-excessive for X for each y,

(p$_3$) $y \mapsto G_r(x,y)$ is r-excessive for $\widehat{X}$ for each x.

Recall that a non-negative measurable function $f : \mathbf{R} \mapsto [0,+\infty]$ is called r-excessive for X if for all $x \in \mathbf{R}$

(e$_1$) $\mathrm{e}^{-rt}\,\mathbf{E}_x(f(X_t)) \leq f(x)$ for all $t \geq 0$,

(e$_2$) $\mathrm{e}^{-rt}\,\mathbf{E}_x(f(X_t)) \to f(x)$ as $t \to 0$.

Notice that r-excessive functions of X are 0-excessive for the process X^o obtained from X by exponential killing with rate r.

From the assumption (h$_1$) that the resolvent kernel of X is absolutely continuous it follows that r-excessive functions are lower semi-continuous.

The Riesz decomposition of excessive functions is of key importance in our approach to optimal stopping. We state the decomposition relying on [11] Theorem 2 p. 505 and Proposition 13.1 p. 523. Indeed, it holds, under the made assumptions, that each r-excessive function u locally integrable with respect to the duality measure m can be decomposed uniquely in the form

$$u(x) = \int_{\mathbf{R}} G_r(x,y)\,\sigma_u(dy) + h_r(x), \tag{2.2}$$

where h_r is an r-harmonic function and σ_u is a Radon measure on $\mathbf{R}$. We remark that the assumption that the dual process is a Hunt process implies that also the spectral measure σ_u is unique (see [11] Proposition 7.11 p. 503).

Conversily (see [11] Proposition 7.6 p. 501), given a Radon measure σ on $(\mathbf{R}, \mathcal{B})$ the function v defined via

$$v(x) := \int_{\mathbf{R}} G_r(x,y)\,\sigma(dy)$$

is an r-excessive function (in fact, a potential).

An r-excessive function u is said to r-harmonic on a Borel subset A of $\mathbf{R}$ if for all open subsets A_c of A with compact closure

$$u(x) = \mathbf{E}_x\left(\mathrm{e}^{-rH(A_c)}\,u(X_{H(A_c)})\right), \tag{2.3}$$

where

$$H(A_c) := \inf\{t \,:\, X_t \notin A_c\}.$$

In our case (see [11] Proposition 6.2 p. 499) it holds that for each fixed $y \in \mathbf{R}$ the function $x \mapsto G_r(x,y)$ is r-harmonic on $\mathbf{R} \setminus \{y\}$. From (2.2) it follows that if there exists an open set A such that the representing measure does not charge A, then the function u is r-harmonic on A i.e.,

$$\sigma_u(A) = 0 \quad \Rightarrow \quad u \text{ is } r\text{−harmonic on } A. \tag{2.4}$$

In fact, (2.4) is an equivalence under mild assumptions; for this see Dynkin [7] Theorem 12.1.

3 Optimal stopping of Hunt processes

Let X be a Hunt process satisfying the assumptions made in Section 2. For simplicity we consider non-negative continuous reward functions. Then the reward function g has the smallest excessive majorant V and

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}_x(\mathrm{e}^{-r\tau} g(X_\tau)), \tag{3.1}$$

where $\mathcal{M}$ denotes the set of all stopping times τ with respect to the natural filtration generated by X. In case $\tau = +\infty$ we define

$$\mathrm{e}^{-r\tau} g(X_\tau) := \limsup_{t \to \infty} \mathrm{e}^{-rt} g(X_t).$$

The result (3.1) can be found, for instance in Shiryayev [22] (Lemma 3 p. 118 and Theorem 1 p. 124) and holds for more general standard Markov processes, and for almost-Borel and $\mathcal{C}_0$-lower semicontinuous reward functions. We can express this result by saying that in an optimal stopping problem the value function and the smallest excessive majorant of the reward function coincide.

From (3.1) and the Riesz decomposition (2.2) we conclude that the problem of finding the value function is equivalent to the problem of finding the representing measure of the smallest excessive majorant (up to harmonic functions). Furthermore, based on (2.4), it is seen, roughly speaking, that the continuation region, that is, the region where it is not optimal to stop, is the "biggest" set not charged by the representing measure σ_V of V. In short, the representing measure gives the value function via (2.2) and the stopping region is by (2.4) the support of the representing measure. In the following result we use the preceeding considerations in order to express the solution of a particular type of optimal stopping problems

Theorem 3.1. *Consider a Hunt process $\{X_t\}$ satisfying the assumptions made in Section 2, a non-negative continuous reward function g, and a discount rate $r \geq 0$ such that*

$$\mathbf{E}_x(\sup_{t\geq 0} e^{-rt} g(X_t)) < \infty. \tag{3.2}$$

Assume that there exists a Radon measure σ with support on the set $[x^*, \infty)$ such that the function

$$V(x) := \int_{[x^*,\infty)} G_r(x,y)\sigma(dy) \tag{3.3}$$

satisfies the following conditions:

(a) *V is continuous,*

(b) *$V(x) \to 0$ when $x \to -\infty$.*

(c) *$V(x) = g(x)$ when $x \geq x^*$,*

(d) *$V(x) \geq g(x)$ when $x < x^*$.*

Let

$$\tau^* = \inf\{t \geq 0\colon X_t \geq x^*\}. \tag{3.4}$$

Then τ^* is an optimal stopping time and V is the value function of the optimal stopping problem for $\{X_t\}$ with the reward function g, in other words,

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}_x \left(\mathrm{e}^{-r\tau} g(X_\tau) \right) = \mathbf{E}_x \left(\mathrm{e}^{-r\tau^*} g(X_{\tau^*}) \right), \quad x \in \mathbf{R}.$$

