# Bounded solutions of quadratic circulant difference equations

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**Abstract.** In this paper we develop some techniques to obtain global hyperbolicity for a certain class of endomorphisms of $(\mathbb{R}^p)^n$ with $p, n \geq 2$. This kind of endomorphisms are obtained from vectorial difference equations where the mapping defining these equations satisfy a circulant condition. In particular, we show that one-parameter families of these quadratic endomorphisms are hyperbolic for large values of the parameter.

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# 1 Introduction

A wide number of mathematical and computational models, as well as those obtained from discretization of partial and ordinary differential equations with constant time step (Euler's method) point out directly to the study of the dynamics of certain class of delayed difference equations

$$X_{n+k} = \varphi(X_k, \cdots, X_{k+n-1}),\ k \geq 0 \tag{1}$$

where $X_n$ is a vector in $\mathbb{R}^p$ and $\varphi$ is a mapping defined on a subset of $(\mathbb{R}^p)^n$.

By a solution of Eq. (1) we mean a sequence $\{X_m\}_{m\in\mathbb{N}}$ in $\mathbb{R}^p$ satisfying (1) for every integer $n \geq 0$. It is clear that for every initial condition $(X_0, \cdots, X_{n-1}) \in (\mathbb{R}^p)^n$ there exists a unique solution of (1) satisfying this initial condition. On the other hand, the endomorphism

$$F(X_0, \cdots, X_{n-1}) = (X_1, \cdots, X_{n-1}, \varphi(X_0, \cdots, X_{n-1})), \tag{2}$$

is called a *delay endomorphism of* $(\mathbb{R}^p)^n$. The orbit by $F$ of each initial condition $(X_0, \cdots, X_{n-1}) \in (\mathbb{R}^p)^n$, that is the set $\{F^k(X_0, \cdots, X_{n-1}) : k \geq 0\}$, describes the evolution of the states $(X_k, \cdots, X_{k+n-1}) = F^k(X_0, \cdots, X_{n-1})$ with $k \geq 0$. Clearly this orbit determines the solution of (1) with such initial condition. Therefore the analysis of the limit points of the solutions of (1) can be done studying the asymptotic behavior of the orbits of the discrete dynamical system given by the delay endomorphism $F$.

**Example 1.** Motivated by different problems in many contexts, such as biological, economic and social sciences, a considerable number of researchers have investigated the dynamics given by endomorphisms like (2) for choices of the function $\varphi$ appropiated to certain problems in many contexts. For example, Z. Zhou and J. Wu [13] considered the system of difference equations

$$\begin{cases} x_k = \beta x_{k-1} + f(y_{k-n}) \\ y_k = \beta y_{k-1} + f(x_{k-n}),\ k \geq 0 \end{cases} \tag{3}$$

to describe some dynamical properties of the interaction of two identical neurons, where $\beta \in (0,1)$ is the internal decay rate, $f$ is the signal transmission function and $n$ is the signal transmission delay. Observe that if $X = (x,y) \in \mathbb{R}^2$ and

$$\begin{aligned} \varphi(X_0, \cdots, X_{n-1}) &= (\varphi_1(X_0, \cdots, X_{n-1}), \varphi_2(X_0, \cdots, X_{n-1})) \\ &= (f(y_0) + \beta x_{n-1}, f(x_0) + \beta y_{n-1}), \end{aligned}$$

then the system (3) is written as

$$X_{n+k} = \varphi(X_k, \cdots, X_{k+n-1}),\ k \geq 0,$$

with $\varphi = (\varphi_1, \varphi_2)$ and satisfying the identity

$$\varphi_2(X_0, \cdots, X_{n-1}) = \varphi_1(\sigma(X_0), \cdots, \sigma(X_{n-1})),$$

where $\sigma(x,y) = (y,x)$ for all $(x,y) \in \mathbb{R}^2$. This example represent an important class of delay difference equations which are characterized by the relationship described between the mapping $\varphi$ and the transformation $\sigma$. Actually this property induces the following definition.

**Definition 1.** A mapping $\varphi : (\mathbb{R}^p)^n \to \mathbb{R}^p$ is *circulant* if satisfies the condition

$$\varphi_i(X_1, \cdots, X_n) = \varphi_1(\sigma^{-i+1}(X_1), \cdots, \sigma^{-i+1}(X_n)),\ 1 \leq i \leq p \tag{C}$$

for all $X_1, \cdots, X_n \in \mathbb{R}^p$ where $\sigma$ is the circular linear transformation given by

$$\sigma(x_1, \cdots, x_p) = (x_p, x_1, \cdots, x_{p-1}). \tag{4}$$

If $\varphi : (\mathbb{R}^p)^n \to \mathbb{R}^p$ is circulant, then the equation (1) is called *circulant difference equation*; and the endomorphism (2) is called *circulant delay endomorphism.*

Clearly the property that a mapping $\varphi : (\mathbb{R}^p)^n \to \mathbb{R}^p$ should verify in order to satisfies (C) is exactly $\sigma \circ \varphi = \varphi \circ \widetilde{\sigma}_n$, where $\widetilde{\sigma}_n : (\mathbb{R}^p)^n \to (\mathbb{R}^p)^n$ is given by $\widetilde{\sigma}_n = (\sigma, \cdots, \sigma)$.

