Hyperbolic Real Quadratic Cellular Automata

J. Delgado[1], N. Romero[2], A. Rovella[3] and F. Vilamajó[4]

[1]Universidade Federal Fluminense. Instituto de Matemática
Rua Mario Santos Braga S.N. CEP 24020-140 Niterói (RJ), Brasil

[2]Universidad Centroccidental Lisandro Alvarado. Departamento de Matemáticas.
Decanato de Ciencias y Tecnología Apdo. 400. Barquisimeto, Venezuela

[3]Universidad de La República. Facultad de Ciencias. Centro de Matemática
Iguá 4225. C.P. 11400. Montevideo, Uruguay

[4]Universitat Politècnica de Catalunya. Departament de Matemàtica Aplicada 2
Colom 11, 08222. Terrasa, Barcelona, Spain

Abstract. In this paper we develop some techniques to obtain global hyperbolicity for a certain class of endomorphisms of $\mathbb{R}^n$ called real cellular automata, which are characterized by the property of commuting with a shift. In particular, we show that one parameter families of generic quadratic cellular automata in $\mathbb{R}^n$ are hyperbolic for large values of the parameter.

Keywords. Hyperbolicity. ω-limit set. Lattice Dynamical Systems. Cellular Automaton. Coupled Map Lattice.

2000 AMS (MOS) subject classification: 37C05 37D05

1 Definitions and statement of results

Several mathematical and computational models, as those that arise from biological networks, image processing or fluid dynamics by discretizing ordinary differential equations and from the qualitative analysis of the evolution of spatially extended dynamical systems given by partial differential equations, take us to the study of a special class of dynamical systems known as Lattices Dynamical Systems (LDS). Roughly speaking, a LDS is an infinite system of ordinary differential equations (continuous time) or difference equations (discrete time).

In order to define a discrete time lattice dynamical system, let Ω be a lattice (with discrete structure) whose elements are called cells (or sites). For each $\omega \in \Omega$, let X_ω be a topological space (in most applications those spaces are the same) and $\mathcal{M} = \prod_{\omega\in\Omega} X_\omega$ endowed with the product topology. A LDS is a pair $(\mathcal{M}, F)$, where $F = \{F_\omega\}_{\omega\in\Omega} : \mathcal{M} \to \mathcal{M}$ is a product structure preserving mapping, also called the global transition function, that is $F(\{x_\omega\}_{\omega\in\Omega}) = \{F_\omega(x)\}_{\omega\in\Omega}$. The state-transition in $(\mathcal{M}, F)$ is given by the difference equation $x(n+1) = F(x(n))$, where $x(n) = \{x_\omega(n)\}_{\omega\in\Omega} \in \mathcal{M}$ for every $n \in \mathbb{Z}_+$. Cellular Automata (CA) are LDS's for which $\Omega = \mathbb{Z}^k$ (integer k dimensional lattice) and, for every $\omega \in \Omega$, $X_\omega = X$ is a finite set (the alphabet in computation theory). The final ingredient of a CA is the action of dynamical systems generated by a group of spatial translations $\sigma : \mathcal{M} \to \mathcal{M}$, $x = \{x_\omega\}_{\omega\in\Omega} \mapsto \sigma(x) = \{x_{\omega'}\}_{\omega\in\Omega}$, that is $(\sigma(x))_\omega = x_{\omega'}$, where the correspondence $\omega \to \omega'$ is a bijection of Ω, and every translation commutes with the global transition function: $F \circ \sigma = \sigma \circ F$, see [5]. Earlier definitions of CA was expressed in terms of a block map f [12]. However, it follows from a result of Hedlund, [4], that both definitions are equivalent.

The natural generalization of CA in terms of the block funtions are the *Coupled Map Lattices* (CML). A definition of CML of d-dimensional and p-component unbounded media is in [9]. The fundamental difference between CA and CML is that the spaces X_ω have an uncountable number of elements and the lattices Ω also can be uncountable. A dissertation about the definition of CML can be found in [3].

We will consider CML's with a finite lattice Ω and $X_\omega = \mathbb{R}$ for every $\omega \in \Omega$. In this setting, one can consider the lattice as a finite set of cells ordered linearly following a circle, subordinate to the action of a group of transitive permutations. This kind of systems are known as *coupled map lattices with periodic boundary condition, circular cellular automata or real cellular automata*, see [11]. Here we use the last terminology.

In [11] real cellular automata appear to describe bifurcations of coupled logistic map together a linear influence from the neighbors; real cellular automata in $\mathbb{R}^2$ are used in [10] to study competitive maps with complicated dynamics and to describe global properties of competition models with riddling and blowout phenomena in [1]. Real cellular automata appear in [6] and in [7] for study a synchronization problem in dissipative physical systems, for a unified framework of this physical phenomenon see [13].

From this time forth, we consider real cellular automata defined on $\mathbb{R}^n$, that is $\Omega = \{1, 2, \ldots, n\}$ and $X_\omega = \mathbb{R}$, for every $\omega \in \Omega$. The group of translations on this lattice are the transitive permutation on n elements, i.e., permutation on Ω with only one periodic orbit.

It is well known that, for every transitive permutations τ_1 and τ_2 of n cells of Ω, there exists a third one, η, such that $\tau_1\eta = \eta\tau_2$. Thus, if $F : \mathbb{R}^n \to \mathbb{R}^n$ commutes with the linear mapping $\overline{\tau_1} : \mathbb{R}^n \to \mathbb{R}^n$ induced by τ_1, there exists $G : \mathbb{R}^n \to \mathbb{R}^n$ commuting with the linear mapping $\overline{\tau_2} : \mathbb{R}^n \to \mathbb{R}^n$ and linearly conjugated to F. Therefore, we will consider a real cellular automaton as a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ communting with the circular shift on $\mathbb{R}^n$:

$$\sigma(x_1, \ldots, x_n) = (x_n, x_1, \ldots, x_{n-1}). \tag{1}$$

The set of real cellular automata of $\mathbb{R}^n$ will be denoted by $\mathcal{A}_n$. A straightforward verification shows that for every cellular automaton $F : \mathbb{R}^n \to \mathbb{R}^n$ there exists a unique function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$F(X) = \left(f(X), f(\sigma^{-1}(X)), \cdots, f(\sigma^{-n+1}(X))\right). \tag{2}$$

In computation theory of cellular automata, the function f is the block function of the cellular automaton given by (2).