Proof. Since V is an r-excessive function (see the discussion after the Riesz decomposition (2.2)) and, from conditions (c) and (d), a majorant of g, it follows by Dynkin's characterization of the value function as the least excessive majorant, that

$$V(x) \geq \sup_{\tau \in \mathcal{M}} \mathbf{E}_x \left(\mathrm{e}^{-r\tau} g(X_\tau) \right) \tag{3.5}$$

In order to conclude the proof, we establish the equality in (3.5). Indeed, consider for each $n \geq 1$ the stopping time

$$\tau_n = \inf\{t \geq 0\colon X_t \notin (-n, x^* - 1/n)\}.$$

For $\omega \in \{\tau^* < \infty\}$ define $\bar\tau = \lim_{n\to\infty} \tau_n$. We have

$$\tau_1 \leq \tau_2 \leq \cdots \leq \bar\tau \leq \tau^*.$$

For n large enough, $X_{\tau_n} \geq x^* - 1/n$, and, as the process is quasi-left continous, $\lim_{n\to\infty} X_{\tau_n} = X_{\bar\tau}$, and, hence, $X_{\bar\tau} \geq x^*$. This give us that $\bar\tau = \tau^*$ a.s.

As V is r-excessive, the sequence $\{\mathrm{e}^{-r\tau_n} V(X_{\tau_n})\}$ is a nonnegative supermartingale, and, in consequence, it converges a.s. to a random variable. Because $X_{\tau_n} \to X_{\tau^*}$ a.s., and V is continuous, we identify the limit as $\mathrm{e}^{-r\tau^*} V(X_{\tau^*})$. From assumptions (a) and (b) it follows that

$$C_V := \sup_{x \leq x^*} V(x) < \infty,$$

Furthermore, as

$$\begin{aligned} e^{-r\tau_n} V(X_{\tau_n}) = e^{-r\tau_n} V(X_{\tau_n}) \mathbf{1}_{\{\tau_n < \tau_*\}} + e^{-r\tau^*} g(X_{\tau^*}) \mathbf{1}_{\{\tau_n = \tau_*\}} \\ \leq C_V + \sup_{t \geq 0} e^{-rt} g(X_t) \end{aligned}$$

we obtain, in view of condition (3.2), using the Lebesgue dominated convergence theorem

$$\mathbf{E}_x \left(\mathrm{e}^{-r\tau_n} V(X_{\tau_n}) \right) \downarrow \mathbf{E}_x \left(\mathrm{e}^{-r\tau^*} V(X_{\tau^*}) \right) \quad \text{as } n \to \infty.$$

Furthermore, as the representing measure σ does not charge the open set $(-\infty, x^*)$, the function V is harmonic on that set (cf. (2.4)), and, as τ_n are exit times from the open sets $(-n, x^* - 1/n)$, we conclude that

$$V(x) = \mathbf{E} \left(\mathrm{e}^{-r\tau_n} V(X_{\tau_n}) \right) \downarrow \mathbf{E} \left(\mathrm{e}^{-r\tau^*} g(X_{\tau^*}) \right),$$

and the proof is complete. □

Under the additional assumption (3.6), valid in many particular cases, we characterize now the optimal threshold x^* as a solution, with a useful uniqueness property, of an equation derived from (3.3). Remember that by the definition of the support of a Radon measure, we have $\sigma(A) > 0$ for all open subsets A of $[x^*, \infty)$ (see for instance page 215 in Folland [8]).

Corollary 3.2. *Let $\{X_t\}, g, V, \sigma$, and x^* be as in Theorem 3.1. Assume that*

$$G_r(x, B) := \int_0^\infty \mathrm{e}^{-rt} \mathbf{P}_x(X_t \in B) dt \; > \; 0 \tag{3.6}$$

for all x and open subsets B of $\mathbf{R}$. Then the equation

$$g(x) = \int_{[x,\infty)} G_r(x, y) \sigma(dy) \tag{3.7}$$

has no solution bigger than x^*.

Proof. Clearly, since $g(x^*) = V(x^*)$ it is immediate from (3.3) that x^* is a solution of (3.7). Let now $x_o > x^*$ be another solution of (3.7), i.e.,

$$g(x_o) = \int_{[x_o,\infty)} G_r(x_o, y) \sigma(dy).$$

From (3.3) we have

$$g(x_o) = V(x_o) = \int_{[x^*,\infty)} G_r(x_o, y) \sigma(dy).$$

Consequently,

$$\int_{[x^*,x_o)} G_r(x_o,y)\sigma(dy) = 0. \tag{3.8}$$

But the function $y \mapsto G_r(x_o,y)$ is lower semi-continuous, and, hence, the set $\{y : G_r(x_o,y) > 0\}$ is open. From (3.8) it follows that $G_r(x_o,\cdot) \equiv 0$ on (x^*,x_o), but this violates (3.6) and the claim is proved. □

The presented method works similarly when the stopping region is not a half line, i.e. when the problem is not a "one-sided" problem. The form of the optimal stopping time (3.4) appears very often in several optimal stopping problems, in particular in mathematical finance, where this sort of the problems are sometimes named *call-like* perpetual problems or options (see e.g. [3]). Furthermore, as exposed in section 4, one sided problems in the context of Lévy process admits a representation in terms of the maximum of the Lévy process.

Minor modifications in the proof of Theorem 3.1 give the following result, that can be considered as a "two-sided" optimal stopping problem.