The set of circulant delay endomorphisms of order $n$ on $\mathbb{R}^p$ will be denoted by $\mathcal{D}_{p,n}$. Observe that $\mathcal{D}_{p,1}$ is the set of endomorphisms $F : \mathbb{R}^p \to \mathbb{R}^p$ commuting with $\sigma$; this kind of endomorphisms is known as *real cellular automata*, see J. Weitkämper [12] and references therein. Real cellular automata appear in different contexts; we mention some of them: Bischi and Gardini [2] utilize real cellular automata to study global properties of competition models with identical competitors; they arrive to a first order difference equation $X_{n+1} = T(X_n)$ where $X_n = (x_n, y_n) \in \mathbb{R}^2$ and $T$ is a real cellular automaton. When the difference equation has superior order, for example $X_{n+1} = T(X_{n-1}, X_n)$, one arrives to a $\mathcal{D}_{2,2}$ endomorphism whenever $T$ is a circulant delay endomorphism. H. Smith, see [11], employs these dynamical systems as competitive and cooperative mappings with chaotic behavior; finally, coupled map lattices with periodic boundary conditions also are examples of real cellular automata; see for example K. Kaneko [5], and W. Lin et al. [6], [7].

**Example 2.** Another example of circulant delay endomorphisms arise naturally when one consider discretizations, for example with time step one, of differential equations. In fact, if $g : \mathbb{R}^p \to \mathbb{R}^p$ is any real cellular automaton, then the second order ordinary differential system

$$X'' = g(X)$$

can be discretized as a delayed circulant endomorphism of order 2 on $\mathbb{R}^p$ as follows: changing variables to make the second order differential system a first order differential system, one has $X' = Y,\ Y' = g(X)$. The elementary discretization with time step one then gives

$$\begin{cases} X(n+1) = X(n) + Y(n) \\ Y(n+1) = Y(n) + g(X(n)); \end{cases}$$

now making $U = X$ and $V = X + Y$ this system of difference equations is equivalent to

$$\begin{cases} U(n+1) = V(n) \\ V(n+1) = 2V(n) - U(n) + g(U(n)), \end{cases}$$

and this system of difference equations is represented by means of the mapping $F(U, V) = (V, 2V - U + g(U))$, which is clearly a circulant delay endomorphism because $\varphi(U, V) = 2V - U + g(U)$ satisfies $\sigma \circ \varphi = \varphi \circ \widetilde{\sigma}_2$.

## 2 Preliminaries and Statement of Results

In this paper we will focus our attention in the description of certain dynamical properties of some discrete dynamical systems given by circulant delay endomorphisms. The first result is related with the ancient problem of find consequences to the hypothesis of absence of points of fixed points for a mapping. In the next section we will prove the following result:

**Theorem 1.** *Let $F$ be a $C^0$ circulant delay endomorphism as given by equation (2). If the function $\sum_{i=1}^{p} \varphi_i$ is strictly convex and $F$ has no fixed points, then the $\omega$-limit set of any point is empty.*

It is very simple to check that if in example 1 we take $f(t) = t^2 + \mu$ where the parameter $\mu$ satisfies $4\mu > (\beta - 1)^2$, then the hypothesis of the theorem hold; therefore every solution of (3) is divergent.

As final part of this paper we will present a result concerning the hyperbolicity of circulant delay endomorphisms, representing, in some sense, a generalization of a theorem for delay endomorphisms in [9]:
*Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ function such that the second derivative with respect to the first variable is bigger than every other second derivative. If $f_\mu = f - \mu$ and $F_\mu(x_1, \cdots, x_n) = (x_2, \cdots, x_n, f_\mu(x_1, \cdots, x_n))$, then for every $\mu$ large the point at $\infty$ is an attractor of $F_\mu$ and the complementary set of the basin of attraction of $\infty$ is an expanding Cantor set.*

We recall that for a mapping $G : \mathbb{R}^m \to \mathbb{R}^m$, $\infty$ is an attractor if there exists $R > 0$ such that

- $\|G(x)\| > R$ if $\|x\| \geq R$, and
- $\|G^k(x)\| \to +\infty$ when $k \to +\infty$ for every $\|x\| > R$.

In this case, the basin of attraction of $\infty$ is the open set

$$B(G) = \{x \in \mathbb{R}^m : \|G^k(x)\| \to +\infty \text{ if } k \to +\infty\}.$$

Here we will find sufficient conditions imposed on certain quadratic circulant delay endomorphism which will imply the same conclusion above. Actually, we are more interested in introduce simple ideas than to arrive to elaborated statements. We will show that the same techniques used in [8] and [9] apply to obtain hyperbolicity for circulant delayed endomorphisms, treating a very particular case to avoid technical developements; nevertheless, we need to introduce some preliminary and notations to establish our second result.