In this paper, we will focus our attention to describe dynamical properties of some real cellular automata. Our first problem consist in determine the consequences of the absence of fixed points for a class of real cellular automata. This question arose from [2] by proving that every orbit of a plane orientation preserving homeomorphism without fixed points diverges (that is, the ω-limit set of every point is empty). In section 3 we will prove the following result:

Theorem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\sum_{i=0}^{n-1} f \circ \sigma^{-i}$ is a C^0 strictly convex function. If the real cellular automaton F given by the block function f has no fixed points, then the ω-limit set of any point is empty.*

We will present, also in section 3, a result concerning with the hyperbolicity of quadratic real cellular automata. It is a simple fact that any quadratic real cellular automaton in $\mathbb{R}^n$ is determined by a unique symmetric linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$, a linear function $L : \mathbb{R}^n \to \mathbb{R}$ and a real number μ, and it is given by:

$$F_\mu(X) = (f_\mu(X), f_\mu(\sigma^{-1}(X)), \cdots, f_\mu(\sigma^{-n+1}(X))), \tag{3}$$

where $f_\mu(X) = f(X) + \mu = \langle A(X), X\rangle + L(X) + \mu$ and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product of $\mathbb{R}^n$.

In order to state our second result, we denote by $\mathcal{S}(n)$ the space of symmetric linear transformations of $\mathbb{R}^n$, and $\mathcal{L}(n)$ the space of linear functions of $\mathbb{R}^n$.

For any quadratic function $f(X) = \langle A(X), X\rangle + L(X)$ with $A \in \mathcal{S}(n)$ and $L \in \mathcal{L}(n)$, we say that the real cellular automaton F_μ given by (3) is *hyperbolic for every $|\mu|$ sufficiently large* if there exist $\mu_1 < \mu_2$ such that, for any $\mu < \mu_1$ the orbit of every point in $\mathbb{R}^n$ is attracted by ∞, and for any $\mu > \mu_2$ there exists an invariant compact set $C(\mu)$ such that:

1. Every point outside $C(\mu)$ has empty ω-limit set, that is, its orbit is attracted by ∞.

2. There exist constants $K > 0$ and $\lambda > 1$ such that for every $X \in C(\mu)$, $v \in \mathbb{R}^n$ and $m \in \mathbb{Z}_+$, $\|(DF_\mu^m)_X(v)\| \geq K\lambda^m \|v\|$.

That is, $C(\mu)$ is an expanding Cantor set whose complementary set is the basin of attraction of ∞.

Once established this concept of hyperbolicity for large enough $|\mu|$, our second result, about hyperbolicity is stated as follows:

Theorem 2. *There exists an open and dense subset $\mathcal{S}'(n) \subset \mathcal{S}(n)$ such that, for every $A \in \mathcal{S}'(n)$ any quadratic real cellular automaton F_μ as given in equation (3) is hyperbolic for every $|\mu|$ sufficiently large.*

To prove theorem 2 we will need some preliminaries on transversality. For each $A \in \mathcal{S}(n)$ and $0 \leq j \leq n-1$ define the quadric

$$\xi_j^A = \{X \in \mathbb{R}^n : \langle A\sigma^{-j}(X), \sigma^{-j}(X)\rangle = 1\},$$

observe that $\xi_j^A = \sigma^j(\xi_0^A)$ for all $j = 0, \cdots, n-1$.

We will use the following argument to obtain expansivity for F_μ. Suppose that D is a matrix whose rows are not only linearly independent, but make an angle bounded away from zero. If in addition, the rows of D have sufficiently large norms, then D expands any vector. Now, the rows of the differential matrix of F_μ at a point X consists of the normal vectors to the level sets of the coordinate functions of F_μ at X. Therefore, the transversality of $\{\xi_0^A, \cdots, \xi_{n-1}^A\}$ is an important ingredient, that accompanied with the fact that the coordinates of F_μ are quadratic and that the nonwandering set of F_μ is far from the origin for large values of $|\mu|$ gives a proof of the result.

The following proposition, which will be proved in section 2, deals with the transversality of the level sets of the cellular automaton F_μ and it has a fundamental role in the proof of the theorem 2.

Proposition 1. *(a) There exists an open and dense subset $\mathcal{S}_{n-1}$ of $\mathcal{S}(n)$, such that for every $A \in \mathcal{S}_{n-1}$ the set $\{\xi_0^A, \cdots, \xi_{n-1}^A\}$ is transverse.*
(b) Let $\mathcal{S}'(n)$ be the subset of $\mathcal{S}_{n-1}$ of transformations A for which the following property is satisfied: $\langle \sigma^j A \sigma^{-j}(X), X\rangle = 0$ for every j, implies $X = 0$. Then $\mathcal{S}'(n)$ is open and dense in $\mathcal{S}(n)$.

Recall that a set of n codimension-one submanifolds of $\mathbb{R}^n$ is said to be transverse if at each point of intersection of these submanifolds, the set of normal vectors is linearly independent. In addition, two submanifolds $\xi_1, \xi_2 \subset \mathbb{R}^n$ are transverse at $X \in \xi_1 \cap \xi_2$ if $\mathbb{R}^n$ is spanned by the tangent spaces of ξ_1 and ξ_2 at X. Observe that if $\xi_0, \cdots, \xi_{n-1}$ are codimension-one submanifolds of $\mathbb{R}^n$ and for every $0 < j < n-1$ the intersection $\bigcap_{i=0}^{j} \xi_i$ is a codimension-$(j+1)$ submanifold transverse to ξ_{j+1}, then $\{\xi_0, \cdots, \xi_{n-1}\}$ is transverse.

2 Preliminaries on transversality

This section is dedicated to the proof of proposition 1; its main ingredient is the use of the well known parametrized transversality theorem, in the following way. Consider the mapping $\Phi : \mathcal{S}(n) \times \mathbb{R}^n \to \mathbb{R}^k$ given by

$$\Phi(A, X) = (\langle A(X), X\rangle, \cdots, \langle A\sigma^{-k+1}(X), \sigma^{-k+1}(X)\rangle),$$

with $1 \leq k \leq n$. Suppose that one can prove that the point $u = (1, \cdots, 1)$ in $\mathbb{R}^k$ is a regular value of Φ. By the above mentioned theorem, it follows that there exists an open and dense set of symmetric linear transformations such that for every A in this set, the mapping Φ_A, defined by $\Phi_A(X) = \Phi(A, X)$, has the point u as a regular value. This means that for such A and any point X satisfying $\langle A\sigma^{-j+1}(X), \sigma^{-j+1}(X)\rangle = 1$ for all $1 \leq j \leq k$, it holds that the set of normal vectors to the ξ_j^A is linearly independent, and this is equivalent to the transversality of $\{\xi_0^A, \cdots, \xi_{k-1}^A\}$. The problem is that the point u is not a regular value of Φ for every A. To obtain this condition one has to take a careful look at the geometry of the transformation σ and then use an inductive argument. For this we will need some elementary results. The first one is a trivial assertion:

Lemma 1. *Let H and L be linear subspaces of $\mathbb{R}^n$ with $\dim H + \dim L \leq n$. Then the set of $A \in \mathcal{S}(n)$ such that $A(H) \cap L = \{0\}$ is open and dense.*

Now we introduce some notations: for $k < n$, let D_k be the set of X in $\mathbb{R}^n$ such that $\{X, \sigma(X), \cdots, \sigma^k(X)\}$ is linearly dependent (ld); the minimal nontrivial invariant subspaces of σ are denoted by $\Delta_0, \cdots, \Delta_{[\![n/2]\!]}$, where $[\![a]\!]$ denotes the integer part of a; Δ_0 is the diagonal of $\mathbb{R}^n$, that is the eigenspace associated to the eigenvalue 1 of σ; when n is even, $\Delta_{n/2}$ is the one-dimensional eigenspace associated to the eigenvalue -1 of σ. When n is odd, we will define $\Delta_{n/2} = \{0\}$ by convenience; any other Δ_j is a plane, and σ restricted to Δ_j is a rotation of angle $\frac{2\pi j}{n}$; finally, for each $1 \leq k \leq n-1$, $H_k = ker(\sigma^k - I)$ and $H_{-k} = ker(\sigma^k + I)$, where I is the identity operator in $\mathbb{R}^n$, and $ker(T)$ denotes the kernel of the linear operator T.

Lemma 2. *Given $1 \leq k \leq n-1$, let $\{v_0, \cdots v_k\} \subset \mathbb{R}^n$ such that:*

1. *$\{v_0, \cdots v_{k-1}\}$ is linearly independent (li), and*
2. *$\{v_j, v_k\}$ is li for every $0 \leq j \leq k-1$.*

Then the transformation $\varphi = \varphi_{\{v_0, \cdots, v_k\}}$ defined in $\mathcal{S}(n)$ and given by:

$$\varphi(V) = (\langle V(v_0), v_0\rangle, \cdots, \langle V(v_k), v_k\rangle),$$

is onto $\mathbb{R}^{k+1}$.

Proof. First we asume that $\{v_0, \cdots, v_k\}$ is ld. Let $U = \{v_0, \cdots, v_{k-1}\}$ and $[U]$ the linear subspace generated by U. For each $0 \leq i \leq j \leq k-1$ and $0 \leq h, l \leq k-1$ define

$$\langle V_{ij}(v_h), v_l\rangle = \frac{1}{2}(\delta_{(i,j),(h,l)} + \delta_{(i,j),(l,h)}),$$

where δ is the Kronecker symbol. It is easy to see that each V_{ij} can be extended in a unique way to a symmetric linear transformation defined in $[U]$, and the set of all V_{ij} with $0 \leq i \leq j \leq k-1$ constitutes a basis for the space of all symmetric linear transformations of $[U]$.

Given $\alpha = (\alpha_0, \cdots, \alpha_k) \in \mathbb{R}^{k+1}$, we will find $V = \sum_{0 \leq i \leq j \leq k-1} \beta_{ij} V_{ij}$ in such a way that $\varphi(V) = \alpha$. So begin defining $\beta_{jj} = \alpha_j$ for every $0 \leq j \leq k-1$. Since $\{v_0, \cdots, v_k\}$ is ld, by hypothesis we have $v_k = \sum_{j=0}^{k-1} \lambda_j v_j$, where at least a pair of numbers λ_{j_0} and λ_{j_1} are not zero. Choose one of these possible pairs with $j_0 < j_1$ and define

$$\beta_{j_0 j_1} = \frac{1}{2}(\lambda_{j_0}\lambda_{j_1})^{-1} \left(\alpha_k - \sum_{j=0}^{k-1} \lambda_j^2 \alpha_j \right).$$

Finally, take $\beta_{ij} = 0$ for every β_{ij} as yet undefined. $\langle V_{j_o j_1}(v_l), v_l\rangle = 0$ for all $l = 0, \cdots, k-1$, it follows that $\langle V(v_l), v_l\rangle = \alpha_l$ for all $l = 0, \cdots, k-1$. It remains to check that $\langle V(v_k), v_k\rangle = \alpha_k$. Indeed, observe that:

$$\begin{aligned}
\langle V(v_k), v_k\rangle &= \sum_{i\le j=0}^{k-1} \beta_{ij}\langle V_{ij}(v_k), v_k\rangle = \sum_{i\le j=0}^{k-1} \beta_{ij} \sum_{p,q=0}^{k-1} \lambda_p\lambda_q\langle V_{ij}(v_p), v_q\rangle \\
&= \sum_{i=1}^{k-1} \alpha_i\lambda_i^2 + \beta_{j_0j_1} \sum_{p,q=0}^{k-1} \lambda_p\lambda_q\langle V_{j_0j_1}(v_p), v_q\rangle \\
&= \sum_{i=1}^{k-1} \alpha_i\lambda_i^2 + \beta_{j_0j_1}\lambda_{j_0}\lambda_{j_1}(\langle V_{j_0j_1}(v_{j_0}), v_{j_1}\rangle + \langle V_{j_0j_1}(v_{j_1}), v_{j_0}\rangle) \\
&= \sum_{i=1}^{k-1} \alpha_i\lambda_i^2 + 2\beta_{j_0j_1}\lambda_{j_0}\lambda_{j_1} = \alpha_k.
\end{aligned}$$

When $\{v_0, \cdots, v_k\}$ is li, we proceed with the same arguments and substituting $U = \{v_0, \cdots, v_{k-1}\}$ by $U = \{v_0, \cdots, v_k\}$. □

Next define L_k as the set of $X \in \mathbb{R}^n$ such that $\{X, \sigma(X), \cdots, \sigma^k(X)\}$ satisfies the hypothesis of lemma 2.

Lemma 3. *For every $0 \le k \le n-1$ the following properties are satisfied:*

(a) $D_k \subset D_{k-1} \cup L_k \cup H_k \cup H_{-k}$.

(b) D_k *is the union of a finite number of subspaces each of dimension at most* k*.*

(c) $\dim H_{\pm k} \le \min\{k, [\![n/2]\!]\}$.