Theorem 3.3. *Consider a Hunt process $\{X_t\}$ satisfying the assumptions made in Section 2 and a non-negative continuous reward function g, and a discount rate $r \geq 0$, such that condition* (3.2) *hold. Assume that there exists a positive Radon measure σ with support on the set $S = (-\infty, x_*] \cup [x^*, \infty)$ such that the function*

$$V(x) := \int_S G_r(x,y)\sigma(dy) = \int_{(-\infty,x_*]} G_r(x,y)\sigma(dy) + \int_{[x^*,\infty)} G_r(x,y)\sigma(dy) \tag{3.9}$$

satisfies the following conditions:

(a) V *is continuous,*

(b) $V(x) = g(x)$ *when* $x \in S$,

(c) $V(x) \geq g(x)$ *when* $x \notin S$.

Let

$$\tau^* = \inf\{t \geq 0 \colon X_t \in S\}.$$

Then τ^* is an optimal stopping time and V is the value function of the optimal stopping problem for $\{X_t\}$ with the reward function g, in other words,

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}_x\left(\mathrm{e}^{-r\tau} g(X_\tau)\right) = \mathbf{E}_x\left(\mathrm{e}^{-r\tau^*} g(X_{\tau^*})\right), \quad x \in \mathbf{R}.$$

4 Optimal stopping and maxima for Lévy processes

As we have mentioned, in several papers explicit solutions to optimal stopping problems for general random walks or Lévy process, and some particular reward functions can be expressed in terms of the maximum of the process, killed at a constant rate r, the discount rate of the problem. The pioneer results in this direction are contained in the paper of Darling, Ligget and Taylor [5], were solutions to optimal stopping problems for rewards $g(x) = (\mathrm{e}^x - 1)^+$ and $g(x) = x^+$ are obtained in the whole class of random walks, in terms of the maximum of the random walk. These results are generalized for Lévy process by Mordecki in [15] and [16], where it is also observed that similar results hold for solutions of optimal stopping problems in terms of the infimum of the process for the payoff $g(x) = (1 - \mathrm{e}^x)^+$. Based on the technique of factorization of pseudo-differential operators, Boyarchenko and Levendorskij (see [3] and the references therein) obtain similar results, in a subclass of Lévy processes, called regular and of exponential type (RLPE), with the important feature that their results are not based on particular properties of the reward function g, and, hence, hold true in a certain class of rewards functions. Results in [3] suggest that any optimal stopping problem for a Lévy process with an increasing payoff can be expressed in terms of the maximum of the process. More recently, Novikov and Shiryaev [18] obtained the solution of the optimal stopping problem for a general random walk, in terms of the maximum, when the reward is $g(x) = (x^+)^n$, (and also when $g(x) = 1 - \mathrm{e}^{-x^+}$). The respective generalization of this problem to the framework of Lévy processes has been carried out by Kyprianou and Surya [12].

4.1 Lévy processes

Let $X = \{X_t\}$ be a Hunt process with independent and stationary increments, i.e. a Lévy process. We denote $\mathbf{E} = \mathbf{E}_0$ and $\mathbf{P} = \mathbf{P}_0$.

If $v \in \mathbf{R}$, Lévy-Khinchine formula states $\mathbf{E}(\mathrm{e}^{ivX_t}) = \mathrm{e}^{t\psi(iv)}$, where, for complex $z = iv$ the characteristic exponent of the process is

$$\psi(z) = az + \frac{1}{2}b^2z^2 + \int_{\mathbf{R}} \big(\mathrm{e}^{zx} - 1 - zh(x)\big)\Pi(dx). \tag{4.1}$$

Here the truncation function $h(x) = x\mathbf{1}_{\{|x|\leq 1\}}$ is fixed, and the parameters characterizing the law of the process are: the drift a, an arbitrary real number; the standard deviation of the Gaussian part of the process $b \geq 0$; and the Lévy jump measure Π, a non negative measure, defined on $\mathbf{R} \setminus \{0\}$ such that $\int (1 \wedge x^2)\Pi(dx) < +\infty$.

Denote by $\tau(r)$ an exponential random variable with parameter $r > 0$, independent of the process X. A key role in this section is played by the following random variables:

$$M_r = \sup_{0 \leq t < \tau(r)} X_t \qquad \text{and} \qquad I_r = \inf_{0 \leq t < \tau(r)} X_t \tag{4.2}$$

called the *supremum* and the *infimum* of the process, respectively, killed at rate r.

A relevant instrument to study these random variables is the Wiener-Hopf-Rogozin factorization, obtained by Rogozin [19], that states

$$\frac{r}{r - \psi(iv)} = \mathbf{E}(\mathrm{e}^{ivM_r})\mathbf{E}(\mathrm{e}^{ivI_r}) \tag{4.3}$$

In our first result we give some simple sufficient conditions in order to hypothesis (3.2) to hold.

Lemma 4.1. *Assume that a non-negative function g satisfies*

$$g(x) \leq A_0 + A_1 \mathrm{e}^{\alpha x}, \tag{4.4}$$

for nonnegative constants A_0, A_1, α. Assume furthermore that

$$\mathbf{E}\left(\mathrm{e}^{\alpha X_1}\right) < \mathrm{e}^r. \tag{4.5}$$

Then, condition (3.2) *holds.*

Proof. Let us first verify that, for $r \geq 0$, the following three statements are equivalent:

(a) $\mathbf{E}\left(\mathrm{e}^{\alpha X_1}\right) < \mathrm{e}^r$.

(b) $\mathbf{E}\left(\mathrm{e}^{\alpha M_r}\right) < \infty$.

(c) $\mathbf{E}\left(\sup_{t \geq 0}\left(\mathrm{e}^{\alpha X_t - rt}\right)\right) < \infty$.