Denote by $\mathcal{A}_p^\ell$ the set of linear cellular automata in $\mathbb{R}^p$, that is, the set of linear transformations of $\mathbb{R}^p$ commuting with $\sigma$. We will always identify every linear transformation with its matrix associated in the cannonical basis. If $a = (a_1, \cdots, a_p) \in \mathbb{R}^p$, let $a^\sigma$ be the unique linear cellular automaton in $\mathbb{R}^p$ having $a$ as first row, that is the circulant matrix

$$a^\sigma = \begin{pmatrix} a \\ \sigma(a) \\ \vdots \\ \sigma^{p-1}(a) \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{p-1} & \alpha_p \\ \alpha_p & \alpha_1 & \cdots & \alpha_{p-2} & \alpha_{p-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_2 & \alpha_3 & \cdots & \alpha_p & \alpha_1 \end{pmatrix}.$$

This identification defines a linear isomorphism between $\mathbb{R}^p$ and $\mathcal{A}_p^\ell$, its inverse is given by $A \to A^t(e_1)$, where $A^t$ is the transpose of the matrix $A$ and $e_1$ is the first vector of the cannonical basis of $\mathbb{R}^p$.

Given vectors $a_1, \cdots, a_n$ in $\mathbb{R}^p$, $\mu \in \mathbb{R}$ and $u = (1, \cdots, 1) \in \mathbb{R}^p$, consider the familiy of endomorphisms $F_\mu : (\mathbb{R}^p)^n \to (\mathbb{R}^p)^n$ given by

$$F_\mu(X_1, \cdots, X_n) = (X_2, \cdots, X_n, \varphi_\mu(X_1, \cdots, X_n)), \tag{5}$$

with $\varphi_\mu : (\mathbb{R}^p)^n \to \mathbb{R}^p$ defined as

$$\varphi_\mu(X_1, \cdots, X_n) = a_1^\sigma X_1^2 + \cdots + a_n^\sigma X_n^2 - \mu u, \tag{6}$$

where $X^2$ denotes the vector $(x_1^2, \cdots, x_p^2)$ whenever $X = (x_1, \cdots, x_p)$. Clearly if we make $\varphi_\mu = (\varphi_{1\mu}, \cdots, \varphi_{p\mu})$, then for all $1 \le j \le p$ is satisfied

$$\varphi_{j\mu}(X_1, \cdots, X_n) = \varphi_{1\mu}(\sigma^{-j+1}(X_1), \cdots, \sigma^{-j+1}(X_n)),$$

that is, $F_\mu$ is an one-parameter family of circulant delay endomorphisms. Under certain hypothesis on the vectors $a_i$ $(i = 1, \cdots, n)$ we will show that for everyy $|\mu|$ large, either $F_\mu$ satisfies the hypothesis of theorem 1 or it is hyperbolic.

To state these conditions precisely we will first introduce a product on $\mathbb{R}^p$ associated with the operator $\sigma$. Given $a, b \in \mathbb{R}^p$ we define the product $a \odot b$ as the vector $(AB)^t(e_1)$ where $A = a^\sigma$ and $B = b^\sigma$. Obviously $(a \odot b)^\sigma = a^\sigma b^\sigma$; moreover, since the inverse of an invertible linear cellular automaton is also a linear cellular automaton, then a vector $a \in \mathbb{R}^p$ is invertible under $\odot$ if and only if $a^\sigma$ is invertible. In this case, $a^{-1} = (A^{-1})^t(e_1)$ is the inverse of $a$, where $A^{-1}$ is the inverse of $a^\sigma$. As $\mathcal{A}_p^\ell$ is a linear space and closed under composition, it follows that $\mathbb{R}^p$ endowed with $\odot$ becomes an algebra with identity $e = (1, 0, \cdots, 0)$. This algebra is commutative since $\sigma$ is self-adjoint.

In $\mathbb{R}^p$ we consider the norm $|a| = \max_{1 \le i \le p}\{|\alpha_i|\}$ with $a = (\alpha_1, \cdots, \alpha_p)$. If $\|a^\sigma\|$ denotes the operator norm associated with the previous norm of $\mathbb{R}^p$, then it is easy to see that $|a| \le \|a^\sigma\| \le p\,|a|$.

Some additional properties of the matrix $a^\sigma$ and $\odot$ are the following:

(a) It is well known, see [3], that the eigenvalues of $a^\sigma$ are

$$\Sigma_j(a) = \sum_{k=1}^{p} \alpha_k x_0^{j(k-1)}, \quad \text{with} \quad 0 \le j \le p-1,$$

where $x_0 = \exp(\frac{2\pi i}{p})$. An eigenvector associated to $\Sigma_j(a)$ with $0 \le j \le p-1$, is $(1, x_0^j, x_0^{2j}, \cdots, x_0^{(p-1)j})$. In particular the vector $u = (1, \cdots, 1) \in \mathbb{R}^p$ is an eigenvector of every $a^\sigma$, its eigenvalue $\Sigma_0(a)$ will be denoted by $\Sigma(a)$ throughout all this work.