Proof. Let $X \in D_k$. If $X \notin D_{k-1}$, then $\{X, \cdots, \sigma^{k-1}(X)\}$ is li. If, in addition, $X \notin H_k \cup H_{-k}$, then $\{X, \sigma^k(X)\}$ is li (because σ^k is orthogonal and can only have 1 and -1 as real eigenvalues). Finally observe that $\{\sigma^j(X), \sigma^k(X)\}$ is also li for every $1 \le j \le k-1$, otherwise $\{X, \cdots, \sigma^{k-1}(X)\}$ is ld. So $\{X, \cdots, \sigma^k(X)\}$ satisfies the hypothesis of lemma 2, that is, $X \in L_k$. This proves part (a).

For the proof of (b) we claim first that the set of invariant subspaces of σ is finite. Indeed, this holds for any linear transformation having no invariant subspace of dimension at least two on which it is a multiple of the identity, and this trivially holds for σ. This proves the claim. It is clear that D_k is invariant under σ; moreover, if $X \in D_k$, then the subspace generated by $\{X, \cdots, \sigma^{k-1}(X)\}$ is invariant under σ, has dimension at most k, and is contained in D_k. It follows that D_k is union of invariant subspaces of σ, each one of dimension at most k. This proves (b).

To prove (c) observe that

$$H_k = \{X = (x_1, \cdots, x_n) \in \mathbb{R}^n : (x_{n-k+1}, \cdots, x_n, x_1, \cdots, x_{n-k}) = X\}.$$

We will show that the dimension of H_k is the great common divisor between k and n. The definition of H_k given above constitutes a set of n equations and one has to determine the maximal number of independent equations contained in it to obtain the dimension of H_k. For any integer t, define $x_j = x_{j-tn}$ when j is an integer belonging to the interval $(tn, (t+1)n]$. Observe that the equations can now be expressed as: $x_{j+k} = x_j$, for every $1 \leq j \leq n$. Then note that the number of independent equations is determined by the permutation $\{1, \cdots, n\} \to \{k, k+1, \cdots, n-k+1\}$ in the following way:

1. the cycles of this permutation have all the same length, say l; so $n = lm$ where m is the number of cycles;
2. each cycle of length l represents $l-1$ independent equations; and
3. different cycles involve different variables.

In conclusion, the number of independent equations is exactly $(l-1)m$, and so the dimension of H_k is equal to $n-(l-1)m = m$. To find m, put $k = rq$ and $n = rp$, where q and p are mutually prime numbers. It is easy to see that the length of a cycle is always p (because p is the smaller positive number t such that kt is a multiple of n). It follows that the dimension of H_k is r. The conclusion of the assertion of the lemma involving H_k follows. The proof for H_{-k} is almost the same. □

Lemma 4. *Let $\tilde{\mathcal{S}}(n)$ be the subset of $\mathcal{S}(n)$ such that $A \in \tilde{\mathcal{S}}$ implies*

(a) $A(\Delta_0 \cup \Delta_{n/2}) \cap D_{n-1} = \{0\}$;

(b) for every $1 \leq k \leq n$, $A(H_k) \cap H_{\pm k} = \{0\}$; *and*

(c) for every $1 \leq k \leq n$, $A(H_k) \cap H_k^{\perp} = \{0\} = A(H_{-k}) \cap H_{-k}^{\perp}$, *where* $H^{\perp}$ *denotes the orthogonal complement of* H.

Then $\tilde{\mathcal{S}}(n)$ is open and dense and for every $A \in \tilde{\mathcal{S}}(n)$, the intersection of $\xi_0^A, \cdots, \xi_{n-1}^A$ with Δ_0 is not empty and transversal as well as the intersection of $\xi_0^A, \cdots, \xi_{n-1}^A$ with $\Delta_{n/2}$ if n is even.

Proof. That the set $\tilde{\mathcal{S}}(n)$ is open and dense is a consequence of lemma 1 and parts (b) and (c) of lemma 3. Take $X \in \Delta_0$ such that $\langle A(X), X \rangle = 1$, that is, $X \in \xi_0^A$. As $\sigma(X) = X$, it follows that $X \in \xi_j^A$ for every j. It remains to show that this intersection is transverse. Indeed, by the first hypothesis on $\tilde{\mathcal{S}}(n)$ we can conclude that $\{A(X), \cdots \sigma^{n-1} A(X)\}$ is li, or, which is the same, $\{\xi_0^A, \cdots, \xi_{n-1}^A\}$ is transverse at X. □

Proof of Proposition 1. As was explained at the beginning of this section, the transversality theorem cannot be applied directly. For this reason, to prove part (a) of proposition 1, an inductive argument will be needed: at each step k the transversality of $\{\xi_0^A, \cdots, \xi_{n-1}^A\}$ is obtained and its intersection (of codimension k) is almost taken off D_{k-1} (union of subspaces of dimension

$\leq k$). The precise formulation of the induction hypothesis is:
Let $1 \leq k \leq n$. There exists an open and dense set $\mathcal{S}_{k-1} \subset \tilde{\mathcal{S}}(n)$ such that, for every $A \in \mathcal{S}_{k-1}$ it holds that:

(a_{k-1}) $\{\xi_0^A, \cdots, \xi_{k-1}^A\}$ *is transversal; and*

(b_{k-1}) $(\bigcap_{j=0}^{k-1} \xi_j^A) \cap D_{k-1} \subset (\Delta_0 \cup \Delta_{n/2})$.

For $k = 1$ this is obvious since we can take A an invertible linear transformation (open and dense condition). Then ξ_0^A is a codimension-one submanifold which does not intersect $D_0 = \{0\}$.

Suppose that (a_{k-1}) and (b_{k-1}) hold. We will first prove that there exists an open and dense subset $\mathcal{S}_{k-1}^0$ of $\mathcal{S}_{k-1}$ such that, for every $A \in \mathcal{S}_{k-1}^0$, (a_k) is satisfied. To do this, define $\Phi : \mathcal{S}_{k-1} \times \mathbb{R}^n \to \mathbb{R}^{k+1}$ as follows:

$$\Phi(A, X) = \left(\langle A(X), X\rangle, \cdots, \langle A\sigma^{-k}(X), \sigma^{-k}(X)\rangle\right).$$

We will prove now that $(1, \cdots, 1) \in \mathbb{R}^{k+1}$ is a regular value of Φ. So let A and X be such that $\Phi(A, X) = (1, \cdots, 1)$, and observe that the differential of Φ at (A, X) satisfies, for $V \in \mathcal{S}(n)$ and a vector $v \in \mathbb{R}^n$:

$$\begin{aligned} (D\Phi)_{(A,X)}(V, v) &= \left(\langle 2A(X), v\rangle, \cdots, \langle 2\sigma^k A\sigma^{-k}(X), v\rangle\right) \\ &+ \left(\langle V(X), X\rangle, \cdots, +\langle V\sigma^k(X), \sigma^k(X)\rangle\right). \end{aligned}$$