First, (a)⇔(b) is Lemma 1 in [15]. The equivalence (a)⇔(c) is a particular case of (a)⇔(b), when considering the Lévy process $\{\alpha X_t - rt\}$, the first constant equal to 1, and the second, the discount rate equal to 0. Now

$$\mathbf{E}_x(\sup_{t\geq 0} \mathrm{e}^{-rt} g(X_t)) \leq \mathbf{E}\left(\sup_{t\geq 0} \mathrm{e}^{-rt}\left(A_0 + A_1 \mathrm{e}^{\alpha(x+X_t)}\right)\right) \\ \leq A_0 + A_1 \mathrm{e}^{\alpha x}\mathbf{E}(\sup_{t\geq 0} \mathrm{e}^{(\alpha X_t - rt)}) < \infty$$

as condition (a)⇒(c). □

Remark 4.2. *Condition* (4.4) *is relatively natural in our context. For instance, if the function is increasing, and* submultiplicative *(as defined in section 25 in [21]) it automatically satisfies our exponential growth condition* (4.4)*. Nevertheless, the submultiplicative property does not seem to be appropiate for optimal stopping problems, as $g(x) = x^+$ is not submultiplicative. Furthermore, condition* (4.5) *is optimal in the following sense: For the reward function $g(x) = (\mathrm{e}^x - 1)^+$, if $\mathbf{E}\left(\mathrm{e}^{X_1}\right) = \mathrm{e}^r$, then condition* (3.2) *does not hold, based on* (a)⇔(c)*.*

Our next result represents the value function of the optimal stopping problem for a Lévy process in terms of the maximum of the process and is a consequence of Theorem 3.1.

Proposition 4.3. *Assume that the conditions of Theorem 3.1 hold, and, furthermore, that $\{X_t\}$ is in fact a Lévy process. Then, there exists a function $Q\colon [x^*, \infty) \to \mathbf{R}$ such that the value function V in* (3.3) *satisfies*

$$V(x) = \mathbf{E}_x\left(Q(M_r)\,;\, M_r \geq x^*\right), \qquad x \leq x^*.$$

Proof. The key ingredient of the proof is formula (4.3), that can be also written as

$$X_{\tau(r)} = M_r + \tilde{I}_r \tag{4.6}$$

where M_r and $\tilde{I}_r = X_{\tau(r)} - M_r$ are *independent* random variables, M_r given in (4.2), and $\tilde{I}_r$ with the same distribution as I_r in (4.2).

From the definition of the Green kernel (2.1), it is clear that

$$rG_r(x, dy) = \mathbf{P}_x(X_{\tau(r)} \in dy),$$

and, in view of (4.6), assuming that M_r and I_r have respective densities f_M and f_I (only for simplicity of exposition), we obtain that

$$rG_r(x,y) = \begin{cases} \int_{-\infty}^{y-x} f_I(t) f_M(y-x-t)dt, & \text{if } y-x<0, \\ \\ \int_{y-x}^{\infty} f_M(t) f_I(y-x-t)dt, & \text{if } y-x>0. \end{cases} \tag{4.7}$$

If we plugg in this formula for the Green kernel in (3.3), when $x < x^*$, and, in consequence, with $y > x$, we obtain

$$\begin{aligned} V(x) &= \int_{x^*}^{\infty} G_r(x,y)\sigma(dy) \\ &= r^{-1}\int_{x^*}^{\infty}\left[\int_{y-x}^{\infty} f_M(t) f_I(y-x-t)dt\right]\sigma(dy) \\ &= r^{-1}\int_{x^*-x}^{\infty} f_M(t)\left[\int_{x^*}^{x+t} f_I(y-x-t)\sigma(dy)\right]dt \\ &= \int_{x^*-x}^{\infty} f_M(t)Q(x+t)dt = \mathbf{E}_x\left(Q(M_r)\,;\, M_r \geq x^*\right), \end{aligned}$$

where, for $z \geq x^*$, we denote

$$Q(z) = r^{-1}\int_{x^*}^{z} f_I(y-z)\sigma(dy). \tag{4.8}$$

This concludes the proof. □

The following results uses Theorem 3.3 to provide a representation of the value function in terms of both the supremum and the infimum of the Lévy process.

Proposition 4.4. *Assume that the conditions of Theorem 3.3 hold, and that $\{X_t\}$ is a Lévy process. Then, there exist two functions*

$$Q_*\colon (-\infty, x_*] \to \mathbf{R}, \qquad Q^*\colon [x^*, \infty) \to \mathbf{R}$$

such that the value function V in (3.3) satisfies

$$V(x) = \mathbf{E}_x\left(Q_*(I_r)\,;\, I_r \leq x_*\right) + \mathbf{E}_x\left(Q^*(M_r)\,;\, M_r \geq x^*\right), \qquad x_* \leq x \leq x^*.$$

Proof. The proof consist in rewriting each summand in (3.3) in terms of the maximum and infimum of the process, respectively. The second identity has been obtained in Proposition 4.3, and states (with Q^* instead of Q), that

$$\int_{[x^*,\infty)} G_r(x,y)\sigma(dy) = \mathbf{E}_x\left(Q^*(M_r)\,;\,M_r \geq x^*\right),$$

where Q^* is defined in (4.8). The first one is obtained from this last equality considering the dual Lévy process $\widehat{X}$, as follows:

$$\begin{aligned}\int_{(-\infty,x_*]} G_r(x,y)\sigma(dy) &= \int_{[-x_*,\infty)} \widehat{G}_r(-x,y)\sigma(-dy)\\ &= \widehat{\mathbf{E}}_{-x}\left(\widehat{Q}^*(\widehat{M}_r)\,;\,\widehat{M}_r \geq -x_*\right) = \mathbf{E}_x\left(Q_*(I_r)\,;\,I_r \leq x_*\right),\end{aligned}$$

where

$$Q_*(z) = \widehat{Q}^*(-z) = r^{-1}\int_{-x_*}^{-z} f_{\widehat{I}}(y+z)\sigma(-dy) = r^{-1}\int_{z}^{x_*} f_M(y-z)\sigma(dy),$$

and the proof is complete. □

5 A case study

5.1 Brownian motion with exponential jumps

Here we illustrate the assumptions made in Section 2 and, in particular, the concept of Green kernel by taking X to be a Brownian motion with drift and compounded with two-sided exponentially distributed jumps.