(b) The function $a \to \Sigma(a)$ defines an algebra homomorphism from $\mathbb{R}^p$ to $\mathbb{R}$. Indeed, the fact that $\Sigma$ is linear is trivial; and to prove that it is multiplicative observe that

$$\Sigma(a \odot b)u = (a \odot b)^\sigma(u) = a^\sigma b^\sigma(u) = a^\sigma(\Sigma(b)u) = \Sigma(a)\Sigma(b)u.$$

(c) To obtain hyperbolicity for the endomorphisms $F_\mu$ with $\varphi_\mu$ given in (6) we need some expansivity property of the matrix $a^\sigma$. To do this we will introduce the concept of transversality of a vector $a \in \mathbb{R}^p$ and the relationship with the expansivity of the matrix $a^\sigma$.

Define the *transversality*, $\tau(a)$, of a vector $a$ in $\mathbb{R}^p$ as follows:

$$\tau(a) = \begin{cases} 0 & \text{if } a \text{ is noninvertible} \\ \frac{1}{|a||a^{-1}|} & \text{if } a \text{ is invertible} \end{cases}.$$

Observe that $\tau$ is a continuous function and invariant under scalar multiplication, that is $\tau(\lambda a) = \tau(a)$ for every $a \in \mathbb{R}^p$ and $\lambda \in \mathbb{R}$. The transversality of $a$ as defined here is equivalent to the transversality of the set of rows of $a^\sigma$, as defined in [8] (cf. definition 1). Moreover, $\tau(a)$ is related to the expansiveness of the linear transformation $a^\sigma$ as follows:

$$|a^\sigma(V)| \geq \frac{1}{\|(a^\sigma)^{-1}\|}|V| \geq \frac{1}{p\,|a^{-1}|}\,|V| = \frac{1}{p}\,\tau(a)\,|a|\,|V|.$$

(d) Define a partial order in $\mathbb{R}^p$ as follows: $a > b$ if each coordinate of $a - b$ is positive. We claim that:
*If $a \in \mathbb{R}^p$ is invertible under $\odot$ and $\Sigma(a) > 0$, then there exists $b \in \mathbb{R}^p$ invertible with $\Sigma(b) > 0$ such that $a \odot b^{-1} > 0$.*
Indeed, the mapping $c \to a \odot c$ is a linear isomorphism if $a$ is invertible. Since invertible elements are dense and the condition is open, it follows that there exists $c \in \mathbb{R}^p$ invertible such that $a \odot c > 0$. Take $b = c^{-1}$ and observe that by property (b), $\Sigma(b) = \Sigma(c^{-1}) = (\Sigma(c))^{-1}$. Then $a \odot c > 0$ and $\Sigma(a) > 0$ imply $\Sigma(c) > 0$.

**Theorem 2.** *Let $F_\mu$ be a one-parameter family of circulant delay endomorphisms as defined in equations (5) and (6).*

- *(i) If $\Sigma(a_j) > 0$ for every $1 \leq j \leq n$, then the $\omega$-limit set of any point $\widehat{X} \in (\mathbb{R}^p)^n$ is empty for every $\mu$ large and negative.*
- *(ii) Given $\epsilon > 0$, there exists $\lambda > 0$ such that if $\tau(a_1) > \epsilon$ and $a_1 > \lambda u$, then for every $\mu$ large and positive, $F_\mu$ has $\infty$ as an attractor and the nonwandering set of $F_\mu$ is an expanding Cantor set; moreover, its complementary set is the basin of attraction of $\infty$.*

There is an obvious symmetric statement when $\Sigma(a_j) < 0$ for $j = 2, \cdots, n$ and $a_1 < \lambda u$ with $\lambda < 0$; also is clear the corresponding translation for quadratic circulant difference equations:

$$X_{n+k} = a_1^\sigma X_k^2 + \cdots + a_n^\sigma X_{k+n-1}^2 - \mu u,\ k \geq 0. \tag{7}$$

# 3 Proofs of the Theorems

In example 2, let $g(x, y) = (ax^2 + by^2 + c, bx^2 + ay^2 + c)$, then it is clear that $F(x_1, y_1, x_2, y_2) = (x_2, y_2, \varphi(x_1, y_1, x_2, y_2))$ with

$$\varphi(x_1, y_1, x_2, y_2) = 2(x_2, y_2) - (x_1, y_1) + g(x_1, y_1)$$

is a circulant delay endomorphism. Moreover, if $a, b, c > 0$, the hypothesis of theorem 1 are satisfied; indeed the endomorphism $F$ has fixed points if and only if the system $\begin{cases} ax^2 + by^2 + c = 0 \\ bx^2 + ay^2 + c = 0 \end{cases}$ has a solution. For the proof of the theorem 1 in this case we observe that the function $L(x_1, y_1, x_2, y_2) = -x_1 - y_1 + x_2 + y_2$

satisfies $L \circ F - L > 0$, so the conclusion of the theorem follows. The proof of the theorem in the general case is based on the construction of a Lyapunov function $L$ such that $L \circ F - L > 0$, which clearly implies the conclusion of the theorem 1, cf. Lemma 2.1 in F. Bofill et al. [1].