To prove that $D\Phi_{(A,X)}$ is onto, take any $\alpha = (\alpha_0, \cdots, \alpha_k) \in \mathbb{R}^{k+1}$. If $X \in L_k$, choose $v = 0$, then $(D\Phi)_{(A,X)}(V, v) = \varphi_{\{X, \cdots, \sigma^{-k}(X)\}}(V)$. From lemma 2 it follows that if $X \in L_k$, then as φ as $D\Phi_{(A,X)}$ are onto. Suppose now that $X \notin L_k$, therefore $X \in D_k$. If $X \in D_{k-1}$, then the hypothesis (b_{k-1}) implies that $X \in \Delta_0 \cup \Delta_{n/2}$ and lemma 4 implies that this intersection is transverse, and consequently (A, X) is a regular point of Φ. It remains to consider the case when $X \in D_k \setminus (D_{k-1} \cup L_k)$. By part (a) of lemma 3 it is known that $X \in H_{\pm k} = H_k \cup H_{-k}$. Suppose first that $X \in H_k$. By lemma 3 part (c), lemma 1 and lemma 4, we know that $A(H_k) \cap H_{\pm k} = \{0\}$. It follows that $A(X), \sigma^k A(X)$ are linearly independent, and as $\sigma^{-k}(X) = X$, we can choose the vector v such that $2\langle A(X), v\rangle = \alpha_0$ and $2\langle \sigma^k A\sigma^{-k}(X), v\rangle = \alpha_k$. Next observe that, as $X \notin D_{k-1}$, then $X \in L_{k-1}$, and so there exists a transformation V such that $\langle V(X), X\rangle = 0$, and for every $1 \leq j \leq k-1$ it holds

$$\langle V\sigma^{-j}(X), \sigma^{-j}(X)\rangle = \alpha_j - 2\langle \sigma^j A\sigma^{-j}(X), v\rangle.$$

It is clear that with this choice, v and V satisfy:

$$\langle 2\sigma^j A\sigma^{-j}(X), v\rangle + \langle V\sigma^{-j}(X), \sigma^{-j}(X)\rangle = \alpha_j, \tag{4}$$

for every $0 \leq j \leq k-1$. As $\langle V(X), X\rangle = 0$ and $\sigma^k(X) = X$, it follows that the equation (4) for $j = k$, is also satisfied. In conclusion, we have proved that $(1, \cdots, 1)$ is a regular value of Φ. Hence, there exists a dense subset $\mathcal{S}_{k-1}^0$

of transformations A for which the mapping Φ_A, $\Phi_A(X) = \Phi(A, X)$, has $(1, \cdots, 1)$ as a regular value. This subset is clearly open, and the regularity of $(1, \cdots, 1)$ for Φ_A is equivalent to the transversality of the sets $\xi_0^A, \cdots, \xi_k^A$. So we have proved (a_k) in the case that $X \in H_k$. The case $X \in H_{-k}$ is treated similarly. This proves (a_k).

For the proof of (b_k), consider a subspace $H \subset D_k$. By lemma 3, the dimension of H is $h \leq k$. Let $M : \mathbb{R}^n \to \mathbb{R}^{n-h}$ be a linear transformation such that $H = ker(M)$, so the rank of M is $n - h$. Consider the mapping

$$\Phi : \mathcal{S}_{k-1}^0 \times \big(\mathbb{R}^n \setminus \big(\Delta_0 \cup \Delta_{n/2} \cup H_k \cup H_{-k}\big)\big) \to \mathbb{R}^{k+1} \times \mathbb{R}^{n-h}$$

given by:

$$\Phi(A, X) = \big(\langle A(X), X\rangle, \cdots, \langle \sigma^k A \sigma^{-k}(X), X\rangle, M(X)\big)\,. \tag{5}$$

We will prove that the point $(\bar{1}, \bar{0}) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-h}$, with $\bar{1} = (1, \cdots, 1)$ and $\bar{0} = (0, \cdots, 0)$, is a regular value of Φ. So let $\Phi(A, X) = (\bar{1}, \bar{0})$. Then for $V \in \mathcal{S}(n)$ and $v \in \mathbb{R}^n$

$$\begin{aligned}(D\Phi)_{(A,X)}(V, v) &= \big(\langle 2A(X), v\rangle, \cdots, \langle 2\sigma^k A\sigma^{-k}(X), v\rangle, M(v)\big)\\ &+ \big(\langle V(X), X\rangle, \cdots, \langle V\sigma^{-k}(X), \sigma^{-k}(X)\rangle, 0\big)\,.\end{aligned}$$

Let $\bar{\alpha} = (\alpha_0, \cdots, \alpha_k)$, and $\bar{\beta} \in \mathbb{R}^{n-h}$. As $\Phi(A, X) = (\bar{1}, \bar{0})$, $X \in \bigcap_{j=0}^k \xi_j^A$, in particular the hypothesis (b_{k-1}) implies that $X \notin D_{k-1}$ (recall that $\Delta_0 \cup \Delta_{n/2}$ were taken off the domain of Φ). Also $H_{\pm k}$ were taken off the domain of Φ, so it follows by lemma 3 part (a), that $X \in L_k$. On the other hand, observe that the rank of M is equal to $n - h$, so there exists a vector $v \in \mathbb{R}^n$ such that $M(v) = \bar{\beta}$. Now as $X \in L_k$, we can apply lemma 2 to conclude that there exists a symmetric transformation V such that for every $0 \leq j \leq k$ it holds that $\langle V\sigma^{-j}(X), \sigma^{-j}(X)\rangle = \alpha_j - 2\langle \sigma^j A \sigma^{-j}(X), v\rangle$. It is clear now that $(D\Phi)_{(A,X)}$ is onto, so $(\bar{1}, \bar{0})$ is a regular value of Φ. This implies that there exists a dense open set $\mathcal{S}_{k-1}^{00} \subset \mathcal{S}_{k-1}^0$, such that every A there satisfies that $\bigcap_{j=0}^k \xi_j^A$ is transversal to $H \setminus (\Delta_0 \cup \Delta_{n/2} \cup H_{\pm k})$. The first one has codimension $k + 1$, while the second one is a submanifold with dimension $h \leq k$, so the transversality implies that the intersection is empty. By lemma 3, D_k is a finite union of subspaces like H, therefore

$$\bigcap_{j=0}^k \xi_j^A \cap \big(D_k \setminus (\Delta_0 \cup \Delta_{n/2} \cup H_{\pm k})\big) = \emptyset.$$