To introduce X, consider a standard Wiener process $W = \{W_t\colon t \geq 0\}$, $N^\lambda = \{N_t^\lambda\colon t \geq 0\}$ and $N^\mu = \{N_t^\mu\colon t \geq 0\}$ two Poisson processes with intensities λ and μ, respectively, $Y^\alpha = \{Y_i^\alpha : i = 1, 2, \dots\}$ and $Y^\beta = \{Y_i^\beta : i = 1, 2, \dots\}$ two sequences of independent exponentially distributed random variables with parameters α and β, respectively. Moreover, $W, N^\lambda, N^\mu, Y^\alpha$ and Y^β are assumed to be independent. The process $X = \{X_t\}$ is now defined via

$$X_t = at + bW_t + \sum_{i=1}^{N_t^\lambda} Y_i^\alpha - \sum_{i=1}^{N_t^\mu} Y_i^\beta, \tag{5.1}$$

where a and $b \geq 0$ are real parameters. Clearly, X is a Lévy process and its Lévy-Khintchine representation is given by

$$\mathbf{E}\left(\exp(z\,X_t)\right) = \exp(t\,\psi(z)) \tag{5.2}$$

with

$$\psi(z) = az + \frac{1}{2}b^2z^2 + \lambda\,\frac{z}{\alpha - z} - \mu\,\frac{z}{\beta + z}. \tag{5.3}$$

It is enough for our purposes to take hereby z real, and then the representation in (5.2) holds for $z \in (-\beta, \alpha)$.

Next we compute the Green kernel of X when all the parameters in (5.3) are positive. It is easily seen that for $r \geq 0$ the equation $\psi(z) = r$ has exactly four solutions ρ_k, $k = 1, 2, 3, 4$. These satisfy

$$\rho_1 < -\beta < \rho_2 \leq 0 < \rho_3 < \alpha < \rho_4 \tag{5.4}$$

and

$$\psi'(\rho_1) < 0, \quad \psi'(\rho_2) < 0, \quad \psi'(\rho_3) > 0, \quad \psi'(\rho_4) > 0. \tag{5.5}$$

Notice that $\rho_2 = 0$ if and only if $r = 0$, in which case it is assumed $\psi'(0) < 0$ implying

$$\lim_{t\to\infty} X_t = -\infty \quad \text{a.s.}$$

Using the general definition of the resolvent, see (2.1), we have for $z \in (\rho_2, \rho_3)$

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{e}^{z\,x}\, G_r(0, dx) &= \int_0^{\infty} dt\, \mathrm{e}^{-rt}\, \mathbf{E}\left(\exp(z\,X_t)\right) = \frac{1}{r - \psi(z)} \\ &= \frac{\psi'(\rho_1)^{-1}}{\rho_1 - z} + \frac{\psi'(\rho_2)^{-1}}{\rho_2 - z} + \frac{\psi'(\rho_3)^{-1}}{\rho_3 - z} + \frac{\psi'(\rho_4)^{-1}}{\rho_4 - z}.\end{aligned}$$

Consequently, inverting the right hand side yields

$$G_r(0, dx) = \begin{cases} -\psi'(\rho_1)^{-1}\mathrm{e}^{-\rho_1 x}dx - \psi'(\rho_2)^{-1}\mathrm{e}^{-\rho_2 x}dx, & x < 0, \\ \psi'(\rho_3)^{-1}\mathrm{e}^{-\rho_3 x}dx + \psi'(\rho_4)^{-1}\mathrm{e}^{-\rho_4 x}dx, & x > 0. \end{cases} \tag{5.6}$$

and, hence, the resolvent is absolutely continuous with respect to Lebesgue measure. With slight abuse of notation, we let $G_r(0, x)$ denote also the Green kernel, i.e., the density of the resolvent G_r with respect to the Lebesgue measure. From the spatial homogeniety of X it follows that $G_r(x, 0) = G_r(0, -x)$.

The absolute continuity of the resolvent can alternatively be verified by checking that the condition (ii) in Theorem II.5.16 in Bertoin [1] holds . We recall also the general result (see [1] p. 25) which says that the absolute continuity of the resolvent is equivalent with the property that $x \mapsto G_r f(x)$ is continuous for all essentially bounded measurable functions f.

As we have noticed, the process $\widehat{X} = \{-X_t\}$ may be viewed as a dual process associated with X. Let $\widehat{G}_r$ denote the resolvent of $\widehat{X}$. Then the duality relationship

$$\int dx\, f(x)\, G_r g(x) = \int dx\, \widehat{G}_r f(x)\, g(x)$$

holds the duality measure being the Lebesgue measure. The Green kernel of the dual process is given by

$$\widehat{G}_r(x,y) = G_r(y,x).$$

Notice that the value of $x \mapsto G_r(0,x)$ at 0 is chosen so that the resulting function is lower semi-continuous (since the Green kernel when considered as a function of the second argument should be excessive for the dual process).

To conclude the above discussion, we have verified Hypothesis (B) in [11], that is, $(\mathrm{h}_1), (\mathrm{h}_2)$ and (h_3) in Section 2 are fullfilled. Consequently, also $(\mathrm{p}_1), (\mathrm{p}_2)$ and (p_3) in Section 2 are valid and the Riesz decomposition (2.2) holds. Moreover, it can be proved, e.g. using the Martin boundary theory, as presented in [11], that the harmonic function h_r appearing in (2.2) is of the form

$$h_r(x) = c_1\, \mathrm{e}^{\rho_2 x} + c_2\, \mathrm{e}^{\rho_3 x},$$

where c_1 and c_2 are non-negative constants.