*Proof of theorem 1.* Let $G : \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$G(t_1, \cdots, t_n) = \frac{1}{p} \sum_{i=1}^{p} \varphi_i(t_1 u, \cdots, t_n u),$$

where $u = (1, \cdots, 1) \in \mathbb{R}^p$.
*Claim:* The graph of $G$ does not intersect the diagonal of $\mathbb{R}^n \times \mathbb{R}$.
In the contrary case, there exists $t \in \mathbb{R}$ such that $G(t, \cdots, t) = t$. It is clear that $\varphi_i(tu, \cdots, tu) = \varphi_1(tu, \cdots, tu)$ for every $1 \leq i \leq p$. In this way $\varphi_1(tu, \cdots, tu) = t$, but this clearly implies that $(tu, \cdots, tu)$ is a fixed point of $F$, contradicting the hypothesis.

Observe that $G$ is a strictly convex function due to the strict convexity of $\sum_{i=1}^{p} \varphi_i$. Hence the claim implies that $G(t, \ldots, t) > t$ for every $t \in \mathbb{R}$. From the same claim it follows that there exists an hyperplane in $\mathbb{R}^n \times \mathbb{R}$ that contains the diagonal and does not intersect the graph of $G$. Furthermore, this hyperplane is the graph of a function $\pi(t_1, \cdots, t_n) = v_1 t_1 + \cdots + v_n t_n$, where $\sum_{i=1}^{n} v_i = 1$ because the hyperplane contains the diagonal of $\mathbb{R}^n \times \mathbb{R}$. As $G$ is strictly convex, we conclude that $G - \pi > 0$.

Let $L : (\mathbb{R}^p)^n \to \mathbb{R}$ be defined by

$$L(X_1, \cdots, X_n) = \sum_{i=1}^{n} (v_1 + \cdots + v_i) \Sigma(X_i) = \sum_{i=1}^{n-1} (v_1 + \cdots + v_i) \Sigma(X_i) + \Sigma(X_n),$$

where $\Sigma(X)$ denotes, as above, the sum of the coordinates of $X$. An easy calculation shows that

$$L(F(X)) - L(X) = \sum_{i=1}^{p} \varphi_i(X_1, \cdots, X_n) - \pi(\Sigma(X_1), \cdots, \Sigma(X_n)).$$

Consider the set $A = \{X \in (\mathbb{R}^p)^n : (LF - L)(X) \leq 0\}$. If we prove that $A$ is empty, then $L$ is a Lyapunov function globally defined and its orbital difference $\Delta L := LF - L$ is positive; this implies the theorem. Observe that $A$ is closed by continuity, is a convex set by the hypothesis on $\sum_{i=1}^{p} \varphi_i$, and $A$ is $\widetilde{\sigma}$-invariant; that is, $(X_1, \cdots, X_n) \in A$ implies that $(\sigma(X_1), \cdots, \sigma(X_n)) \in A$.

By contradiction, suppose that there exists $X = (X_1, \cdots, X_n) \in A$, then

$$\widetilde{X} = \frac{1}{p} \sum_{i=1}^{p} (\sigma^i(X_1), \cdots, \sigma^i(X_n)) = (t_1 u, \cdots, t_n u)$$

also belongs to $A$, by $\widetilde{\sigma}$ invariance and convexity of $A$; here $t_j = \frac{1}{p} \Sigma(X_j)$ for $j = 1, \cdots, n$. Now this fact implies that

$$G(t_1, \ldots, t_n) - \pi(t_1, \ldots, t_n) = \frac{1}{p}(L(F(\tilde{X})) - L(\tilde{X})) \leq 0,$$

contradicting $G - \pi > 0$. This proves the theorem. □

**Remark 1.** This theorem was proved for $n = 1$ in [4] and for $p = 1$ in [10].

To prove the theorem 2 we will need the following lemma:

**Lemma 1.** *Let $b$ and $c$ be vectors in $\mathbb{R}^p$.*

*(i) If $\Sigma(b) > 0$ and $\mu$ is a real number sufficiently large, then the equation $b^\sigma X^2 + c^\sigma X + \mu u = 0$ has no solution $X \in \mathbb{R}^p$.*

*(ii) If $b > 0$, then $b^\sigma X^2 + c^\sigma X + \mu u > 0$ for every $X \in \mathbb{R}^p$ and $\mu$ sufficiently large.*

*Proof.* (i) Recall that if $a \in \mathbb{R}^p$, then $a^\sigma(u) = \Sigma(a)u$ for $u = (1, \cdots, 1) \in \mathbb{R}^p$. So, it is enough to prove that $\Sigma(b^\sigma X^2 + c^\sigma X + \mu u) > 0$ for every $X \in \mathbb{R}^p$ and $\mu$ sufficiently large.