To conclude the proof of (b_k), it remains to find an open dense subset of $\mathcal{S}_{k-1}^{00}$ such that for any A in this set it holds that:

$$\bigcap_{j=0}^k \xi_j^A \cap H_{\pm k} \subset \Delta_0 \cup \Delta_{n/2}.$$

To prove this, consider the same function Φ as defined in equation (5), but now the domain of Φ is $\mathbb{R}^n \setminus (\Delta_0 \cup \Delta_{n/2})$, and the linear transformation M now satisfies $M(v) = 0$ iff $v \in H_k$. The rows of M are linearly independent and constitute a basis of $H_k^\perp$ (the orthogonal complement of H_k). Suppose that $\Phi(A, X) = (\bar{1}, \bar{0})$, which implies $X \in H_k$, and so $A(X) \notin H_{\pm k}$ by lemma 4 and the fact that $A \in \tilde{\mathcal{S}}$. It follows that $\{A(X), \sigma^{-k} A(X)\}$ is li. Another application of lemma 4 permits to affirm that $A(X) \notin H_k^\perp$, and so (as $H_k^\perp$ is invariant under σ^k) it follows that $A(X)$, $\sigma^k A(X)$, and the rows of M, form a li set with $2 + n - h$ elements. Then there exists a vector v such that

$$\langle 2A(X), v\rangle = \alpha_0,\ \langle 2\sigma^k A(X), v\rangle = \alpha_k \ \text{ and } M(v) = \bar{\beta}.$$

Since $X \notin D_{k-1}$ we can choose V satisfying

$$\langle V(X), X\rangle = 0 \ \text{ and } \ \langle V\sigma^{-j}(X), \sigma^{-j}(X)\rangle = \alpha_j - 2\langle \sigma^j A \sigma^{-j}(X), v\rangle,$$

for $1 \le j \le k-1$. Then $(D\Phi)_{(A,x)}$ is onto, which implies that the codimension-$(k+1)$ submanifold $\bigcap_{j=0}^{k} \xi_j^A$ is transversal to $H_k \setminus (\Delta_0 \cup \Delta_{n/2})$. As the $\dim H_k \le \min\{k, [\![n/2]\!]\}$ it follows that $H_k \setminus (\Delta_0 \cup \Delta_{n/2})$ does not intersect $\bigcap_{j=0}^{k} \xi_j^A$. The proof for H_{-k} is similar, so we have proved (b_k), completing the induction and the proof of the first part of the proposition 1.

To prove parte (b) of proposition 1 first observe that this condition is open. For every $j = 0, \cdots, n-1$, let

$$\tau_j^A = \{X \in \mathbb{R}^n : \langle A\sigma^{-j}(X), \sigma^{-j}(X)\rangle = 0\}.$$

The proof is again by induction. Now the induction hypothesis is:
There exists an open and dense set $\mathcal{T}_{k-1} \subset \mathcal{S}(n)$ such that for every $A \in \mathcal{T}_{k-1}$ it holds:

(a_{k-1}) $\displaystyle\bigcap_{j=0}^{k-1} \tau_j^A \cap S^{n-1}$ *is a codimension-k submanifold of the sphere* S^{n-1}.

(b_{k-1}) $\displaystyle\bigcap_{j=0}^{k-1} \tau_j^A \cap (D_k \setminus \{0\}) = \emptyset.$

Fix $A \in \mathcal{S}(n)$; let $X_1, \cdots, X_n$ be eigenvectors of A and $\lambda_1, \cdots, \lambda_n$ its eigenvalues. Let $V = \sum_{j=0}^n \alpha_j X_j$. Observe that $V \in \tau_0^A$ if and only if $\sum_{j=0}^n \lambda_j \alpha_j^2 = 0$. It is clear that without changing the vectors X_j one can perturb the eigenvalues λ_j, $j = 0, \cdots, n$, in such a way to obtain all the values $\sum_j \lambda_j \alpha_j^2 \neq 0$. This can be done for V generating Δ_0 and for V generating $\Delta_{n/2}$. As $D_1 = \Delta_0 \cup \Delta_{n/2}$, this implies (b_0). Clearly (a_0) holds for every A with eigenvalues different from zero.

Now we prove that (b_{k-1}) implies (a_k). Define $\Phi : \mathcal{T}_{k-1} \times \mathbb{R}^k \to \mathbb{R}^{k+1}$ by

$$\Phi(A, X) = (\langle A(X), X\rangle, \cdots, \langle A\sigma^{-k}(X), \sigma^{-k}(X)\rangle, \|x\|^2).$$

Let A, X be such that $\Phi(A, X) = (\bar{0}, 1)$, where $\bar{0} = (0, \ldots, 0) \in \mathbb{R}^{k+1}$. This condition is equivalent to $X \in \bigcap_{j=0}^{k} \tau_j^A \cap S^{n-1}$, hence $X \notin D_k$. We will prove that $(\bar{0}, 1)$ is a regular value of Φ. It is easy to see that for $V \in \mathcal{S}(n)$ and $v \in \mathbb{R}^n$

$$\begin{aligned}(D\Phi)_{(A,X)}(V, v) &= \big(\langle 2A(X), v\rangle, \cdots, \langle 2\sigma^{-1+k} A\sigma^{k-1}(X), v\rangle, \langle 2X, v\rangle\big) \\ &+ \big(\langle V(X), X\rangle, \cdots, \langle V\sigma^{-k}(X), \sigma^{-k}(X)\rangle, 0\big).\end{aligned}$$

To prove that $(D\Phi)_{(A,X)}$ is onto, take any $(\alpha_0, \ldots, \alpha_k, \beta) \in \mathbb{R}^{k+2}$. Consider a vector $v = \frac{\beta X}{2\|X\|^2}$, so that $2\langle X, v\rangle = \beta$. Since $X \notin D_k$ we can apply again lemma 2 to obtain a symmetric transformation V such that $\langle V\sigma^j(X), \sigma^j(X)\rangle = \alpha_j - 2\langle A\sigma^j(X), \sigma^j(v)\rangle$ for every $j = 0, \ldots, k-1$. This proves that $D\Phi_{(A,x)}$ is onto. Then it follows that $\bigcap_{j=0}^{k} \tau_j^A \cap S^{n-1}$ is a submanifold of S^{n-1} with codimension at least $k+1$ in S^{n-1}. This proves (a_k). To prove (b_k), define $\Phi : (\mathcal{T}_{k-1} \cap \tilde{\mathcal{S}}(n)) \times \mathbb{R}^{k+1} \to \mathbb{R}^{k+1} \times \mathbb{R}^{n-h}$, as in equation (5), where, as before, M is a linear transformation which kernel is H, a subspace of dimension $h \leq k$ contained in D_k. Observe that the domain of Φ was taken contained in $\tilde{\mathcal{S}}(n)$. Such condition implies that $A(H_{\pm k}) \cap H_{\pm k}^{\perp} = \{0\}$. So the proof that $(D\Phi)_{(A,X)}$ is onto follows as in the final part of the proof of (b_k) in the first part of this proposition. In conclusion, the intersection of all the τ_j^A with S^{n-1} is empty. So, the proof of proposition 1 is complete. □

3 Proof of main results

Proof of Theorem 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function satisfying the hypothesis of theorem 1 and F the real cellular automaton generated by f.