It is interesting to note that when multiplying both sides of (5.6) by z and letting $z \to \infty$ we obtain, in case $b > 0$ (cf. (5.3)),

$$\frac{1}{\psi'(\rho_1)} + \frac{1}{\psi'(\rho_2)} + \frac{1}{\psi'(\rho_3)} + \frac{1}{\psi'(\rho_4)} = 0,$$

which implies that the Green kernel is continuous at $x = 0$. But, when $b = 0$, the Green kernel may be discontinuous. This happens, for instance, when X is a compound Poisson process with negative drift and exponentially distributed positive jumps. More precisely, taking $b = \mu = 0$, and $a < 0$ in (5.1) the characteristic exponent reduces to

$$\psi(z) = az + \lambda\, \frac{z}{\alpha - z}.$$

Now there are only two roots ρ_1 and ρ_2 and these satisfy

$$\rho_1 \leq 0 < \rho_2.$$

Consequently,

$$\frac{1}{r - \psi(z)} = \frac{\psi'(\rho_1)^{-1}}{\rho_1 - z} + \frac{\psi'(\rho_2)^{-1}}{\rho_2 - z}, \tag{5.7}$$

and we have the Green kernel

$$G_r(0,x) = \begin{cases} -\psi'(\rho_1)^{-1}\,\mathrm{e}^{-\rho_1 x}, & x < 0, \\ \psi'(\rho_2)^{-1}\mathrm{e}^{-\rho_2 x}, & x \geq 0. \end{cases} \tag{5.8}$$

From (5.7) it is seen that

$$\psi'(\rho_1)^{-1} + \psi'(\rho_2)^{-1} = \frac{1}{a},$$

and, hence, $x \mapsto G_r(0,x)$ is discontinuous at 0 (but lower semi-continuous since $a < 0$ implies $-\psi'(\rho_1) < \psi'(\rho_2)$).

5.2 Optimal stopping of processes with two sided exponential Green kernel

We consider here a subclass of processes introduced in Section 5.1 the aim being to apply results in Theorem 3.1 and 3.3. Indeed, let X be a Lévy process having a Green kernel with the following simple exponential structure:

$$G_r(x) := G_r(0,x) = \begin{cases} -A_1\,\mathrm{e}^{-\rho_1 x}, & x < 0, \\ A_2\,\mathrm{e}^{-\rho_2 x}, & x \geq 0, \end{cases} \tag{5.9}$$

where $\rho_{1,2}$ are the roots of the equation $\psi(z) = r$ such that $\rho_1 \leq 0 < \rho_2$ and $A_{1,2} = 1/\psi'(\rho_{1,2})$. In the case $r = 0$ it is assumed that the process drifts to $-\infty$ and, hence, we have $\rho_1 = 0$.

The Green kernel of form (5.9) appears in two basic cases which, using the notation in (5.2) and (5.3), are:

- Wiener process with drift, i.e., $b > 0$, $\lambda = \mu = 0$,
- compound Poisson process with negative drift and positive exponential jumps, i.e., $a < 0$, $b = 0$, $\lambda > 0$ and $\mu = 0$.

The point we want to make here is that our approach to optimal stopping treats these processes similarly. Recall that in the case of Wiener process usually smooth pasting is valid when moving from the continuation region to the stopping region but in the compound Poisson case there is "only" continuous pasting. In other words, our approach does not use smooth pasting as a tool, but this property can, of course, be checked (when valid) from the calculated explicit form of the value function.

Proposition 5.1. *For a given $x^* \in \mathbf{R}$ let σ be a measure on $[x^*, +\infty)$ with a continuously differentiable density σ' on $(x^*, +\infty)$. Then the function*

$$V(x) := \int_{x^*}^{\infty} G_r(y - x)\sigma(dy)$$

is two times continuously differentiable on $D := \{x \in \mathbf{R}\colon x \neq x^*\}$ and satisfies on D the ordinary differential equation (ODE)

$$\begin{aligned} V''(x) - (\rho_2 + \rho_1)V'(x) + \rho_1\rho_2 V(x) \\ = -(A_2 + A_1)\sigma''(x) + (\rho_2 A_1 + \rho_1 A_2)\sigma'(x), \end{aligned} \tag{5.10}$$

where $\sigma''(x) = \sigma'(x) = 0$ for $x \in (-\infty, x^*)$.

Proof. From the definition of V, taking into account the form of the Green kernel, we have for $x > x^*$

$$V(x) = -A_1 e^{\rho_1 x} \int_{x^*}^{x} e^{-\rho_1 y}\sigma(dy) + A_2 e^{\rho_2 x} \int_{x}^{\infty} e^{-\rho_2 y}\sigma(dy).$$

The right hand side of this equation can be differentiated twice proving that V'' exist in D, and the claimed ODE is obtained after some straightforward manipulations. □

Corollary 5.2. *Let X be a Wiener process with drift. Then the ODE in (5.10) takes the form*

$$\frac{b^2}{2}\, V''(x) + a\, V'(x) - r\, V(x) = -(a^2 + 2b^2 r)\, \sigma'(x). \tag{5.11}$$

Proof. The quantities needed to derive (5.11) from (5.10) are

$$\rho_1 = -\frac{1}{b^2}\left(\sqrt{a^2 + 2b^2 r} + a\right), \quad \rho_2 = \frac{1}{b^2}\left(\sqrt{a^2 + 2b^2 r} - a\right)$$

and

$$A_1 = -\sqrt{a^2 + 2b^2 r}, \quad A_2 = \sqrt{a^2 + 2b^2 r}.$$

In particular, notice that $A_2 + A_1 = 0$ which reflects the fact that the Green kernel is continuous. □

In Novikov and Shiryayev [18] the optimal stopping problem for a general random walk with reward function $\max\{0, x^n\}$, $n = 1, 2, \ldots$, is considered, and the solution is characterized via the Appell polynomials associated with the distribution of the maximum of the process. In the next example we present explicit reults for a more general reward function, that is, $\max\{0, x^\gamma\}$, $\gamma \geq 1$, but for a more particular Lévy process studied in the subsection.