Observe that for every $X \in \mathbb{R}^p$ it holds

$$(b^\sigma X^2 + c^\sigma X + \mu u)^\sigma u = (\Sigma(b)\Sigma(X^2) + \Sigma(c)\Sigma(X) + p\mu)u,$$

and each of the entries of the right side vector is equal to

$$\sum_{k=1}^{p} (\Sigma(b)x_k^2 + \Sigma(c)x_k + \mu), \tag{8}$$

where, as above, $X = (x_1, \cdots, x_p)$ and $X^2 = (x_1^2, \cdots, x_p^2)$. Clearly each term of the sum (8) is positive for every $X \in \mathbb{R}^p$ and $\mu$ sufficiently large.
(ii) This part follows with the same arguments of part (i). □

*Proof of theorem 2.* To prove (i) we will show that if $\mu$ is sufficiently negative, then $F_\mu$ satisfies the hypothesis of theorem 1, that is, the function $\sum_{j=1}^p \varphi_{j\mu}$ is strictly convex and $F_\mu$ has no fixed points, then the assertion in part (i) follows.

It easy to see that

$$\sum_{j=1}^{p} \varphi_{j\mu}(X_1, \cdots, X_n) = \Sigma(a_1)\Sigma(X_1^2) + \cdots + \Sigma(a_n)\Sigma(X_n^2) - p\mu;$$

moreover, since $\Sigma(a_j) > 0$ for all $1 \le j \le p$, then the convexity of $\sum_{j=1}^p \varphi_{j\mu}$ follows obviously.

On the other hand, observe that $(X_1, \cdots, X_n) \in (\mathbb{R}^p)^n$ is a fixed point of $F_\mu$ if and only if $X_1 = X_2 = \cdots = X_n = X$ and $(\sum_{j=1}^n a_j^\sigma)X^2 - X - \mu u = 0$. Since $\sum_{j=1}^n a_j^\sigma = (\sum_{j=1}^n a_j)^\sigma$ and $\Sigma(\sum_{j=1}^n a_j) > 0$, then from part (i) of lemma 1 it follows that $F_\mu$ has no fixed points if $\mu$ is sufficiently negative.

To prove (ii) suppose $b_1, \cdots, b_{n-1} \in \mathbb{R}^p$ are given and define $L : (\mathbb{R}^p)^n \to \mathbb{R}^p$ by

$$L(X_1, \cdots, X_n) = X_n + b_1^\sigma X_1^2 + \cdots + b_{n-1}^\sigma X_{n-1}^2. \tag{9}$$

Denote by $\widehat{X} = (X_1, \cdots, X_n) \in (\mathbb{R}^p)^n$. We will show that the orbital difference $\Delta L(\widehat{X}) = L(F_\mu(\widehat{X})) - L(\widehat{X})$ is positive whenever $L(\widehat{X}) > \rho\mu u$ and $\mu$ is large enough, where $\rho$ is a small positive number to be determined.

A straightforward computation shows that

$$\Delta L(\widehat{X}) = (a_1^\sigma - b_1^\sigma)X_1^2 + \sum_{j=2}^{n-1}(a_j^\sigma + b_{j-1}^\sigma - b_j^\sigma)X_j^2 + (a_n^\sigma + b_{n-1}^\sigma)X_n^2 - X_n - \mu u.$$

Now suppose that $L(\widehat{X}) > \rho\mu u$. Let $\theta > \rho^{-1}$ and choose $b_1 > 0$ such that $a_1 > (\theta+1)b_1$. Substituting in equation (9), this implies that

$$(a_1^\sigma - b_1^\sigma)X_1^2 > \theta b_1^\sigma X_1^2 > \theta\rho\mu u - \theta\sum_{j=2}^{n-1} b_j^\sigma X_j^2 - \theta X_n.$$

So,

$$\Delta L(\widehat{X}) > \sum_{j=2}^{n-1} d_j^\sigma X_j^2 + (a_n^\sigma - b_{n-1}^\sigma)X_n^2 - (1+\theta)X_n + (\rho\theta - 1)\mu u,$$

where $d_j = a_j + b_{j-1} - (1+\theta)b_j$ for $j = 2, \cdots, n-1$.
*Claim:* The vectors $b_1, \cdots, b_{n-1}$ can be chosen in such a way that $b_i > 0$ for all $1 \le i \le n-1$, $a_n + b_{n-1} > 0$ and $d_j > 0$ for every $2 \le j \le n-1$.

If we prove this claim and the fact that was assumed above ($a_1 > (1+\theta)b_1$), then lemma 1 part (ii) implies that $\Delta L(\widehat{X})$ is positive when $L(\widehat{X}) > \rho\mu u$ and $\mu$ sufficiently large.

Begin taking any $b_{n-1} > 0$ such that $b_{n-1} > -a_n$. Then observe that for $2 \le j \le n-1$, the vector $d_j$ is positive if and only if $b_{j-1} > -a_j + (\theta+1)b_j$ for all $2 \le j \le n-1$. Therefore a sequence $b_j$ can be chosen by recurrence in such a way to satisfy all the conditions $b_j > 0$ and $d_j > 0$ for $2 \le j \le n-1$. This shows that $b_1$ has to verify $b_1 > -\sum_{j=0}^{n-2}(\theta+1)^j a_{j+2}$. We conclude that all the choices are possible if $a_1 > (\theta+1)b_1 > -\sum_{j=2}^{n}(\theta+1)^{j-1}a_j$, which can be done by hypothesis provided $\lambda$ is sufficiently large. This finishes the proof of the claim.