Let $C = \{X \in \mathbb{R}^n : \sum_{i=0}^{n-1} f \circ \sigma^{-i}(X) \leq \Sigma(X)\}$, where $\Sigma(X)$ denotes the sum of the coordinates of X. Observe that:

1. $\sigma(C) = C$; this is obvious since σ^{-n} is the identity map.
2. C is convex: by hypothesis the function $X \to \Sigma(f \circ \sigma^{-i}(X))$ is convex and Σ is linear.
3. C is closed by continuity of $X \to \Sigma(F(X)) - \Sigma(X)$.
4. $\partial C = \{X \in \mathbb{R}^n : \Sigma(F(X)) = \Sigma(X)\}$, the boundary of C, does not intersect the diagonal of $\mathbb{R}^n$. In the contrary case, there exists a real number t such that $tu = (t, \cdots, t) \in \mathbb{R}^n$ belongs to ∂C. As tu is fixed by σ, it follows that $f(tu) = t$. Then $F(tu) = tu$, contradicting the hypothesis of abscence of fixed points for F.
5. C does not intersect the diagonal of $\mathbb{R}^n$: Indeed, the function $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(t) = f(tu) - t$ is strictly convex. If there exists a point $t_0 u \in C$, then $\phi(t_0) < 0$. By the strict convexity of ϕ there exists points t_1 with $\phi(t_1) > 0$. This means that there exists at least a point where ϕ vanishes, contradicting the previous item.

We conclude that C is empty: by contradiction, if a point $X \in C$, then $\sigma^i(X)$ belongs to C for every i by item 1. Then $\frac{1}{n}\sum_{i=0}^{n-1}\sigma^i(X)$ also belongs to C by item 2, but this point belongs to the diagonal, so this contradicts the previous item. It follows that $\Sigma(F(X)) - \Sigma(X) > 0$; this implies that Σ is a Lyapunov function positively defined, so $\|F^m(X)\| \to +\infty$ when $m \to +\infty$, consequently the ω-limit set of any point is empty. $\square$

Proof of Theorem 2. Let $\mathcal{S}'(n)$ be the open and dense set obtained in proposition 1. Take $A \in \mathcal{S}'(n)$ and $L \in \mathcal{L}(n)$. Let F_μ be the quadratic real cellular automaton given by equation (3). We will prove that F_μ is hyperbolic for every $|\mu|$ sufficiently large. From part (b) of proposition 1 we have that $\bigcap_{j=0}^{n-1}\{X : \langle A\sigma^{-j}(X), \sigma^{-j}(X)\rangle = 0\} = \{0\}$, then there exists $\delta > 0$ such that

$$\bigcap_{j=0}^{n-1}\{X : |\langle A\sigma^{-j}(X), \sigma^{-j}(X)\rangle| < \delta\} \cap S^{n-1} = \emptyset,$$

where S^{n-1} is the $(n-1)$-dimensional sphere. So, for every $X \in \mathbb{R}^n \setminus \{0\}$ there exists $j \in \{0, \cdots, n-1\}$ such that $|\langle A\sigma^{-j}(X), \sigma^{-j}(X)\rangle| \geq \delta\|X\|^2$. Then it follows the existence of a constant $\delta_0 > 0$ such that

$$\|F_\mu(X)\| \geq \delta_0\|X\|^2 - \mu. \tag{6}$$

Here comes the argument showing that the nonwandering set of F_μ is located at a distance of the order $\sqrt{|\mu|}$ from the origin whenever $|\mu|$ is large. We claim that there are constants $r_2 > r_1 > 0$ such that:

(i) $\|F_\mu^m(X)\| \to +\infty$ *when* $m \to +\infty$ *for all* $X \notin D(r_2\sqrt{|\mu|})$, *where* $D(r)$ *denotes the open disk centered at the origin and radius* r*; and*

(ii) *if* $|\mu|$ *is large and* $X \in D(r_1\sqrt{|\mu|})$, *then* $F_\mu(X) \notin D(r_2\sqrt{|\mu|})$.

In fact, from (6) and taking r_2 sufficiently large ($r_2 > \frac{3}{\delta_0}$ is enough) it follows that $\|F_\mu(X)\| > 2\|X\|$ for all $X \notin D(r_2\sqrt{|\mu|})$, which implies (i). On the other hand, since $F_\mu = (f_0 - \mu, \cdots, f_{n-1} - \mu)$ with $|f_i(X)| \leq K_1\|X\|^2 + K_2$ for some positive constants, then for all $X \in \mathbb{R}^n$ we have

$$\|F_\mu(X)\| \geq \max_{0\leq j\leq n-1}|f_j(X) - \mu| \geq |\mu| - K_1\|X\|^2 - K_2.$$

Hence, if $X \in D(r_1\sqrt{|\mu|})$ and $\frac{1}{K_1} > r_1^2$, the last inequality implies

$$\|F_\mu(X)\| \geq |\mu|(1 - K_1r_1^2) - K_2,$$

which is bigger than $r_2\sqrt{|\mu|}$ for all $|\mu|$ sufficiently large. This proves (ii).