Example 5.3. Let X denote a compound Poisson process with negative drift and positive exponential jumps, i.e., take $a < 0$, $b = 0$ and $\mu = 0$ in (5.1). For simplicity, we consider optimal stopping problem without discounting:

$$\sup_{\tau \in \mathcal{M}} \mathbf{E}_x \left(g(X_\tau) \right),$$

where $g(x) := \max\{0, x^\gamma\}$ with $\gamma \geq 1$. For $r = 0$ the Green kernel of X is

$$G(x, 0) := G_0(x, 0) = \begin{cases} A_2 \, \mathrm{e}^{\rho \, x} \, dx, & x \leq 0, \\ -A_1, & x > 0, \end{cases} \tag{5.12}$$

where

$$\rho := \rho_2 = \alpha + \frac{\lambda}{a} > 0 \tag{5.13}$$

and

$$A_1 = \frac{\alpha}{\lambda + a\alpha} < 0, \qquad A_2 = \frac{\lambda}{a(\lambda + a\alpha)} > 0.$$

Notice that $\rho > 0$ means that a.s. $\lim_{t \to \infty} X_t = -\infty$.

Our aim is to find a measure σ and a number x^* such that the function V defined via

$$V(x) = \int_{[x^*, +\infty)} G(x, y) \sigma(dy) \tag{5.14}$$

has properties (a), (b), (c) and (d) given in Theorem 3.1.

To begin with, consider equation (5.10) for $x > x^*$ and $V(x) = x^\gamma$, that is,

$$-\sigma''(x) + \alpha\sigma'(x) = a\gamma(\gamma - 1)x^{\gamma - 2} - (a\alpha + \lambda)\, \gamma x^{\gamma - 1}.$$

Assuming $\lim_{x\to+\infty} \mathrm{e}^{-\alpha x}\,\sigma'(x) = 0$ we obtain the solution

$$\sigma'(x) = -a\gamma\, x^{\gamma-1} - \lambda \mathrm{e}^{\alpha x} \int_x^\infty \mathrm{e}^{-\alpha y}\gamma\, y^{\gamma-1}\, dy.$$

If $\gamma = 1$ then $\sigma'(x) = -a - (\lambda/\alpha) > 0$. For $\gamma > 1$ it is easily seen that $\sigma'(0) < 0$ and $\sigma'(x) \to +\infty$ as $x \to \infty$.

The claim is that the equation $\sigma'(x) = 0$, that is

$$x^{\gamma-1} = \frac{\lambda}{(-a)}\, \mathrm{e}^{\alpha x} \int_x^\infty \mathrm{e}^{-\alpha z}\, z^{\gamma-1}\, dz, \tag{5.15}$$

has a unique solution for $x > 0$, which we denote by $x^*_{\gamma-1}$. Equation (5.15) is equivalent to

$$F(x, \gamma - 1) = 1, \tag{5.16}$$

if we define

$$F(x; u) := \frac{\lambda}{(-a)} \int_0^\infty \mathrm{e}^{-\alpha y} \left(1 + \frac{y}{x}\right)^u dy. \tag{5.17}$$

We revise some properties of the function just introduced.

Lemma 5.4. *The function $F(x, u)$ in* (5.17) *defines an implicit function $\varphi\colon [1, \infty) \to \mathbf{R}$ such that $F(\varphi(u), u) = 1$ for each $u \geq 1$. Furthermore, the function φ is increasing, and satisfies the inequality*

$$\varphi(1) < \varphi(u) < \frac{u}{\rho}. \tag{5.18}$$

Proof. It is not difficult to verify that, for fixed $u > 0$, the function F is decreasing in x, and that

$$\lim_{x\to 0+} F(x, u) = \infty, \qquad \lim_{x\to\infty} F(x, u) = \frac{\lambda}{(-a)\alpha} < 1.$$

This means that for any $u \geq 1$ the equation $F(x, u) = 1$ has a unique solution $x := \varphi(u)$. Furthermore, it is also clear that, for fixed $x > 0$, the function $F(x, u)$ is increasing in u. This means, that φ is increasing, as

$$\frac{\partial \varphi}{\partial u} = -\frac{\partial F}{\partial x} \left(\frac{\partial F}{\partial u}\right)^{-1} > 0.$$

Finally, multiplying the inequality

$$\left(1+u\frac{y}{x}\right) \le \left(1+\frac{y}{x}\right)^u \le \mathrm{e}^{uy/x}$$

by $\mathrm{e}^{-\alpha y}$ and integrating we obtain that

$$F_1(x,u) := \frac{\lambda}{(-a)\alpha}\left(1+\frac{u}{\alpha x}\right) < F(x,u) < \frac{\lambda}{(-a)}\frac{1}{\alpha - u/x} =: F_2(x,u).$$

and (5.18) follows as the bounds are the respective roots of the equations $F_1(x,u)=1$, $F_2(x,u)=1$, and, in particular, the root x_1 of the first equation is

$$x_1 = \frac{\gamma\lambda}{(-a)\alpha\rho},$$

and $x_1^* = -\lambda/a\alpha\rho$. This last value can be computed from the equation $F(x,1)=1$, and was found in [14]. This concludes the proof of the Lemma. □