Take $b_1, \cdots, b_{n-1}$ as in the claim. Observe that any point in the preimage, under $F_\mu$, of $\{\widehat{X} : L(\widehat{X}) > \rho\mu u\}$ is also attracted to $\infty$ for every $\mu$ large enough, and this preimage is given by

$$\xi_\mu = \{\widehat{X} = (X_1, \cdots, X_n) : a_1^\sigma X_1^2 + \sum_{j=2}^{n}(a_j^\sigma + b_{j-1}^\sigma)X_j^2 > (\rho+1)\mu u\}.$$

Since $(\mathbb{R}^p)^n \setminus \xi_\mu$ is a compact set, it follows that $F_\mu$ has $\infty$ as an attractor when $\mu$ is sufficiently large. In this case, denote by $B_\infty(\mu)$ the basin of attraction of $\infty$.

Now suppose $\widehat{X} \notin B_\infty(\mu)$ and $\mu$ large enough. As $\widehat{X} \notin \xi_\mu$, $a_1^\sigma X_1^2 \le (\rho+1)\mu u$, which implies that $abs(X_1) \le \sqrt{\frac{(\rho+1)\mu}{|a_1|}}u$, where $abs(X) = (|x_1|, \cdots, |x_p|)$

whenever $X = (x_1, \cdots, x_p) \in \mathbb{R}^p$. On the other hand, as $F_\mu^j(\widehat{X}) \notin B_\infty(\mu)$ for all $j \geq 0$, then in particular for $2 \leq j \leq n$ it holds

$$abs(X_j) \leq \sqrt{\frac{(\rho+1)\mu}{|a_1|}}\, u, \tag{10}$$

and

$$abs(\varphi_\mu(\widehat{X})) \leq \sqrt{\frac{(\rho+1)\mu}{|a_1|}}\, u.$$

From this last inequality and equation (6) it follows that

$$\sum_{j=1}^{n} a_j^\sigma X_j^2 - \mu.u \geq -\sqrt{\frac{(\rho+1)\mu}{|a_1|}}\, u.$$

Therefore,

$$a_1^\sigma X_1^2 \geq \left(\mu - \sqrt{\frac{(\rho+1)\mu}{|a_1|}}\right) u - \sum_{j=2}^{n} a_j^\sigma X_j^2.$$

Using the upper bounds obtained in equation (10), it follows that $a_1^\sigma X_1^2 > R\mu u$, where

$$R = 1 - (\rho+1)\sum_{j=2}^{n} \frac{\Sigma(abs(a_j))}{|a_1|} - \sqrt{\frac{\rho+1}{\mu|a_1|}}.$$

Clearly if the positive number $\lambda$ is taken sufficiently large, then $R > 1-\rho$ for all $\mu$ large enough. Putting this together with the lower bound obtained above, it comes that

$$\mu(u - \rho u) \leq a_1^\sigma X_1^2 \leq \mu(u + \rho u).$$

Hence, there exists $\delta \in \mathbb{R}^p$ with $abs(\delta) \leq \rho$ such that $a_1^\sigma X_1^2 = \mu(u+\delta)$. In this way $X_1^2 = \mu(a_1^\sigma)^{-1}(u+\delta)$. By property (3) of $\tau(a_1)$ it follows that

$$\left|(a_1^\sigma)^{-1}(\delta)\right| \leq \frac{|\delta|\, p}{|a_1|\tau(a_1)} \leq \frac{\rho\, p}{|a_1|\tau(a_1)}.$$

Then from the fact that $(a_1^\sigma)^{-1}(u) = (\Sigma(a_1))^{-1} u$, we have that $X_1^2 \geq \mu\kappa u$, where $\kappa = \frac{1}{\Sigma(a_1)} - \frac{\rho\, p}{|a_1|\tau(a_1)}$; that is positive if $\rho > 0$ is taken such that $\rho < \frac{\epsilon}{p^2} \leq \frac{|a_1|\tau(a_1)}{\Sigma(a_1)}$. This proves that $abs(X_1) \geq \sqrt{\kappa\mu}u$ whenever $\widehat{X} \notin B_\infty$. In particular this implies that the set of critical points of $F_\mu$ is contained in $B_\infty(\mu)$. Recall that the set of critical points of a mapping $G : \mathbb{R}^m \to \mathbb{R}^m$ is the set of those $X \in \mathbb{R}^m$ such that the differential of $G$ at $X$, $(DG)_X$, is noninvertible. It is easy to see that for $F_\mu$ this set coincides with the set of points $\widehat{X} = (X_1, \cdots, X_n) \in (\mathbb{R}^p)^n$ for which at least one of the coordinates of $X_1$ is zero.

To finish with the proof of the theorem, we will prove that if $\mu$ is large enough, then $(DF_\mu)_{\widehat{X}}$ is an expanding map in the complementary set of $B_\infty(\mu)$.