Observe that (i) implies that ∞ is an attractor for F_μ, that is, there exists $R > 0$ such that $\|X\| > R$ is an invariant region, and $\|F_\mu^m(X)\| \to +\infty$ when $m \to +\infty$, for all $\|X\| > R$. If $B_\infty(\mu)$ denotes the basin of attraction of ∞, i.e. the set of $X \in \mathbb{R}^n$ such that $\|F_\mu^m(X)\| \to +\infty$ when $m \to +\infty$, then from the claim it follows that

$$\mathbb{R}^n \setminus B_\infty(\mu) \subset D(r_2\sqrt{|\mu|}) \setminus D(r_1\sqrt{|\mu|})$$

for all $|\mu|$ sufficiently large. Moreover, from the invariance of $\mathbb{R}^n \setminus B_\infty(\mu)$ it can be proved that $B_\infty(\mu)$ contains the complementary set of points satisfying

$$r_1\sqrt{|\mu|} \leq |\langle A\sigma^{-j}(X), \sigma^{-j}(X)\rangle + L(X) + \mu| \leq r_2\sqrt{|\mu|}$$

for every $0 \leq j \leq n-1$ and all $|\mu|$ large enough. Now it follows that there exist constants s_1, s_2 such that for every $X \notin B_\infty(\mu)$ and $Y = \frac{X}{\sqrt{|\mu|}}$ it holds that

$$|\langle A\sigma^{-j}(Y), \sigma^{-j}(Y)\rangle| \in \left[1 - \frac{s_1}{\sqrt{|\mu|}}, 1 + \frac{s_2}{\sqrt{|\mu|}}\right] \tag{7}$$

for all $|\mu|$ sufficiently large. Observe that no point $Y \in \mathbb{R}^n$ satisfies (7) if $|\mu|$ is large and $\bigcap_{j=0}^{n-1} \xi_j^A = \emptyset$. Therefore we conclude in this case that the basin of attraction of ∞ is $\mathbb{R}^n$.

Now we introduce a quantified notion of tranversality, which together with proposition 1 will be used to obtain expansivity of F_μ. First we introduce some notation. For a linear subspace V of $\mathbb{R}^n$, $P_V^\perp$ denotes the orthogonal projection of $\mathbb{R}^n$ onto V; recall that $[U]$ denotes the linear subspace generated by $U \subset \mathbb{R}^n$.

Definition 1. Given $\epsilon > 0$ we say that $\{v_1, \cdots, v_n\} \subset \mathbb{R}^n \setminus \{0\}$ is ϵ-*transverse* if for each $V_i = [\{v_1, \cdots, v_n\} \setminus \{v_i\}]$ with $i = 1, \cdots, n$, it holds that $\|P_{V_i}^\perp(v_i)\| \geq \epsilon\|v_i\|$.

This notion is associated with the angle between the vector v_i and the subspace V_i for each $i = 1, \cdots, n$. Moreover, the concept of ϵ-transversality can be extended to the intersection of n codimension-one manifolds, saying that the set $\{\xi_j : 0 \leq j \leq n-1\}$ is ϵ-transverse if the set of normal vectors at any point of intersection is ϵ-transverse.

We will use the following criterion for expansiveness using ϵ-transversality.

Lemma 5 (cf. Lemma 3, [8]). *Given $\epsilon > 0$ there exists a constant $c(\epsilon) > 0$ such that if the set of unit vectors $\{v_1, \cdots, v_n\} \subset \mathbb{R}^n$ is ϵ-transverse, then the matrix A whose rows are the vectors $v_1, \cdots, v_n$ is $c(\epsilon)$-expanding:*

$$\|A(v)\| \geq c(\epsilon)\|v\| \;\; \textit{for every} \;\; v \in \mathbb{R}^n.$$

By proposition 1 the set $\{\xi_j^A : j = 0, \cdots, n-1\}$ is ϵ-transverse for some ϵ. This implies that $\{2\sigma^j A\sigma^{-j}(Z) : j = 0, \cdots, n-1\}$ is ϵ-transverse for every point $Z \in \bigcap_{j=0}^{n-1} \xi_j^A$. Using continuity arguments, there exists $\delta_1 > 0$ such that if $d_j \in (1-\delta_1, 1+\delta_1)$, then $\{\xi_j(d_j) : j = 0, \cdots, n-1\}$ is $\frac{\epsilon}{2}$-transverse, where $\xi_j(d_j) = \{Z : \langle \sigma^j A\sigma^{-j}(Z), Z\rangle = d_j\}$. Consider $X \notin B_\infty(\mu)$, $Y = \frac{X}{\sqrt{|\mu|}}$

and $h_j = \langle \sigma^j A \sigma^{-j}(Y), Y \rangle$. From (7) we have that $h_j \in (1-\delta_1, 1+\delta_1)$ for all $j = 0, \cdots, n-1$ and $|\mu|$ large enough. So, $\{\xi_j(h_j) : j = 0, \cdots, n-1\}$ is $\frac{\epsilon}{2}$-transverse; in particular $\{2\sigma^j A \sigma^{-j}(Y) : j = 0, \cdots, n-1\}$ is $\frac{\epsilon}{2}$-transverse because $Y \in \bigcap_{j=0}^{n-1} \xi_j(h_j)$ and $2\sigma^j A \sigma^{-j}(Y)$ is normal to $\xi_j(h_j)$ at Y for all $j = 0, \cdots, n-1$.

Finally, and following the sketch of proof indicated after the statement of the teorema 2 in section 1, observe that $(DF_\mu)_X$ is the matrix whose $(j+1)$-th row $(j = 0, \cdots, n-1)$ is the vector

$$w_j = 2\sigma^j A \sigma^{-j}(X) + L\sigma^{-j} = \sqrt{|\mu|}\left(2\sigma^j A \sigma^{-j}(Y) + \frac{1}{\sqrt{|\mu|}} L\sigma^{-j}\right).$$

If $|\mu|$ is large, the vector $v_j = 2\sigma^j A \sigma^{-j}(Y) + \frac{1}{\sqrt{|\mu|}} L\sigma^{-j}$ is a small perturbation of $2\sigma^j A \sigma^{-j}(Y)$ for all $j = 0, \cdots, n-1$, it comes that $\{v_0, \cdots, v_{n-1}\}$ is $\frac{\epsilon}{4}$-transverse. In addition, as

$$(DF_\mu)_X(u) = \begin{pmatrix} \langle w_0, u \rangle \\ \vdots \\ \langle w_{n-1}, u \rangle \end{pmatrix} = \sqrt{|\mu|} \begin{pmatrix} \|v_0\| \langle \frac{v_0}{\|v_0\|}, u \rangle \\ \vdots \\ \|v_{n-1}\| \langle \frac{v_{n-1}}{\|v_{n-1}\|}, u \rangle \end{pmatrix}$$

it follows that

$$\|(DF_\mu)_X(u)\| \geq \sqrt{|\mu|} \min_{0 \leq j \leq n-1} \|v_j\| c(\epsilon/4) \|u\|, \tag{8}$$

where $c(\epsilon/4)$ is the constant obtained from lemma 5 applied to the matrix with rows $\frac{v_0}{\|v_0\|}, \cdots, \frac{v_{n-1}}{\|v_{n-1}\|}$. Clearly (8) implies that $(DF_\mu)_X$ expands vectors at a rate of $\sqrt{|\mu|} \min_{0 \leq j \leq n-1} \|v_j\| c(\epsilon/4)$, which can be taken greater than 1 (if $|\mu|$ is large); so the proof of theorem 2 is complete. □

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