Observe now that for $x > x^*_{\gamma-1} =: x^o_\gamma$ the function σ' induces a positive Radon measure on (x^o_γ, ∞). However, since, for any constant c, the function $x^\gamma + c$ induces the same measure as the function x^γ it remains to find, the support of σ of the form (x^*, ∞) such that for all $x > x^*$

$$x^\gamma = \int_{[x^*,+\infty)} G(x,y)\,\sigma'(y)\,dy.$$

Therefore, consider

$$\begin{aligned}\int_x^\infty G(x,y)\sigma(dy) &= A_2\,\mathrm{e}^{\rho x}\int_x^\infty \mathrm{e}^{-\rho y}\,\sigma'(y)\,dy\\ &= \gamma\,A_2\,\mathrm{e}^{\rho x}\int_x^\infty \mathrm{e}^{-\rho y}\left(-a\,y^{\gamma-1} - \lambda \mathrm{e}^{\alpha y}\int_y^\infty \mathrm{e}^{-\alpha z}\,z^{\gamma-1}dz\right)dy.\end{aligned}$$

Applying Fubini's theorem for the latter term yields

$$\begin{aligned}&\int_x^\infty dy\,\mathrm{e}^{(\alpha-\rho)y}\int_y^\infty dz\,\mathrm{e}^{-\alpha z}\,z^{\gamma-1}\\ &\qquad = \frac{1}{\alpha-\rho}\left(\int_x^\infty \mathrm{e}^{-\rho z}\,z^{\gamma-1}\,dz - \mathrm{e}^{(\alpha-\rho)x}\int_x^\infty \mathrm{e}^{-\alpha z}\,z^{\gamma-1}\,dz\right).\end{aligned}$$

Observing that $-a = \lambda/(\alpha - \rho)$ we have

$$\int_x^\infty G(x,y)\sigma(dy) = \frac{\alpha - \rho}{\rho}\,\mathrm{e}^{\alpha x}\int_x^\infty \mathrm{e}^{-\alpha z}\,\gamma\, z^{\gamma-1}\,dz$$

Consequently, after an integration by parts, the equation

$$x^\gamma = \frac{\alpha - \rho}{\rho}\,\mathrm{e}^{\alpha x}\int_x^\infty \mathrm{e}^{-\alpha z}\,\gamma\, z^{\gamma-1}\,dz$$

is seen to be equivalent with

$$x^\gamma = (\alpha - \rho)\,\mathrm{e}^{\alpha x}\int_x^\infty \mathrm{e}^{-\alpha z}\, z^{\gamma}\,dz \tag{5.19}$$

which coincides with equation (5.15) if therein $\gamma - 1$ is changed to γ. Hence, equation (5.19) has a unique solution which is, using the notation introduced above, x^*_γ. As the function $x = \varphi(u)$ is increasing, we know that $x^o_\gamma = \varphi(\gamma - 1) < \varphi(\gamma) = x^*_\gamma$.

Next step is to verify that the value function obtained from (5.14) is continuous and satisfies $V(x) > x^\gamma$ for $x < x^*_\gamma$. Therefore consider for $x < x^*_\gamma$

$$V(x) = \int_{[x^*_\gamma, +\infty)} G(x,y)\sigma(dy) = \mathrm{e}^{\rho(x - x^*_\gamma)}(x^*_\gamma)^\gamma. \tag{5.20}$$

Consequently, V is continuous and

$$V(x) > x^\gamma \quad \Leftrightarrow \quad \mathrm{e}^{-\rho x^*_\gamma}(x^*_\gamma)^\gamma > \mathrm{e}^{-\rho x} x^\gamma. \tag{5.21}$$

The right hand side of (5.21) holds if $x \mapsto G(x) := \mathrm{e}^{-\rho x}x^\gamma$ is increasing for $x < x^*_\gamma$. Clearly, G' is positive if

$$-\rho x + \gamma > 0,$$

and this holds since $x^*_\gamma < \gamma/\rho$ by the second inequality in (5.18).

To conclude, the optimal stopping time τ^* is given by

$$\tau^* := \inf\{t : X_t \geq x^*_\gamma\},$$

and the value function V is for $x < x^*_\gamma$ as in (5.20). Since $V'(x^*_\gamma -) = \rho(x^*_\gamma)^\gamma$ and $g'(x^*_\gamma) = \gamma(x^*_\gamma)^{\gamma-1}$ there is no smooth fit at x^*_γ.

We conclude by presenting the following table with some numerical results. The computations are done with Mathematica-package where one can find a subroutine for incomplete gamma-function and programs for numerical solutions of equations based on standard Newton-Raphson's method and the secant method. A good starting value for Newton-Raphson's method seems to be γ/ρ. It is interesting to notice from the table that if $\rho << \alpha$ then $x_\gamma^* \simeq \gamma/\rho$.

α	ρ	$-\lambda/a$	γ	γ/ρ	x^*_γ	x^o_γ
10	1	9	20	20	19.8896	18.8896
10	1	9	10	10	9.8902	8.8904
10	1	9	5	5	4.8915	3.8921
10	1	9	2.5	2.5	2.3939	1.3968
10	1	9	1	1	.9	–
10	9	1	20	2.2222	1.7613	1.6579
10	9	1	10	1.1111	.7511	.6547
10	9	1	5	.5555	.2881	.2045
10	9	1	2.5	.2789	.0917	.0319
10	9	1	1	.1111	.0111	–
1	.5	.5	20	40	38.1592	36.166
1	.5	.5	10	20	18.2726	16.2942
1	.5	.5	5	10	8.4369	6.5011
1	.5	.5	2.5	5	3.6529	1.8398
1	.5	.5	1	2	1	–

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