In general, a sufficient condition for a mapping $G : \mathbb{R}^m \to \mathbb{R}^m$ to be expanding in a compact and invariant set $\Lambda$ is the following:

- for every $X$ in $\Lambda$ and every vector $V$, $|(DG)_X(V)| \geq |V|$, and
- there exists a constant $\gamma > 1$ and a positive integer $k$ such that for every point $X \in \Lambda$ and any vector $V$ there exists an integer $q$, $1 \leq q \leq k$ such that $|(DG^q)_X(V)| \geq \gamma\,|V|$.

Indeed, it is clear that this condition implies that for every $X \in \Lambda$ and any vector $V$ it holds that $|(DG^k)_X(V)| \geq \gamma\,|V|$; this implies that $G$ is expanding on $\Lambda$.

To prove the condition above, observe first that if $\widehat{V} = (V_1, \cdots, V_n) \in \mathbb{R}^p$, $\widehat{X} = (X_1, \cdots, X_n) \notin B_\infty(\mu)$ and $\mu$ sufficiently large, then

$$(DF_\mu)_{\widehat{X}}(\widehat{V}) = (V_2, \cdots, V_n, 2a_1^\sigma \tilde{X}_1 V_1 + \cdots + 2a_n^\sigma \tilde{X}_n V_n),$$

where $\tilde{X}_j$ is the $p \times p$ diagonal matrix whose diagonal is the vector $X_j$. Observe that

$$|\tilde{X}_1 V_1| \geq \sqrt{\kappa\mu}\,|V_1| \;\text{ and }\; |\tilde{X}_j V_j| \leq \sqrt{\frac{(\rho+1)\mu}{|a_1|}}\,|V_j|, \;\text{ for all }\; 2 \leq j \leq n. \tag{11}$$

For $j = 1, \cdots, n$, let $T_j = |2a_j^\sigma \tilde{X}_j V_j|$. It is clear that from property (c) of the function $\tau$ and (11) it follows that

$$T_1 \geq \frac{2}{p}\,\tau(a_1)\,|a_1|\,|\tilde{X}_1 V_1| \geq \frac{2}{p}\,\epsilon\,\lambda\,\sqrt{\kappa\mu}\,|V_1|; \tag{12}$$

and for each $j = 2, \cdots, n$

$$T_j \leq 2\,\|a_j^\sigma\|\,|\tilde{X}_j V_j| \leq 2\,|a_j|\,\sqrt{\frac{(\rho+1)\mu}{|a_1|}}\,|V_j|. \tag{13}$$

Now take $\widehat{V} = (V_1, \cdots, V_n) \in (\mathbb{R}^p)^n$ with $|\widehat{V}| = 1$. If $|V_1| = 1$, it follows that

$$\begin{aligned} |(DF_\mu)_{\widehat{X}}(\widehat{V})| &\geq \left|2a_1^\sigma \tilde{X}_1 V_1 + \cdots + 2a_n^\sigma \tilde{X}_n V_n\right| \\ &\geq |2a_1^\sigma \tilde{X}_1 V_1| \left(1 - \frac{\sum_{j=2}^n |2a_j^\sigma \tilde{X}_j V_j|}{|2a_1^\sigma \tilde{X}_1 V_1|}\right) = T_1 \left(1 - \frac{\sum_{j=2}^n T_j}{T_1}\right). \end{aligned}$$

From (12) and (13), if $\lambda$ is taken large with respect to $|a_j|$ for $j = 2, \cdots, n$, then $|(DF_\mu)_{\widehat{X}}(\widehat{V})| > 2$. On the other hand, if $|V_1| < 1$, then there exists $j = 2, \cdots, n$ such that $|V_j| = 1$, so $|(DF_\mu)_{\widehat{X}}(\widehat{V})| \geq 1$. In this case, if the norm of the last $\mathbb{R}^p$-entry of $(DF_\mu)_{\widehat{X}}(\widehat{V})$ is less than 2, then take $|(DF_\mu^2)_{\widehat{X}}(\widehat{V})|$ and argue as above. In this way, by recurrence we will find that there exists an integer $q \leq n$ such that $|(DF_\mu^q)_{\widehat{X}}(\widehat{V})| \geq 2$. This proves that the sufficient condition for expansiveness holds for $F_\mu$ in the complementary set of $B_\infty(\mu)$ and proves the theorem. □

**Remark 2.** The proof of this theorem is almost the same when the mapping $\varphi$ has also linear parts because the fundamental hypothesis is the relationship between the quadratic parts. See also the proof of theorem 2 in [9].

**Example 3.** In example 2, taking $g(x, y) = \alpha x^2 + \beta y^2$ with $\alpha, \beta > 0$ it comes that $a_1 = (\alpha, \beta)$ and $a_2(0, 0)$, so the hypothesis of the theorem are satisfied.

In example 1, if $f(t) = t^2$, then $a_1 = (0, 1)$, $a_2 = a_3 = (0, 0)$ and the hypothesis of the theorem hold. In this case the Cantor set has four symbols.

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